ON MACH REFLECTION AND STRENGTH OF REFLECTED SHOCK

H. S. TAN
CORNELL UNIVERSITY

JULY 1952
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ABSTRACT

The configuration of non-stationary Mach reflection has a conical similarity in time-space, since there is no fundamental time or space interval involved. This property is easily shown (Part Ib), and can be used to simplify the determination of various disturbance quantities.

For this type of reflection the strength of the reflected shock depends on that of the incident shock and also on the deflection angle. To the first approximation, this reflected wave is a sonic front. Thus, the resulting boundary-value problem in linearized theory is relatively simple, and has been attacked by several investigators. The extension to second-order theory is discussed (Part Id) but is not carried out in detail.

Based on the results obtained from linear theory, it is shown how the shock strength can actually be determined to the second order (Part II). The strength of this reflected shock is found to be of second order, and it vanishes at the triple point.

PUBLICATION REVIEW

The publication of this report does not constitute approval by the Air Force of the findings or the conclusions contained therein. It is published only for the exchange and stimulation of ideas.

FOR THE COMMANDING GENERAL:

[Signature]

Leslie B. Williams, Colonel, USAF
Chief, Flight Research Laboratory
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I. Mach Reflection

(a) General Consideration

Let us consider a plane shock \( I \) propagating into still air with a constant velocity \( \mathbf{U}_s \) in the positive \( x \) direction; the air behind the shock has a uniform velocity \( \mathbf{U}_r \). Let us further denote the flow region ahead of the shock by \( 0 \), and that behind by \( 1 \).

With respect to a frame attached to shock \( I \), this is a stationary flow field. The strength of shock \( I \) is measured by the pressure ratio \( \mathcal{I} = \frac{p_r}{p_o} \), or by the Mach number in flow \( 1 \) referred to frame \( I \), i.e.,

\[
\mathcal{M} = \frac{\mathbf{U}_s - \mathbf{U}_r}{a},
\]

Since

\[
\mathcal{M} = \frac{2 \sigma^2 \sigma - 1}{\sigma + 1}, \quad 1 - \sigma = \frac{\sigma^2 - 1}{2 \sigma} \left( 1 - \frac{1}{\mathcal{M}} \right)
\]

\[
\mathbf{U}_r = \frac{2 (1 - \sigma^2)}{(\sigma + 1) \sigma} a, \quad \mathbf{U}_s = \frac{2 + (\sigma - 1) \sigma}{(\sigma + 1) \sigma} a,
\]

two immediate conclusions can be drawn.

i) Since \( \mathcal{M} > 1 \), i.e., \( \mathcal{M} = \frac{\mathbf{U}_s}{a} \), with respect to flow in \( 1 \), shock \( I \) always propagates with subsonic speed, so that any disturbance in \( 1 \) catches up with the shock. Thus when the shock hits a wall corner, the appearance of Mach reflection is natural unless the turning angle \( \theta \) be so large that the so-called "regular reflection" occurs.
ii) The Mach number of flow is given by

\[ M_1 = \frac{c_1}{a_1} = \frac{2 \left( \frac{1}{\gamma} - \frac{\gamma}{2} \right)}{(2 \gamma - 1)} \sigma \]

Thus the flow behind shock is supersonic \((M_1 > 1)\) when

\[ \sigma < \frac{\sqrt{\left( \frac{1}{\gamma-1} \right)} - \frac{\gamma}{4}}{\frac{\gamma}{2}} = \sigma_1 = 0.566 \]

or

\[ 3 > \frac{2.4}{2.8 \sigma^2 - 4} = \frac{2.4}{2.8 \sigma^2 - 4} = 4.84 \]

and subsonic \((M_1 < 1)\) when \(\sigma > 0.566\) or \(3 < 4.84\).

When such a propagating plane shock hits a corner, the configuration of resulting Mach reflection can be stated as follows:

There is a triple point \(T\); part \(PT\) of shock \(I\) remains straight, being unaffected by the signal from corner \(A\); part \(TQ\) is curved, due to effect of deflection by \(A\); \(TP\) is the reflected wave signaling the presence of \(A\) (in the first approximation, the entire front is sonic in subsonic case, while only part \(TP'\) is sonic in supersonic case); and a slip stream \(TS\) appears, across which entropy and density (but not pressure) are discontinuous. (Fig. 1, 2). The whole region \(C_0\) propagates to the right with velocity \(\sigma\), and at the same time expands radially about origin \(O\) with speed of sound \(a\).

Inside region \(C_0\), the flow is non-steady. However, by a proper transformation of coordinates it can be shown that there exists a conical property, with \(O\) as apex, and time \(t\) as axis of the cone, so that the problem can be reduced to a two dimensional steady one.

The flow will then be signified by two parameters in its conical plane: strength of the incident shock, and angle of wall corner.
An exact solution to the problem is mathematically not practical, so a linear solution is first sought. This has been done by different authors under different assumptions.

If $\mathcal{J}$ is very small, reflected wave is weak. Vorticity $\mathcal{N}$ above the slipstream can be neglected (indeed, $\mathcal{N} \ll O(\mathcal{J}^2)$). However, unless incident shock is also weak, vorticity in region $\mathcal{O} \mathcal{P} \mathcal{T}$ is by no means negligible.

By assuming a weak shock and small corner angle, Bargmann investigated the disturbance-velocity distribution in the whole region $\mathcal{C}_o$ by introducing a velocity potential. This, however, necessitates an assumption that the shock be so weak that the existence of the slip stream and the region of rotational flow can be neglected. Later, not assuming weak shock, Lighthill showed that the disturbance pressure $\mathcal{P}$ in whole region $\mathcal{C}_o$ satisfies one single conical equation in spite of the presence of the slip stream and the rotational flow, and so he calculated the pressure distribution along both the wall and Mach shock. Recently, Ludloff, using the Lorentz transformation, determined both pressure and density field in $\mathcal{C}_o$. His analysis showed the slip stream lies along the radial line $\mathcal{O} \mathcal{T}$, as expected, flow in region $\mathcal{O} \mathcal{T} \mathcal{P}$ being rotational, while that in region $\mathcal{O} \mathcal{T} \mathcal{P}$ irrotational.
Utilizing the results of previous authors, it should be possible to determine the disturbance velocity $u', u'$ in region $0 \leq \phi$, taking into consideration the slip stream, so that the incident shock need not be weak, although the deflection angle would still be assumed small. This solution should prove an immediate improvement over Bargmann's.

Although we allow the shock strength to be not small, we shall in the present report still limit it so that the flow behind the shock is subsonic; the case where the after-flow is supersonic will be left to further investigation.

Essentially, the procedure followed will consist in taking the uniform flow behind the straight shock as our basic flow, and deflection angle $\phi$ as a perturbation. Thus all the perturbing quantities will be expressed in orders of $\phi$. Based on the results of the present investigation, further approximations can be carried out using a technique suggested by Lighthill.

(b) Conical Similarity

Designate the uniform flow field behind the propagating straight shock by 1, the disturbed flow behind the curved Mach shock and inside the reflected front $R$ by 2. Let velocity, pressure, density and entropy be denoted by $u, p, \rho, s$ (i.e., 1, 2), and the corresponding non-dimensional perturbed quantities by $(u', p', \rho', s')$, so that
\[ q_2 = \rho, (\rho u, \rho v) \]
\[ p_2 = \rho, + q, \rho, \rho, \rho \]
\[ p_2 = \rho, (\rho + \rho) \]
\[ M_i = \rho, / q, \]

Then referred to the fixed coordinate system with corner as origin, the equation of continuity, equations of motion, and equation of energy (isentropy along flow line) take on the following forms:

\[
\frac{\partial \rho}{\partial t} + 9, (\rho u) \frac{\partial \rho}{\partial x} + 9, v \frac{\partial \rho}{\partial y} + 9, (\frac{\partial \rho}{\partial x} + \frac{\partial \rho}{\partial y}) = 0
\]

\[
\frac{\partial \rho}{\partial t} + 9, (\rho u) \frac{\partial \rho}{\partial x} + 9, v \frac{\partial \rho}{\partial y} + \frac{\rho \rho}{\partial x} \frac{\delta \rho}{\partial y} = 0
\]

\[
\frac{\partial \rho}{\partial t} + 9, (\rho u) \frac{\partial \rho}{\partial x} + 9, v \frac{\partial \rho}{\partial y} + \frac{\rho \rho}{\partial y} \frac{\delta \rho}{\partial x} = 0
\]

\[
\frac{\partial \rho}{\partial x} + 9, (\rho u) \frac{\partial \rho}{\partial x} + 9, v \frac{\partial \rho}{\partial y} = 0
\]

On introducing the transformation:

\[ x = \frac{3 - q, t}{q, t} \quad \gamma = \frac{3}{q, t} \]

we are referring the flow to a new reference frame which has its origin at \( o \) and moves together with flow, and we are contracting the whole flow field by a scale \( q, t \). This means that, at any instant, the sonic front that originated from \( o \) in the physical plane:

\[ (x - q, t)^2 + \rho^2 = (q, t)^2 \]

reduces to the unit circle in the transformed plane:

\[ x^2 + \gamma^2 = 1 \]

If by this transformation we can eliminate the time dependence, then our problem simplifies to a two-dimensional steady one. That is, the non-steady motion will have a conical property in space-time; the apex of the cone being at \( o \), the axes of the cone being the time ordinate, normal to the plane of the motion.
Now since
\[
\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \frac{1}{c^2} \frac{\partial^2}{\partial x^2} - \frac{1}{c^2} \frac{\partial^2}{\partial y^2}
\]
we have, on introducing the transformation, the following system of equations of continuity, motion and energy
\[
D \rho = M_c (1 + \rho) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]
\[
D u = i \rho \frac{\partial p}{\partial x}
\]
\[
D v = i \rho \frac{\partial p}{\partial y}
\]
\[
D \rho = (1 + \sigma M \rho) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]
where
\[
D = -t \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] - M_c \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]
The form of the system of equations clearly shows that the motion is indeed conical in time-space.

Using vector notation \( \mathbf{q} = (u, v), \mathbf{x} = (x, y) \), these take the form
\[
(\mathbf{x} - M \mathbf{q}) \cdot \nabla \mathbf{q} = i \rho \nabla p
\]
\[
(\mathbf{x} - M \mathbf{q}) \cdot \nabla p = M_c (1 + \rho) \mathbf{v} \cdot \mathbf{q}
\]
\[
(\mathbf{x} - M \mathbf{q}) \cdot \nabla \mathbf{q} = (1 + \sigma M \rho) \mathbf{v} \cdot \mathbf{q}
\]
(\( \nabla \) refers to \( \mathbf{r} \) plane hereafter)

(c) Linearization

Let us suppose that all the perturbation quantities and their derivatives are small, i.e.
\[ u, v, \rho, \rho, \text{etc.} \ll 1 \]
Then we have

\[ r \cdot \nabla \rho = \mathbf{v} \cdot \nabla \mathbf{q} \]
\[ r \cdot \nabla \mathbf{q} = \nabla \rho \]
\[ r \cdot \nabla \mathbf{p} = \nabla \cdot \mathbf{q} \]

or

\[ x \frac{\partial \mathbf{q}}{\partial x} + y \frac{\partial \mathbf{q}}{\partial y} = \mathbf{v} \cdot \nabla \mathbf{q} \]
\[ x \frac{\partial \mathbf{q}}{\partial x} + y \frac{\partial \mathbf{q}}{\partial y} = \frac{\partial \mathbf{p}}{\partial x} \]
\[ x \frac{\partial \mathbf{q}}{\partial x} + y \frac{\partial \mathbf{q}}{\partial y} = \frac{\partial \mathbf{p}}{\partial y} \]
\[ x \frac{\partial \mathbf{q}}{\partial x} + y \frac{\partial \mathbf{q}}{\partial y} = \frac{\partial^2 \mathbf{p}}{\partial x^2} + \frac{\partial^2 \mathbf{p}}{\partial y^2} \]

In conformity with the approximation, it may be noted that the reflected front \( \mathbf{F} \) coincides with the sonic front, i.e., \( r = 1 \), and slip stream \( \mathbf{S} \) coincides with radial line \( \theta = \cos^{-1}(M_s - M) \).

By eliminating the undesired variables, it is easily obtained.

i) Pressure perturbation

\[ \mathbf{v} \cdot \nabla \mathbf{q} = (r \cdot \mathbf{v} + 1) \nabla \mathbf{q} = \mathbf{v} \cdot (r \cdot \nabla \mathbf{q}) \]

i.e.

\[ \nabla^2 \rho = \mathbf{v} \cdot \left( \frac{\partial \mathbf{p}}{\partial r} \right) \left( \frac{\partial \mathbf{p}}{\partial r} \right) \]

Thus even when vorticity is present, \( \rho \) satisfies a differential equation which is well known from usual steady conical flow problem. The problem is immediately solved by introducing the Chaplygin transformation, with boundary values determined by consideration of Rankine-Hugoniot conditions. Accordingly we have

\[ \nabla_s \rho = 0 \]
where

\[ s = \frac{1 - \sqrt{1-r^2}}{r} \]  \hspace{1cm} (10)

is the Chaplygin transformation. This is the approach by which (2) Lighthill calculated the pressure distribution.

(ii) Density perturbation

\[ \nabla \rho = \rho_0 \nabla \rho \]  \hspace{1cm} (11)

so in a region where there is no vorticity, it is immediately obtained that \( \rho = \rho_0 \rho_0 \). However, it should be noted here that \( \rho_0 \) is now divided into two separate regions by the slip stream, and this nice simple relation holds only in region \( \sigma \rho \). For region \( \sigma \rho \), \( \rho \) has different boundary condition than \( \rho \) and the solution will accordingly be different. This fact is clearly borne out by Ludloff's calculation. (3)

(iii) Velocity perturbation

or

\[ \nabla \times \nabla \rho = \nabla \cdot \rho \]  \hspace{1cm} (12)

By elimination, it is easily obtained that

\[ (\nabla \cdot \rho + 2) \left[ \nabla \cdot (\nabla \cdot (\nabla \cdot \rho) - \rho \nabla ^2 \right] \rho = 0 \]  \hspace{1cm} (12)

Now

\[ (\nabla \cdot \rho + 2) \rho = 0 \]

gives

\[ \rho = \frac{1}{\sigma^2} \phi(\theta) \]

So on introducing Chaplygin transformation (10), we obtain in general

\[ \nabla_s^2 \rho = f(s) \phi(\theta) \]  \hspace{1cm} (13)

i.e., \( u, v \) satisfies a Poisson differential equation.
However, if the flow is irrotational, elimination again results in equation (9).

\[ \nabla^2 \mathcal{F} = (\mathbf{r} \cdot \nabla + 1) (\mathbf{r} \cdot \nabla) \mathcal{F} \]

or

\[ \mathcal{S} \mathcal{F} = 0 \]

Thus, if we confine our attention to the vorticity free region \( \sigma \sigma \), it is seen that the problem simplifies to solving the equation (13') subject to following boundary conditions:

i) along wall \( \sigma \sigma \), \( u = 0 \), \( \frac{\partial \mathcal{F}}{\partial n} = 0 \);
ii) over \( \mathcal{F} \), i.e.,

\[ r = 1 \], \( u = v = 0 \); and
iii) on \( \sigma \sigma \), normal velocity component vanishes since this is a slipstream.

(d) Higher Approximations

From above, it is clear that with boundary conditions properly determined, all first order quantities \( u, v, \rho, \rho \) can be calculated. Now suppose these have been found. A second approximation can be readily carried out in following manner:

Putting \( \mathcal{S} = \mathcal{S}^{(0)} + \mathcal{S}^{(1)} + \mathcal{S}^{(3)} \)

where \( \mathcal{S} \) denotes any of \( u, v, \rho, \rho \), and collecting the second order terms in eqs. (6), we obtain

\[ \begin{align*}
\mathcal{S} \mathcal{S}^{(2)} & - \mathcal{M}, \mathcal{G}^{(2)} : \mathcal{O} \rho^{(2)} = \mathcal{M}, [\rho^{(2)} \mathcal{G}^{(2)} + \mathcal{O} \rho^{(2)}] \\
\mathcal{S} \mathcal{S}^{(2)} & - \mathcal{M}, \mathcal{G}^{(2)} : \mathcal{O} v^{(2)} = - \rho^{(2)} \mathcal{O} v^{(2)} + \mathcal{O} \rho^{(2)} \\
\mathcal{S} \mathcal{S}^{(2)} & - \mathcal{M}, \mathcal{G}^{(2)} : \mathcal{O} u^{(2)} = \mathcal{M}, \rho^{(2)} \mathcal{G}^{(2)} + \mathcal{O} \rho^{(2)}
\end{align*} \]
Thus the equation for the second-order term in $p$ becomes

$$\nabla^2 p^{(2)} = (r \frac{2}{\partial r} + s) \left(r \frac{\partial p^{(0)}}{\partial r}\right)$$

$$= \mathcal{M}_s \left\{ \begin{array}{l}
\nabla^2 p^{(0)} + \nabla p^{(0)} \cdot \nabla p^{(0)} - \chi^{(0)} \cdot \nabla \chi^{(0)} \\
- \left[ \chi^{(0)} \cdot \nabla \left( r \frac{\partial^2 \chi^{(0)}}{\partial r^2} + s \chi^{(0)} \right) \right] \left( r \frac{\partial p^{(0)}}{\partial r} \right)
\end{array} \right\}$$

or

$$\nabla^2 p^{(2)} = \mathcal{M}_s \left( p^{(0)} \cdot p^{(0)} \right)$$

The corresponding boundary conditions over the wall can be obtained by standard procedure, those along Mach and reflected shocks should be obtained by first determining the second order values on the original boundary using Lighthill's technique. Thus the work involved in a second order solution will in general be prohibitive, and accordingly not attempted. Only the shock strength of reflected wave, which is of physical interest, is investigated in next part of this report.
II Strength of Reflected Shock

Since for small wall deflection angle $\theta$ the flow behind straight shock and slip stream is irrotational,* we are justified to introduce a velocity potential $\Phi$, which, on account of the conical property shown before, further reduces to the form

$$\Phi (r^*, \theta, t) = a_i t f (r, \theta) \quad (1)$$

where we have put $a_i = \frac{v_{i}}{\rho_i}$, $r = \frac{r^*}{a_i t}$.

Now denoting partial derivatives by subscripts, the radial and tangential velocity components $u$, $v$ can be written

$$u = \Phi_r = f_r a_i, \quad v = \frac{1}{\rho_i} \Phi_\theta = \frac{a_i}{\rho_i} f_\theta$$

and since

$$(\Phi_t)_r = (\Phi_r)_r + \Phi_r r_t = (\dot{f} - r f_r) a_i^2$$

$$(\Phi_{tt})_r = \frac{r^2}{t^2} f_{rr} a_i^2$$

So we have

i) $\Phi_t + \frac{1}{t} (u^2 + v^2) + \frac{a_i^2}{\rho_i} = \frac{a_i^2}{\rho_i}$

$$a_i = \left\{ - (\rho_i t) [f_r r_r + \frac{1}{t} (f_t r + r^2 f_\theta^2)] \right\} a_i \quad (2)$$

ii) $a_i^2 \frac{\partial}{\partial t} \frac{\partial}{\partial r} \Phi = \Phi_{tt} + 2 \sum u_i \frac{\partial u_i}{\partial t} + \sum \Sigma u_i v_j \frac{\partial u_j}{\partial r}$

$$\frac{a_i^2}{t} \frac{\partial^2 \Phi}{\partial r^2} = a_i \left[ \frac{r^2}{t^2} f_{rr} + 2 \left[ f_r r_r (- \frac{v}{t}) + \frac{1}{t} f_\theta (r f_\theta - r f_\theta) \right] \right]$$

$$+ \frac{1}{t} \left[ - f_r r_r + f_\theta r_\theta + \frac{r f_\theta \left( \frac{\partial}{\partial r} (r f_\theta) - \frac{1}{r} \frac{\partial}{\partial \theta} f_\theta \right) \right]$$

* $\sqrt{\mathcal{R}} = O (\mathcal{V}^3)$

Bargmann, p. 16

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as \[ \frac{\partial^2 f}{\partial x^2} = \frac{1}{u} \frac{\partial^2 f}{\partial t^2} \]

\[ u_t = -\frac{a}{r} f_{rr} \]
\[ v_t = -\frac{a}{r^2} \left( -\frac{a}{r^2} + \frac{1}{r} f_{rr} \right) \]

i.e.

\[ \frac{a^2}{r^2} \frac{\partial^2 f}{\partial r^2} = r^2 f_{rr} + 2 \left[ -rf_{rr} + \frac{1}{r^2} f_{r}^2 - \frac{1}{r} f_{r} f_{r} \right] \]

\[ + \left[ r^2 f_{rr} + \frac{1}{r^2} f_{r}^2 f_{\theta} - \frac{r f_{r} f_{\theta}^2}{r^2} + \frac{1}{r} f_{r} f_{\theta} f_{\theta} \right] \]

Eliminating \( \frac{a}{r} \) from i) and ii)

\[ \left[ f_{rr} + \frac{1}{r} f_{r} + \frac{1}{r^2} f_{\theta} \right] \left\{ 1 - (r^2 - 1) \left[ r f_{rr} + \frac{1}{r} \left( r^2 + r^2 f_{\theta}^2 \right) \right] \right\} \]

\[ = r^2 f_{rr} - 2 rf_{rr} f_{rr} + \frac{1}{r^2} f_{r}^2 - \frac{1}{r} f_{r} f_{r} \]
\[ + f_{r}^2 f_{rr} + \frac{f_{r}^2 f_{\theta}^2}{r^2} - \frac{f_{r} f_{r} f_{\theta}}{r^2} + \frac{1}{r^2} f_{r} f_{\theta} f_{\theta} \]

So

\[ \left\{ 1 - (r^2 - 1) \left[ f + \frac{1}{r} \frac{1}{r} f_{r} \right] + (r^2 - 1) \left[ r f_{r} - \frac{1}{r} f_{r} \right] \right\} f_{rr} \]
\[ + \left\{ \text{terms involving } f_{r}, f_{r}, f_{\theta}, f_{\theta} \right\} = 0 \]

(4)

Referring to coordinates fixed with respect to shock, the Rankine-Hugoniot condition can be written

i) \[ \frac{\partial f}{\partial t} = \text{continuous} \] \hspace{1cm} (5)

ii) \[ \Delta \frac{\partial f}{\partial t} = \frac{1}{r} \left( \frac{a^2}{r^2} - \frac{\partial^2 f}{\partial r^2} \right) \]

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where \( \mathbf{g}^{(t)} \) and \( \mathbf{g}^{(n)} \) denote velocity components tangential and normal to the shock respectively, and \( \Delta(\cdot) \) denotes the increase of a quantity across the shock front.

Now since \( \mathbf{F} \) exists, and flow is steady in the \((r, \theta)\) plane, let the reflected shock front be \( r = \eta(\theta) \), then

\[
\mathbf{u} = a_{r}(-r + f_{r}) \quad \mathbf{v} = \frac{\varphi_{r}}{r} \mathbf{f}_{\theta}
\]

\[
\mathbf{g}^{(t)} = \frac{f_{\theta} + \eta'(\theta)(f_{r} - \eta)}{\sqrt{1 + r^{-2} \eta'^{2}}} a_{r},
\]

\[
\mathbf{g}^{(n)} = \frac{-\eta f_{r} + r^{-2} \eta' f_{\theta}}{\sqrt{1 + r^{-2} \eta'^{2}}} a_{r},
\]

i) gives

\[
\Delta \left[ f_{\theta} + \eta'(\theta)(f_{r} - \eta) \right] = 0
\]

ii) \( \Delta \mathbf{g}^{(n)} = \frac{-\Delta f_{r} - r^{-2} \eta' \Delta f_{\theta}}{\sqrt{1 + r^{-2} \eta'^{2}}} a_{r} = -\frac{1}{r_{+1}^{2}} \frac{\varphi_{r}}{r} \mathbf{f}_{\theta}
\]

i.e.,

\[
\frac{2}{r_{+1}^{2}} \left( \frac{a_{r}^{2}}{\mathbf{g}^{(n)}} - \mathbf{g}^{(m)} \right) = \frac{2}{r_{+1}^{2}} \left[ \frac{-f_{r} - r^{-2} \eta' f_{\theta}}{\mathbf{g}^{(m)}} - \mathbf{g}^{(n)} \right]
\]

Therefore we have

\[
\Delta f_{r} = -\frac{2}{r_{+1}^{2}} \left[ \frac{a_{r}^{2}}{\mathbf{g}^{(m)}} (f_{r} - r^{-2} \eta' f_{\theta}) - \frac{\eta f_{r}}{1 + r^{-2} \eta'^{2}} \right]
\]

\[
= -\frac{2}{r_{+1}^{2}} \left[ \frac{1 - (r^{-1})(f_{r} - r f_{r})}{1 + (r^{-1}) - f_{r}} - f_{r} \right]
\]

\[
= \frac{2}{r_{+1}^{2}} \left[ (r^{-1})(f_{r} - r f_{r}) - 2 f_{r} + (r^{-1}) \right]
\]

(6)

Since \( f_{r} = f_{rR} = 0 \) on \( R=1 \) and \( f_{rR} \sim A(\theta) \sqrt{1 - R} \)

in linear theory.
But
\[
\frac{\phi - \phi_i}{\rho, \sigma^2} = -\frac{d^2}{d\xi^2} = -\frac{f + Rf}{M_i} \sim A\theta \sqrt{1 - R} \quad R \to 1
\] (7)

So
\[
f_{i, R} = M_i^{-2} A\theta \sqrt{1 - R}
\] (8)

Following Lighthill by putting
\[
r = R + r(\theta) = R + r_1(\theta) + r_2(\theta) + \ldots
\]
\[
f = f_1(R, \theta) + f_2(R, \theta) + \ldots
\] (9)

in (4) and condition on \( R = 1 \), that the linear coefficients in (4) must vanish; and noting that although since
\[
\frac{\partial f}{\partial R} \rightarrow \frac{\partial f}{\partial R} \quad \frac{\partial f}{\partial \theta} \rightarrow \frac{\partial f}{\partial \theta} - r'(\theta) \frac{\partial f}{\partial R}
\]
terms \( f \theta \) and \( f \theta \theta \) may produce terms in \( f_{RR} \), they are however all not of dimension 1; we have
\[
-2R r - (\theta - 1) (f_{i - Rf_{iR}}) + 2R f_{iR} = 0 \quad \text{on} \quad R = 1
\] (10)
i.e. \( r_1(\theta) = 0 \)

Now
\[
\lim_{R \to 1} [(\theta - 1) R f_{iR} + 2R f_{iR}] f_{RR} = (\theta - 1) R f_{iR} f_{RR}
\]
So
\[
f_{2R} + f_2 \theta = \frac{\theta - 1}{2} M_i^{-1} A_2(\theta)
\]
\[-2r_2 - (\theta - 1) (f_2 - f_{2R}) + 2f_{2R} \] \( R = 1 \) = 0
\[
r_2(\theta) \sim A^2(\theta)
\] (11)
The shock condition at \( r = \gamma(\theta) \) are: Since \( f \equiv 0 \) ahead of shock, so directly behind it

\[
f = 0 \quad f_R = \frac{u (\gamma - 1)}{\gamma} \quad (12)
\]

Shock cones in range \( R < \gamma \), \( \gamma - 1 + \gamma_1 + \gamma_2 + \cdots \)

so \( \gamma - 1 \) of dimension 2, \( \gamma = 1 + \gamma_1 + \gamma_2 + \cdots \), \( 0 \leq \gamma_2 \leq \gamma_1 \).

In condition \( f = 0 \) on \( r = \gamma(\theta) \), terms \( f_1, f_2, \ldots \) can be expanded by Taylor expansion (at \( R = 1 \)) using \( R = 1 + (\gamma_2 - \gamma_1) + (\gamma_3 - \gamma_2) + \cdots \)

Terms of dimension 1 and 2 give

\[
\begin{align*}
(f_1)_R &= 0 \quad (f_2)_R = 0 \\
(f_3)_R &= \frac{r_3}{2} M_1^4 A^2(\theta) \\
\gamma_2(\theta) &= \left(\frac{r_3}{4}\right) M_1^4 A^2(\theta) \\
\end{align*}
\]

(13)

To express the second condition of (2), \( f_R \) is needed when

\( R = 1 + (\gamma_2 - \gamma_1) + (\gamma_3 - \gamma_2) + \cdots \)

The largest terms here are of dimension 2, and these come from

\( f_3 R \sim M_1^2 A(\theta) \sqrt{1 - \gamma} \quad \text{and from} \quad (f_2 R) \) \( R = 1 \)

so they are

\[
M_1^2 A(\theta) \sqrt{1 - \gamma} + (f_2 R) \quad R = 1
\]

by (12) and (13)

\[
M_1^2 A(\theta) \sqrt{\left(\frac{r_3}{4}\right) M_1^4 A^2(\theta) - \gamma} + \frac{r_3}{2} M_1^4 A^2(\theta) = \frac{u \gamma_2}{\gamma - 1}
\]

(16)

Putting \( \gamma_2 = \left(\frac{r_3}{4}\right) M_1^4 K(\theta) \), this becomes

\[
A(\theta) \sqrt{A^2(\theta) - K(\theta)} = 2 K(\theta) - A^2(\theta)
\]

(15)
The solution of (15) gives

\[ K(\theta) = \frac{3}{4} A^2(\theta) \quad \text{for} \quad A(\theta) > 0 \]
\[ = 0 \quad \text{for} \quad A(\theta) < 0 \] (16)

For the case of compression, \( A(\theta) > 0 \), the shock is thus located at

\[ r = 1 + \frac{3}{16} (n-1)^2 M_1^2 A^2(\theta) \] (17)

Now the pressure change across the shock is

\[ \frac{p_1}{P_0} = \frac{\rho_0}{M_1^2} (r \gamma - 1) = \rho_0 \frac{\gamma^2 n - 1}{(n-1) M_1^2} \]

Since \( \rho_0 P_0 = \alpha \rho P \), the shock strength is given by

\[ \frac{A P}{P_0} = \frac{\gamma^2 n - 1}{(n-1) M_1^2} \quad \tau_n(\theta) = \frac{3}{4} n(n+1) M_1^2 A^2(\theta) \] (18)

where

\[ A(\theta) = \lim_{r \to 1} \left( \frac{P - P_1}{\rho_0 \frac{\gamma^2 n - 1}{(n-1) M_1^2}} \right) \]

on linear theory

\[ = \frac{\partial P}{\partial \eta} \quad \text{in Chaplygin plane} \]

and \( \frac{\partial P}{\partial \eta} \) as given by Ref. 2 (59) is

\[ \frac{\partial P}{\partial \eta} = \int \frac{d^2 f}{d \xi^2} \left[ \frac{c}{D(x_1 - x_2) - 1} \right] \frac{\left[ D(x_1 - x_2) - 1 \right] \left[ D(x_1 - x_2) - 1 \right]}{\left[ D(x_1 - x_2) - 1 \right] \left[ D(x_1 - x_2) - 1 \right]} \]

\[ at \text{triple point} \quad (x = r, \quad x_1 = \infty) \quad \text{is} \quad 0, \quad \text{so} \]

strength of reflected shock vanishes at the triple point.
CONCLUSIONS

The conical property in time-space of diffraction of propagating plane shock is discussed. Second order solution for pressure $p$ has been suggested, but since this requires more information regarding $\alpha$, $\beta$ and the second-order boundary conditions than is presently available, actual carrying out of the solution is not attempted.

The strength of reflected shock, however, has been investigated; and it is found to be generally of order $\mathcal{O}$: $\bullet$. In particular, this vanishes at the triple point. The immediate conclusion is that while the strength is of second order in general, it can be at most of third order at triple point in particular.


The Reflected-Shock Angle at the Triple Point

A slight extension of this investigation discloses that at the triple point where the incident, reflected and Mach shocks meet, not only is the reflected shock of vanishing strength but also is it locally tangent to the acoustic circle. The latter statement can be proved as follows:

Denoting by \( \psi \) the angle included between the reflected shock and acoustic circle, then

\[
\tan \psi = \frac{dr}{rd\theta} \tag{19}
\]

where \( r \) is given by Eq. (17).

Now the mapping from the \( S = \rho e^{i\theta} \) plane to the \( z = x, y \) plane, in Lighthill's paper, is everywhere analytic, so \( \left| \frac{dz'}{dS} \right| \) and its derivatives are finite. Even though the mapping from the physical conical plane \( R = e^{i\theta} \) to the \( S \) plane is singular on the unit circle, the following relations hold:

\[
\frac{\partial}{\partial \theta} = \frac{\partial}{r \partial \theta} \quad \frac{\partial}{\partial r} = \frac{\partial}{\partial \rho} \left( \frac{1}{\sqrt{1-r^2}} \right) \tag{20}
\]

So it is immediately seen that this singular behavior will not affect the determination of \( \psi \). Furthermore, since

\[
\frac{\partial}{\partial \theta} = \frac{\partial}{\partial \xi} = \frac{\partial}{\partial x} \left| \frac{dz'}{dS} \right|
\]

we eventually get

\[
\tan \psi = \frac{\partial r}{\rho \partial \theta} = \frac{2}{\kappa} \left( \delta^2 \right)^{1/4} \frac{d^2}{dx^2} \frac{\partial A}{\partial x} \tag{21}
\]

and the conclusion follows at once.