INTRINSIC GEODESY

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OHIO STATE UNIVERSITY RESEARCH FOUNDATION

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Ohio State University Research Foundation

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Wright Air Development Center
Air Research and Development Command
United States Air Force
Wright-Patterson Air Force Base, Ohio

FOREWORD

This report was prepared by the Mapping and Charting Research Laboratory of the Ohio State University Research Foundation under USAF Contract No. AF 18(600)-90. The contract is administered by the Mapping and Charting Branch of the Photographic Reconnaissance Laboratory, Weapons Components Division, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio. Mr. A.S. Rosing is Project Engineer on the project applicable to the subject of this report.

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This report was originally initiated at the Ohio State University Research Foundation as OSURF Technical Paper No. 159.

ABSTRACT

In the present Technical Paper are given the outlines of <u>Intrinsic Geodesy</u>, a theory by which the gravity field of the Earth may be analyzed by using only the results of actual measurements, and without any additional assumption on the structure of the field itself. Any geometrical or mechanical element that would not have a physical reality is therefore excluded.

After having specified the aims of the problem and the means by which it may be solved (Introduction and Section I), short summaries on the elements of vector and tensor calculus are given (Section II and III).
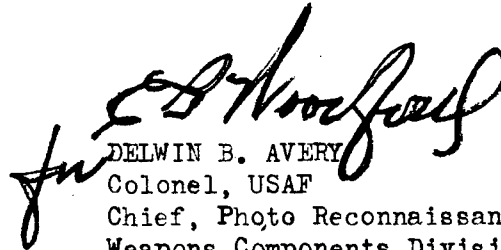
By the help of this calculus, the study of the gravity field of the Earth is performed, using coordinates innate in the field itself (Section IV.), and several applications to practical problems are shown.

Somigliana's field is finally suggested as the most appropriate for generalizing the usual ellipsoidal Geodesy in a three dimensional scheme (Section V).

PUBLICATION REVIEW

The publication of this report does not constitute approval by the Air Force of the findings or conclusions contained therein. It is published only for the exchange and stimulation of ideas.

FOR THE COMMANDING GENERAL:

DELWIN B. AVERY
Colonel, USAF
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INTRINSIC GEODESY

INTRODUCTION

As for any other Science, the development of Geodesy has been greatly influenced by the historical process; in our Science it led us from the primitive idea of a flat Earth to the actual concept on the shape and size of our planet, and its internal structure.

In early times it was discovered that the Earth could be approximated by a sphere, and already the Greeks and the Arabs tried to determine its radius. Only much later the theoretical works of Galileo, Newton, Huygens, and Clairaut led to the ellipsoidal conception of the shape of the Earth that was definitely proved by the celebrated expeditions of the Academy of France in Lapland and in Peru as early as the second quarter of the 18th century.

As geodetical work progressed and methods and instruments were greatly improved, it was soon discovered that even the ellipsoid could only be regarded as a further approximation for the shape of the Earth; and the idea of the Geoid (Listing, 1873) became essential.

Notwithstanding, and since the times of the expeditions of the Academy of France, the ellipsoid was still the dominating feature in Geodesy; not only it was and is still universally used in practical Geodesy and surveying, but it also affects most of theoretical research.

It may be pointed out at once that the function of the ellipsoid in Geodesy may be regarded as two-fold. Sometimes it is accepted as a good approximation to the Geoid, and therefore confused with it. That

happens mostly in problems of practical Geodesy, such as triangulation. Sometimes it is simply regarded as a standard surface of reference, with respect to which the "anomalies" or the "deflections" of the Geoid are referred. But several times its use is promiscuous, and it is not easy to discern what in the results is due to the actual structure of the Earth's gravity field, and what is affected by the ellipsoidal assumption.

It seems, therefore, of some interest to establish a theory leading to the study of geodetical and geophysical problems, abstracting from any hypothesis on the shape of any level surface in the gravity field of the Earth, and leaning, therefore, only on the data of actual measurements.

Moreover, since the time Geodesy did no longer confine to the study of geometrical problems on the shape and the size of one particular level surface, and started studying the dynamical problems of gravity, its concern became more ample, and its aim more precise. <u>Geodesy is the Science devoted to the study of Earth's gravity field.</u> The study of the geoid became a particular problem in this general definition.

Our study must, therefore, not be confined to one particular level surface, but must extend to the third dimension in such a way as to link in an overall picture both the geometrical and the dynamical aspects of geodetic problems.

The difficulty to overcome at first lies in the fact that Operative Geodesy must always refer to coordinates suitable for carrying out actual measurements; if independence from any exterior, arbitrary system of reference (like the ellipsoid, or cartesian baricentric axes, etc.) is

claimed, then the coordinates to be used must be innate in the field itself. We will therefore call such coordinates _intrinsic_, and our study _Intrinsic Geodesy_.

Also, in such study we will only consider entities having a physical reality and quantities susceptible to undergo actual direct or indirect measurements which Operative Geodesy is already able to perform, or would be able to perform if appropriate theoretically conceivable instruments should be available. A standard surface of reference (like the ellipsoid) or a standard field of reference (like Somigliana's field) should be sometimes used for testing the actual gravitational field of the Earth; but the particular assumption of such standards would never affect the general results.

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We already remarked that the ellipsoid may be correctly regarded as a standard reference surface in two-dimensional Geodesy for referring the discrepancies between it and the Geoid. The most natural extension of such concept in three dimensions is given by Pizzetti-Somigliana's field of gravity, which was accepted in 1930 by the Stockholm meeting of the International Association of Geodesy on the suggestion of Dr. Lambert, Prof. Cassinis, and Prof. Somigliana himself. Somigliana's field is fully determined, on account of the celebrated Stoke's theorem, by giving the ellipsoid as one of its level surfaces (usually Hayford's International Ellipsoid), by the well-known angular speed of rotation of the Earth, and by the equatorial value of gravity on the ellipsoid itself (usually g_e = 978.049 gal). On Somigliana's field is based the international formula for normal gravity, the coefficients of which have been determined by Prof. Silva and Cassinis.

The principles of Intrinsic Geodesy may be very easily particularized for Somigliana's field, and the results used as a standard of reference or as a good first approximation in whatever geodetic problem the third dimension is involved; and that, in a perfectly similar way to that in which the ellipsoid is used in two-dimensional Geodesy.

Modern techniques tend indeed each day more to extend geodetic exploration in space, as is shown, e.g., by the impressive fact that in classical Geodesy only measurements along geodesics and lines on a surface (Geoid or Ellipsoid) were considered, and that at present times optical and electronic devices allow, instead, the measurement of distances along lines in space, which may be very closely confused with geodesics in space, i.e., straight lines.

INTRINSIC GEODESY

SECTION I

PRELIMINARY

1.1 Object of Geodesy

We will first point out the object of Geodesy, as it is referred to in the present lectures: it is the study of the gravity field of the Earth.

As it is very well known, the field of gravity (= gravitation + centrifugal forces), in the same way as all conservative (non-dissipative) fields, may be thoroughly described by means of a single space function, the potential function as introduced first by Laplace, in all points exterior to the volume occupied by attracting masses. The object of Geodesy may be also formulated, therefore, as the study of the potential function, or also of the shape and size of its level surfaces (equipotential surfaces), and of their orthogonal trajectories (lines of force).

It is also very well known that the study of a potential field involves geometrical and mechanical aspects; for instance, the problem of determining the size and shape of each of two near equipotential surfaces is of geometrical nature, but that of specifying the value of the corresponding potential difference (the work to be done by transferring the unit mass from the one surface to the other) is a mechanical one, which involves the concept of "force" (gravity).

We may add that in this scheme the aim of Topography (Surveying) is merely the study of the shape of the actual physical surface of the Earth, referred to the above gravity field, or in a more simple way, to a particular level surface of the same.

1.2   Two Aspects of the Gravity Field of the Earth

The study of a potential field may be considered from two completely different points of view. The first (also historically) is based on Newton's law of gravitation and on elementary laws of centrifugal forces; i.e., it is based on the consideration of the "cause" of gravity itself. It admits, therefore, an interaction between attracting masses and is called, therefore, by Hermann Weyl [1] the "Fernwirkungsgesetz" (principle of action in distance). Its fundamental equations are, as it is easily seen,

$$W = f \int_{(V)} \frac{\mu \, dV}{r} + \frac{\omega^2 R^2}{2} \; ; \; \bar{g} = \text{grad } W = \nabla W \qquad (I-1)$$

where W is the potential of gravity in a point P of the space, due to the attracting masses of density μ distributed in the volume (V), and to a rotation of angular velocity ω; f is Newton's constant of gravitation, r the distance between P and any point whatever of (V), R the distance of P from the axis of rotation. Moreover $\bar{g}$ is the vector of gravity (the gradient of the potential W). (Here and in the following, overlined letters like $\bar{g}$ will stand for vectors.)

We may also refer to the above principle as an integral principle, owing to the fact that the knowledge of the whole distribution of attracting masses is required for computing the integral in (I-1)

It is the principle followed in the classical works of Clairaut, Helmert, Bruns, etc.

The second principle may be referred to, on the other hand, as a differential principle, no longer involving the knowledge of the distribution of attracting masses. Weyl refers to this principle as to the

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[1] Hermann Weyl, Raum, Zeit, Materie - Vorlesungen über allgemeine Relativitätstheorie, (Berlin, 1921). (Also in French.)

"Nahewirkungsgesetz" (principle of action in proximity). This denomination is due to the fact that only the knowledge of the field's structure in the immediate neighborhood of a given point is wanted.

In other words, the gravity field may be studied here as an "effect", independently from its "cause" (the distribution of masses); and it is well known that the gravity vector $\bar{g}$ must satisfy the following two fundamental (differential) equations:

$$\text{rot } \bar{g} = \nabla \times \bar{g} = 0 \; ; \quad \text{div } \bar{g} = \nabla \cdot \bar{g} = 2\omega^2 - 4\pi f\mu \; . \quad (\text{I-2})$$

The first equation affirms that the field is conservative, and the second that a rotation is superposed, and that in general in the point P where $\bar{g}$ is considered, masses of density μ are present (the last term disappears in vacuo, and to the greatest approximation in free air).

The differential principle has been followed by Stokes primarily, and his theorems are of utmost importance in Geodesy; and later on by Pizzetti, Somigliana, and many other prominent geodesists and physicists.

The outstanding importance of the second principle lies in the fact that it is completely independent of the distribution of attracting masses, and it may therefore be applied with great advantage in Geodesy, where the distribution of density in the interior of the Earth is unknown.

1.3   Geodetic Position of the Problem

We have so far outlined the problem of gravity from the standpoint of mathematical physics. Our task is now to examine it from the point of view of Geodesy, i.e., of experimental science, and to specify suitable methods for the practical exploration and numerical definition of the field itself.

On account of its independence of the knowledge of the Earth's interior, in the following we will confine ourselves to considering only the second principle stated above.

Our problem may be simply formulated as the problem of studying, in a given domain, the vector $\bar{g}$ of gravity, and other vectors connected with it in a very simple way. Such vectors are, for instance, the unit vector $\bar{n}$ of $\bar{g}$ ($\bar{g} = -g\,\bar{n}$), where g (a scalar) is the intensity of gravity, or "gravity"; the derivatives

$$\frac{d\bar{g}}{dS}, \frac{d^2\bar{g}}{dS^2}, \ldots ; \frac{d\bar{n}}{dS}, \frac{d^2\bar{n}}{dS^2}, \ldots$$

of the same vectors in given directions, the vector

$$\bar{G} = \text{grad } g \qquad\qquad (I\text{-}3)$$

(the gradient of gravity), etc., and the only difficulty is to find a way of defining numerically the above absolute (not connected with a particular reference system) entities.

A vector itself is not a measurable entity. Only its components (or projections) with respect to a given system of three cartesian axes are numerically definable and therefore apt to be measured by means of suitable instruments and methods, and to be used in numerical (or algebraic) computations. We are therefore forced to choose, in Geodesy, reference systems of three axes (orthogonal or not, unit or not, but not coplanar) in respect to which to consider the components (or the projections) of our vectors, and it is a question of fundamental importance for the practical use in Geodesy to have in mind that:

(1) the axes of reference must have an immediate physical reality;

(2) no additional hypothesis must be made in connection with the gravity field itself;

(3) the components (or projections) of all vectors to be considered may be easily measured.

The three conditions above exclude therefore, for instance, an absolute cartesian system of reference for the whole Earth, which would not have a physical reality and would make it therefore impossible to measure coordinates and components.

So far as the possibility of having a single system of reference for the whole Earth, satisfying the above conditions, is obviously to be excluded, we are obliged to choose suitable local reference systems, one for each point of the field in the region to be considered.

As a first solution, reference systems of this kind would be given by orthogonal unit systems of axes in any point, one axis being directed towards the Zenith, a second towards the North, and a third towards the East. Such systems are often used in local geodetic surveys of limited extension, and we will refer to them as local astronomical systems. Each system satisfies obviously the three fundamental conditions stated above and has, therefore, an intrinsic definition.

But we must always have in mind that the extension to a wider domain, as is often considered in Geodesy, requires the different systems of reference to be connected each to the other in such a way as to make it possible and easy to transfer all mathematical properties from one system to the next, or also to compare the components (or projections) of the same entity referred to different points in space.

We will see that the local astronomical systems are not convenient for such purposes, in spite of their very simple definition, and this is due to the fact that it is not easy to connect them each to another. We will see, on the contrary, that another system is much more suitable for the same purpose, though its definition is a little more complicated than the foregoing.

1.4   Object of Intrinsic Geodesy

The task of our study, Intrinsic Geodesy, may be: we must define in any point of our gravity field a system of three non-coplanar axes, each satisfying the conditions stated in 1.3 ,. and give a rule as simple as possible for connecting each system to the next one.

If we are able to do this, we will also be able to develop every geodetic research on an actual basis without referring to any arbitrary hypothesis (like that of the spheroid or the ellipsoid), nor to any system of reference not accessible to our measurements.

An easy way to reach this result, as we will see in more detail, is to establish in the whole gravity field an intrinsic system of curvilinear coordinates, i.e., coordinates whose definition is implicit in the field itself and do not need therefore any additional assumption. We will see that the most suitable coordinates to be used for this purpose are those defined by the following coordinate surfaces:

(1) the family of equipotential surfaces themselves; we attribute to each surface as parametric coordinate the corresponding value of the potential;

(2) the family of the surfaces connecting all points in space having the same astronomical latitude Φ; Φ is at the same time the parametric coordinate connected with this family;

(3) the family of the surfaces connecting all points in space having the same astronomical longitude λ; λ is at the same time the parametric coordinate connected with this family.

The coordinate lines are therefore:

(1) the isozenithal lines (intersection of Φ and λ surfaces), connecting in space all points having the same Zenith (same Φ and λ);

(2) the geodetic meridians (intersection of W and λ surfaces);

(3) the geodetic parallels (intersection of W and Φ surfaces).

All entities so defined have a physical reality; the equipotential surfaces can be materialized by a level; astronomical latitudes and longitudes may be also easily measured; the difference of potential W is, with abstraction of an insignificant numerical factor, the dynamic difference of height.

Attention is recalled to the fact that geodetic meridians and parallels should not be confounded with North and East Lines, i.e., the lines whose tangents are directed in each point towards the North or East, and also that geodetic parallels and meridians do not cut in general at right angles.

As it may be easily seen, the system of coordinates so defined is intrinsic; in fact, we did not use any additional hypothesis, or any exterior element other than the axis of the World to which astronomical latitudes and longitudes are referred. Moreover, the direction in space of the axis has a physical reality and may be used for experimental measures.

The coordinate lines so defined give us now the possibility of fixing the position of any point in space by means of its three parametric coordinates Φ, λ, and W. And moreover, the coordinate lines defined above enable us to draw at each point a system of three fundamental vectors tangent to the lines themselves. Furthermore, as we will see in more detail afterwards, the same coordinate lines will allow us to connect the systems of fundamental vectors in neighboring points and enable us, therefore, to compare vector entities at points far apart.

SECTION II

VECTOR CALCULUS (ELEMENTS AND NOTATIONS)

2.1 Preliminary

In the following we shall consider absolute entities, like points, vectors, and homographies, which all are independent from any system of reference; and, on the other hand, relative entities, like coordinates, components, and projections, which all depend not only upon the absolute entity they represent, but also upon a reference system of coordinates.

The absolute entities are studied by the Vector Calculus, and the relative ones by Cartesian Geometry in the elementary case of rectilinear reference systems of coordinates, and by Tensor Calculus in the case of general curvilinear coordinates. (See reference 4, 5, 6, 11.)

The difference of two points $P_2 - P_1 = \bar{a}$ defines a vector; we may write, therefore, also $P_2 = P_1 + \bar{a}$. From this point of view, $\bar{a}$ may be also regarded as an operator leading from P_1 to P_2.

A vector is completely defined by a direction (of the straight line joining P_1 and P_2), a sense on it (from P_1 to P_2), and a magnitude, or modulus, or length (of the segment P_1P_2). A vector may be represented by an arrow. All vectors specified by the same three qualities stated above are said to be equipollent. The point from which the vector starts is immaterial.

We will always use an overlined letter for indicating vectors. The same not overlined letter will represent the modulus (length) of the vector (a scalar).

A unit vector is sometimes called a versor; so, for the instance, $\bar{u} = \frac{\bar{a}}{a}$ is a unit vector; we have therefore $\bar{a} = a \, \bar{u}$. A vector may also be defined by its modulus and its unit vector.

Let us have two infinitesimally near points P and P', and put P' - P = dP; dP is therefore an infinitesimal vector, and we may say that the differential of a point is a vector. If ds be the length PP', then $\bar{t} = \frac{dP}{ds}$ is a unit vector. If P and P' are two infinitesimally near points of a curve, then $\bar{t}$ is the tangent unit vector to the curve in P.

We assume that the operations of sum and subtraction, and multiplication by a scalar of vectors are well known.

Scalar product of two vectors $\bar{a}$ and $\bar{b}$ is indicated by $\bar{a}.\bar{b}$ (Italian notation, $\bar{a} \times \bar{b}$) (read $\bar{a}$ dot $\bar{b}$, or also $\bar{a}$ scalar $\bar{b}$) and gives a scalar (a pure number). If θ be the angle between the (positive directions of the) two vectors, then we have
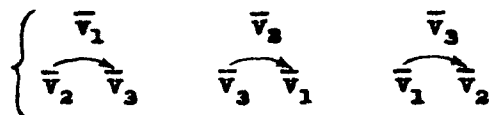
$$\bar{a}.\bar{b} = a \, b \, \cos \theta$$

Scalar or dot product is distributive and commutative.

Condition for orthogonality of two vectors: $\bar{a}.\bar{b} = 0$

Vector or cross product of two vectors is indicated by $\bar{a} \times \bar{b}$ (Italian notation, $\bar{a} \wedge \bar{b}$) (read $\bar{a}$ cross $\bar{b}$, or also $\bar{a}$ vector $\bar{b}$). For full definition of vector product a preliminary orientation of space is necessary. If we are given three (non-coplanar) vectors $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, (in the given order), such orientation is fixed by the rule that an observer situated along one of the three vectors would see the other two (considered in cyclic order), the one at his left, the second at his right, following one of the two schemes:

I. scheme $\left\{\begin{array}{ccc} \overline{v}_1 \\ \overline{v}_2 \quad \overline{v}_3 \end{array} \quad \begin{array}{c} \overline{v}_2 \\ \overline{v}_3 \quad \overline{v}_1 \end{array} \quad \begin{array}{c} \overline{v}_3 \\ \overline{v}_1 \quad \overline{v}_2 \end{array}\right.$

II. scheme $\left\{\begin{array}{ccc} \overline{v}_1 \\ \overline{v}_3 \quad \overline{v}_2 \end{array} \quad \begin{array}{c} \overline{v}_2 \\ \overline{v}_1 \quad \overline{v}_3 \end{array} \quad \begin{array}{c} \overline{v}_3 \\ \overline{v}_2 \quad \overline{v}_1 \end{array}\right.$
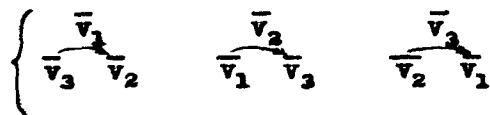
In the first case we will say that the orientation of our space is dextrorsum, or clockwise, whereas in the second case it is sinistrorsum, or anticlockwise. We will assume for our space clockwise orientation.

Vector or cross product of two vectors $\overline{a}$ and $\overline{b}$ is a new vector $\overline{c}$ the direction of which is perpendicular to both the directions of $\overline{a}$ and $\overline{b}$. Its sense is such that $(\overline{a}, \overline{b}, \overline{c})$ be a positive trihedron (dextrorsum in our case), and its modulus ab sin θ.

Vector product is **distributive,** but not commutative:

$$\overline{a} \times \overline{b} = - \overline{b} \times \overline{a}$$

Condition of parallelism of two vectors: $\overline{a} \times \overline{b} = 0$.

Triple product, or mixed product, or box product of three vectors $\overline{a}$, $\overline{b}$, $\overline{c}$ is the scalar

$$V = [\overline{a} \; \overline{b} \; \overline{c}] = \overline{a} \times \overline{b} . \overline{c} = \overline{c} \times \overline{a} . \overline{b} = \overline{b} \times \overline{c} . \overline{a} =$$
$$= \overline{a} . \overline{b} \times \overline{c} = \overline{c} . \overline{a} \times \overline{b} = \overline{b} . \overline{c} \times \overline{a} = \quad\quad (\text{II-1})$$
$$= -\overline{b} \times \overline{a} . \overline{c} = \ldots\ldots \;.$$

We give here some useful elementary formulae:

$$(\bar{a} \times \bar{b}) \times \bar{c} = \bar{a}.\bar{c}\,\bar{b} - \bar{b}.\bar{c}\,\bar{a},$$

$$\bar{a} \times (\bar{b} \times \bar{c}) = \bar{a}.\bar{c}\,\bar{b} - \bar{a}.\bar{b}\,\bar{c},$$

$$(\bar{a} \times \bar{b}) \times \bar{c} + (\bar{b} \times \bar{c}) \times \bar{a} + (\bar{c} \times \bar{a}) \times \bar{b} = 0,$$

$$(\bar{a} \times \bar{b}).(\bar{c} \times \bar{d}) = \begin{vmatrix} \bar{a}.\bar{c} & \bar{a}.\bar{d} \\ \bar{b}.\bar{c} & \bar{b}.\bar{d} \end{vmatrix},$$

$$[\bar{a}\,\bar{b}\,\bar{c}]\,[\bar{u}\,\bar{v}\,\bar{w}] = \begin{vmatrix} \bar{u}.\bar{a} & \bar{u}.\bar{b} & \bar{u}.\bar{c} \\ \bar{v}.\bar{a} & \bar{v}.\bar{b} & \bar{v}.\bar{c} \\ \bar{w}.\bar{a} & \bar{w}.\bar{b} & \bar{w}.\bar{c} \end{vmatrix}, \tag{II-2}$$

$$(\bar{a} \times \bar{b}) \times (\bar{c} \times \bar{d}) = [\bar{a}\,\bar{c}\,\bar{d}]\bar{b} - [\bar{b}\,\bar{c}\,\bar{d}]\bar{a} =$$

$$= [\bar{a}\,\bar{b}\,\bar{d}]\bar{c} - [\bar{a}\,\bar{b}\,\bar{c}]\bar{d},$$

$$\bar{a}\,[\bar{b}\,\bar{c}\,\bar{d}] - \bar{b}\,[\bar{c}\,\bar{d}\,\bar{a}] + \bar{c}\,[\bar{d}\,\bar{a}\,\bar{b}] - \bar{d}\,[\bar{a}\,\bar{b}\,\bar{c}] = 0,$$

$$\bar{d} = \frac{[\bar{d}\,\bar{b}\,\bar{c}]\,\bar{a} + [\bar{a}\,\bar{d}\,\bar{c}]\,\bar{b} + [\bar{a}\,\bar{b}\,\bar{d}]\,\bar{c}}{[\bar{a}\,\bar{b}\,\bar{c}]} \quad ([\bar{a}\,\bar{b}\,\bar{c}] \neq 0).$$

Condition of coplanarity of three vectors: $[\bar{a}\,\bar{b}\,\bar{c}] = 0$.

Rules for differentiation of products are the same as for ordinary products:

$$d(\bar{a}.\bar{b}) = \bar{a}.d\bar{b} + \bar{b}.d\bar{a},$$

$$d(\bar{a} \times \bar{b}) = \bar{a} \times d\bar{b} + d\bar{a} \times \bar{b}, \tag{II-3}$$

$$d[\bar{a}\,\bar{b}\,\bar{c}] = d\bar{a} \times \bar{b}.\bar{c} + \bar{a} \times d\bar{b}.\bar{c} + \bar{a} \times \bar{b}.d\bar{c}.$$

2.2 Cartesian Components; Differential Formulae

If we are given a positive system of three orthogonal unit vectors ($\bar{i}_1$, $\bar{i}_2$, $\bar{i}_3$), we may always write

$$\bar{a} = a_1\bar{i}_1 + a_2\bar{i}_2 + a_3\bar{i}_3, \tag{II-4}$$

a_1, a_2, a_3 are the components (and also, in this case, the projections) of $\bar{a}$ on the three axes: $a_r = \bar{a}.\bar{i}_r$.

We have:

$$\bar{a}\cdot\bar{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 ; \quad \bar{a}\times\bar{b} = \begin{vmatrix} \bar{i}_1 & \bar{i}_2 & \bar{i}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} ,$$

$$[\bar{a}\ \bar{b}\ \bar{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} , \tag{II-5}$$

Positional vector of a point P referred to an origin O and the system $(\bar{i}_1,\ \bar{i}_2,\ \bar{i}_3)$:

$$P - O = x_1 \bar{i}_1 + x_2 \bar{i}_2 + x_3 \bar{i}_3 , \tag{II-6}$$

$(x_1,\ x_2,\ x_3)$ are the cartesian coordinates of P.

A function $W(x_1,\ x_2,\ x_3) = W(P)$ is a function of position.

We call gradient of W and write grad W or also ∇W (read nabla W) the following vector:

$$\text{grad } W = \nabla W = \frac{\partial W}{\partial x_1}\bar{i}_1 + \frac{\partial W}{\partial x_2}\bar{i}_2 + \frac{\partial W}{\partial x_3}\bar{i}_3 . \tag{II-7}$$

"grad" is an operator between scalars and vectors.

It is immediately seen that grad W . $dP = dW$; it follows that grad W is always perpendicular to the surfaces $W(P) = $ const.

If we are given a vector $\bar{u} = \bar{u}(P)$ function of position (the components of which are functions of P), we shall call <u>curl</u> or <u>rotation</u> of $\bar{u}$ and write rot $\bar{u}$ or also $\nabla \times \bar{u}$ the following vector

$$\text{rot } \bar{u} = \nabla\times\bar{u} = \bar{i}_1\times\frac{\partial\bar{u}}{\partial x_1} + \bar{i}_2\times\frac{\partial\bar{u}}{\partial x_2} + \bar{i}_3\times\frac{\partial\bar{u}}{\partial x_3} =$$

$$= \bar{i}_1\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \bar{i}_2\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) + \bar{i}_3\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \tag{II-8}$$

$$= \begin{vmatrix} \bar{i}_1 & \bar{i}_2 & \bar{i}_3 \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_3} \\ u_1 & u_2 & u_3 \end{vmatrix} .$$

"rot" is an operator between vectors and vectors.

It follows immediately that

$$\text{rot grad } W = \nabla \times \nabla \, W = 0 \qquad\qquad (II\text{-}9)$$

A field of vectors $\bar{u}(P)$ such that rot $\bar{u} = 0$ identically is called irrotational or conservative. In such case, a scalar function $W(P)$ exists, such that $\bar{u}(P) = $ grad $W(P)$. $W(P)$ is called scalar potential, or potential of the field.

If we are given a field of vectors $\bar{u}(P)$, we shall call divergence of $\bar{u}$ and write div $\bar{u}$ or also $\nabla \cdot \bar{u}$ the following scalar:

$$\text{div } \bar{u} = \nabla \cdot \bar{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} . \qquad\qquad (II\text{-}10)$$

"div" is an operator between vectors and scalars.

We have identically div rot $\bar{v} = 0$; and, conversely, if div $\bar{u} = 0$, it follows $\bar{u} = $ rot $\bar{v}$.

A field such that div $\bar{u} = 0$ identically is called solenoidal; it follows that every solenoidal field is the rotation of a new field of vectors (vector potential).

Combining the two operators div and grad we put

$$\text{div grad } W = \Delta_2 W = \nabla \cdot \nabla W = \nabla^2 W, \qquad\qquad (II\text{-}11)$$

the new operator Δ_2 (between scalars and scalars) is called the Laplacian operator.

A given field $\bar{u}(P)$ may always be written as the sum of an irrotational and a solenoidal field. We have

$$\bar{u} = \text{grad } W + \text{rot } \bar{v} , \qquad\qquad (II\text{-}12)$$

and also it may be written as

$$\bar{u} = \text{grad } W + m \text{ grad } n , \qquad\qquad (II\text{-}13)$$

where W, m, n, and $\bar{v}$ are suitable scalar or vector functions of P.

It is very important to point out at once that although the foregoing differential operators (grad, div, rot) have been defined for the sake of simplicity by the help of cartesian coordinates, they are absolute operators, as we can see from the following vector formulae which may define as well the operators themselves:

grad W . dP = dW ,

rot $\bar{u}$. $dP \times \delta P$ = $d\bar{u}$. δP - $\delta \bar{u}$. dP ,   (II-14)

$\left(\text{div } \bar{u}\right) dP \times \delta P$. ∂P = $\delta P \times \partial P$. $d\bar{u}$ + $\partial P \times dP$. $\delta \bar{u}$ + $dP \times \delta P$. $\partial \bar{u}$,

where d, δ, ∂ are differentials in three non parallel nor coplanar directions.

Following formulae may be derived immediately from the foregoing:

grad$(V + W)$ = grad V + grad W , rot $(\bar{u} + \bar{v})$ = rot $\bar{u}$ + rot $\bar{v}$,

grad mW = m grad W + W grad m ,

rot $m\bar{u}$ = m rot $\bar{u}$ + grad $m \times \bar{u}$,   (II-15)

div $m\bar{u}$ = m div $\bar{u}$ + grad m . $\bar{u}$,

div $\bar{u} \times \bar{v}$ = $\bar{v}$. rot $\bar{u}$ - $\bar{u}$. rot $\bar{v}$.

2.3   Homographies

A linear operator (transformation) α between vectors and vectors, such that

$\alpha \bar{a}$ = $\bar{u}$,

$\alpha(\bar{a} + \bar{b})$ = $\alpha \bar{a}$ + $\alpha \bar{b}$, (distributive property)   (II-16)

$\alpha(m \bar{a})$ = $m\alpha\bar{a}$,   (commutative property)

will be called a homography.

Each homography transforms parallel vectors into parallel vectors, and vectors parallel to a plane into vectors parallel to a plane.

Moreover, each homography may be fully determined by the corresponding vectors of three (independent-non parallel to a plane) given vectors.

Proper homography is a homography transforming vectors non parallel to a plane into vectors non parallel to a plane.

Singular or degenerative homography is a homography transforming a group whatever of independent vectors into other vectors parallel to a plane, or even parallel to a direction.

Some special homographies are:

(1) homothety (proper homography)

$$\alpha = m \text{ (real number)}, \quad \alpha\bar{x} = m\bar{x} \ (\bar{x} \text{ arbitrary}). \tag{II-17}$$

(2) axial homography (singular homography transforming a group whatever of independent vectors into a group of vectors parallel to a plane)

$$\alpha = \bar{u} \times \quad , \quad \alpha\bar{x} = \bar{u} \times \bar{x} \tag{II-18}$$

(3) dyad (of Gibbs) (singular homography transforming each vector into a vector parallel to a direction)

$$\alpha = H(\bar{u}, \bar{v}) , \quad \alpha\bar{x} = (\bar{u} \cdot \bar{x})\bar{v} \tag{II-19}$$

(4) dilatation (proper homography)

Homography satisfying to the condition

$$\bar{x} \cdot \alpha\bar{y} = \bar{y} \cdot \alpha\bar{x} . \tag{II-20}$$

A homography may be indicated also by the following notation, which defines it fully:

$$\alpha = \begin{pmatrix} \bar{u} & \bar{v} & \bar{w} \\ \bar{a} & \bar{b} & \bar{c} \end{pmatrix} \quad \text{i.e.,} \qquad \begin{aligned} \alpha\bar{a} &= \bar{u} \\ \alpha\bar{b} &= \bar{v} \\ \alpha\bar{c} &= \bar{w} . \end{aligned} \tag{II-21}$$

In the case of a dilatation, there may always be found three unit vectors (principal directions) $\bar{i}_1$, $\bar{i}_2$, $\bar{i}_3$ such that

$$a = \begin{pmatrix} m\,\bar{i}_1 & m\,\bar{i}_2 & m\,\bar{i}_3 \\ \bar{i}_1 & \bar{i}_2 & \bar{i}_3 \end{pmatrix}. \tag{II-22}$$

The three invariants of an homography are $I_1 a$, $I_2 a$, $I_3 a$ (first, second and third invariant) and are defined by the following formulae:

$$I_1 a \cdot \bar{u} \times \bar{v} \cdot \bar{w} = \bar{v} \times \bar{w} \cdot a\,\bar{u} + \bar{w} \times \bar{u} \cdot a\,\bar{v} + \bar{u} \times \bar{v} \cdot a\,\bar{w} ,$$

$$I_2 a \cdot \bar{u} \times \bar{v} \cdot \bar{w} = (a\,\bar{v}) \times (a\,\bar{w}) \cdot \bar{u} + (a\,\bar{w}) \times (a\,\bar{u}) \cdot \bar{v} + (a\,\bar{u}) \times (a\,\bar{v}) \cdot \bar{w} ,$$

$$I_3 a \cdot \bar{u} \times \bar{v} \cdot \bar{w} = (a\,\bar{u}) \times (a\,\bar{v}) \cdot (a\,\bar{w}) . \tag{II-23}$$

The vector Va of a homography a is defined by

$$2V\,a \cdot \bar{u} \times \bar{v} = \bar{v} \cdot a\,\bar{u} - \bar{u} \cdot a\,\bar{v} \quad (\bar{u},\ \bar{v},\ \bar{w}\ \text{arbitrary vectors}). \tag{II-24}$$

The dilatation Da of a is defined by

$$Da = a - Va \times . \tag{II-25}$$

The conjugate Ka of a is defined by

$$Ka = Da - Va \times . \tag{II-26}$$

The cyclic Ca of a is defined by

$$Ca = I_1 a - a . \tag{II-27}$$

Theorem 1: Da is always a dilatation.

Theorem 2: A homography may always be written uniquely as the sum of a dilatation and an axial homography, as follows:

$$a = Da + Va \times , \qquad Ka = Da - Va \times . \tag{II-28}$$

We have

$$a\ \text{is a dilatation if}\ \begin{cases} Va = 0 , \\ Ka = a ; \end{cases}$$

$$a\ \text{is axial if}\quad \begin{cases} Da = 0 , \\ Ka = -a . \end{cases}$$

Some important formulae:

$$I_1(\alpha + \rho) = I_1\alpha + I_1\rho, \qquad\qquad I_1 m\alpha = m I_1\alpha \ ,$$

$$V(\alpha + \rho) = V\alpha + V\rho \ , \qquad\qquad V m\alpha = m V\alpha \ ,$$

$$D(\alpha + \rho) = D\alpha + D\rho \ , \qquad\qquad D m\alpha = m D\alpha \ ,$$

$$K(\alpha + \rho) = K\alpha + K\rho \ , \qquad\qquad K m\alpha = m K\alpha \ ,$$

$$C(\alpha + \rho) = C\alpha + C\rho \ , \qquad\qquad C m\alpha = m C\alpha \ ,$$

(II-29)

$$I_1 m \quad = 3m \quad , \quad I_2 m \quad = 3m^2 \ , \ I_3 m \quad = m^3 \ ,$$

$$V m \quad = 0 \quad , \quad D m \quad = m \ , \ K m \quad = m \ , \ (m\text{-scalar})$$

$$I_1(\bar{u} \times \bar{v}) = 0 \quad , \quad I_2(\bar{u} \times \bar{v}) = \bar{u}^2 \ , \ I_3(\bar{u} \times \bar{v}) = 0 \ , \ \Big\} \ \text{axials} \quad \text{(II-30)}$$

$$V(\bar{u} \times \bar{v}) = \bar{u} \quad , \quad D(\bar{u} \times \bar{v}) = 0 \ , \ K(\bar{u} \times \bar{v}) = -\bar{u} \ , \ \Big)$$

$$I_1 H(\bar{u},\bar{v}) = \bar{u}.\bar{v} \quad , \quad I_2 H(\bar{u},\bar{v}) = 0 \ , \ I_3 H(\bar{u},\bar{v}) = 0 \ , \ \Big\} \ \text{dyads}$$

$$2VH(\bar{u},\bar{v}) = \bar{u} \times \bar{v} \ , \quad 2DH(\bar{u},\bar{v}) = H(\bar{u},\bar{v}) + H(\bar{v},\bar{u}) \ , \ \Big)$$

$$KH(\bar{u},\bar{v}) = H(\bar{v},\bar{u}).$$

Commutation theorem of Jacobi:

$$\bar{x}.\alpha\bar{y} = \bar{y}.K\alpha\bar{x} \qquad\qquad \text{(II-31)}$$

(α, $\bar{x}$, and $\bar{y}$ arbitrary homography and vectors).

$$I_1(\rho\alpha) = I_1\alpha \ I_1\rho + I_2\alpha + I_2\rho - I_2(\alpha + \rho) \ ,$$

$$I_3(\rho\alpha) = I_3\alpha \ I_3\rho .$$

(II-32)

2.4 Spatial Derivatives of Vectors

We will define the operator $\dfrac{d\bar{u}}{dP}$ and call it spatial deri-

vative of the vector $\bar{u} = \bar{u}(P)$, as the homography which applied to a

displacement δP, gives us the corresponding increment $\delta\bar{u}$ undergone

by $\bar{u}$:

$$\frac{d\bar{u}}{dP} \ \delta P = \delta\bar{u} \ . \qquad\qquad \text{(II-33)}$$

If we apply the operator $\frac{d\bar{u}}{dP}$ to a given unit vector $\bar{a}$,

$\frac{d\bar{u}}{dP}\bar{a}$ gives us the derivative of $\bar{u}$ in the direction $\bar{a}$. For this

reason $\frac{d\bar{u}}{dP}$ may be called the spatial derivative of $\bar{u}$.

We have the following absolute expressions for rot $\bar{u}$ and div $\bar{u}$:

$$\text{rot } \bar{u} = 2V\frac{d\bar{u}}{dP} , \qquad \text{div } \bar{u} = I_1\frac{d\bar{u}}{dP} . \qquad\qquad \text{(II-34)}$$

Furthermore, the following important formulae may be noticed:

$$\text{grad } \bar{u} \cdot \bar{v} = K\frac{d\bar{u}}{dP}\bar{v} + K\frac{d\bar{v}}{dP}\bar{u} . \qquad\qquad \text{(II-35)}$$

If $\bar{i}_1$, $\bar{i}_2$, $\bar{i}_3$ be a constant orthogonal system of vectors, and

$$\bar{u} = u_1\bar{i}_1 + u_2\bar{i}_2 + u_3\bar{i}_3 ,$$

then

$$\text{rot } \bar{u} = \sum_{r=1}^{3} \text{grad } u_r \times \bar{i}_r ,$$

$$\qquad\qquad \text{(II-36)}$$

$$\text{div } \bar{u} = \sum_{r=1}^{3} \text{grad } u_r \cdot \bar{i}_r .$$

Some theorems follow:

Theorem 1: If everywhere in a given field $\bar{u} \cdot \text{rot } \bar{u} = 0$, then three scalar functions of position m, n, and μ exist, such that

$$\bar{u} = m \text{ grad } n ,$$

$$\text{rot}(\mu\, \bar{u}) = 0 , \qquad\qquad \text{(II-37)}$$

$$(\mu = 1/m) .$$

Theorem 2: If everywhere in a given field div $\bar{u} = 0$, then two scalars m and n exist such that

$$\bar{u} = \text{grad } m \times \text{grad } n \qquad\qquad \text{(II-38)}$$

Theorem 3: For an arbitrary vector field $\bar{u}$, scalars m and n exist such that

$$\text{rot } \bar{u} = \text{grad } m \times \text{grad } n .\tag{II-39}$$

2.5   Cartesian Components of Homographies

We may consider the cartesian components of a homography. If $(\bar{i}_r)$ are the orthogonal unit vectors of our reference system and a the homography considered, we may put

$$a\bar{i}_1 = a_{11}\,\bar{i}_1 + a_{12}\,\bar{i}_2 + a_{13}\,\bar{i}_3 ,$$
$$a\bar{i}_2 = a_{21}\,\bar{i}_1 + a_{22}\,\bar{i}_2 + a_{23}\,\bar{i}_3 ,\tag{II-40}$$
$$a\bar{i}_3 = a_{31}\,\bar{i}_1 + a_{32}\,\bar{i}_2 + a_{33}\,\bar{i}_3 .$$

The matrix $\|\, a_{rs} \,\|$ gives us the components of a homography and defines it completely. In fact, if we put

$$\bar{u} = \sum_{r=1}^{3} u_r \bar{i}_r , \text{ we will have}$$

$$\begin{aligned}
a\bar{u} &= u_1\,a\bar{i}_1 + u_2 a\bar{i}_2 + u_3 a\bar{i}_3 =\\
&= (u_1 a_{11} + u_2 a_{21} + u_3 a_{31})\bar{i}_1 +\\
&\quad +(u_1 a_{12} + u_2 a_{22} + u_3 a_{32})\bar{i}_2 +\\
&\quad +(u_1 a_{13} + u_2 a_{23} + u_3 a_{33})\bar{i}_3 .
\end{aligned}\tag{II-41}$$

We have, therefore:

$$a_{rs} = a\bar{i}_r \cdot \bar{i}_s .\tag{II-42}$$

We have, furthermore:

$$\alpha = \begin{Vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{Vmatrix} \quad ,$$

$$K\alpha = \begin{Vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{Vmatrix} \quad ,$$

$$D\alpha = \begin{Vmatrix} a_{11} & 1/2(a_{12} + a_{21}) & 1/2(a_{13} + a_{31}) \\ 1/2(a_{21} + a_{12}) & a_{22} & 1/2(a_{23} + a_{32}) \\ 1/2(a_{31} + a_{13}) & 1/2(a_{32} + a_{23}) & a_{33} \end{Vmatrix} \quad ,$$

$$2V\alpha = (a_{23} - a_{32})\bar{I}_1 + (a_{31} - a_{13})\bar{I}_2 + (a_{12} - a_{21})\bar{I}_3 \quad ,$$

$$V\alpha X = \begin{Vmatrix} 0 & 1/2(a_{12} - a_{21}) & 1/2(a_{13} - a_{31}) \\ 1/2(a_{21} - a_{12}) & 0 & 1/2(a_{23} - a_{32}) \\ 1/2(a_{31} - a_{13}) & 1/2(a_{32} - a_{23}) & 0 \end{Vmatrix} \quad ,$$

$$\bar{a}X = \begin{Vmatrix} 0 & +a_3 & -a_2 \\ -a_3 & 0 & +a_1 \\ +a_2 & -a_1 & 0 \end{Vmatrix} \quad ,$$

$$(II-43)$$

$$H(\bar{u}, \bar{v}) = \begin{Vmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{Vmatrix} \quad ,$$

$$I_1\alpha = I_1K\alpha = a_{11} + a_{22} + a_{33} \quad ,$$

$$I_2\alpha = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{33} & a_{31} \\ a_{13} & a_{11} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = I_2K\alpha \quad ,$$

$$I_3\alpha = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = I_3K\alpha \quad ,$$

$$\alpha = \sum_{r,s} a_{rs} H(\bar{i}_r, \bar{i}_s) \quad \text{(decomposition of a homography into dyads)} \quad .$$

Furthermore:

$$\frac{d\bar{u}}{dP} = \begin{pmatrix} \dfrac{\partial \bar{u}}{\partial x_1} & \dfrac{\partial \bar{u}}{\partial x_2} & \dfrac{\partial \bar{u}}{\partial x_3} \\[2mm] \bar{i}_1 & \bar{i}_2 & \bar{i}_3 \end{pmatrix} = \begin{Vmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_3}{\partial x_1} \\[2mm] \dfrac{\partial u_1}{\partial x_2} & \dfrac{\partial u_2}{\partial x_2} & \dfrac{\partial u_3}{\partial x_2} \\[2mm] \dfrac{\partial u_1}{\partial x_3} & \dfrac{\partial u_2}{\partial x_3} & \dfrac{\partial u_3}{\partial x_3} \end{Vmatrix} \quad ,$$

$$\frac{d\bar{u}}{dP}\,\bar{x} = \bar{x}\cdot\bar{i}_1\,\frac{\partial \bar{u}}{\partial x_1} + \bar{x}\cdot\bar{i}_2\,\frac{\partial \bar{u}}{\partial x_2} + \bar{x}\cdot\bar{i}_3\,\frac{\partial \bar{u}}{\partial x_3} \quad ,$$

$$K\frac{d\bar{u}}{dP}\,\bar{x} = \bar{x}\cdot\frac{\partial \bar{u}}{\partial x_1}\,\bar{i}_1 + \bar{x}\cdot\frac{\partial \bar{u}}{\partial x_2}\,\bar{i}_2 + \bar{x}\cdot\frac{\partial \bar{u}}{\partial x_3}\,\bar{i}_3 \quad ,$$

$$\text{rot }\bar{u} = \bar{i}_1 \times \frac{\partial \bar{u}}{\partial x_1} + \bar{i}_2 \times \frac{\partial \bar{u}}{\partial x_2} + \bar{i}_3 \times \frac{\partial \bar{u}}{\partial x_3} \quad ,$$

$$\text{div }\bar{u} = \bar{i}_1 \cdot \frac{\partial \bar{u}}{\partial x_1} + \bar{i}_2 \cdot \frac{\partial \bar{u}}{\partial x_2} + \bar{i}_3 \cdot \frac{\partial \bar{u}}{\partial x_3} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} \quad ,$$

$$\Delta_2 W = \text{div grad } W = I_1 \frac{d \text{ grad } W}{dP} = \frac{\partial^2 W}{\partial x_1^2} + \frac{\partial^2 W}{\partial x_2^2} + \frac{\partial^2 W}{\partial x_3^2} \tag{II-44}$$

$$I_3 \frac{d \ (\text{grad } W)}{dP} = \begin{vmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{vmatrix} \qquad W_{rs} = \frac{\partial^2 W}{\partial x_r \partial x_s} \quad (\text{Hessian})$$

$$\text{grad } m \times \text{grad } n \cdot \text{grad } p = \begin{vmatrix} \dfrac{\partial m}{\partial x_1} & \dfrac{\partial m}{\partial x_2} & \dfrac{\partial m}{\partial x_3} \\[2ex] \dfrac{\partial u}{\partial x_1} & \dfrac{\partial u}{\partial x_2} & \dfrac{\partial u}{\partial x_3} \\[2ex] \dfrac{\partial p}{\partial x_1} & \dfrac{\partial p}{\partial x_2} & \dfrac{\partial p}{\partial x_3} \end{vmatrix} \qquad (\text{Jacobian})$$

SECTION III

TENSOR CALCULUS (ELEMENTS AND NOTATIONS)

3.1   General Rectilinear Reference Systems

We have studied so far, absolute entities like vectors and homographies, and also absolute operators like grad, rot, div, $\frac{d}{dP}$, etc., either without using any reference system of axes or coordinates, or only using cartesian orthogonal unit systems.

As we will see, it is instead often very important to refer the same absolute entities and operators to general cartesian systems (i.e., not unit, and not orthogonal systems), as happens in the infinitely small domain around a point using general curvilinear coordinates. (See reference 3, 4, 6, 7, 8, 9, 10, 11, 12.)

As a first step, we shall confine our study to rectilinear cartesian coordinates, neither orthogonal nor isometric, as they may be specified by a point 0 (the origin) and three (not coplanar and not unit) vectors $\bar{v}_1$, $\bar{v}_2$, $\bar{v}_3$, our fundamental (or base) vectors.

If we are given an arbitrary vector $\bar{u}$ and we want to define $\bar{u}$ by means of our reference system, we always may express it as a linear combination of the fundamental vectors, as follows:

$$\bar{u} = u^1\bar{v}_1 + u^2\bar{v}_2 + u^3\bar{v}_3 = \sum_s u^s \bar{v}_s \qquad \text{(III-1)}$$

(lower case Roman indices have here and in the following the range from 1 to 3); or also we may define it by means of the three scalar products

$$u_1 = \bar{u}.\bar{v}_1 \;;\; u_2 = \bar{u}.\bar{v}_2 \;;\; u_3 = \bar{u}.\bar{v}_3 . \qquad \text{(III-2)}$$

Both systems of numbers $u^r = (u^1, u^2, u^3)$, or also $u_r = (u_1, u_2, u_3)$ allows us, with the help of our reference system of vectors, to reconstruct $\bar{u}$; u^r or also u_r are therefore called the components of $\bar{u}$ in our system.

More precisely, u^r are called the contravariant components of $\bar{u}$, u_r the covariant components. Contravariant and covariant components are in general different. They coincide only in the elementary case of cartesian orthogonal unit reference systems.

The reason for the denominations covariant and contravariant given to the two kinds of components is due to the fact that, by transforming the original fundamental vectors $\bar{v}_r$ into other new fundamental vectors $\bar{v}'_r$ by means of a linear transformation

$$\bar{v}'_r = \alpha_r^1 \bar{v}_1 + \alpha_r^2 \bar{v}_2 + \alpha_r^3 \bar{v}_3 = \Sigma \alpha_r^s \bar{v}_s \quad, \tag{III-4}$$

the two kinds of components of the same vector do not transform by the same rule, as we will see at once.

(Summation Convention). We will use in the following the convention to dispense with the sign Σ, all times an index is repeated in a monome twice, once as a superscript, and once as a subscript. Thus, for instance, we will write the following formulae:

$$\bar{u} = u^s \bar{v}_s \; ; \; \bar{v}'_r = \alpha_r^s \bar{v}_s \quad. \tag{III-5}$$

The letter used for the summation index (dummy index) is immaterial (like s in the foregoing formulae), and any letter may be substituted at will for it. An index which is not repeated (like r in the foregoing), is called a free index.

Let us now have, in the first system of reference $\bar{u} = u^r \bar{v}_r$, $u_r = \bar{u} \cdot \bar{v}_r$, and $\bar{u} = u'^r \bar{v}'_r$, $u'_r = \bar{u} \cdot \bar{v}'_r$ in the second; then

$$\bar{u} = u^s \bar{v}_s = u'^r \bar{v}'_r = u'^r \alpha_r^s \bar{v}_s \tag{III-6}$$

and therefore

$$u^s = u'^r \alpha_r^s \quad \text{(law of contravariance)} \quad, \tag{III-7}$$

and furthermore

$$u_r' = \bar{u} \cdot \bar{v}_r' = \alpha_r^s \, \bar{u} \circ \bar{v}_s = \alpha_r^s \, u_s \quad \text{(law of covariance).} \quad \text{(III-8)}$$

As we see, u_r' transforms like the fundamental vectors $\bar{v}_s$ themselves (they are therefore cogradient to the same, and the law of transformation is covariant); the u^r transform conversely by a different, contragradient law with respect to the fundamental vectors, i.e. they are contravariant to the same vectors.

In a quite similar way we may consider the components of a homography α, and put

$$a_{rs} = \alpha \, \bar{v}_r \cdot \bar{v}_s \quad \text{(covariant components).} \quad \text{(III-9)}$$

It may easily be seen that these components transform by the following generalized law of covariance:

$$a_{rs}' = \alpha_r^p \, \alpha_s^q \, a_{pq} . \quad \text{(III-10)}$$

3.2   Reciprocal Vectors

We will define a set of reciprocal vectors $\bar{v}^s$ with respect to our fundamental system $\bar{v}_r$ by putting

$$\bar{v}_r \cdot \bar{v}^s = \delta_r^s , \quad \text{(III-11)}$$

where δ_r^s (Kronecker's symbol) has the following meaning:

$$\delta_r^s \begin{cases} = 0 & \text{if } s \neq r \\ = 1 & \text{if } s = r \end{cases} . \quad \text{(III-12)}$$

Thus for instance $\bar{v}^1$ is perpendicular to both $\bar{v}_2$ and $\bar{v}_3$.

It may be easily seen that the general expression of $\bar{v}^s$ is given by

$$\bar{v}^s = \frac{\bar{v}_{s+1} \times \bar{v}_{s+2}}{[\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3]} \quad \text{(III-13)}$$

(we identify the indices differing by multiples of 3); and reciprocally

$$\bar{v}_r = \frac{\bar{v}^{r+1} \times \bar{v}^{r+2}}{[\bar{v}^1\ \bar{v}^2\ \bar{v}^3\ ]} \qquad (III-14)$$

Moreover

$$[\bar{v}^1\ \bar{v}^2\ \bar{v}^3] = \frac{1}{[\bar{v}_1\ \bar{v}_2\ \bar{v}_3]} \qquad (III-15)$$

By the help of the reciprocal vectors we also may write

$$\bar{u} = u_r \bar{v}^r \qquad\qquad u^r = \bar{u} \cdot \bar{v}^r \qquad (III-16)$$

In a similar way we may put

$$a^{rs} = \alpha\bar{v}^r \cdot \bar{v}^s \qquad \text{(contravariant components of } \alpha\text{)}$$

$$a^{r\cdot}_{\cdot s} = \alpha\bar{v}^r \cdot \bar{v}_s \qquad (III-17)$$

$$\text{(mixed components of } \alpha\text{)} .$$

$$a^{\cdot s}_{r\cdot} = \alpha\bar{v}_r \cdot \bar{v}^s$$

By a transformation of the fundamental vectors $\bar{v}_r$, the reciprocal vectors transform contravariantly.

We may put

$$\bar{v}'^s = A^s_{p}\bar{v}^p , \qquad (III-18)$$

and have

$$\bar{v}'_r \cdot \bar{v}'^s = \delta^s_r = \alpha^t_r A^s_p\bar{v}_t \cdot \bar{v}^p = \alpha^t_r A^s_p \delta^p_t = \alpha^t_r A^s_t , \qquad (III-19)$$

and finally

$$\alpha^t_r A^s_t \doteq \delta^s_r . \qquad (III-20)$$

This formula shows that $\| A_t^s \|$ is the inverse of the matrix $\| \alpha_s^t \|$. Our formulae read now:

$$\bar{v}'_r = \alpha_r^s \bar{v}_s \quad , \quad \bar{v}_s = A_s^r \bar{v}'_r$$

(covariance) ,

$$u'_r = \alpha_r^s u_r \quad , \quad u_s = A_s^r u'_r$$

(III-21)

$$\bar{v}'^r = A_s^r \bar{v}^s \quad , \quad \bar{v}^s = \alpha_r^s \bar{v}^r$$

(contravariance) .

$$u'^r = A_s^r v^s \quad , \quad u^s = \alpha_r^s u^r$$

3.3 Tensors; the Fundamental Metric Tensor

3.3.1 All systems (single, double,...) of members like u_r, u^r, a_{rs}, a^{rs}, $a^r_{\cdot s}$,... having the meaning specified above, connected to the fundamental system of vectors (or its reciprocal), and transforming accordingly to the above rules of covariance or contravariance, are called tensors (of order one, of order two, etc.).

If we are given two vectors

$$\bar{u} = u_r \bar{v}^r = u^s \bar{v}_s \quad , \quad \bar{w} = w_r \bar{v}^r = w^s \bar{v}_s \quad ,$$

(III-22)

we may form their scalar product in several forms:

$$\bar{u} \cdot \bar{w} = u_r w_s g^{rs} = u^r w^s g_{rs} = u_r w^s \delta_s^r = u^r w_r = u^r w_s \delta_r^s = u^r w_r \quad ,$$

having put

$$g_{rs} = \bar{v}_r \cdot \bar{v}_s = g_{sr} \quad , \quad g^{rs} = \bar{v}^r \cdot \bar{v}^s = g^{sr} \quad .$$

(III-23)

It may easily be seen that g_{rs} and g^{rs} are, respectively, a covariant and a contravariant double tensor. We call them the fundamental metric tensor in covariant and in contravariant form.

We also have

$$\bar{u} = u_r \bar{v}^r = \bar{u}.\bar{v}_r v^r = u^s \bar{v}_s . \bar{v}_r \bar{v}^r = g_{sr} u^s \bar{v}^r , \qquad \text{(III-24)}$$

and therefore

$$u_r = g_{sr} u^s . \qquad \text{(III-25)}$$

In a similar way we may find

$$u_r = g_{rs} u^s , \quad u^r = g^{rs} u_s , \quad a_{rs} = g_{rp} g_{sq} a^{pq} , \quad .... \qquad \text{(III-26)}$$

The fundamental metric tensor allows us to compute the scalar product of two vectors, and therefore to establish a linear and angular metric; moreover, it allows us to "change the variance" of the components of a given tensor, i.e., to deduce the covariant components from the contravariant ones, and conversely.

We also have the formulae:

$$\bar{v}_r = \delta^s_r \bar{v}_s = g_{rs} \bar{v}^s ,$$
$$\qquad\qquad\qquad\qquad \text{(III-27)}$$
$$\bar{v}^s = \delta^s_r \bar{v}^r = g^{rs} \bar{v}_r ,$$

i.e., g_{rs}, δ^s_r, g^{rs}, are respectively the covariant, mixed, and contravariant components of the fundamental system of vectors and its reciprocal.

3.4 Operations on Tensors, Invariants

Addition of tensors. The addition of the corresponding components of two tensors of the same kind and order gives another tensor of that order and kind; thus for instance:

$$a_r + b_r = c_r , \quad a^r_{st} + b^r_{st} = c^r_{st} .$$

Product of tensors. The product of the components of (two) tensors of any kind or order gives another tensor the order of which is the sum of the orders of the two given tensors; thus for instance:

$$a_{r}\,b_{s} = c_{rs} \ , \ a_{r}\,b^{s} = c_{r}^{s} \ , \ a_{rst}^{pq}\,b_{m}^{n} = c_{rstm}^{pqn}$$

Contraction of tensors. If we make an index of contravariance and an index of covariance the same in a given mixed tensor, so that it becomes a dummy index, and we must sum for it from 1 to 3, the result is a new tensor, the order of which is reduced by two; thus:

$$a_{rm}^{rst} = c_{m}^{st} \ , \ a_{rs}^{rst} = d^{t} \ , \ a_{sr}^{rst} = e^{t} \ , \ \text{etc.}$$

Invariants. If we contract completely a tensor of even order with as many indices of covariance and contravariance, the result would be an invariant (a scalar function of position, the value of which does not depend upon the fundamental system of axes); thus:

$$a_{r}^{r} = b \ , \ a_{rs}^{rs} = c \ , \ a_{sr}^{rs} = d \ .$$

Contracted multiplication. We may multiply two tensors of any order and contract at the same time with respect to a couple (or more than a couple) of indices. The result is a tensor, or an invariant, thus:

$$a_{r}\,b^{r} = c \ (= \bar{a}.\bar{b}, \text{ scalar product of } \bar{a} \text{ and } \bar{b}; \text{ invariant}),$$

$$a_{r}\,a^{r} = (a)^{2} \ (\text{square of the length of } \bar{a}; \text{ invariant}),$$

$$a_{.s}^{r.}\,b^{s} = c^{s} \ (\text{homography } \alpha \text{ of mixed components } a_{.s}^{r.} \text{ applied to vector}$$
$$\bar{b} = b^{r}\bar{v}_{r} = \text{vector } \bar{c} = c^{s}\bar{v}_{s}) \ ,$$

$$a_{rs}^{p}\,b_{pt}^{r} = c_{st} \ \text{etc.}$$

Quotient law of tensors. If the result of the contracted
multiplication of an arbitrary tensor by a given system gives us a
tensor or an invariant, then the given system is a tensor.

Thus, for instance, if we are given a system of the third order
$A(r,s,t)$ and we take an arbitrary tensor b^s_r , and we know that

$$A(r,s,t)b^s_r = c^t ,$$

where c^t is a tensor, then $A(r,s,t)$ is also a tensor and must therefore
be written as

$$A(r,s,t) = a^{rt}_s$$

The foregoing rule allows us to recognize the tensor character
of a given system. It is very important to stress that the tensor
used as factor (in our instance b^s_r) must be completely arbitrary.

A particular case is to use as test tensor the product of tensors
of the first order; thus, for instance, if

$$A(r,s,t)x^r y^s z_t = c \quad \text{(an invariant)}$$

$A(r,s,t)$ is a tensor and must be written as a^t_{rs} . We may also say
that the coefficients of a multilinear invariant form in the
variables x^r, y^s, z_t are the components of a tensor.

We may also observe that the contraction of a tensor of the
first order and the fundamental (or reciprocal) vectors give us a
vector, i.e., an absolute entity; thus $u^i \bar{v}_i = \mathbb{U}$.

3.5 General Curvilinear Coordinates

In the foregoing we have merely used cartesian rectilinear
coordinates, i.e., the same system of reference for the whole space
considered. Thus, for instance, if we were given a vector $\bar{a}$ the

components of which would be a^r (or a_r) at a given point P of space, and we transport by equipollence the vector $\bar{a}$ to another point Q, the components of $\bar{a}$ would remain unaltered. The same may be stated for tensors of any order and kind.

In many instances it is, however, not possible or convenient to use cartesian rectilinear coordinates. Let us confine our attention for the sake of simplicity to a plane; it is always possible to use on it cartesian rectilinear systems of reference, and the foregoing theories may be very easily applied with some slight modifications (summation from 1 to 2 instead of from 1 to 3, obvious modification as to the reciprocal vectors). But this would no longer be possible if we consider a curved surface (not developable) on which no cartesian system of coordinates is possible.

But even in three dimensional space it may happen that we were led to use curvilinear coordinates (for instance, spherical coordinates, cylindrical coordinates, ellipsoidal coordinates, etc.), as may be suggested by the particular problem we are dealing with. In more advanced geometrical theories we may even consider curved three dimensional spaces in which no cartesian system of reference would be possible.

In such cases the foregoing simple theories cannot be applied and we must establish other rules for the definition of the components of our absolute entities, like vectors, homographies, etc. in any point of space, and for operation on them.

Let us consider, for the sake of simplicity, the ordinary space and imagine to have established a system of general curvilinear coordinates y^1, y^2, y^3, such that we have a one-to-one correspondence between

the sets $y^r = (y^1, y^2, y^3)$ and the points P of space. It is well

known that the equations $y^r =$ constant give us the three families of

coordinate surfaces; each surface will be specified by one and only one

value of the parameter y^r. The intersection of the surfaces of the

three families give us the "congruence" of lines, the coordinate

lines, each of which is specified by a pair of coordinates (y^r, y^{r+1}).

Along each coordinate line only the third coordinate y^{r+2} may vary.

Let us consider, furthermore, a point $P(y_0^1, y_0^2, y_0^3) = P(y_0^r)$, and

at this point the three vectors

$$\bar{v}_1 = \frac{\partial P}{\partial y_0^1} \quad , \quad \bar{v}_2 = \frac{\partial P}{\partial y_0^2} \quad , \quad \bar{v}_3 = \frac{\partial P}{\partial y_0^3} \qquad (\text{III-28})$$

These vectors have obviously the directions of the coordinate

lines through the point, in the order. Thus, for instance, $\bar{v}_1$ is

directed along the intersection of the surfaces $y^2 = y_0^2$ and $y^3 = y_0^3$.

The direction of them is that of the increasing y's, and the length

is given by the ratio $\frac{ds}{dy^i}$ of an infinitesimal displacement along the

line and the increment of the corresponding parameter.

Thus if a general system of curvilinear coordinates is given in

space, we are able to define in a very simple and natural way a funda-

mental system of vectors, one for each point. It is obvious that the

systems so established are different each from the other not only as

their orientation is concerned, but also in their interior structure.

By this way, we are now able to define in each point the components

of our absolute entities, i.e., tensors, by simply applying the theories

stated above for rectilinear coordinates. The only difference is that

the same entity (a vector, for instance,) will not have the same

components at different points of space, as was the case in rectilinear

coordinates.

The further question is to see how it is possible to connect the different fundamental systems of vectors we have established at each point, each to another, so as to make it possible to compare the same absolute entity at different points, although its components are different from point to point.

Before meeting this problem we may observe that we may obviously write

$$dP = \frac{\partial P}{\partial y^1}dy^1 + \frac{\partial P}{\partial y^2}dy^2 + \frac{\partial P}{\partial y^3}dy^3 = \frac{\partial P}{\partial y^i}dy^i = \bar{v}_i dy^i , \qquad (III-29)$$

and thus dy^i are the contravariant components of the elementary displacement dP. (In the case of rectilinear coordinates y^i would be the contravariant components of the positional vector $P-0$; this is no longer true in general coordinates, and is replaced by the foregoing differential relation.)

Also we have

$$dP.dP = ds^2 = \bar{v}_i.\bar{v}_j dy^i dy^j = g_{ij}dy^i dy^j , \qquad (III-30)$$

having written g_{ij} for $\bar{v}_i.\bar{v}_j$. g_{ij} is therefore our fundamental metric tensor. It defines completely the internal structure of the fundamental system of vectors $(\bar{v}_i)$ at each point. As we can see, it depends upon the position of the point considered (in the case of cartesian rectilinear coordinates it was constant for the whole space).

3.6 Connection Coefficients

Let us consider a point $P(y^r)$ of space, to which the fundamental system of vectors $(\bar{v}_r) = \frac{\partial P}{\partial y^r}$ is related, and an infinitely near point $P'(y^r + dy^r) = P + dP$ ($dP = \bar{v}_j dy^j$) to which the fundamental system of vectors $(\bar{v}'_r) = \frac{\partial P'}{\partial y^r}$ may be related. We may put

$$\bar{v}'_i = \bar{v}_i + d\bar{v}_i \qquad (III-31)$$

and we want to evaluate $d\bar{v}_i$.

As a first step, and as far as $d\bar{v}_i$ is a vector attached to P, we may write

$$d\bar{v}_i = \omega_i^k \, \bar{v}_k \, , \tag{III-32}$$

ω_i^k being thus the contravariant components of $d\bar{v}_i$ at P.

Moreover, ω_i^k depends upon the displacement dP, i.e., upon its (contravariant) components dy^j; and so far as we may confine ourselves to a first approximation (we neglect infinitesimals of higher order) we may always write ω_i^k as a linear combination of dy^j :

$$\omega_i^k = \Gamma_{ij}^k \, dy^j \, , \tag{III-33}$$

and therefore

$$d\bar{v}_i = \Gamma_{ij}^k \, \bar{v}_k \, dy^j \, . \tag{III-34}$$

The Γ_{ij}^k are systems of the third order, called connection coefficients. Their knowledge gives us, in fact, the means of connecting our fundamental system of vectors to the neighboring.

The connection coefficients are not tensors, as may be easily understood considering that a tensor is essentially an entity attached to one point of space only, whereas the Γ's are attached to the point and to the neighboring.

We may find an algebraic expression for the connection coefficients if we only consider that

$$d\bar{v}_i = \frac{\partial \bar{v}_i}{\partial y^j} \, dy^j \, , \tag{III-35}$$

and thus

$$\frac{\partial \bar{v}_i}{\partial y^j} = \Gamma_{ij}^h \, \bar{v}_h \, . \tag{III-36}$$

If we multiply this equation by $\bar{v}^k$ we get

$$\frac{\partial \bar{v}_i}{\partial y^j} \cdot \bar{v}^k = \Gamma^h_{ij} \, \delta^k_h = \Gamma^k_{ij} \quad , \tag{III-37}$$

which gives us the requested formula.

Let us remember now that $\bar{v}_i \cdot \bar{v}^k = \delta^k_i$ and differentiate

partially with respect to y^j:

$$\frac{\partial \bar{v}_i}{\partial y^j} \cdot \bar{v}^k + \frac{\partial \bar{v}^k}{\partial y^j} \cdot \bar{v}_i = 0 \, ;$$

we may therefore write the following expressions for the Γ's:

$$\Gamma^k_{ij} = \frac{\partial \bar{v}_i}{\partial y^j} \cdot \bar{v}^k = - \frac{\partial \bar{v}^k}{\partial y^j} \cdot \bar{v}_i = \frac{\partial \bar{v}_j}{\partial y^i} \cdot \bar{v}^k = - \frac{\partial \bar{v}^k}{\partial y^i} \cdot \bar{v}_j =$$

$$\tag{III-38}$$

$$= \frac{\partial^2 P}{\partial y^i \partial y^j} \cdot \bar{v}^k = \Gamma^k_{ji} \, .$$

The Γ's are also called, if expressed algebraically,
Christoffel symbols of the second kind. The Christoffel symbols of
the first kind are expressed by the following relations

$$(ij,h) = g_{hk} \Gamma^k_{ij} = (ji,h) = \frac{\partial \bar{v}_i}{\partial y^j} \cdot \bar{v}_h = \frac{\partial \bar{v}_j}{\partial y^i} \cdot \bar{v}_h \tag{III-39}$$

Both the symbols of Christoffel of the first and of the second
kind may easily be expressed in terms of the fundamental tensor g_{ij}
and its first partial derivatives. But we are not interested in
developing these formulae, that are to be found in each book dealing
with tensor calculus.

We may only give a formula that will be useful in the following and which establishes a simple relation between the Christoffel symbols of the second kind and the value $D = |g_{ij}|$ of the determinant of the fundamental tensor.

We have

$$\frac{\partial g_{ij}}{\partial y^k} = \frac{\partial \bar{v}_i \cdot \bar{v}_j}{\partial y^k} = \bar{v}_j \cdot \frac{\partial \bar{v}_i}{\partial y^k} + \bar{v}_i \cdot \frac{\partial \bar{v}_j}{\partial y^k} = (ik,j) + (jk,i) ,$$

and moreover, by contracted multiplication by g^{ij}:

$$g^{ij} \frac{\partial g_{ij}}{\partial y^k} = \Gamma^i_{ik} + \Gamma^j_{jk} = 2 \Gamma^r_{rk} .$$

We now observe that $g^{ij} \frac{\partial g_{ij}}{\partial y^k}$ might be written as follows:

$$g^{ij} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{D} G^{ij} \frac{\partial g_{ij}}{\partial y^k} ,$$

G^{ij} being, by definition of g^{ij}, the algebraic complement of g_{ij} in $|g_{ij}|$ ($G^{ij} = D\, g^{ij}$).

But $G^{ij} \frac{\partial g_{ij}}{\partial y^k}$ is nothing else than the sum of three determinants obtained from g_{ij} by substituting one of the rows with the derivatives of the corresponding elements, and therefore, as it is well known, this development gives us the derivative of D:

$$\frac{1}{D} G^{ij} \frac{\partial g_{ij}}{\partial y^k} = \frac{1}{D} \frac{\partial D}{\partial y^k} = \frac{\partial \lg D}{\partial y^k} .$$

And finally we get the requested formula:

$$\Gamma^r_{rk} = \frac{1}{2} \frac{\partial \lg D}{\partial y^k} = \frac{\partial \lg \sqrt{D}}{\partial y^k} . \tag{III-40}$$

In a three dimensional space the number of Christoffel symbols of each kind is 27 ($= 3^3$), but only 18 are distinct, owing to the fact that the symbols are symmetrical with respect to the two lower indices.

We will still observe that by multiplying $dP = \bar{v}_r \, dy^r$ by $\bar{v}^s$, we get

$$\bar{v}^s . dP = \bar{v}_r . \bar{v}^s dy^r = \delta_r^s dy^r = dy^s \, ,$$

and therefore, by the definition of the gradient, we may write

$$\bar{v}^s = \text{grad } y^s \, , \qquad\qquad\qquad \text{(III-41)}$$

i.e., the reciprocal vectors of each fundamental system are given by the gradients of the parametric coordinates at that point.

3.7   Absolute Differentiation

Let us have a field of vectors $\bar{u}(P) = u_r \bar{v}^r$, each vector being attached to the corresponding point P; we may compare $\bar{u}(P)$ to the neighboring vector $\bar{u}(P+dP) = \bar{u}(P) + d\bar{u}$. As a particular case, we may have identically $d\bar{u} = 0$, i.e., the field of vectors would be formed by equipollent vectors.

From $\bar{u} = u_r \bar{v}^r$ we have by differentiation (interchanging some indices)

$$d\bar{u} = du_r \, \bar{v}^r + u_r \, d\bar{v}^r = \left(\frac{\partial u_k}{\partial y^h} \bar{v}^k + u_r \frac{\partial \bar{v}^r}{\partial y^h} \right) dy^h =$$

$$= \left(\frac{\partial u_k}{\partial y^h} \bar{v}^k + u_r \frac{\partial \bar{v}^r}{\partial y^h} . \bar{v}_k \, \bar{v}^k \right) dy^h = \left(\frac{\partial u_k}{\partial y^h} - u_r \left. \right\lceil_{hk}^r \right) \bar{v}^k \, dy^h \, , \qquad \text{(III-42)}$$

and not $d\bar{u} = \frac{\partial u_k}{\partial y^h} \bar{v}^k dy^h$ as would be in the case of cartesian rectilinear coordinates.

The presence of the term $-u_r \left\lceil_{hk}^r \right.$ is due to the fact that in in passing from P to P + dP not only $\bar{u}$ has varied, but also the reference system of vectors.

We put

$$\frac{\partial u_k}{\partial y^h} - u_r \Gamma^r_{hk} = u_{k/h} \tag{III-43}$$

(read u_k derived h) and call $u_{k/h}$ the covariant derivative of
u_k ; we also may call

$$\delta u_k = \left(\frac{\partial u_k}{\partial y^h} - u_r \Gamma^r_{hk} \right) dy^h \tag{III-44}$$

the absolute differential of u_r .

In a quite similar way we may obtain for contravariant components
and for double tensors the following formulae

$$u^k_{/h} = \frac{\partial u^k}{\partial y^h} + u^r \Gamma^k_{rh} \ ,$$

$$a_{hk/l} = \frac{\partial a_{hk}}{\partial y^l} - a_{hj} \Gamma^j_{kl} - a_{jk} \Gamma^j_{hl} \ , \ \text{etc.} \tag{III-45}$$

Going back to our first differential formula, it is very worthy
to be noted that it may be written as follows:

$$d\overline{u} = u_{k/h} \overline{v}^k dy^h \ , \tag{III-46}$$

and that this formula is derived from the expression $\overline{u} = u_k \overline{v}^k$ by
the ordinary rule of differentiation in cartesian rectilinear coordi-
nates, only substituting the covariant derivative $u_{k/h}$ in place of
the partial derivative $\frac{\partial u_k}{\partial y^h}$. We may also say, therefore, that the
fundamental vectors behave as constants with respect to the covariant
(tensor) differentiation.

This is a way of expressing the fundamental theorem of Ricci.

We may now consider the particular case where $d\overline{u} = 0$ identically,
i.e., the case of a parallel (equipollent) field of vectors. In this
case we have

$$u_{k/h} = 0 \ \text{identically,}$$

and therefore

$$\frac{\partial u_k}{\partial y^h} = u_r \Gamma^r_{hk} \tag{III-47}$$

3.8   Equation of Straight Lines

If we are given an arbitrary curve (Γ) in parametric form $y^i = y^i(s)$, s being the arc measured on the curve, we may put for the sake of simplicity $\frac{dy^i}{ds} = \lambda^i$. Thus, if we call $\bar{t}$ the tangent unit vector to our curve at P, we have

$$\bar{t} = \frac{dP}{ds} = \frac{\bar{v}_i dy^i}{ds} = \lambda^i \bar{v}_i \; ; \tag{III-48}$$

the λ^i's are therefore the contravariant components of the tangent unit vector to a curve.

As it is well known, Frenet's formula gives us

$$\frac{d\bar{t}}{ds} = k\bar{n} = k\nu^i \bar{v}_i \; , \tag{III-49}$$

k being the first curvature (or flexion) of (Γ), and $\bar{k} = \nu^i \bar{v}_i$ the unit vector of its principal normal. The same formula may be written in tensor form as follows:

$$\lambda^i/_r \; \lambda^r = k\nu^i \;\; (\text{because } \frac{d\bar{t}}{ds} = \frac{\partial \bar{t}}{\partial y^r} \frac{dy^r}{ds}) \; . \tag{III-50}$$

If $k = 0$ identically, our curve is a straight line, and therefore

$$\lambda^i/_r \; \lambda^r = 0 \tag{III-51}$$

may be regarded as the tensor differential equation of a straight line in space referred to general curvilinear coordinates. We may also write more explicitly

$$(\frac{\partial \lambda^i}{\partial y^r} + \lambda^k \left\lceil^i_{kr} \right.)\lambda^r = 0 \; ;$$

and moreover, remembering that $\frac{\partial \lambda^i}{\partial y^r} \lambda^r = \frac{\partial \lambda^i}{\partial y^r} \frac{dy^r}{ds} = \frac{d^2 y^i}{ds^2}$,

$$\frac{d^2 y^i}{ds^2} + \lambda^k \lambda^r \left\lceil^i_{kr} \right. = 0. \tag{III-52}$$

This is the final form of the differential equation of a straight line which we will use in the following;

3.9   Some Complements

The great help we are able to find in tensor calculus is essentially due to the fact that formulae may be written in a very compact and transparent form even though they contain a great number of terms, and moreover on the fact that the following rules hold:

(1) A zero tensor (the components of which are zero) in a given system of coordinates, is identically zero in each other system of coordinates (in fact, the transformation between different systems is linear and homogeneous).

(2) Two tensors (of the same kind and order) identical in one system of reference are identical in any other system.

(3) The properties of symmetry and of skew-symmetry of a tensor are irvariant for any transformation of the base vectors and indicate therefore an intrinsic property of the entity represented.

Therefore a tensor equation presents an absolute character, whatever may be the system of coordinates used for writing it.

We will at once show some consequences of the above rules.

First of all, we may very easily recognize by another way that the Christoffel symbols are not tensors; in fact, they are all zero in cartesian coordinates, and this is no longer true in whatever other system of coordinates.

Secondly, let us consider a vector $\bar{u} = u^i \bar{v}_i$ and the homography $\frac{d\bar{u}}{dP}$. In cartesian coordinates we may write, as we have already seen,

$$\frac{d\bar{u}}{dP} = \begin{Vmatrix} \dfrac{\partial u^1}{\partial x^1} & \dfrac{\partial u^2}{\partial x^1} & \dfrac{\partial u^3}{\partial x^1} \\[2mm] \dfrac{\partial u^1}{\partial x^2} & \dfrac{\partial u^2}{\partial x^2} & \dfrac{\partial u^3}{\partial x^2} \\[2mm] \dfrac{\partial u^1}{\partial x^3} & \dfrac{\partial u^2}{\partial x^3} & \dfrac{\partial u^3}{\partial x^3} \end{Vmatrix} \qquad (III-53)$$

and

$$\text{div } \bar{u} = L_i \frac{d\bar{u}}{dP} = \frac{\partial u^1}{\partial x^1} + \frac{\partial u^2}{\partial x^2} + \frac{\partial u^3}{\partial x^3} = \frac{\partial u^i}{\partial x^i} \quad . \qquad \text{(III-54)}$$

If we wish to preserve the tensor character of the formulae written, we must replace the ordinary partial derivatives by the covariant or tensor derivatives, and we will therefore have in any system of coordinates

$$\text{div } \bar{u} = u^i{}_{/i} \quad . \qquad \text{(III-55)}$$

Also remembering the expression for $u^i{}_{/j}$ we have

$$u^i{}_{/i} = \frac{\partial u^i}{\partial y^i} + u^r \left\lceil{}^{-i}_{ri} \right. \quad ,$$

and using the formula recalled in 3.6:

$$\text{div } \bar{u} = u^i{}_{/i} = \frac{\partial u^i}{\partial y^i} + \frac{u^r}{\sqrt{D}} \frac{\partial \sqrt{D}}{\partial y^r} = \frac{1}{\sqrt{D}} \left(\sqrt{D} \frac{\partial u^i}{\partial y^i} + u^i \frac{\partial \sqrt{D}}{\partial y^i} \right) =$$

$$\qquad \text{(III-56)}$$

$$= \frac{1}{\sqrt{D}} \frac{\partial \sqrt{D} u^i}{\partial y^i} \quad .$$

3.10   Curves in Space

Let us have a curve in space defined by its parametric equation $P = P(s)$, s being the arc of the curve. We immediately have that

$$\bar{t} = \frac{dP}{ds} \qquad \text{(III-57)}$$

is the tangent unit vector to the curve.

Considering that $\bar{t}.\bar{t} = 1$, we have $\bar{t}.\frac{d\bar{t}}{ds} = 0$, and therefore $\bar{t}$ and $\frac{d\bar{t}}{ds} = \frac{d^2 P}{ds^2}$ are orthogonal. We put

$$\frac{d\bar{t}}{ds} = \frac{d^2 P}{ds^2} = \frac{\bar{n}}{\rho} \quad , \qquad \text{(III-58)}$$

$\bar{n}$ being the unit vector of the principal normal to the given curve, and ρ being the radius of first curvature (or flexion) of the curve.

Let us next call $\bar{b}$ the unit vector normal to both $\bar{n}$ and $\bar{t}$, and such that the system $(\bar{t},\bar{n},\bar{b})$ may be positive. We have from $\bar{b}.\bar{b} = 1$ and $\bar{b}.\bar{t} = 0$:

$$\frac{d\bar{b}}{ds}.\bar{b} = 0 \; , \quad \frac{d\bar{b}}{ds}.\bar{t} + \frac{\bar{b}.\bar{n}}{\rho} = 0, \text{ and therefore also } \frac{d\bar{b}}{ds}.\bar{t} = 0.$$

That means that $\frac{d\bar{b}}{ds}$ is perpendicular to both $\bar{b}$ and $\bar{t}$ and is directed therefore along $\bar{n}$. We put

$$\frac{d\bar{b}}{ds} = \frac{\bar{n}}{\tau}$$

and call τ the radius of second curvature (or torsion) of the curve.

From $\bar{n}.\bar{n} = 1$, $\bar{n}.\bar{t} = 0$, $\bar{n}.\bar{b} = 0$ we get after this by differentiation, taking into account the foregoing formulae,

$$\frac{d\bar{n}}{ds}.\bar{n} = 0, \quad \frac{d\bar{n}}{ds}.\bar{t} = -\frac{1}{\rho}, \quad \frac{d\bar{n}}{ds}.\bar{b} = \frac{1}{\tau} \; ,$$

and therefore

$$\frac{d\bar{n}}{ds} = -\frac{\bar{t}}{\rho} - \frac{\bar{b}}{\tau} \; .$$

The set of formulae which we summarize

$$\frac{d\bar{t}}{ds} = \frac{\bar{n}}{\rho} \; , \quad \frac{d\bar{n}}{ds} = -\frac{\bar{t}}{\rho} - \frac{\bar{b}}{\tau} \; , \quad \frac{d\bar{b}}{ds} = \frac{\bar{n}}{\tau} \qquad \text{(III-59)}$$

are known as Frenet's or Serret's formulae.

The plane of $\bar{t}$ and $\bar{n}$ is called tangent plane to the curve; the plane of $\bar{t}$ and $\bar{b}$ is called the rectifying plane; and the plane of $\bar{n}$ and $\bar{b}$ is called the normal plane.

3.11   Curves on a Surface

Let us now consider a curve (τ) on a surface Σ; we will indicate by $\bar{N}$ the normal unit vector to Σ, and by $\bar{n}$ the principal normal to (τ). $\bar{t}$ being always the tangent unit vector to (τ)

(perpendicular to both $\bar{n}$ and $\bar{N}$), we may also consider the positive trihedron (surface trihedron) $(\bar{t},\bar{N},\bar{B})$ of normal unit vectors, and put

$$\bar{N}.\bar{n} = \cos\theta\,, \qquad \bar{B}.\bar{b} = \cos\theta\,,$$
$$\bar{n}.\bar{B} = \sin\theta\,, \qquad \bar{N}.\bar{b} = -\sin\theta\,,$$

θ being the angle of obliquity of (γ) on Σ.

Therefore:

$$\bar{n} = \bar{N}\cos\theta + \bar{B}\sin\theta\,,$$
$$\bar{b} = -\bar{N}\sin\theta + \bar{B}\cos\theta\,,$$
$$\bar{N} = \bar{n}\cos\theta - \bar{b}\sin\theta\,,$$
$$\bar{B} = \bar{n}\sin\theta + \bar{b}\cos\theta\,.$$

$$(\text{III-60})$$

By substituting these formulae in Frenet's formulae we easily get

$$\frac{d\bar{t}}{ds} = \frac{\bar{N}}{\rho_n} + \frac{\bar{B}}{\rho_g}\,,$$

$$\frac{d\bar{N}}{ds} = -\frac{\bar{B}}{\tau_g} - \frac{\bar{t}}{\rho_n}\,,$$

$$(\text{III-61})$$

$$\frac{d\bar{B}}{ds} = \frac{\bar{N}}{\tau_g} - \frac{\bar{t}}{\rho_g}\,,$$

which are called the generalized Frenet's formulae for the surface trihedron; and where we have put

$$\frac{1}{\rho_n} = \frac{\cos\theta}{\rho} \qquad (\text{normal curvature})\,,$$

$$\frac{1}{\rho_g} = \frac{\sin\theta}{\rho} \qquad (\text{tangential or geodesic curvature})\,, \qquad (\text{III-62})$$

$$\frac{1}{\tau_g} = \frac{1}{\tau} + \frac{d\theta}{ds} \qquad (\text{geodesic torsion})\,.$$

For geodesic curves on the surface we have identically $\bar{n} = \bar{N}$, and therefore $\theta = \dfrac{d\theta}{ds} = 0$; thus for geodesic curves:

$$\frac{1}{\rho} = \frac{1}{\rho_n}\,, \qquad \frac{1}{\rho_g} = 0\,, \qquad \frac{1}{\tau_g} = \frac{1}{\tau}\,. \qquad (\text{III-63})$$

3.12 Surfaces

Let us consider a surface Σ and its normal $\bar{N}$, and take an arbitrary displacement dP on Σ. If we give the point P the displacement dP, $\bar{N}$ would become $\bar{N} + d\bar{N}$; and from $\bar{N}.\bar{N} = 1$ we have that $\bar{N}.d\bar{N} = 0$, i.e., $d\bar{N}$ is a tangential vector.

The homography (of Burali-Forti)

$$\sigma = \frac{d\bar{N}}{dP} \qquad (III-64)$$

such that $\sigma\, dP = d\bar{N}$ gives us the law of variation of $\bar{N}$ in any direction and defines therefore the shape of Σ in space.

If we consider the parametric form of Σ,

$$P = P(y^1, y^2),$$

y^1 and y^2 being two parameters, we have

$$dP = \frac{\partial P}{\partial y^1} dy^1 + \frac{\partial P}{\partial y^2} dy^2 = \bar{v}_1 dy^1 + \bar{v}_2 dy^2 = \bar{v}_\alpha dy^\alpha = g_{\alpha\beta} \bar{v}^\alpha dy^\beta, \qquad (III-65)$$

and therefore

$$ds^2 = dP.dP = E dy^{1^2} + 2F dy^1 dy^2 + G dy^{2^2} = g_{\alpha\beta} dy^\alpha dy^\beta,$$

(first fundamental form of Σ)

having put

$$g_{11} = E = \bar{v}_1 . \bar{v}_1 , \quad g_{12} = g_{21} = F = \bar{v}_1 . \bar{v}_2 = \bar{v}_2 . \bar{v}_1, \quad g_{22} = G = \bar{v}_2 . \bar{v}_2. (III-66)$$

The $g_{\alpha\beta}$ ($\alpha, \beta = 1, 2$) are the components of the metric (first) fundamental tensor of Σ in our coordinates y^α.

We may observe that $F = \sqrt{EG} \cos\theta$, where θ is the angle of the considered coordinates lines on Σ; and therefore

$$\cos\theta = \frac{F}{\sqrt{EG}} , \quad \sin\theta = \sqrt{\frac{EG - F^2}{EG}} . \qquad (III-67)$$

Considering now σ, we may put

$$\sigma\, \bar{v}_\alpha . \bar{v}_\rho = - b_{\alpha\rho} \qquad (III-68)$$

and call $b_{\alpha\rho}$ the second fundamental tensor of Σ. We have

$$\sigma \bar{v}_\alpha = \frac{\partial \bar{N}}{\partial y^\alpha} = -b_{\alpha\rho}\,\bar{v}^\rho \;;\; d\bar{N} = -b_{\alpha\rho}\,\bar{v}^\alpha\,dy^\rho \qquad \text{(III-69)}$$

and moreover

$$d\bar{N}.dP = -b_{\alpha\rho}\,dy^\alpha\,dy^\rho \text{ (second fundamental form of } \Sigma). \qquad \text{(III-70)}$$

Finally we have

$$d\bar{N}.d\bar{N} = b_{\alpha\beta}\,\bar{v}^\alpha\,dy^\beta.b_{\gamma\delta}\,\bar{v}^\gamma\,dy^\delta = g^{\alpha\gamma}\,b_{\alpha\rho}\,b_{\gamma\delta}\,dy^\rho\,dy^\delta = c_{\beta\delta}\,dy^\beta\,dy^\delta \qquad \text{(III-71)}$$

(third fundamental form of Σ)

and $c_{\beta\delta} = g^{\alpha\gamma}\,b_{\alpha\beta}\,b_{\gamma\delta}$ is called the third fundamental tensor of Σ.

We may observe that $-d\bar{N}.dP = \bar{N}.d^2P$ because $d(\bar{N}.dP) =$
$= d\bar{N}.dP + \bar{N}.d^2P = 0$, and also that from

$$\bar{N}.dP = \bar{N}.\bar{v}_\alpha\,dy^\alpha$$

we get by partial differentiation and interchanging the indices α and β :

$$\frac{\partial \bar{N}}{\partial y^\beta}.\bar{v}_\alpha\,dy^\alpha + \bar{N}.\frac{\partial^2 P}{\partial y^\alpha \partial y^\beta}\,dy^\alpha = 0 ,$$

$$\qquad \text{(III-72)}$$

$$\frac{\partial \bar{N}}{\partial y^\alpha}.\bar{v}_\beta\,dy^\beta + \bar{N}.\frac{\partial^2 P}{\partial y^\alpha \partial y^\beta}\,dy^\beta = 0 ,$$

and therefore

$$\frac{\partial \bar{N}}{\partial y^\beta}.\bar{v}_\alpha = -b_{\alpha\beta} = \frac{\partial \bar{N}}{\partial y^\alpha}.\bar{v}_\beta = -b_{\alpha\beta} . \qquad \text{(III-73)}$$

The second fundamental tensor is therefore symmetric and is usually indicated as $b_{\alpha\beta} = (D,D',D'')$. Also, it follows that the homography σ is a dilatation, owing to the fact that the property of symmetry of tensors is invariant with respect to any transformation.

We remember now that

$$\frac{\partial \bar{N}}{\partial y^\alpha} = -b_{\alpha\rho} \bar{v}^\rho$$

and differentiate this formula covariantly with respect to y^γ

Remembering also that the fundamental base vectors (and their

reciprocals) may be regarded as constants with respect to covariant

differentiation, we get

$$\frac{\partial \bar{N}}{\partial y^\alpha}{}_{/\beta} = \frac{\partial^2 \bar{N}}{\partial y^\alpha \partial y^\beta} - \frac{\partial \bar{N}}{\partial y^\delta} \Gamma^\delta_{\alpha\beta} = -b_{\alpha\rho/\beta} \bar{v}^\rho$$

Interchanging α and β, the central part of the equation

written remains unaltered (on account of the symmetry of the second

partial derivatives, and of the Christoffel symbols of the second kind

with respect to the lower indices); and therefore

$$b_{\alpha\rho/\beta} = b_{\alpha\beta/\rho} \,. \tag{III-74}$$

This expresses in tensor form the Mainardi-Codazzi equations,

which give the integrability conditions which the second fundamental

tensor of a surface must satisfy.

Moreover, if we take a displacement dP in the direction of the

unit vector $\bar{t}$ on Σ, from

$$\frac{\sigma \, dP}{ds} = \sigma \bar{t} = \frac{d\bar{N}}{ds} = -\frac{\bar{B}}{\tau_g} - \frac{\bar{t}}{\rho_n} \,, \tag{III-75}$$

we immediately have that

$$\sigma \, \bar{t}.\bar{t} = -\frac{1}{\rho_n} \quad \text{(normal curvature of } \Sigma \text{ in the direction } \bar{t} \text{)}$$

$$\tag{III-76}$$

$$\sigma \, \bar{t}.\bar{B} = -\frac{1}{\tau_g} \quad \text{(geodesic torsion of } \Sigma \text{ in the direction } \bar{t}\text{)}.$$

The united directions $\bar{c}_1$ and $\bar{c}_2$ of Σ, such that

$$\sigma \bar{c}_1 = \frac{\bar{c}_1}{\rho_1} \quad , \quad \sigma \bar{c}_2 = \frac{\bar{c}_2}{\rho_2} \qquad \text{(III-77)}$$

(which always exist since σ is a dilatation) give us the principal directions of σ; ρ_1 and ρ_2 are the principal radii of curvature of Σ. Along these directions the geodesic torsion is zero.

Also $I_1\sigma$ gives us the mean curvature $\frac{1}{\rho_1} + \frac{1}{\rho_2}$, and $I_2\sigma$ the total curvature $\frac{1}{\rho_1 \rho_2}$ of Σ.

INTRINSIC GEODESY

(See reference 44, 51, 52, 53)

4.1 <u>Intrinsic Geographic Coordinates on a Surface (See reference 26, 27)</u>

If we take a constant anticlockwise (sinistrorsum) trihedron of orthogonal unit vectors $(\bar{a}_1, \bar{a}_2, \bar{a}_3)$ and a variable clockwise (dextrorsum) trihedron of orthogonal unit vectors $(\bar{i}_1, \bar{i}_2, \bar{i}_3)$, we may always write

$$\bar{i}_1 = -\sin\Phi\cos\lambda . \bar{a}_1 - \sin\Phi\sin\lambda . \bar{a}_2 + \cos\Phi . \bar{a}_3 \ ,$$

$$\bar{i}_2 = \qquad -\sin\lambda . \bar{a}_1 \qquad + \cos\lambda . \bar{a}_2 \qquad ,$$

$$\bar{i}_3 = \cos\Phi\cos\lambda . \bar{a}_1 + \cos\Phi\sin\lambda . \bar{a}_2 + \sin\Phi . \bar{a}_3 \ ,$$

$$\text{(IV-1)}$$

$$\bar{a}_1 = -\sin\Phi\cos\lambda . \bar{i}_1 - \sin\lambda . \bar{i}_2 + \cos\Phi\cos\lambda . \bar{i}_3 \ ,$$

$$\bar{a}_2 = -\sin\Phi\sin\lambda . \bar{i}_1 + \cos\lambda . \bar{i}_2 + \cos\Phi\sin\lambda . \bar{i}_3 \ ,$$

$$\bar{a}_3 = \qquad \cos\Phi . \bar{i}_1 \qquad + \sin\Phi . \bar{i}_3 \ ,$$

where Φ and λ are the latitude and longitude (respectively) of $\bar{i}_3$ with reference to $\bar{a}_3$ as an axis (world's axis directed towards the North) and the plane $(\bar{a}_2 , \bar{a}_3)$ as fundamental meridian; and we understand that $\bar{i}_1$ must always be in the plane $(\bar{i}_3 , \bar{a}_3)$, i.e., directed towards the North.

By differentiation we get immediately

$$d\bar{i}_1 = \qquad -\bar{i}_2 \sin\Phi . d\lambda - \bar{i}_3 . d\Phi \ ,$$

$$d\bar{i}_2 = \bar{i}_1 \sin\Phi . d\lambda \qquad -\bar{i}_3 . \cos\Phi . d\lambda \ , \qquad \text{(IV-2)}$$

$$d\bar{i}_3 = \bar{i}_1 . d\Phi \qquad + \bar{i}_2 \cos\Phi . d\lambda \ ,$$

and therefore

$$d\bar{i}_3 . d\bar{i}_3 = (d\bar{i}_3)^2 = d\Phi^2 + \cos^2\Phi d\lambda^2 \ . \qquad \text{(IV-3)}$$

Let us now take a surface Σ, and its normal $\overline{N}$ at a point P and establish a correspondence (Gauss's spherical correspondence) such that $\overline{N} = \overline{I}_3$. Under a certain hypothesis of convexity for the surface which we will assume, the correspondence will be one-to-one; and we also may therefore attribute to P the coordinates Φ and λ of $\overline{I}_3$. We will call them the intrinsic geographical coordinates of P on Σ. In the case in which Σ is the geoid, the intrinsic geographical coordinates coincide with the astronomical coordinates.

We then have

$$\sigma dP = d\overline{N} = \overline{I}_1 \, d\Phi + \overline{I}_2 \cos \Phi \, d\lambda \; ;$$

$\overline{i}_1$ and $\overline{i}_2$ give us the North and East directions on Σ.

If we now consider the fundamental vectors on Σ connected with the intrinsic geographic coordinates, $\overline{v}_1 = \frac{\partial P}{\partial \Phi}$, $\overline{v}_2 = \frac{\partial \dot{P}}{\partial \lambda}$ (the directions along which the longitude, and latitude do not vary, i.e. the geodetic meridians and parallels on Σ), we immediately have from the foregoing formulae for our second fundamental tensor:

$$b_{11} = D = -\frac{\partial \overline{N}}{\partial \Phi} \cdot \overline{v}_1 = -\overline{I}_1 \cdot \overline{v}_1 \quad , \quad b_{21} = D' = -\frac{\partial \overline{N}}{\partial \lambda} \cdot \overline{v}_1 = -\overline{I}_2 \cdot \overline{v}_1 \cos \Phi ,$$

$$b_{12} = D' = -\frac{\partial \overline{N}}{\partial \Phi} \cdot \overline{v}_2 = -\overline{i}_1 \cdot \overline{v}_2 \quad , \quad b_{22} = D'' = -\frac{\partial \overline{N}}{\partial \lambda} \cdot \overline{v}_2 = -\overline{i}_2 \cdot \overline{v}_2 \cos \Phi ,$$

$$\text{(IV-4)}$$

and therefore

$$\overline{v}_1 = \quad \overline{I}_1 \, D - \overline{i}_2 \frac{D'}{\cos \Phi} \quad ,$$

$$\overline{v}_2 = -\overline{i}_1 \, D' - \overline{i}_2 \frac{D''}{\cos \Phi} \quad .$$

$$\text{(IV-5)}$$

If we multiply these relations and remember that $g_{11} = E = \bar{v}_1 \cdot \bar{v}_1$
.... we immediately get

$$g_{11} = E = D^2 + \frac{D'^2}{\cos^2 \phi} \ ,$$

$$g_{12} = g_{21} = F = D'(D + \frac{D''}{\cos^2 \phi}) \ , \qquad \text{(IV-6)}$$

$$g_{22} = G = D'^2 + \frac{D''^2}{\cos^2 \phi} \ .$$

Although in general the first and the second fundamental forms
are not related by equations in finite terms, in this case (of geographic
intrinsic coordinates) they are, owing to the particular spatial meaning
of the coordinates themselves.

As may be seen, the total curvature of Σ is given by

$$K = \frac{1}{\rho_1 \rho_2} = \frac{\cos^2 \phi}{DD'' - D'} \ . \qquad \text{(IV-7)}$$

Two vectors $\bar{a}$ and $\bar{b}$ such that

$$\bar{a} \cdot \sigma \bar{b} = 0 \qquad \text{(IV-8)}$$

and said to be conjugate with respect to σ; it follows from the fact
that $\sigma = K\sigma$ (because σ is a dilatation), that

$$\bar{a} \cdot \sigma \bar{b} = \bar{b} \cdot \sigma \bar{a} \ ,$$

and therefore the property of being conjugate is reciprocal.

We have from the foregoing that

$$d\bar{N} = \bar{i}_1 \, d\phi + \bar{i}_2 \cos \phi \, d\lambda = \sigma \, dP = \sigma \bar{v}_1 \, d\phi + \sigma \bar{v}_2 \, d\lambda \ ,$$

and therefore

$$\sigma \bar{v}_1 \cdot \bar{i}_2 = \bar{i}_1 \cdot \bar{i}_2 = \bar{v}_1 \cdot \sigma \bar{i}_2 = 0 \ ,$$
$$\sigma \bar{v}_2 \cdot \bar{i}_1 = \bar{i}_2 \cdot \bar{i}_1 \cos \phi = \bar{v}_2 \cdot \sigma \bar{i}_1 = 0 \ , \qquad \text{(IV-9)}$$

i.e., the directions of $\bar{v}_1$ and $\bar{v}_2$ are conjugate respectively to the directions of $\bar{i}_2$ and $\bar{i}_1$. This leads to the theorem first formulated by Pizzetti: the direction of the meridian (parallel) on a given surface is conjugate to the East (North) direction with respect to the homography of Burali-Forti (or also with respect to the indicatrix of Dupin). (See reference 31, 32)

We also may observe that

$$\bar{v}_1 \cdot \bar{i}_1 = \sqrt{E}\ \cos \mu = -D\ , \quad \cos \mu = -\frac{D}{\sqrt{E}}\ , \quad \text{tg } \mu = -\frac{D'}{D \cos \phi}\ ,$$

$$\bar{v}_2 \cdot \bar{i}_2 = \sqrt{G}\ \sin \pi = -\frac{D''}{\cos \phi}\ , \quad \text{tg } \pi = -\frac{D''}{D' \cos \phi} \tag{IV-10}$$

give us the azimuths μ and π of the meridian and of the parallel, respectively, and that

$$\bar{v}_1 \cdot \bar{v}_2 = \sqrt{EG}\ \cos \theta = D'(D + \frac{D''}{\cos^2 \phi})\ , \quad \cos \theta = \frac{D'}{\sqrt{EG}}(D + \frac{D''}{\cos \phi})\ \tag{IV-11}$$

give us the angle θ between the meridian and the parallel.

4.2 Curves on a Surface Referred to Geographic Coordinates

If we remember the generalized formulae of Frenet and put

$$\bar{t} = \bar{i}_1 \cos A + \bar{i}_2 \sin A \quad (A = \text{azimuth of } \bar{t})$$

$$\frac{d\bar{t}}{ds} = -\bar{i}_1 \sin A \frac{dA}{ds} + \bar{i}_2 \cos A \frac{dA}{ds} + \frac{d\bar{i}_1}{ds} \cos A + \frac{d\bar{i}_2}{ds} \sin A\ , \tag{IV-12}$$

we immediately get the important formulae valid for an arbitrary curve on Σ :

$$\frac{1}{\rho_g} = -\sin \phi \frac{d\lambda}{ds} + \frac{dA}{ds}\ , $$

$$-\frac{1}{\rho_n} = \cos \phi \sin A \frac{d\lambda}{ds} + \cos A \frac{d\phi}{ds}\ . \tag{IV-13}$$

For a geodesic we have therefore

$$\frac{1}{\rho_g} = 0 \quad , \quad \frac{dA}{ds} = \sin \phi \, \frac{d\lambda}{ds} \quad . \tag{IV-14}$$

Moreover, we may write

$$\bar{t} = \lambda^\alpha \bar{v}_\alpha = \lambda^\alpha \frac{\partial P}{\partial y^\alpha} = \frac{\partial P}{\partial y^\alpha} \frac{dy^\alpha}{ds} \quad , \tag{IV-15}$$

and therefore $\lambda^\alpha = \dfrac{dy^\alpha}{ds}$ are the contravariant components of the tangent unit vector $\bar{t}$ to a curve $P = P(s)$ on Σ . If this curve would be a geodesic, the y^α's would satisfy to an equation similar to that we found in 3.8, i.e.,

$$\frac{d^2 y^\alpha}{ds^2} = - \lambda^\beta \lambda^\gamma \left\lceil \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right. \quad . \tag{IV-16}$$

This formula may be differentiated successively; remembering that, for instance,

$$\frac{d\lambda^\beta}{ds} = \frac{d^2 y^\beta}{ds^2} \quad ,$$

we easily get

$$\frac{d^3 y^\alpha}{ds^3} = (2 \left\lceil \begin{matrix} \alpha \\ \beta \rho \end{matrix} \right. \left\lceil \begin{matrix} \rho \\ \gamma\delta \end{matrix} \right. - \frac{\partial \left\lceil \begin{smallmatrix} \alpha \\ \beta\gamma \end{smallmatrix} \right.}{\partial y^\delta})\lambda^\beta \lambda^\gamma \lambda^\delta \quad , \tag{IV-17}$$

These developments allow us to generalize for a geodesic line drawn on a surface, Legendre's series for the computation of geographic coordinates from polar geodesic coordinates, (see reference 47) i.e.,

$$y^\alpha = y_0^\alpha + s \, \lambda_0^\alpha + \frac{s^2}{2!} \left(\frac{d\lambda^\alpha}{ds} \right)_0 + \frac{s^3}{3!} \left(\frac{d^2 \lambda^\alpha}{ds^2} \right)_0 + \cdots \quad . \tag{IV-18}$$

For the azimuth we get in a similar way the following expansion:

$$\lambda^\alpha = \lambda_0^\alpha + s \left(\frac{d\lambda^\alpha}{ds} \right)_0 + \frac{s^2}{2!} \left(\frac{d^2 \lambda^\alpha}{ds^2} \right)_0 + \cdots \quad . \tag{IV-19}$$

The contravariant components λ^{α} are easily expressed in terms of the azimuth A as follows:

$$\overline{t} = \overline{v}_{\alpha}\lambda^{\alpha} \; ; \; \overline{t} \cdot \overline{i}_1 = \cos A = \lambda^1 \overline{v}_1 \cdot \overline{i}_1 + \lambda^2 \overline{v}_2 \cdot \overline{i}_1 =$$

$$= - D\lambda^1 - D^{\dagger}\lambda^2 \; ,$$

$$\overline{t} \cdot \overline{i}_2 = \sin A = - \frac{D^{\prime}\lambda^1 + D^{\prime\prime}\lambda^2}{\cos \Phi} \quad ,$$

and therefore

$$\lambda^1 = \frac{1}{\Delta} \, (- D^{\prime\prime} \cos A + D^{\prime} \cos \Phi \sin A) \; ,$$

$$\lambda^2 = \frac{1}{\Delta} \, (D^{\prime} \cos A - D \cos \Phi \sin A) \; , \qquad (IV-20)$$

$$\Delta = D D^{\prime\prime} - D^{\prime 2} \; .$$

Formulae perfectly similar to these may also be established for a curve on Σ , provided the geodesic curvature be known as a function of the arc s.

4.3   Intrinsic Geodesy in Space; Absolute Formulae. (See references 16, 44)

Let us consider now the vector $\overline{g}$ of gravity in space and put

$$\overline{g} = - g\overline{n} \; ; \qquad (IV-21)$$

as it is very well known, $\overline{n}$ is the normal unit vector to each equipotential surface of gravity, and g is the intensity of gravity. We take $\overline{g}$ and $\overline{n}$ in opposite directions.

If we differentiate the foregoing formula with respect to P , we obtain a homography (Eötvös's homography)

$$\mathcal{W} = \frac{d\overline{g}}{dP} = \frac{d}{dP} \, \text{grad } W, \qquad (IV-22)$$

W being obviously the potential of gravity; we also will write

$$\sigma = \frac{d\bar{n}}{dP} \qquad\qquad (IV\text{-}23)$$

and call σ the generalized homography of Burali-Forti (generalized, because here σ is no longer related to a single surface, but to a family of surfaces, the equipotential surfaces).

We immediately get

$$\mathcal{W} = \frac{d\overline{g}}{dP} = - g\,\frac{d\bar{n}}{dP} \;-\; \bar{n}\,\frac{dg}{dP} = - g\,\sigma - \bar{n}\,\frac{dg}{dP} \qquad\qquad (IV\text{-}24)$$

where the meaning of $\dfrac{dg}{dP}$ is given by the formula:

$$\frac{dg}{dP}\,dP = dg \ .$$

The operator $\dfrac{dg}{dP}$ is therefore identical to the operator grad g, an hyperhomography operating between vectors and scalars,

$$\frac{dg}{dP} = \text{grad } g \ ; \qquad\qquad (IV\text{-}25)$$

grad g is called the gradient of the intensity of gravity (gradient of gravity).

Our formula may now be written, remembering the dyads, as follows:

$$\mathcal{W} = - g\,\sigma - H(\text{grad } g,\ \bar{n}) \ , \qquad\qquad (IV\text{-}26)$$

and therefore also

$$\sigma = - \frac{1}{g}\left\{ \mathcal{W} + H(\text{grad } g,\ \bar{n}) \right\} \ , \qquad\qquad (IV\text{-}27)$$

The relations between $\mathcal{W}$ and σ now established give us in the simplest way the relation between the mechanical ($\mathcal{W}$) and the geometrical (σ) structure of the gravity field if we confine our attention to its characteristics of the second order.

The gravimetric gradient may immediately be expressed by means of $\mathcal{W}$; we observe, therefore, that for any two vectors $\bar{x}$ and $\bar{y}$ we have, remembering Jacobi's theorem (II-31) and (II-35),

$$\text{grad}(\bar{x}.\bar{y}).dP = d(\bar{x}.\bar{y}) = \bar{y}.d\bar{x} + \bar{x}.d\bar{y} = \bar{y} \cdot \frac{d\bar{x}}{dP} dP + \bar{x} \cdot \frac{d\bar{y}}{dP} dP = \tag{IV-28}$$

$$= K\frac{d\bar{x}}{dP}\bar{y} \cdot dP + K\frac{d\bar{y}}{dP}\bar{x} \cdot dP = (K\frac{d\bar{x}}{dP}\bar{y} + K\frac{d\bar{y}}{dP}\bar{x}) \cdot dP$$

($K\frac{d\bar{x}}{dP}$ is the conjugate of $\frac{d\bar{x}}{dP}$). If $\bar{x}$ and $\bar{y}$ are the gradients of two scalar functions, then

$$K\frac{d\bar{x}}{dP} = \frac{d\bar{x}}{dP} \quad ; \quad K\frac{d\bar{y}}{dP} = \frac{d\bar{y}}{dP} \quad , \tag{IV-29}$$

and we have in this case

$$\text{grad}(\bar{x}.\bar{y}) = \frac{d\bar{x}}{dP}\bar{y} + \frac{d\bar{y}}{dP}\bar{x} \quad . \tag{IV-30}$$

Thus

$$\text{grad}(\bar{g}.\bar{g}) = \text{grad } g^2 = 2g \text{ grad } g = 2\frac{d\bar{g}}{dP}\bar{g} = 2\mathcal{W}\bar{g} \quad ,$$

and therefore

$$\text{grad } g = \frac{1}{g}\mathcal{W}\bar{g} = -\mathcal{W}\bar{n} \quad . \tag{IV-31}$$

We also have in consequence

$$\sigma = -\frac{1}{g}\left\{\mathcal{W} - H(\mathcal{W}\bar{n},\bar{n})\right\} \quad . \tag{IV-32}$$

We may observe that the homography σ is a dilatation if considered on a surface (as already seen), but it is no longer a dilatation in space. We have in fact

$$K\sigma = -\frac{1}{g}\left\{\mathcal{W} + H(\bar{n},\text{grad } g)\right\} \neq \sigma \quad , \tag{IV-33}$$

because $K\mathcal{W} = \mathcal{W}$; and, therefore,

$$K\sigma\bar{n} = 0 \quad , \tag{IV-34}$$

and $\bar{n}$ is the null direction of $K\sigma$ (but not of σ).

The vector grad g is always perpendicular to the surfaces g = constant, which are called isogravitational surfaces.

Since $\bar{g}$ is a gradient, we have furthermore

$$\text{rot } \bar{g} = 0 = -\text{ rot } g\,\bar{n} = -g \text{ rot } \bar{n} + \bar{n} \times \text{grad } g;$$

$$\text{rot } \bar{n} = \frac{1}{g}\,\bar{n} \times \text{grad } g.$$

(IV-35)

We may, moreover, observe that

$$\digamma\bar{n} = -\frac{1}{g}\left\{ \varpi\,\bar{n} + H(\text{grad } g, \bar{n}) \right\}$$

$$= \frac{1}{g}\left\{ \text{grad } g - (\text{grad } g \cdot \bar{n})\,\bar{n} \right\} = \frac{1}{g}\,\text{grad}_\Sigma\,g\,,$$

(IV-36)

$\text{grad}_\Sigma\,g$ being the component vector of grad g tangent to the equipotential surface Σ (also called the surface gradient of g, or the horizontal gradient of gravity).

Owing to the fact that $\bar{n} \times \bar{n} = 0$, we may therefore also write

$$\text{rot } \bar{n} = \frac{1}{g}\,\bar{n} \times \text{grad}_\Sigma\,g = \bar{n} \times \sigma\bar{n}\,.$$

(IV-37)

The vector $\sigma\bar{n} = \frac{d\bar{n}}{dP}\,\bar{n}$ gives us obviously the derivative of the normal unit vector $\bar{n}$ along the line of force; its modulus is therefore $\frac{1}{R} = F$ (curvature of the line of force) and its direction is that of the principal normal $\bar{N}'$ to the line of force itself. If we call furthermore $\bar{B}'$ the binormal to the line of force, we will have

$$\bar{n} \times \sigma\bar{n} = \text{rot } \bar{n} = \frac{\bar{B}'}{R}\,,$$

$$\frac{1}{R} = \frac{1}{g}\,\text{grad}_\Sigma\,g \cdot \bar{N}'\,.$$

(IV-38)

Considering now the divergence of $\bar{n}$, we would have

$$\text{div } \bar{n} = -\text{ div }\frac{\bar{g}}{g} = -\frac{1}{g}\,\text{div }\bar{g} - \bar{g} \cdot \text{grad}\frac{1}{g}$$

$$= \frac{1}{g}\,(4\pi f\delta - 2\omega^2 - \text{grad } g \cdot \bar{n}\,)$$

(IV-39)

(f = constant of gravitation, δ = density of matter).

The divergence of $\bar{n}$ has a precise meaning; let us consider the homography

$$\sigma_{\Sigma} = \sigma - H(\bar{n}, \sigma\bar{n}) = \sigma[1 - H(\bar{n},\bar{n})] \quad ; \qquad \text{(IV-40)}$$

we have

$$\sigma_{\Sigma}\bar{n} = 0 \quad ; \quad \sigma_{\Sigma}\bar{t} = \sigma\bar{t} \; (\bar{t} = \text{tangential vector}); \qquad \text{(IV-41)}$$

i.e., σ_{Σ} operates on tangential vectors like σ and gives a null vector if operating on normal vectors to Σ. We call therefore σ_{Σ} a surface homography; we immediately see that it coincides with the homography of Burali-Forti for Σ which we had already considered.

We also have

$$I_1\sigma_{\Sigma} = I_1\sigma - I_1 H(\bar{n}, \sigma\bar{n}) = I_1\sigma - \bar{n} \cdot \sigma\bar{n} = I_1\sigma \qquad \text{(IV-42)}$$

since $\bar{n}$ and $\sigma\bar{n}$ are orthogonal vectors. $I_1\sigma_{\Sigma}$ equals now the mean curvature $H = \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$ of Σ, therefore

$$H = \text{div } \bar{n} = \frac{1}{g}(4\pi f \delta - 2\omega^2 - \text{grad } g \cdot \bar{n}) \qquad \text{(IV-43)}$$

This is the celebrated formula of Bruns, which may also be written as follows

$$\text{grad } g \cdot \bar{n} = H - \frac{1}{g}(2\omega^2 - 4\pi f \delta) \, . \qquad \text{(IV-44)}$$

The first member is often called (improperly) the vertical gradient of gravity.

4.4   Local Astronomical Reference Systems

Let us consider at a point P the local system of orthogonal unit vectors $(\bar{i}_1, \bar{i}_2, \bar{i}_3)$, at which as usual $\bar{i}_3$ is directed (like $\bar{n}$) towards the zenith of the equipotential surface Σ at P, $\bar{i}_1$ towards the North, and $\bar{i}_2$ towards the East. Along the axes $(\bar{i}_1, \bar{i}_2, \bar{i}_3)$ we imagine to measure the cartesian isometrical coordinates x^1, x^2, x^3.

We will furthermore put

$$\frac{\partial W}{\partial x^r} = W_r \; , \; \frac{\partial^2 W}{\partial x^r \partial x^s} = \frac{\partial W_r}{\partial x^s} = \frac{\partial W_s}{\partial x^r} = W_{rs} = W_{sr} \; , \qquad (IV\text{-}45)$$

and also we will write

$$w_{rs} = \frac{- W_{rs}}{g} \; . \qquad (IV\text{-}46)$$

We immediately have that $g = - W_3$, and

$$\bar{g} = \text{grad } W = - g \, \bar{i}_3 \; ,$$

$$\mathcal{W} = \frac{d \text{ grad } W}{dP} = \frac{d\bar{g}}{dP} = \left\| \begin{array}{ccc} W_{11} & W_{12} & W_{13} \\ \\ W_{21} & W_{22} & W_{23} \\ \\ W_{31} & W_{32} & W_{33} \end{array} \right\| \; , \qquad (IV\text{-}47)$$

$$W_{rs} = \mathcal{W} \bar{i}_r \cdot \bar{i}_s = \mathcal{W} \bar{i}_s \cdot \bar{i}_r \; ,$$

$$\mathcal{W} \bar{i}_1 = W_{11}\bar{i}_1 + W_{12}\bar{i}_2 + W_{13}\bar{i}_3 \; ,$$

$$\mathcal{W} \bar{i}_2 = W_{21}\bar{i}_1 + W_{22}\bar{i}_2 + W_{23}\bar{i}_3 \; ,$$

$$\mathcal{W} \bar{i}_3 = W_{31}\bar{i}_1 + W_{32}\bar{i}_2 + W_{33}\bar{i}_3 \; .$$

The formulae written justify the name of Eötvös homography given to $\mathcal{W}$, as Eötvös was the first geodesist who considered systematically the second derivatives of the potential.

Furthermore

$$I_1\mathcal{W} = \text{div } \bar{g} = W_{11} + W_{22} + W_{33} = 2\omega^2 - 4\pi f \delta \; ,$$

$$\text{grad } g = -\mathcal{W} \bar{i}_3 = - W_{31}\bar{i}_1 - W_{32}\bar{i}_2 - W_{33}\bar{i}_3 \; ,$$

$$\sigma = \left\| \begin{array}{ccc} w_{11} & w_{12} & 0 \\ \\ w_{21} & w_{22} & 0 \\ \\ w_{31} & w_{32} & 0 \end{array} \right\| \; , \qquad (IV\text{-}48)$$

and therefore we get

$$H = I_1\sigma = w_{11} + w_{22} \; , \; K = I_2\sigma = w_{11}w_{22} - w_{12}^2 \qquad (IV\text{-}49)$$

4.5   General Intrinsic Coordinates

The directions forming the local astronomical trihedron $(\bar{i}_1, \bar{i}_2, \bar{i}_3)$ have already an intrinsic character and may therefore be materialized by actual physical measurements proper to Astronomy and Geodesy; but this does not imply that the same trihedrons are the principal trihedrons of a triple congruence of coordinate lines. In fact, we would not be able to find three congruences of curves intersecting in a point and having at this point the given directions of $(\bar{i}_1, \bar{i}_2, \bar{i}_3)$.

We will overlook the proof of this general proposition and will confine ourselves to observing that if we consider any equipotential surface and draw on it in a point the North and East directions, the normal $\bar{i}_3$ along these directions would not cut in general the infinitely near surface along North and East directions.

The astronomical local systems of vectors are therefore not suitable for defining a general system of coordinates in space.

We will instead chose as general coordinates in space the following:

(1) the intrinsic (astronomical) latitude on each equipotential surface, ϕ or y^1 ;

(2) the intrinsic (astronomical) longitude on each equipotential surface, λ or y^2 ;

(3) the potential W or y^3 .

The three families of coordinate surfaces are therefore

(1) the surfaces of equal latitude ,

(2) the surfaces of equal longitude ,

(3) the equipotential surfaces.

We may observe that the parameter W (the potential) is measured, with the exception of an inessential additive quantity and a factor, by the dynamic height.

The three congruences of coordinate curves are

1. the geoidal meridians (along which only Φ is varying) ,

2. the geoidal parallels (along which only λ is varying) ,

3. the isozenithal (or isovertical) lines (along which only W is varying) .

The three fundamental vectors are therefore

$$\bar{v}_1 = \frac{\partial P}{\partial \Phi} = \frac{\partial P}{\partial y^1} , \quad \bar{v}_2 = \frac{\partial P}{\partial \lambda} = \frac{\partial P}{\partial y^2} , \quad \bar{v}_3 = \frac{\partial P}{\partial W} = \frac{\partial P}{\partial y^3} . \quad (IV\text{-}50)$$

The direction of $\bar{v}_3$ is the null direction of σ ; in fact, $\sigma dP = d\bar{n}$, and therefore, since $\bar{v}_3$ is the isozenithal direction, $\sigma \bar{v}_3 = 0$.

If we would attempt to express the fundamental vectors ($\bar{v}_r$) in terms of the local astronomical unit vectors ($\bar{i}_r$), we only want to remember that $\sigma \bar{v}_1 = \bar{i}_1$, $\sigma \bar{v}_2 = \bar{i}_2 \cos \Phi$, $\sigma \bar{v}_3 = 0$, and moreover, that

$$\text{grad } W : \frac{\partial P}{\partial W} = \bar{g} \cdot \frac{\partial P}{\partial W} = - g \, \bar{i}_3 \cdot \bar{v}_3 = 1 ,$$

because

$$\text{grad } W \cdot dP = dW .$$

If we, therefore, put

$$\bar{v}_1 = a_{11}\bar{i}_1 + a_{12}\bar{i}_2 ,$$

$$\bar{v}_2 = a_{21}\bar{i}_1 + a_{22}\bar{i}_2 , \quad\quad (IV\text{-}51)$$

$$\bar{v}_3 = a_{31}\bar{i}_1 + a_{32}\bar{i}_2 + a_{33}\bar{i}_3 ,$$

and remember that

$$\sigma \, \bar{i}_1 = w_{11}\bar{i}_1 + w_{12}\bar{i}_2 \quad ,$$

$$\sigma \, \bar{i}_2 = w_{21}\bar{i}_1 + w_{22}\bar{i}_2 \quad , \qquad (IV-52)$$

$$\sigma \, \bar{i}_3 = w_{31}\bar{i}_1 + w_{32}\bar{i}_2 \quad ,$$

we will have

$$\sigma \bar{v}_1 = \bar{i}_1 = (a_{11}w_{11} + a_{12}w_{21})\bar{i}_1 + (a_{11}w_{12} + a_{12}w_{12})\bar{i}_2$$

$$\sigma V_2 = \bar{i}_2 \cos \Phi = (a_{21}w_{11} + a_{22}w_{21})\bar{i}_1 + (a_{21}w_{12} + a_{22}w_{22})\bar{i}_2 \ , \qquad (IV-53)$$

$$\sigma \bar{v}_3 = 0 = (a_{31}w_{11} + a_{32}w_{21} + a_{33}w_{31})\bar{i}_1 + (a_{31}w_{12} + a_{32}w_{22} + a_{33}w_{32})\bar{i}_2 \ ,$$

and, therefore, immediately

$$a_{11} = + \frac{w_{22}}{K} = - D \quad ; \quad a_{12} = - \frac{w_{12}}{K} = \frac{- D'}{\cos \Phi} \quad ,$$

$$a_{21} = - \frac{w_{12} \cos \Phi}{K} = - D' \ ; \ a_{22} = + \frac{w_{11} \cos \Phi}{K} = \frac{- D''}{\cos \Phi} \ , (IV-54)$$

$$K = w_{11}w_{22} - w_{12}^2 = \frac{\cos^2 \Phi}{\Delta} \quad ; \quad \Delta = D\,D'' - D'^2 \ ,$$

and, moreover,

$$a_{31}w_{11} + a_{32}w_{21} + a_{33}w_{31} = 0 \ ,$$

$$a_{31}w_{12} + a_{32}w_{22} + a_{33}w_{32} = 0 \ , \qquad (IV-55)$$

$$- a_{33}g = 1 \ ,$$

and thus

$$a_{31} = \frac{w_{13}w_{22} - w_{23}w_{12}}{gD} = - H_1 \ ,$$

$$a_{32} = \frac{w_{23}w_{11} - w_{13}w_{12}}{gD} = - H_2 \ , \qquad (IV-56)$$

$$a_{33} = - \frac{1}{g} \quad .$$

We thus have finally:

$$\bar{v}_1 = -D\,\bar{i}_1 - \frac{D'}{\cos \Phi}\,\bar{i}_2 \ ,$$

$$\bar{v}_2 = -D'\bar{i}_1 - \frac{D''}{\cos \Phi}\,\bar{i}_2 \ , \qquad \qquad \text{(IV-57)}$$

$$\bar{v}_3 = -H_1\bar{i}_1 - H_2\bar{i}_2 - \frac{1}{g}\,\bar{i}_3 \ .$$

The meaning of H_1 and H_2 may be seen by observing that

$$\sigma\,\bar{v}_3 = 0 = -H_1\,\sigma\bar{i}_1 - H_2\,\sigma\bar{i}_2 - \frac{1}{g}\sigma\bar{i}_3 \ ,$$

and that

$$\sigma\bar{i}_3 = \sigma\bar{n} = \frac{1}{g}\,\mathrm{grad}_\Sigma\ g \ .$$

Therefore

$$H_1\,\sigma\bar{i}_1 + H_2\,\sigma\bar{i}_2 = \mathrm{grad}_\Sigma\,\frac{1}{g}$$

and multiplying by $\bar{v}_1$ and $\bar{v}_2$, remembering that $\mathrm{grad}_\Sigma\,\frac{1}{g} \cdot \bar{v}_i = \frac{\partial\frac{1}{g}}{\partial y^i}$

and, furthermore, that $\sigma\bar{i}_1 \cdot \bar{v}_1 = \bar{i}_1 \cdot \sigma\bar{v}_1 = 1,\ \sigma\bar{i}_1 \cdot \bar{v}_2 = \sigma\bar{i}_2 \cdot \bar{v}_1 = 0,$

$\sigma\bar{v}_2 \cdot \bar{i}_2 = \cos \Phi$,

we get

$$H_1 = \frac{\partial\frac{1}{g}}{\partial \Phi} \quad , \quad H_2 = \frac{1}{\cos \Phi}\,\frac{\partial\frac{1}{g}}{\partial \lambda} \quad , \qquad \qquad \text{(IV-58)}$$

and, therefore,

$$\bar{v}_1 = -D\,\bar{i}_1 - \frac{D'}{\cos \Phi}\,\bar{i}_2 \ ,$$

$$\bar{v}_2 = -D'\bar{i}_1 - \frac{D''}{\cos \Phi}\,\bar{i}_2 \ , \qquad \qquad \text{(IV-59)}$$

$$\bar{v}_3 = -\frac{\partial\frac{1}{g}}{\partial \Phi}\,\bar{i}_1 - \frac{1}{\cos \Phi}\,\frac{\partial\frac{1}{g}}{\partial \lambda}\,\bar{i}_2 - \frac{1}{g}\,\bar{i}_3 \ ,$$

and also

$$\bar{i}_1 = - \frac{D''}{\Delta} \bar{v}_1 + \frac{D'}{\Delta} \bar{v}_2 \quad,$$

$$\bar{i}_2 = \frac{D' \cos \Phi}{\Delta} \bar{v}_1 - \frac{D \cos \Phi}{\Delta} \bar{v}_2 \quad,$$
(IV-60)

$$\bar{i}_3 = \frac{g \, h \cos \Phi}{\Delta} \bar{v}_1 + \frac{g \, k \cos \Phi}{\Delta} \bar{v}_2 - g \, \bar{v}_3 \quad,$$

having put for simplicity

$$h = \frac{D'' H_1}{\cos \Phi} - D' H_2 = \frac{w_{13} \, \Delta}{g \cos \Phi} = \frac{1}{\cos \Phi} \left(D'' \frac{\partial}{\partial \Phi} \frac{1}{g} - D' \frac{\partial}{\partial \lambda} \frac{1}{g} \right),$$

(IV-61)

$$k = D H_2 - \frac{D' H_1}{\cos \Phi} = \frac{w_{23} \, \Delta}{g \cos \Phi} = \frac{1}{\cos \Phi} \left(D \frac{\partial}{\partial \lambda} \frac{1}{g} - D' \frac{\partial}{\partial \Phi} \frac{1}{g} \right) \quad.$$

4.6   Reciprocal Vectors; Components

We will now consider the reciprocal vectors $\bar{v}^r$, given by the relations $\bar{v}^r \cdot \bar{v}_s = \delta^r_s$; we get by easy computation

$$\bar{v}^1 = \text{grad } \Phi = - \frac{D''}{\Delta} \bar{i}_1 + \frac{D' \cos \Phi}{\Delta} \bar{i}_2 + \frac{g \, h \cos \Phi}{\Delta} \bar{i}_3 \quad,$$

$$\bar{v}^2 = \text{grad } \lambda = \frac{D'}{\Delta} \bar{i}_1 - \frac{D \cos \Phi}{\Delta} \bar{i}_2 + \frac{g \, k \cos \Phi}{\Delta} \bar{i}_3 \quad,$$
(IV-62)

$$\bar{v}^3 = \text{grad } W = \hspace{6cm} - g \, \bar{i}_3 \quad,$$

and also

$$\bar{i}_1 = \hspace{1cm} - D \, \bar{v}^1 \hspace{2cm} - D' \bar{v}^2 - \frac{\partial \frac{1}{g}}{\partial \Phi} \bar{v}^3 \quad,$$

$$\bar{i}_2 = \frac{- D'}{\cos \Phi} \bar{v}^1 - \frac{D''}{\cos \Phi} \bar{v}^2 - \frac{1}{\cos \Phi} \frac{\partial \frac{1}{g}}{\partial \lambda} \bar{v}^3 \quad,$$
(IV-63)

$$\bar{i}_3 = \hspace{6cm} - \frac{1}{g} \bar{v}^3 \quad.$$

The contravariant components $dy^r = (d\Phi, d\lambda, dW)$ of a displacement $dP = \bar{i}_1 dx^1 + \bar{i}_2 dx^2 + \bar{i}_3 dx^3$ are related to the local cartesian coordinates dx^r by the formulae

$$d\Phi = dy^1 = -\frac{D''}{\Delta} dx^1 + \frac{D'\cos\Phi}{\Delta} dx^2 + \frac{g\, h\cos\Phi}{\Delta} dx^3 \quad,$$

$$d\lambda = dy^2 = \frac{D'}{\Delta} dx^1 - \frac{D\cos\Phi}{\Delta} dx^2 + \frac{g\, k\cos\Phi}{\Delta} dx^3 \quad,$$

$$dW = dy^3 = \qquad\qquad\qquad\qquad\qquad - g dx^3 \quad,$$

$$dx^1 = \qquad - D\, dy^1 \qquad - D' dy^2 \qquad - \frac{\partial\frac{1}{g}}{\partial\Phi} dy^3 \quad,$$

$$dx^2 = -\frac{D'}{\cos\Phi} dy^1 - \frac{D''}{\cos\Phi} dy^2 - \frac{1}{\cos\Phi}\frac{\partial\frac{1}{g}}{\partial\lambda} dy^3 \quad,$$

$$dx^3 = \qquad\qquad\qquad\qquad\qquad - \frac{1}{g} dy^3 \quad.$$

(IV-64)

The metric fundamental tensor is given therefore by

$$\left\|g_{rs}\right\| = \left\|\begin{array}{ccc} D^2 + \dfrac{D'^2}{\cos^2\Phi} & D'\left(D + \dfrac{D''}{\cos^2\Phi}\right) & D\, H_1 + \dfrac{D' H_2}{\cos\Phi} \\[3ex] D'\left(D + \dfrac{D''}{\cos^2\Phi}\right) & D'^2 + \dfrac{D''^2}{\cos^2\Phi} & D' H_1 + \dfrac{D'' H_2}{\cos\Phi} \\[3ex] D\, H_1 + \dfrac{D' H_2}{\cos\Phi} & D' H_1 + \dfrac{D'' H_2}{\cos\Phi} & H_1^2 + H_2^2 + \dfrac{1}{g^2} \end{array}\right\| \quad. \quad (IV-65)$$

4.7   Mainardi-Codazzi Equations; Christoffel Symbols

It is now very easy to compute Christoffel's symbols of the second kind, only remembering their definition as given in (3.6):

$$\Gamma_{ij}^{-h} = \frac{\partial\bar{v}_i}{\partial y^j}\cdot\bar{v}^h = \frac{\partial\bar{v}_j}{\partial y^i}\cdot\bar{v}^h = -\frac{\partial\bar{v}^h}{\partial y^i}\cdot\bar{v}_j = -\frac{\partial\bar{v}^h}{\partial y^j}\cdot\bar{v}_i \quad.$$

Before doing this, we must, however, assure that the second derivatives of the point P be reversible, i.e. ,

$$\frac{\partial \bar{v}_i}{\partial y^j} = \frac{\partial \bar{v}_j}{\partial y^i} \quad (= \frac{\partial^2 P}{\partial y^i \partial y^j} = \frac{\partial^2 P}{\partial y^j \partial y^i})$$

This leads us to a particular form of the so-called Mainardi-Codazzi equations (integrability conditions) for the coordinate surfaces.

We will confine ourselves to the computation of only one pair of derivatives; we have thus, for instance,

$$\frac{\partial \bar{v}_1}{\partial \lambda} = \frac{\partial \bar{v}_1}{\partial y^2} = - (\frac{\partial D}{\partial \lambda} + D' \, tg \, \phi) \bar{i}_1 - (\frac{\partial D'}{\partial \lambda} - D \sin \phi \cos \phi) \frac{\bar{i}_2}{\cos \phi} + D' \bar{i}_3 \, ,$$

$$\frac{\partial \bar{v}_2}{\partial \phi} = \frac{\partial \bar{v}_2}{\partial y^1} = - \frac{\partial D'}{\partial \phi} \bar{i}_1 - (\frac{\partial D''}{\partial \phi} + D'' \, tg \, \phi) \frac{\bar{i}_2}{\cos \phi} + D' \bar{i}_3 \, ,$$

and, therefore,

$$\frac{\partial D''}{\partial \phi} - \frac{\partial D'}{\partial \lambda} + D \sin \phi \cos \phi + D'' \, tg \, \phi = 0 \, . \qquad (IV-66)$$

Operating in a similar way on the other fundamental vectors, we easily get the required integrability conditions:

for the ϕ surfaces

$$\frac{\partial H_1}{\partial \lambda} - \frac{\partial D'}{\partial W} + H_2 \sin \phi = 0$$

$$\frac{\partial H_2}{\partial \lambda} - \frac{\partial D''}{\partial W} \frac{1}{\cos \phi} - H_1 \sin \phi + \frac{1}{g} \cos \phi = 0 \, ,$$

for the λ surfaces

$$\frac{\partial H_1}{\partial \phi} - \frac{\partial D}{\partial W} + \frac{1}{g} = 0$$

$$\frac{\partial H_2}{\partial \phi} - \frac{\partial D'}{\partial W} \frac{1}{\cos \phi} = 0 \, , \qquad (IV-67)$$

for the W surfaces

$$\frac{\partial D}{\partial \lambda} - \frac{\partial D'}{\partial \phi} + D' \, tg \, \phi = 0 \quad ,$$

$$\frac{\partial D''}{\partial \phi} - \frac{\partial D'}{\partial \lambda} + D \sin \phi \cos \phi + D'' \, tg \, \phi = 0 \quad ,$$

The Mainardi-Codazzi equations, or integrability conditions, simplify in the following if we remember the meaning of H_1 and H_2 :

$$\frac{\partial D}{\partial \lambda} - \frac{\partial D'}{\partial \phi} + D' \, tg \, \phi = 0 \quad ,$$

$$\frac{\partial D''}{\partial \phi} - \frac{\partial D'}{\partial \lambda} + D \sin \phi \cos \phi + D'' \, tg \, \phi = 0 \quad ,$$

$$\frac{\partial D}{\partial W} = \frac{1}{g} + \frac{\partial^2}{\partial \phi^2} \frac{1}{g} \quad , \qquad\qquad (IV-68)$$

$$\frac{\partial D'}{\partial W} = tg \, \phi \frac{\partial}{\partial \lambda} \frac{1}{g} + \frac{\partial^2}{\partial \phi \partial \lambda} \frac{1}{g} \quad ,$$

$$\frac{\partial D''}{\partial W} = \frac{1}{g} \cos^2 \phi - \sin \phi \cos \phi \frac{\partial}{\partial \phi} \frac{1}{g} + \frac{\partial^2}{\partial \lambda^2} \frac{1}{g} \quad .$$

The last three equations yield us the rate of variation with the dynamic height of the second fundamental tensor of the equipotential surface, and allow us, therefore, to extend in the third dimension the properties of our equipotential surface.

It should be noted at once that the same rate of variation for the intensity of gravity is yielded by Poisson-Brun's formula, as will be seen.

Moreover, for the Christoffel symbols of the second kind

we get:

$$\Gamma^1_{11} = \frac{1}{\Delta}\left(D''\frac{\partial D}{\partial \Phi} - 2D'\frac{\partial D'}{\partial \Phi} + D'\frac{\partial D}{\partial \lambda} + D\,gh\cos\Phi \right) \ ,$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \frac{1}{\Delta}\left(D''\frac{\partial D}{\partial \lambda} - D'\frac{\partial D''}{\partial \Phi} + D'\,gh\cos\Phi \right) ,$$

$$\Gamma^1_{22} = \frac{1}{\Delta}\left(-D''\frac{\partial D''}{\partial \Phi} + 2D''\frac{\partial D'}{\partial \lambda} - D'\frac{\partial D''}{\partial \lambda} + D''\,gh\cos\Phi \right) - \sin\Phi\cos\Phi \ ,$$

$$\Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{\Delta}\left(D''\frac{\partial D}{\partial W} - D'\frac{\partial D'}{\partial W} \right) \ ,$$

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{\Delta}\left(D''\frac{\partial D'}{\partial W} - D'\frac{\partial D''}{\partial W} \right) ,$$

$$\Gamma^1_{33} = \frac{1}{\Delta}\left(D''\frac{\partial H_1}{\partial W} - D'\frac{\partial H_2}{\partial W}\cos\Phi - g\frac{\partial}{\partial W}\frac{1}{g}\cdot h\cos\Phi \right) \ ,$$

$$\Gamma^2_{11} = \frac{1}{\Delta}\left(-D'\frac{\partial D}{\partial \Phi} + 2D\frac{\partial D'}{\partial \Phi} - D\frac{\partial D}{\partial \lambda} + D\,gk\cos\Phi \right) , \qquad (\text{IV-69})$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{\Delta}\left(D\frac{\partial D''}{\partial \Phi} - D'\frac{\partial D}{\partial \lambda} + D'\,gk\cos\Phi \right) + tg\,\Phi \quad ,$$

$$\Gamma^2_{22} = \frac{1}{\Delta}\left(D'\frac{\partial D''}{\partial \Phi} - 2D'\frac{\partial D'}{\partial \lambda} + D\frac{\partial D''}{\partial \lambda} + D''\,gk\cos\Phi \right) ,$$

$$\Gamma^2_{13} = \Gamma^2_{31} = \frac{1}{\Delta}\left(D\frac{\partial D'}{\partial W} - D'\frac{\partial D}{\partial W} \right) ,$$

$$\Gamma^2_{23} = \Gamma^2_{32} = \frac{1}{\Delta}\left(D\frac{\partial D''}{\partial W} - D'\frac{\partial D'}{\partial W} \right) ,$$

$$\Gamma^2_{33} = \frac{1}{\Delta}\left(D\frac{\partial H_2}{\partial W}\cos\Phi - D'\frac{\partial H_1}{\partial W} - g\frac{\partial}{\partial W}\frac{1}{g}\cdot k\cos\Phi \right) ,$$

$$\Gamma^3_{11} = -g\,D; \quad \Gamma^3_{12} = \Gamma^3_{21} = -g\,D'; \quad \Gamma^3_{22} = -g\,D'' \ ;$$

$$\Gamma^3_{13} = \Gamma^3_{31} = \Gamma^3_{23} = \Gamma^3_{32} = 0; \quad \Gamma^3_{33} = -\frac{\partial}{\partial W}\,\lg g =$$

$$= g\,\frac{\cos\Phi}{\Delta}\,(H_1 h + H_2 k \cos\Phi) + \frac{W_{33}}{g} =$$

$$= g\,\frac{\cos^2\Phi}{\Delta^2}\,(D h^2 + 2 D' h k + D'' k^2) + \frac{W_{33}}{g} \ .$$

Also, by expanding the values of h and k,

$$\Gamma^1_{11} = \frac{1}{\Delta}\left[g\left(D''\frac{\partial}{\partial\Phi} - D'\frac{\partial}{\partial\lambda}\right)\frac{D}{g} - 2 D'^2\,\mathrm{tg}\,\Phi\right] \ ,$$

$$\Gamma^1_{21} = \frac{1}{\Delta}\left[g\left(D''\frac{\partial}{\partial\Phi} - D'\frac{\partial}{\partial\lambda}\right)\frac{D'}{g} + D D'\sin\Phi\cos\Phi\right] \ ,$$

$$\Gamma^1_{22} = \frac{1}{\Delta}\left[g\left(D''\frac{\partial}{\partial\Phi} - D'\frac{\partial}{\partial\lambda}\right)\frac{D''}{g} + \frac{1}{2}(D D'' + D'^2)\sin 2\Phi + 2 D''^2\,\mathrm{tg}\,\Phi\right] \ ,$$

$$\Gamma^1_{13} = \frac{1}{\Delta}\left[\left(D''\frac{\partial}{\partial\Phi} - D'\frac{\partial}{\partial\lambda}\right)\frac{\partial}{\partial\Phi}\frac{1}{g} + \left(D'' - \mathrm{tg}\,\Phi\frac{\partial}{\partial\lambda}\right)\frac{1}{g}\right] \ ,$$

$$\Gamma^1_{23} = \frac{1}{\Delta}\left[\left(D''\frac{\partial}{\partial\Phi} - D'\frac{\partial}{\partial\lambda}\right)\frac{\partial}{\partial\lambda}\frac{1}{g} + \left(D''\,\mathrm{tg}\,\Phi\frac{\partial}{\partial\lambda} - \frac{1}{2}D'\sin 2\Phi\frac{\partial}{\partial\Phi}\right)\frac{1}{g}\right.$$
$$\left. - \frac{D'\cos^2\Phi}{g}\right] \ ,$$

$$\Gamma^1_{33} = \frac{1}{\Delta}\left[D''\left(\frac{\partial}{\partial\Phi} - g\frac{\partial\frac{1}{g}}{\partial\Phi}\right) - D'\left(\frac{\partial}{\partial\lambda} - g\frac{\partial}{\partial\lambda}\frac{1}{g}\right)\right]\frac{\partial\frac{1}{g}}{\partial W} \ ,$$

$$\Gamma^2_{11} = \frac{1}{\Delta}\left[g\left(D\frac{\partial}{\partial\lambda} - D'\frac{\partial}{\partial\Phi}\right)\frac{D}{g} + 2 D D'\,\mathrm{tg}\,\Phi\right] \ , \tag{IV-70}$$

$$\Gamma^2_{12} = \frac{1}{\Delta}\left[g\left(D\frac{\partial}{\partial\lambda} - D'\frac{\partial}{\partial\Phi}\right)\frac{D'}{g} - \frac{1}{2}D^2\sin 2\Phi\right] \ ,$$

$$\Gamma^2_{22} = \frac{1}{\Delta} \left[g \left(D \frac{\partial}{\partial \lambda} - D' \frac{\partial}{\partial \phi} \right) \frac{D''}{g} - DD' \sin 2\phi - 2 D'D'' \, tg \, \phi \right] ,$$

$$\Gamma^2_{13} = \frac{1}{\Delta} \left[\left(D \frac{\partial}{\partial \lambda} - D' \frac{\partial}{\partial \phi} \right) \frac{\partial}{\partial \phi} \frac{1}{g} + \left(D \, tg \, \phi \frac{\partial}{\partial \lambda} - D' \right) \frac{1}{g} \right] ,$$

$$\Gamma^2_{23} = \frac{1}{\Delta} \left[\left(D \frac{\partial}{\partial \lambda} - D' \frac{\partial}{\partial \phi} \right) \frac{\partial}{\partial \lambda} \frac{1}{g} - \left(\frac{1}{2} D \sin 2\phi \frac{\partial}{\partial \phi} + D' \, tg \, \phi \frac{\partial}{\partial \lambda} \right) \frac{1}{g} \right.$$
$$\left. + \frac{D}{g} \cos^2 \phi \right] ,$$

$$\Gamma^2_{33} = \frac{1}{\Delta} \left[D \left(\frac{\partial}{\partial \lambda} - g \frac{\partial \frac{1}{g}}{\partial \lambda} \right) - D' \left(\frac{\partial}{\partial \phi} - g \frac{\partial \frac{1}{g}}{\partial \phi} \right) \right] \frac{\partial \frac{1}{g}}{\partial W} ,$$

$$\Gamma^3_{11} = - g D \quad ; \quad \Gamma^3_{12} = - g D' \quad ; \quad \Gamma^3_{22} = - g D'' \quad ,$$

$$\Gamma^3_{33} = - \frac{\partial}{\partial W} \, tg \, g = \frac{2\omega^2 - 4\pi f \, \delta}{g^2} - \frac{g}{\Delta} \left[D'' \left(\frac{\partial \frac{1}{g}}{\partial \phi} \right)^2 - 2 D' \frac{\partial \frac{1}{g}}{\partial \phi} \frac{\partial \frac{1}{g}}{\partial \lambda} + D \left(\frac{\partial \frac{1}{g}}{\partial \lambda} \right)^2 \right]$$
$$- \frac{1}{g\Delta} \left(D + \frac{D''}{\cos^2 \phi} \right) .$$

4.8   Components of Some Vectors and Homographies

It is now easy to give the components of some important vectors and homographies in our fundamental system of intrinsic coordinates; we get, for instance,

for the vector $\bar{g}$ of gravity, $\bar{g} = \bar{v}^3 = grad \, W$

covariant components: $g_i = \bar{g} \cdot \bar{v}_i = \bar{v}^3 \cdot \bar{v}_i = \delta^3_i = (0,0,1)$,

contravariant components: $g^i = \bar{g} \cdot \bar{v}^i = \bar{v}^3 \cdot \bar{v}^i = g^{3i} = (g^{31}, g^{32}, g^{33}) =$
$$= \left(- \frac{g^2 h \cos \phi}{\Delta}, - \frac{g^2 k \cos \phi}{\Delta}, g^2 \right) ;$$

for the gravimetric gradient grad g

covariant components: $g/_i = \frac{\partial g}{\partial y^i}$;

for the homography $\mathcal{W} = \dfrac{d\,\text{grad } W}{dP}$

covariant components:

$$W_{ij} = g_{i/j} = \left(\frac{\partial g_i}{\partial y^j} - g_r \left\lceil \begin{matrix} r \\ ij \end{matrix} \right) = - \left\lceil \begin{matrix} 3 \\ ij \end{matrix} = \mathcal{W}\bar{v}_i \cdot \bar{v}_j = \mathcal{W}\bar{v}_j \cdot \bar{v}_i = \right.$$

$$= \left\| \begin{matrix} gD & gD' & 0 \\ gD' & gD'' & 0 \\ 0 & 0 & \dfrac{\partial \lg g}{\partial W} \end{matrix} \right\| \quad ; \qquad \qquad (\text{IV-71})$$

for the homography $\mathcal{H} = H(\text{grad } g, \bar{g})$

$$H_{ij} = \left\| \begin{matrix} 0 & 0 & \dfrac{\partial g}{\partial \varphi} \\ 0 & 0 & \dfrac{\partial g}{\partial \lambda} \\ 0 & 0 & \dfrac{\partial g}{\partial W} \end{matrix} \right\| \quad ; \qquad \qquad (\text{IV-72})$$

for the generalized Burali-Forti's homography $\mathcal{G} = \dfrac{d\bar{n}}{dP} = -\dfrac{1}{g}\mathcal{W} + \dfrac{1}{g^2}\mathcal{H}$

$$S_{ij} = \left\| \begin{matrix} -D & -D' & \dfrac{\partial g}{\partial \varphi} \\ -D' & -D'' & \dfrac{\partial g}{\partial \lambda} \\ 0 & 0 & 0 \end{matrix} \right\| . \qquad \qquad (\text{IV-73})$$

Brun's equation may be written, therefore, as follows:

$$\frac{\partial g}{\partial W} = \frac{2\omega^2 - 4\pi f \delta}{g} - \frac{D\cos^2\varphi + D''}{\Delta} - \frac{1}{g^2\Delta}\left\{ D'' \left(\frac{\partial g}{\partial \varphi}\right)^2 - 2D' \frac{\partial g}{\partial \varphi}\frac{\partial g}{\partial \lambda} + D\left(\frac{\partial g}{\partial \lambda}\right)^2 \right\} ,$$

and it yields us the rate of variation of the intensity of gravity with

respect to the dynamic height.

APPLICATIONS OF INTRINSIC GEODESY

5.1   Generalized Theory of Eötvös's Torsion Balance (See reference 53)

We consider at a given point P of space, referred to the local astronomical unit vectors $(\bar{i}_r)$, a beam of unit length symbolized by the unit vector $\bar{a}$:

$$\bar{a} = \sin \nu \cos \alpha \, \bar{i}_1 + \sin \nu \sin \alpha \, \bar{i}_2 + \cos \nu \, \bar{i}_3 \qquad (V\text{-}1)$$

(ν = zenith distance, α = azimuth of the beam). The beam may be suspended at P at its baricenter; the acting couple will therefore be given by the moment vector $\bar{M} = (\mathcal{W}\bar{a}) \times \bar{a}$, $\mathcal{W}\bar{a}$ being the (vector) difference between the gravity at the ends of the beam. We easily get

$$(\mathcal{W}\bar{a}) \times \bar{a} = \left\{ \frac{1}{2} (W_{33} - W_{22}) \sin 2\nu \sin \alpha + W_{23}(\sin^2\nu \sin^2 \alpha - \cos^2\nu) + \right.$$

$$\left. + \frac{1}{2} (W_{13} \sin^2\nu \sin 2\alpha - W_{12} \sin 2\nu \cos \alpha) \right\} \bar{i}_1 +$$

$$+ \left\{ \frac{1}{2}(W_{11} - W_{33}) \sin 2\nu \cos \alpha + W_{13}(\cos^2\nu - \sin^2\nu \cos^2 \alpha) + \right.$$

$$\left. + \frac{1}{2} (W_{12} \sin 2\nu \sin^2\alpha - W_{23} \sin^2\nu \sin 2\alpha) \right\} \bar{i}_2 + \qquad (V\text{-}2)$$

$$+ \left\{ \frac{1}{2} (W_{22} - W_{11}) \sin^2\nu \sin 2\alpha + W_{12} \sin^2\nu \cos 2\alpha + \right.$$

$$\left. + \frac{1}{2} \sin 2\nu (W_{23} \cos \alpha - W_{13} \sin \alpha) \right\} \bar{i}_3 \; ,$$

where W_{rs} stands for $\dfrac{\partial^2 W}{\partial x^r \partial x^s}$, and the x^r are measured along the axes $\bar{i}^r$.

We thus see that any instrument based on the measure of moments, like torsion balances, would not separate the second derivatives of W with equal indices; these derivatives may only be determined separately, remembering, in addition to the results of the balance measurements, Laplace's equation

$$W_{11} + W_{22} + W_{33} = 2\omega^2 \qquad (V-3)$$

The theory of all types of balance may easily be derived from the foregoing formulae by particularizing the direction of the beam and the direction along which the scalar moment is measured. We thus get:

Eötvös balance of the first kind (horizontal beam, vertical swinging axis) ,

$$(\omega \bar{a}) \times \bar{a} \cdot \bar{i}_3 = \frac{1}{2} (W_{22} - W_{11}) \sin 2\alpha + W_{12} \cos 2\alpha ; \qquad (V-4)$$

Eötvös balance of the second kind (oblique beam, vertical swinging axis) ,

$$(\omega \bar{a}) \times \bar{a} \cdot \bar{i}_3 = \frac{1}{2} (W_{22} - W_{11}) \sin^2 \nu \sin 2\alpha + $$

$$\qquad (V-5)$$

$$+ W_{12} \sin^2 \nu \cos 2\alpha + \frac{1}{2} \sin 2\nu (W_{23} \cos \alpha - W_{13} \sin \alpha) .$$

Berroth's balance (oblique beam, horizontal swinging axis perpendicular to the vertical plane containing the beam):

$$(\omega \bar{a}) \times \bar{a} \cdot (- \sin \alpha \, \bar{i}_1 + \cos \alpha \, \bar{i}_2) = \frac{1}{2} \sin 2\nu [(W_{11} - W_{33}) \cos^2 \alpha + $$

$$\qquad (V-6)$$

$$+ (W_{22} - W_{33}) \sin^2 \alpha + W_{12} \sin 2\alpha] + \cos 2\nu (W_{13} \cos \alpha + W_{23} \sin \alpha) .$$

5.2   Legendre's Developments in Space (See reference 48, 49)

We already have generalized in 4.2 on an arbitrary surface Legendre's developments for computation of geographical coordinates and azimuth. We are now able to further generalize the same developments in space, and compute both the geographic coordinates and the dynamical height, assuming that the path is a straight line of which the azimuth, the zenith distance, and the length are given.

We immediately have, remembering the differential equation of a straight line (3.8), and using the same procedure as in 4.2,

$$y^r = y_o^r + s\lambda_o^r - \frac{s^2}{2}\left(\Gamma_{ij}^r\right)_o \lambda_o^i \lambda_o^j + \frac{s^3}{3!}\left(2\Gamma_{hi}^r \Gamma_{jk}^i - \frac{\partial \Gamma_{jh}^r}{\partial y^k}\right)_o \lambda_o^j \lambda_o^h \lambda_o^k + - \quad ,$$

$$\text{(V-7)}$$

$$\lambda^r = \lambda_o^r - s\left(\Gamma_{ij}^r\right)_o \lambda_o^i \lambda_o^j + \frac{s^2}{2}\left(2\Gamma_{hi}^r \Gamma_{jk}^i - \frac{\partial \Gamma_{jh}^r}{\partial y^k}\right)_o \lambda_o^j \lambda_o^h \lambda_o^k + - \cdots \, ;$$

and the relationships between λ^r and the azimuth and zenith distance of the path are given by the formulae:

$$\lambda^1 = -\frac{D''}{\Delta} \sin z \cos \alpha + \frac{D'}{\Delta} \cos \phi \sin z \sin \alpha + \frac{1}{\Delta}\left(D' \frac{\partial \lg g}{\partial \lambda} - D'' \frac{\partial \lg g}{\partial \phi}\right) \cos z,$$

$$\text{(IV-8)}$$

$$\lambda^2 = \frac{D'}{\Delta} \sin z \cos \alpha - \frac{D}{\Delta} \cos \phi \sin z \sin \alpha + \frac{1}{\Delta}\left(D' \frac{\partial \lg g}{\partial \phi} - D \frac{\partial \lg g}{\partial \lambda}\right) \cos z ,$$

$$\lambda^3 = - g \cos z ,$$

$$\text{tg } \alpha = \frac{1}{\cos \phi} \frac{D'\lambda^1 + D''\lambda^2 + \lambda^3 \frac{\partial}{\partial \lambda} \frac{1}{g}}{D \lambda^1 + D'\lambda^2 + \lambda^3 \frac{\partial}{\partial \phi} \frac{1}{g}} \quad ,$$

$$\cos z = -\frac{1}{g} \lambda^3 \quad .$$

In classical Geodesy, length was usually measured along geodesics on a surface; in modern Geodesy there is a tendency to measure length along geodesics in space (straight lines) by means of electronic or interferometric devices (Shoran, Hiran, Geodimeter, ...); the foregoing expansions solve, therefore, the first problem of Geodesy in this case.

5.3   Computation of Gravity at a Point of Given Intrinsic Coordinates

Let us have two points in space $P_o(y_o^r)$ and $P(y^r)$, and put $\Delta y^r = y^r - y_o^r$ (Δy^r are not the components of a vector!); let us call, moreover, $\bar{g}$ and $\bar{g}_o$ the gravity vectors at P and P_o respectively; we then will have:

$$\bar{g} = \bar{g}_o + \left(\frac{\partial \bar{g}}{\partial y^i}\right)_o \Delta y^i + \frac{1}{2!} \left(\frac{\partial^2 \bar{g}}{\partial y^i \partial y^j}\right)_o \Delta y^i \Delta y^j + \cdots . \quad (V\text{-}9)$$

Multiplying the foregoing equation by the fundamental vectors $\bar{v}_r$ in P_o, we get the covariant components g_r of $\bar{g}$ in P_o; we furthermore remember that

$$\frac{\partial \bar{g}}{\partial y^i} \cdot \bar{v}_r = \omega \bar{v}_i \cdot \bar{v}_r = g_{r/i} = g_{i/r} = -\Gamma_{ri}^3 , \quad (V\text{-}10)$$

$$\frac{\partial^2 \bar{g}}{\partial y^i \partial y^j} \cdot \bar{v}_r = g_{r/ij} = -\frac{\partial \Gamma_{ri}^3}{\partial y^j} + \Gamma_{rh}^3 \Gamma_{ij}^h + \Gamma_{ih}^3 \Gamma_{rj}^h ,$$

and therefore

$$g_r - \delta_r^3 = -\left(\Gamma_{ri}^3\right)_o \Delta y^i - \frac{1}{2!} \left(\frac{\partial \Gamma_{ri}^3}{\partial y^j} - \Gamma_{rh}^3 \Gamma_{ij}^h - \Gamma_{ih}^3 \Gamma_{rj}^h\right)_o \Delta y^i \Delta y^j + \cdots . \quad (V\text{-}11)$$

By using this formula we may compare the gravity vectors at two different points of space, the intrinsic coordinates of which are given.

If only the value g of the intensity of gravity is required, we simply would have

$$g = g_o + \left(\frac{\partial g}{\partial y^i}\right)_o \Delta y^i + \frac{1}{2!} \left(\frac{\partial^2 g}{\partial y^i \partial y^j}\right)_o \Delta y^i \Delta y^j + \cdots . \quad (V\text{-}12)$$

5.4   Computation of Gravity at a Point of Given Polar Coordinates.

We imagine now that the two given points P_0 and P are connected by a line $P = P(s)$ of length s; we have in this case

$$\bar{g} = \bar{g}_0 + s\left(\frac{d\bar{g}}{ds}\right)_0 + \frac{s^2}{2}\left(\frac{d^2\bar{g}}{ds^2}\right)_0 + \cdots . \qquad (V-13)$$

We get

$$\frac{d\bar{g}}{ds} = \frac{\partial \bar{g}}{\partial y^i}\frac{dy^i}{ds} = \frac{\partial \bar{g}}{\partial y^i}\lambda^i ,$$

$$\frac{d^2\bar{g}}{ds^2} = \frac{\partial^2 \bar{g}}{\partial y^i \partial y^j}\lambda^i\lambda^j + \frac{\partial \bar{g}}{\partial y^i}\lambda^i/_j$$

etc.

where obviously λ^i gives us the components of the tangent unit vector $\bar{t}$ to our line at P_0, and $\lambda^i/_j$ depends upon the nature of the line.

If this line be a geodesic (a straight line), then $\lambda^i/_j = 0$, and, therefore, we have for a straight line

$$g_r - \delta_r^3 = -s\left(\Gamma_{ri}^3\right)_0\lambda_0^i - \frac{s^2}{2!}\left(\frac{\partial \Gamma_{ri}^3}{\partial y^j} - \Gamma_{rh}^3\Gamma_{ij}^h - \Gamma_{ih}^3\Gamma_{rj}^h\right)_0\lambda_0^i\lambda_0^j + \cdots . (V-14)$$

In a perfectly similar way we get for the intensity of gravity

$$g - g_0 = s\left(\frac{\partial g}{\partial y^i}\right)_0\lambda_0^i + \frac{s^2}{2!}\left(\frac{\partial^2 g}{\partial y^i \partial y^j}\right)_0\lambda_0^i\lambda_0^j + \cdots . \qquad (V-15)$$

In all of the above formulae the λ^i's are related to the azimuth and the zenith distance of our path by the same formulae as in 5.2.

5.5   The Local Cartesian Equation of the Geoid

We have for any two points P_0 and P

$$P = P_0 + \left(\frac{\partial P}{\partial y^i}\right)_0\Delta y^i + \frac{1}{2!}\left(\frac{\partial^2 P}{\partial y^i \partial y^j}\right)_0\Delta y^i \Delta y^j + \cdots . \qquad (V-16)$$

We remember that

$$\frac{\partial P}{\partial y^i} = \bar{v}_i \; ; \quad \frac{\partial^2 P}{\partial y^i \partial y^j} = \frac{\partial \bar{v}_i}{\partial y^j} = \frac{\partial \bar{v}_j}{\partial y^i} = -\left.\Gamma_{ij}^r \right. \bar{v}_r \quad \text{etc.,} \quad \text{(V-17)}$$

and, therefore

$$P = P_o + (\bar{v}_i)_o \Delta y^i - \frac{1}{2!} \left(\Gamma_{ij}^r \, \bar{v}_r \right)_o \Delta y^i \Delta y^j + \cdots \quad . \quad \text{(V-18)}$$

By multiplying the foregoing formula by the local astronomic unit vectors $(\bar{i}_s)$ in P_o we get

$$(P - P_o) \cdot \bar{i}_s = \mathcal{X}_s = (\bar{v}_i \cdot \bar{i}_s)_o \Delta y^i - \frac{1}{2!} \left(\bar{v}_r \cdot \bar{i}_s \, \Gamma_{ij}^r \right)_o \Delta y^i \Delta y^j + \cdots, \text{(V-19)}$$

where the scalar expressions of $\bar{v}_i \cdot \bar{i}_s$ may be found in 4.5; we thus have the cartesian coordinates referred to a local system at P_o for a point of given intrinsic coordinates. If we take $\Delta y^3 = 0$, we get the cartesian parametric equation of the level surface through P_o.

The inverse problem of getting the intrinsic coordinates from the cartesian has been already solved by the generalized developments of Legendre given in 5.2; we only need to compute previously the polar coordinates, which is immediate.

5.6   Application to Somigliana's Field (See reference 45, 46, 50)

It is well known from Stoke's theorem that a gravitational field is fully determined in the empty space if one equipotential surface of the field and the total acting mass (or also the value of gravity in one point of the surface) are given.

Pizzetti and Somigliana have fully solved the problem of determining the gravity field (gravitation + centrifugal forces) if one of the equipotential surfaces is an ellipsoid of revolution. Somigliana, in particular, has furnished a formula giving the value of gravity along the ellipsoid itself.

The formulae of Somigliana's field are following (on the ellipsoid):

$$\rho = \frac{a(1-e^2)}{(1-e^2 \sin^2\Phi)^{3/2}} \; , \; N = \frac{a}{(1-e^2 \sin^2\Phi)^{1/2}} \; ,$$

$$g = \frac{a \, g_e \cos^2\Phi + b \, g_p \sin^2\Phi}{\sqrt{a^2 \cos^2\Phi + b^2 \sin^2\Phi}} \; , \; (e^2 = \frac{a^2 - b^2}{a^2}), \qquad (V\text{-}20)$$

$$g_p = \frac{b}{a} g_e + \frac{5}{2} b\omega^2 (1 + \frac{9}{35} e^2 + \frac{47}{180} e^4 + \cdots) \; .$$

(Pizzetti's function)

As we see, only the knowledge of the major axis a, the minor axis b, the velocity of rotation ω, and the gravity g_e at the equator (equivalent to the knowledge of the total mass) are required.

The formula giving the value of g has been expanded into a trigonometrical series of the latitude. The first terms of the series have been determined by Cassinis and Silva, and give the international formula for gravity accepted by the International Association of Geodesy at the Stockholm meeting in 1930.

The acceptance of an ellipsoidal field as a standard field for gravity, as proposed by Somigliana himself, by Dr. W. D. Lambert, and Professor G. Cassinis has eliminated the discrepancy which existed up to 1930 between geometrical and dynamical Geodesy.

The international formula, based on the International Ellipsoid, is the following

$$g = 978,049 \, (1 + 0.005,288,4 \sin^2\Phi - 0.000,005,9 \sin^2 2\Phi) \; .$$

The knowledge of gravity along the ellipsoid of reference allows, as we are going to see, the easy computation of all elements needed at any other point of the space.

We have **on the basic ellipsoid:**

$$g_{11} = \rho^2 \ , \qquad g_{12} = 0 \qquad g_{13} = \frac{\rho}{g}\,\frac{\partial \lg g}{\partial \phi}\ ,$$

$$g_{21} = 0 \ , \qquad g_{22} = N^2 \cos^2\phi \ , \quad g_{23} = 0 \ , \qquad\qquad (V\text{-}21)$$

$$g_{31} = \frac{\rho}{g}\,\frac{\partial \lg g}{\partial \phi}\ , \quad g_{32} = 0 \qquad g_{33} = \frac{1}{g^2}\left\{1 + \left(\frac{\partial \lg g}{\partial \phi}\right)^2\right\}.$$

Moreover, the integrability conditions we have found give us immediately

$$\frac{\partial \rho}{\partial W} = -\frac{1}{g} - \frac{\partial^2}{\partial \phi^2}\,\frac{1}{g}\ , \quad \frac{\partial N}{\partial W} = -\frac{1}{g} + \operatorname{tg}\phi\,\frac{\partial}{\partial \phi}\,\frac{1}{g}\ ,$$

$$\frac{\partial N}{\partial \phi} = (N - \rho)\operatorname{tg}\phi\ , \qquad\qquad (V\text{-}22)$$

and Laplace's equation gives, furthermore,

$$\frac{\partial \lg g}{\partial W} = \frac{2\omega^2}{g} + \frac{1}{g}\left(\frac{1}{\rho} + \frac{1}{N}\right) + \frac{1}{\rho g}\left(\frac{\partial \lg g}{\partial \phi}\right)^2\ . \qquad (V\text{-}23)$$

We therefore get immediately for the Christoffel symbols of the second kind:

$$\Gamma^1_{11} = \frac{\partial}{\partial \phi}\lg\frac{\rho}{g}\ , \quad \Gamma^1_{22} = \frac{N\cos^2\phi}{\rho}\left(\operatorname{tg}\phi - \frac{\partial \lg g}{\partial \phi}\right)\ ,$$

$$\Gamma^1_{13} = \frac{\partial \lg \rho}{\partial W}\ , \quad \Gamma^1_{33} = \frac{1}{g\rho}\,\frac{\partial^2 \lg g}{\partial \phi\,\partial W}\ , \quad \Gamma^2_{12} = -\frac{\rho}{N}\operatorname{tg}\phi\ , \qquad (V\text{-}24)$$

$$\Gamma^2_{23} = \frac{\partial \lg N}{\partial W}\ , \quad \Gamma^3_{11} = g\rho\ , \quad \Gamma^3_{22} = g\,N\cos^2\phi\ ,$$

$$\Gamma^3_{33} = -\frac{\partial \lg g}{\partial W}\ .$$

The quantities denoted by h, k, H_1 and H_2 have the following values:

$$h = \frac{N \cos \phi}{g} \frac{\partial \lg g}{\partial \phi} \quad , \quad k = 0 \quad ,$$

(V-25)

$$H_1 = - \frac{1}{g} \frac{\partial \lg g}{\partial \phi} \quad , \quad H_2 = 0 \ .$$

The covariant components of the gravimetric gradient are given by

$$g_{/1} = \frac{\partial g}{\partial \phi} = - \frac{\rho g}{R} \quad , \quad g_{/2} = 0 \quad , \quad g_{/3} = - g \left\lceil \frac{3}{33} \right. \quad , \qquad \text{(V-26)}$$

and the fundamental and reciprocal vectors are

$$\bar{v}_1 = \rho \bar{i}_1 \quad , \quad \bar{v}_2 = N \cos\phi\, \bar{i}_2 \quad , \quad \bar{v}_3 = \frac{1}{g} \left(\frac{\partial \lg g}{\partial \phi} \bar{i}_1 - \bar{i}_3 \right) \quad ,$$

(V-27)

$$\bar{v}^1 = \frac{1}{\rho} \left(\bar{i}_1 - \frac{\partial \lg g}{\partial \phi} \bar{i}_3 \right) \quad , \quad \bar{v}^2 = \frac{\bar{i}_2}{N \cos \phi} \quad , \quad \bar{v}^3 = - g\, \bar{i}_3 \ .$$

The foregoing formulae give us the most natural generalization of a two-dimensional geometry on the ellipsoid, as usually adopted in Geodesy, to a three dimensional scheme.

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