SPATIAL CORRELATION FUNCTION FOR A FREQUENCY BAND FOR VERTICAL --ETC(U)
The expression for the single frequency spatial correlation function reported in Reference 1 as equation 2 of Section II-B has been modified by Mr. B. Cron (Reference 2) to comply with the revised general expression given in Reference 3. The expression for the single frequency spatial correlation function with directional noise \( g(\alpha) = \cos \alpha \) for vertical receivers as given by Reference 2 is:

\[
P(d, \tau) = 2 \left[ \frac{\sin kd}{kd} + \frac{(\cos kd - 1)}{(kd)^2} \right] \cos \omega \tau
\]

\[
+ 2 \left[ \frac{\sin ka}{ka} - \frac{\cos ka}{kd} \sin \omega \tau \right]
\]

The definitions of the terms in this expression and those in subsequent equations are found in the glossary at the end of the memorandum. Most of these definitions come from References 1 and 3.
**Title:** SPATIAL CORRELATION FUNCTION FOR A FREQUENCY BAND FOR VERTICAL RECEIVERS AND DIRECTIONAL NOISE

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**Report Date:** 27 AUG 65

**Distribution Statement:** Approved for public release; distribution unlimited.

**Supplementary Notes:**

**Keywords:**

**Abstract:**

**Security Classification:** UNCLASSIFIED

**DD Form 1473 Edition:** 1 Jan 73, 1473

**Security Classification of This Page:** UNCLASSIFIED
In this memorandum the expression for the spatial correlation for a frequency band was obtained by forming the product of the single frequency spatial correlation function and the power spectrum and then integrating this product over a bandwidth. This expression for the spatial correlation for a frequency band was checked by letting \( b \) approach unity and obtaining equation (1) again.

DEVIATION OF THE SPATIAL CORRELATION FOR A FREQUENCY BAND

The expression for the spatial correlation for a frequency band, \( \rho(d, \tau, b) \), as outlined in the introduction is:

\[
\rho(d, \tau, b) = \int_{s_1}^{s_2} \left\{ 2 \left[ \frac{\sin \beta d}{\beta d} + \frac{(\cos \beta d - 1)}{\beta d} \right] \cos \omega T 
+ 2 \left[ \frac{\sin \beta d}{(\beta d)^2} - \frac{\cos \beta d}{\beta d} \right] \sin \omega T \right\} ds
\]

In equation (2) \( \rho(d, \tau, b) \) can be thought of as being composed of two parts \( \rho_1(d, \tau, b) \) and \( \rho_2(d, \tau, b) \) where \( \rho_1(d, \tau, b) \) and \( \rho_2(d, \tau, b) \) are given by:

\[
\rho_1(d, \tau, b) = \int_{s_1}^{s_2} \left[ 2 \left[ \frac{\sin \beta d}{\beta d} + \frac{(\cos \beta d - 1)}{\beta d} \right] \cos \omega T \right] ds
\]

Equation (3) has been evaluated in reference 1 and is given by:

\[
\rho_2(d, \tau, b) = \int_{s_1}^{s_2} \left[ 2 \left[ \frac{\sin \beta d}{(\beta d)^2} - \frac{\cos \beta d}{\beta d} \right] \sin \omega T \right] ds
\]

Equation (3) has been evaluated in reference 1 and is given by:

\[
\rho_1(d, \tau, b) = \int_{s_1}^{s_2} \left[ 2 \left[ \frac{\sin \beta d}{\beta d} + \frac{(\cos \beta d - 1)}{\beta d} \right] \cos \omega T \right] ds
\]

\[
\rho_2(d, \tau, b) = \int_{s_1}^{s_2} \left[ 2 \left[ \frac{\sin \beta d}{(\beta d)^2} - \frac{\cos \beta d}{\beta d} \right] \sin \omega T \right] ds
\]
which now leaves the solution of the integral in equation 4 to be evaluated.

In evaluating the integral of equation (4) the following notation is used

\[ I_1 = \int_{s_1}^{s_2} \frac{2 \sin \omega T}{(kd)^2} \sin \omega T \, ds \]  
\[ I_2 = \int_{s_1}^{s_2} \frac{2 \cos \omega T}{(kd)^2} \cos \omega T \, ds \]  

and

\[ I_1 (d, \frac{b}{b}) = I_1 - I_2 \]  

By letting \( u = kd = 2x f / \sigma \); then \( du = 2\sigma d f / \sigma \) and \( \omega T = u c / d T = \gamma u \). Equation (6) can now be written as

\[ I_1 = \frac{1}{2\pi b^2} \int_{u_1}^{u_2} \frac{\cos u (1 - y)}{u^2} - \frac{\cos u (1 + y)}{u^2} \, du \]  

where \( u_1 = 2x x / b \) and \( u_2 = 2x bx \)

Using trigonometric formulas, equation (9) becomes

\[ I_1 = \frac{1}{2\pi b^2} \int_{u_1}^{u_2} \left[ \frac{\cos u (1 - y)}{u^2} - \frac{\cos u (1 + y)}{u^2} \right] \, du \]
The values of the integrals are given by

\[
\int_{u_1}^{u_2} \frac{\cos[u(1-y)]}{u^2} \, du = - \left[ \frac{\cos[u(1-y)]}{u} \right]_{u_1}^{u_2} - (1-y) \int_{u_1}^{u_2} \frac{\sin[u(1-y)]}{u} \, du
\]

\[
= - \frac{\cos[(1-y)\pi x/b]}{2\pi x/b} + \cos \left[ \frac{(1-y)\pi x/b}{2n_x/b} \right]
\]

\[
- (1-y) \left[ \ln \left[ (1-y)\pi x/b \right] - \ln \left[ (1-y)\pi x/b \right] \right]
\]

and

\[
\int_{u_1}^{u_2} \frac{\cos[u(1+y)]}{u^2} \, du = \cos \left[ \frac{2\pi x/b (1+y)}{2\pi x/b} \right] - \cos \left[ \frac{2\pi x/b (1+y)}{2\pi x/b} \right]
\]

\[
+ (1+y) \left[ \ln \left[ (1+y)\pi x/b \right] - \ln \left[ (1+y)\pi x/b \right] \right].
\]

Using the same substitutions used in arriving at equation (9), equation (7) becomes

\[
I_0 = \frac{1}{2\pi \alpha (b-1/b)} \int_{u_1}^{u_2} \frac{\cos[u(1+y)]}{u} \, du
\]

Using trigonometric formulas equation (13) becomes

\[
I_0 = \frac{1}{2\pi \alpha (b-1/b)} \int_{u_1}^{u_2} \left[ \sin[u(1+y)] \, du + \sin[u(1+y)] \, du \right]
\]

Integrating this gives

\[
I_0 = \frac{-1}{2\pi \alpha (b-1/b)} \left\{ \ln \left[ (1+y)\pi x/b \right] - \ln \left[ (1+y)\pi x/b \right] \right\}
\]

\[
- \ln \left[ (1-y)\pi x/b \right] + \ln \left[ (1-y)\pi x/b \right] \right\]

(15)
From equations (11), (12), and (15) equation (8) becomes:

\[
P_2(\theta, b) = \frac{1}{2\pi b} \left\{ \cos \left[ \frac{3\pi b (1 + \nu)}{2\pi b} \right] \cos \left[ \frac{3\pi b (1 - \nu)}{2\pi b} \right] - \cos \left[ \frac{3\pi b (1 + \nu)}{2\pi b} \right] - \cos \left[ \frac{3\pi b (1 - \nu)}{2\pi b} \right] \right\} \\
+ \frac{\cos \left[ \frac{(1-\nu) 2\pi b}{2\pi b} \right] + (1+\nu) \left\{ \text{li} \left[ (1+\nu) 2\pi b \right] - \text{li} \left[ (1-\nu) 2\pi b \right] \right\}}{\pi b} \\
- (1-\nu) \left\{ \text{li} \left[ (1-\nu) 2\pi b \right] - \text{li} \left[ (1-\nu) 2\pi b \right] \right\} \\
- \text{li} \left[ (1+\nu) 2\pi b \right] + \text{li} \left[ (1+\nu) 2\pi b \right] \\
- \text{li} \left[ (1-\nu) 2\pi b \right] + \text{li} \left[ (1-\nu) 2\pi b \right] \right\}
\]

(16)

Regrouping terms and using appropriate trigonometric formulas, equation (16) can be rewritten as:

\[
P_2(\theta, b) = \frac{1}{2\pi (b-1/\nu)} \left\{ - \frac{\sin 2\pi b \sin 2\pi b}{2\pi b} \right\} \\
+ \frac{\sin 2\pi b \sin 2\pi b}{2\pi b} + \nu \left\{ \text{li} \left[ (1-\nu) 2\pi b \right] \right\} \\
- \text{li} \left[ (1-\nu) 2\pi b \right] + \text{li} \left[ (1+\nu) 2\pi b \right] \\
- \text{li} \left[ (1+\nu) 2\pi b \right] \right\}
\]

(17)
The expression for \( \rho(d, \tau, b) \) can be regrouped and written as:

\[
\rho(d, \tau, b) = \frac{1}{2\pi x(b' - b)} \left\{ \frac{\cos \frac{2\pi x}{b} \cos \frac{2\pi x}{b'}}{2\pi x/b} - \frac{2 \cos \frac{2\pi x}{b} \cos \frac{2\pi x}{b'}}{2\pi x/b} \right. \\
- \frac{2 \cos \frac{2\pi x}{b}}{2\pi x/b} + \frac{2 \cos \frac{2\pi x}{b'}}{2\pi x/b} + 24 \left[ \frac{\sin (2\pi b') - \sin (2\pi b)}{2\pi x/b} \right] \\
- \pi \left\{ \frac{\sin [(1 + \nu)2\pi x] - \sin [(1 - \nu)2\pi x]}{2\pi x/b} \right\} \\
+ \frac{\sin [(1 - \nu)2\pi x]}{2\pi x/b} \left\}
\]

Equations (17) and (18) can be combined to give:

\[
\rho(d, \tau, b) = \rho_1(d, \tau, b) + \rho_2(d, \tau, b)
\]

\[
= \frac{1}{2\pi x(b' - b)} \left\{ \frac{\cos \frac{2\pi x}{b} \cos \frac{2\pi x}{b'}}{2\pi x/b} - \frac{2 \cos \frac{2\pi x}{b} \cos \frac{2\pi x}{b'}}{2\pi x/b} \right. \\
+ \frac{2 \cos \frac{2\pi x}{b}}{2\pi x/b} - \frac{2 \cos \frac{2\pi x}{b'}}{2\pi x/b} + 24 \left[ \frac{\sin (2\pi b') - \sin (2\pi b)}{2\pi x/b} \right] \\
- \pi \left\{ \frac{\sin [(1 + \nu)2\pi x] - \sin [(1 - \nu)2\pi x]}{2\pi x/b} \right\} \\
+ \frac{\sin [(1 - \nu)2\pi x]}{2\pi x/b} \left\}
\]

(19)
An alternate form of equation (19) is

\[
\rho(d, T, b) = \frac{1}{2\pi(b-\frac{1}{2})} \left\{ \frac{2 \sin \pi b \cdot \sin \left[ \pi b \left( 1-2x \right) \right]}{\pi b} - 2 \sin \pi b \cdot \sin \left[ \pi b \left( 1-2x \right) \right] + \left( \frac{\pi b}{\pi b} \right) + \left( \frac{2}{2\pi b} \right) \right\}
\]

\[
(20)
\]

EVALUATION OF THE SPATIAL CORRELATION FOR A FREQUENCY BAND AS \( b \) APPROACHES UNITY

As a check on the validity of the spatial correlation for a frequency band, \( b \) was allowed to approach unity in equation (19). By letting \( 2\pi x = A \), \( 2\pi x(1-x) = B \), and \( 2\pi xy = C \) equation (19) becomes

\[
\rho(d, T, b) = \frac{2}{\pi^2 (b-\frac{1}{2})} \left[ \frac{\cos \frac{d}{b} - \cos \frac{B}{b}}{b} + \frac{\cos \frac{C}{b} - \cos \frac{C}{b}}{b} \right]
\]

\[
+ \frac{A}{\left( \frac{d}{b} \right)} \left[ \frac{d}{b} - \frac{d}{b} \right] - \frac{B}{\left( \frac{d}{b} \right)} \left[ \frac{d}{b} - \frac{d}{b} \right]
\]

\[
(21)
\]
By expressing the cosine and sine integral in the following series
(References (4) and (5))
\[
\text{Ci} Z = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} Z^{2n+1}}{(2n+1)(2n+2)!}
\]
(23)
equation (21) becomes
\[
\rho(d, \gamma, b) = \frac{2}{A(b \cdot \eta)^2} \sum_{n=0}^{d} \left\{ \frac{(-1)^{n+1}}{(3n+2)(3n+3)!(3n+4)} \right\}
\[
\left[ C^\gamma 2^\eta + B^\gamma 2^\eta \right] \left[ b^\gamma 2^\eta \right]
\]
(24)
Applying L'Hospital's rule and letting b approach unity
\[
\rho(d, \gamma, b) = \frac{1}{A^2} \sum_{n=0}^{d} \left\{ \frac{(-1)^{n+1}}{(3n+2)(3n+3)!} \right\}
\left[ C^\gamma 2^\eta + B^\gamma 2^\eta \right] \left[ a(3n+1) \right]
\]
(25)
which can be rewritten as
\[
\rho(d, \tau, b) = \frac{2}{A} \sum_{m=0}^{2} \left\{ \cos \left[ 2 \pi x (i-m) \right] - \cos \left[ 3 \pi x (i-m) \right] \right\} \\
+ \frac{3}{A} \sin B
\]  
(26)

Substituting the appropriate values of A, B, and C equation (26) becomes
\[
\rho(d, \tau, b) = \frac{2}{(2 \pi x)} \left\{ \cos \left[ 2 \pi x (i-m) \right] - \cos \left[ 3 \pi x y \right] \right\} \\
+ \frac{3}{2 \pi x} \left\{ \sin \left[ 2 \pi x (i-m) \right] \right\}
\]  
(27)

Equation (27) can be written in the form
\[
\rho(d, \tau, b) = \frac{2 \left\{ \cos \left[ 2 \pi x (i-m) \right] - \cos \left[ 3 \pi x y \right] \right\}}{(d \delta)} + \frac{\sin (d \delta - \omega t)}{d} 
\]  
(28)

Using the proper trigonometric formulas, equation (28) becomes
\[
\rho(d, \tau, b) = 2 \left[ \frac{\cos \left[ \frac{3 \pi x y - \delta}{d \delta} \right]}{(d \delta)} + \frac{\sin \left[ \frac{3 \pi x y - \omega t}{d} \right]}{d} \right] \\
+ 2 \left[ \frac{\sin \left[ \frac{3 \pi x y - \delta}{d \delta} \right]}{(d \delta)} - \frac{\cos \left[ \frac{3 \pi x y - \omega t}{d} \right]}{d} \right] \sin \omega t
\]  
(29)
Equation (29) is the desired result. Thus, equation (19) has been verified by letting \( b \) approach unity with the expected result of obtaining equation (1) again.

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GLOSSARY OF TERMS

\( \phi \) is the angle between the line connecting the receiver and noise source and the vertical line passing through the center of the circular area of noise sources.

\( d \) is the distance between receivers.

\( \Upsilon \) is the electrical time delay

\[ \Upsilon = \frac{2\pi}{\omega} \]

\( \lambda = c/f \) wavelength

\( \omega = 2\pi f \) angular frequency.

\( c \) is velocity of sound

\[ b = \sqrt{\frac{3\lambda}{\pi f_s}} \]

\[ x = \frac{d}{\lambda g} \]

\( \lambda g = \frac{c}{\sqrt{f_s f_w}} \) geometric wavelength

\[ \mu = \frac{\Upsilon}{d/c} \]

\[ \sqrt{f_s f_w} \] is geometric mean frequency of flat bandwidth \( f_s \) to \( f_w \).

\[ Si(x) = \int_{0}^{x} \frac{\sin(u)}{u} \, du \] sine integral
REFERENCES


2. Private discussion with Mr. B. F. Cron.


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