<table>
<thead>
<tr>
<th>REPORT DOCUMENTATION PAGE</th>
<th>READ INSTRUCTIONS BEFORE COMPLETING FORM</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. REPORT NUMBER</td>
<td>5. GOV'T ACCESSION NO.</td>
</tr>
<tr>
<td>2. TYPE OF REPORT &amp; PERIOD COVERED</td>
<td>7. RECIPIENT'S CATALOG NUMBER</td>
</tr>
<tr>
<td>SURFACE REPRESENTATION - SYMBOLIC COMPUTATION</td>
<td>Technical Report</td>
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<tr>
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<tr>
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<tr>
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<td></td>
</tr>
</tbody>
</table>

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Abstract

Recent research in Computer Aided Geometric Design is surveyed in this paper. A demonstration of the feasibility and utility of some of the new methods of curve and surface representation for interactive design is given in a movie which will be shown during the presentation.

PART I

by R. E. Barnhill

SURFACE REPRESENTATION

The representation of surfaces is needed for computer aided ship design and for the finite element analysis of ship hulls.

This module of our paper is a very broad survey of recent methods for representing and approximating surfaces. For more details consult the survey by Barnhill (1977), which also contains the relevant references.

There are three geometric units for surfaces: rectangles, triangles, and points. The approximation used depends upon which of these three correspond to the geometry of the data.

Rectangular Methods

Data are often given along orthogonal lines, either all along the lines or at the meshpoints of the corresponding rectangular grid. Let \( F = F(u,v) \) be a "general coordinate", i.e., \( F \) stands for \( x \), \( y \), and \( z \) successively. The simplest Coons patch interpolates to the data \( F(0,v) \), \( F(1,v) \), \( F(u,0) \), and \( F(u,1) \) and is the following:

\[
\begin{align*}
B(u,v) &= (1-u)\begin{bmatrix} F(0,v) \\ F(1,v) \end{bmatrix} + \begin{bmatrix} F(u,0) \\ F(u,1) \end{bmatrix}(1-v) \\
&- (1-u)\begin{bmatrix} F(0,0) \\ F(0,1) \end{bmatrix}(1-v) - \begin{bmatrix} F(1,0) \\ F(1,1) \end{bmatrix}(1-v)
\end{align*}
\]

where

\[
\begin{align*}
0 &< u, v < 1 \\
B &< 1
\end{align*}
\]

The bilinearly blended Coons patch defined by (1.1), considered over a mesh of rectangles, provides a continuous, locally defined interpolant. Useful surfaces usually must be at least continuously differentiable, i.e., \( C^1 \). The bicubically blended Coons patch is a start on the problem of defining a \( C^1 \) surface and it is given by the following:

\[
\begin{align*}
PF(u,v) &= \begin{bmatrix} h_0(u) \\ h_1(u) \\ F_0(u) \\ F_1(u) \end{bmatrix}(1-v) + \begin{bmatrix} F(u,0) \\ F(u,1) \\ F_0(0) \\ F_1(0) \end{bmatrix}(1-v) \\
&- \begin{bmatrix} F_0(0) \\ F_1(0) \\ F_0(1) \\ F_1(1) \end{bmatrix}(1-v) - \begin{bmatrix} F_0(0) \\ F_1(0) \\ F_0(1) \\ F_1(1) \end{bmatrix}(1-v)
\end{align*}
\]

The relevant references.
The $h_i$ and $F_i$ are the univariate cubic Hermite basis functions

\[(1.4)\]

\[h_0(t) = (1-t)^2(2t+1)F_0(t) + t(1-t)^2\]

\[h_1(t) = t^2(-2t+3)F_1(t) + t^2(t-1).\]

For some time people thought that (1.2) provided a $C^1$ interpolant to the data, $F$ and the normal derivative $\nabla F$, all around the boundary of $\{(u,v) \mid 0 < u, v < 1\}$. However, $PF$ is $C^1$ but fails to interpolate to $F_0(1,u,0)$ or to $F_0(0,1,u)$, in general. A “compatibly corrected” Coons patch that does interpolate to the data is (1.2) with the twist partition (the lower right partition) of $B$ replaced by:

\[(1.5)\]

\[\frac{\partial^2 F}{\partial u \partial v}(0,0) + v \frac{\partial^2 F}{\partial u \partial v}(0,0)\]

\[-u \frac{\partial^2 F}{\partial u \partial v}(0,1) \times (v-1) \frac{\partial^2 F}{\partial u \partial v}(0,1)\]

\[-u \frac{\partial^2 F}{\partial u \partial v}(0,1) \times (v-1) \frac{\partial^2 F}{\partial u \partial v}(0,1)\]

\[\frac{\partial^2 F}{\partial u \partial v}(1,1) \times (v-1) \frac{\partial^2 F}{\partial u \partial v}(1,1)\]

\[\text{(enumerated row-wise)}\]

All of these Coons patches can be discretized to form finite dimensional interpolants. For example, in (1.2), the positions such as $F(u,0)$ could be cubic Hermite polynomials and the normal derivatives such as $F_{0,1}(u,0)$ could be linear polynomials.

The (1,1) derivatives are called “twists”. These are awkward geometric handles which are best avoided. Two sets of solutions to avoiding the specification of twists have recently been developed (Barnhill (1977)):

1. Construct a preprocessor that calculates the twists from an intermediate $C^0$ surface.
2. Construct interpolants whose twists are calculated in terms of $F$, $F_{1,0}$, and $F_{0,1}$.

**Triangular Methods**

Triangular Coons patches were initiated by Barnhill, Birkhoff, and Gordon (1973). These interpolants can be discretized like the rectangular ones, to obtain finite dimensional schemes. A variety of 9-parameter $C^1$ triangular interpolants have been devised and implemented in our SURFED system. Some of these interpolants are illustrated in the movie which goes with this paper. Seven sets of triangular interpolants are given in Barnhill (1977). Before interpolating over triangles, the triangles themselves must be available. F. F. Little has devised an effective method of triangulating given $(x_i, y_i)$, and then of optimizing the triangulation. This is an example of a two-level process: (1) preprocess by means of a fast first triangulation and (2) improve the first triangulation by optimizing according to a certain criterion.

An example of a $C^1$ interpolant over a triangulation is the Barnhill, Birkhoff, and Gordon scheme given in Kluczewicz (1977). The “transfinite” triangular scheme analogous to (1.2) for the standard triangle with vertices $(1,0)$, $(0,1)$, and $(0,0)$ is the following:

\[(2.1)\]

\[PF(p,q) = h_0(p^{1/2})F(0,q) + F_0(1-p^{1/2})(1-q).\]

\[F_1(0,0) + P_2F - h_0(1-p^{1/2})F(0,q) + F_0(1-p^{1/2})(1-q)\]

\[
\frac{\partial^2 F}{\partial u \partial v}(1,0) \times (v-1) \frac{\partial^2 F}{\partial u \partial v}(0,1)\]

\[\text{where}\]

\[P_2F = h_0(1-p^{1/2})F(p,0) + F_0(1-p^{1/2})(1-p)F_0(0,q)\]

\[h_1(1-p^{1/2}) F(p,1-p) + F_1(1-p^{1/2})\]

\[F_0(1,p,1-p) - p^2F(p+q-1)^2(\frac{\partial^2 F}{\partial u \partial v})(0,0)\]

\[-(\frac{\partial^2 F}{\partial u \partial v})(0,0).\]

Broadly speaking, rectangular patches should be used where possible, e.g., along the simpler regions of ship hulls. Triangular patches should be used for the more complicated regions which do not have rectangular-like symmetry. Of course, for arbitrarily spaced point data, triangular patches or else the methods of the next Section must be used. Rectangular and triangular patches can be blended smoothly together, with common parameters.

**Arbitrarily Spaced Data**

For applications such as mapping the bottom of a harbor, the available data is likely to be irregularly spaced. Such data can be interpolated either: (1) by triangulating and then using a triangular interpolant, as in Section 2, or else (2) by using a variant of Shepard’s Formula. Shepard’s Formula for the data $(x_i, y_i, F_i)$ is the following:

\[\text{For applications such as mapping the bottom of a harbor, the available data is likely to be irregularly spaced. Such data can be interpolated either: (1) by triangulating and then using a triangular interpolant, as in Section 2, or else (2) by using a variant of Shepard’s Formula. Shepard’s Formula for the data $(x_i, y_i, F_i)$ is the following:}\]
\[ \forall i \leq 1, (x, y) \neq (x_i, y_i), i \neq 1 \]
\[ SF = \sum_{i=1}^{n} w_i \]
\[ SF = F_i, (x, y) = (x_i, y_i) \text{ for some } i. \]

We recommend letting \( u = 2 \), in which case Shepard's Formula is an inverse square distance method. Shepard's formula itself has some defects, such as the property that

\[ \frac{\partial SF}{\partial x} = \frac{\partial SF}{\partial y} = 0 \text{ at each } (x_i, y_i). \]

Therefore, we have the following Theorem of Barnhill and Gregory to find improved interpolants: If \( P \) and \( Q \) are two interpolation operators, then their Boolean sum defined by

\[ P \oplus Q = P \cap Q - PQ \]

has the following properties:

1. \( P \oplus Q \) has at least the interpolation properties of \( P \) and \( Q \).
2. \( P \oplus Q \) has at least the function precision of \( Q \).

This Theorem yields many possible improvements of Shepard's Formula. For example, let \( PF \) be SF defined in (3.1) and let \( QF \) be the quadratic least squares approximation to the \( F_i \). Then \( (P \oplus Q)F \) has the following properties:

1. interpolates to the \( F_i \).
2. does not have the flat spots due to (3.2).
3. is in continuity class \( C^2 \).
4. has quadratic precision.

This Theorem has been implemented by D. Feng so that symbolic \( P \) and \( Q \) can be entered in \( P \oplus Q \). R. F. Riesenfeld's module of this paper reports on Feng's work.

Acknowledgments

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PART II

SYMBOLIC COMPUTATION

Although scientific computation traditionally has had the connotation of numerical computation, advances in the area of symbolic computation have brought its use increasingly into the domain of scientific and engineering computation. Classical numerical analysis gives rise to algorithms and procedures that, when carried out with suitable input values, produce a single number of small set of numbers which we would call "the answer" to a given problem. It is typical of these numerical methods that any change in the input values results in a new problem whose answer costs as much as the previous one. When a problem is solved symbolically, it is solved in such a way that the "answer," typically an expression involving variables, has many valid interpretations that it can take on, being restricted only by the domain of definition of the variables in the expression. This kind of computation is what we humanly perform when we manipulate equations algebraically or when we solve for an indefinite integral. A specific answer is obtained when we bind the variables by substituting specific values for them in the final expression. A similar problem can be solved simply by making a new evaluation of the previously obtained symbolic answer, a process which is often insignificant compared to the amount of work required to compute the original expression.

In 1969 Gordon published a milestone paper which described a general framework for classifying and studying multivariate interpolation schemes. That paper introduced the use of idempotent linear operators, called
proectors, to develop an elegant
theory in which the Coons patch with
its extraordinary interpolation
properties is merely the boolean sum
$P_1 + P_2$ of two appropriate projectors.

At the time of its publication, however,
the principal value of the Gordon
proectors probably did not become
an actual research tool until people
began to study interpolants defined
over nonrectilinear regions like
triangles. These interpolants are as
manifoldly more complex in their
structure as they are in their algebraic
embodiment. The formula of a triangular
patch, written out as algebraic combinations
of parametric variables
and point evaluations of functions and
function derivatives, can easily occupy
a full page of a technical report
devoted to its description. As an
example, see equation (2.1) in Part I.

On the other hand, in projector notation,
the representation of the same surface
interpolant may be very simply expressed
as the boolean sum of two or the
projectors whose univariate definition
is straightforward. It was this impetus
that moved D. Feng to investigate the
applicability of symbolic computation
to computer aided geometric design.

Making some reasonable restrictions,
Feng implemented a symbolic processor
that allows the user to specify surfaces
as boolean combinations of the commonly
used projectors. This capability is
interfaced with SURFED, the surface
editor developed and used by the CAGD
Group to analyze and inspect visually
various new surface forms. This
implementation was realized in FORTRAN
by writing the necessary stack support
routines in FORTRAN, so that the code
is portable, compilable, and compatible
with the rest of our system. The
addition of this symbolic capability
to the surface system is essentially
the addition of an interactive symbolic
surface specification feature, for it
is possible to request the display of a
surface that was specified only in
projector form. The necessary symbolic
computations, and subsequent numerical
evaluations are carried out in order to
produce a realtime picture of the
surface being investigated. Modifica-
tions to the parameters of the surface
are effected by numerically re-evaluat-
ing the same symbolic expression, just
as one would in the case in which the
formula has been explicitly typed in
from a terminal.

One major advantage of this
symbolic processor is that the algebraic
computations necessary to yield lengthy
interpolation formulas can be mechanized
so that one has much greater confidence
in the correctness of the answer. Even

If the computation is performed
correctly by hand, entering the formula
is also a very error-prone operation.
The fact that surface specification can
be an effortless interactive activity
of forming expressions in a natural
high-level language, instead of an
oncious low-level chore, means that the
researcher is far more likely to
experience with new surface forms and
new ideas, a general benefit that inter-
active computing systems are supposed
to engender. Indeed, Feng employed his
processor to devise some new surface
forms.

Smooth Curves and Total Positivity

Designing curves that are aesthetic,
graceful, and "sweet" has been part of
geometric design as long as this has
been considered an area of human
endeavor. The field of CAGD has inheri-
ted this problem, and consequently it
has been concerned with making quanti-
fied statements in this hitherto
qualitative, nonscientific process.
Judgment and experience were the
essential ingredients of geometric
design before the introduction of the
computer. Recently there has been
considerable research devoted to
mathematical analysis of approximations
that are "wiggle free", and more
recently there have been efforts to
apply these ideas toward developing
improved human-machine graphical inter-
faces for CAGD. In 1977 J. Lane linked
the mathematical theory of total
positivity to this application, a
relationship which will be outlined in
this section.

P. Bézier of Renault attracted
widespread attention in the early 1970's
with a method of curve design that
seemed to assure the user that the
output curve had satisfactory shape
characteristics. Using a polygon to
roughly specify the curve, Bézier
succeeded in gaining interactive control
over shape. An analysis of Bézier
curves revealed that they are mathemati-
cally related to Bernstein approximation.
In particular Bézier curves shared the
exceptional property of being variation
diminishing.

Definition (1): An approximation
scheme is variation diminishing if the
number of intersections of the approxi-
mation with any straight line does not
exceed the number of crossings of that
straight line by the primitive function.

This definition captures the
intuitive notion that the approximation
to a primitive function should have no
more wiggles in it than the original
primitive function itself has.

An analytical approach to the
theory of variation diminishing
approximations has been developed by
Schoenberg, Karlin, and others. They
have shown that, if a set of basis
functions (blending functions) is
totally positive, then the associated
approximation method is variation
diminishing. Although the connection
between the theory of total positivity
and the variation diminishing property
is not intuitive, the definition of
total positivity is given below as an
indication of the kind of mathematical
analysis that it invokes.

**Definition TP2:** A set of basis
functions \( \{ \phi_i(x) \} \) is totally positive
provided that, for all \( 1 \leq i_1 < i_2 < \ldots < i_n \)
and \( x_1 < x_2 < \ldots < x_n \), we have the inequality

\[
\begin{vmatrix}
\phi_{i_1}(x_1) & \cdots & \phi_{i_1}(x_n) \\
\vdots & \ddots & \vdots \\
\phi_{i_n}(x_1) & \cdots & \phi_{i_n}(x_n)
\end{vmatrix} > 0,
\]

for all finite values of \( n \).

The application of total positivity
has shown in a straightforward manner
that the Bézier approximation method is
totally positive, thus variation
diminishing. Furthermore, it has been
used to establish that B-spline methods
are also variation diminishing. By
using total positivity, Lane has developed
new methods for curve design which
incorporate the tension-like properties
studied by Nielson and Dube and others,
while maintaining the valuable variation
diminishing property. This feature
permits the designer to "tighten" a
curve within a certain region where its
appearance is unsatisfactory.

There are several composition
theorems that assure the total positivity
of compound functions that are composed
of more elementary totally positive
functions. These theorems make it
possible to devise many new schemes
that enjoy the variation diminishing
property. It also helps one to analyze
various ad hoc schemes that have been
proposed.

The proofs and deductions in the
theory of total positivity tend to be
somewhat specialized to the area, but
as researchers in CAGD become more
accustomed to them, it is likely to
become a more widely used concept and
tool.

**Hayes Surface Form**

The standard tensor product defini-
tion of a surface is given by

\[
N \times N \sum_{i=1}^{N} \sum_{j=1}^{N} P_{ij} \phi_i(u) \phi_j(v)
\]

Hayes observed that, while it was
traditional for the points \( P_{ij} \) to fall
on a rectangular grid, it was not
essential for the application of the
tensor product formula. It is only
necessary that there be a \( P_{ij} \) defined
for all valid index values. For the
case of tensor product B-spline surfaces,
special effects are possible if the
Cox-deBoor Algorithm is used to evaluate
them.

What are the knots or points of
derivative discontinuity in univariate
spline curves become whole lines of
derivative discontinuities or knot lines
in the tensor product surface. Normally
these knot lines correspond to an
orthogonal set of lines in the parameter
domain, but Hayes noticed that so long
as the order of the knot lines was not
violated, the tensor product B-spline
formula was formally defined and made
sense. In fact the lines can be allowed
to coalesce and the Cox-deBoor algorithm
yields a surface with diminished or
deficient continuity across the coales-
cent knot lines, just as coalescent
knots yield deficient splines in the
univariate case. If enough knot lines
are brought together, a knot line of
sufficient deficiency to produce a cusp
is produced. This situation is depicted
in the figure.

Finally, Hayes pointed out that the
knot lines themselves can be defined in
the parameter space using the B-spline
method of designing curves. The result-
ing power of this more general form is
that one can control cusps in B-spline
surfaces, feathering them out in a very
smooth and aesthetic manner. This is
one of the most useful methods of design
that involve manipulating the locations
of the knots of a spline in the parameter
space. Perhaps this approach will
encourage people to try other variations
that employ parameters related to the
domain of definition of the surface. A
generalization and extension of the
curved knot line technique to transfinite
interpolants is thoroughly given by
Nielson and Wixom (4).

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A knot-set producing a fading discontinuity

A surface containing a fading discontinuity in slope

(Courtesy of J. G. Hayes [2, p. 12-13]. Figures from original Hayes NPL Rep. NAC 58)
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