Among the requirements for future sonar transmitting arrays will undoubtedly be three items: broader bands, lower frequencies, and higher power. The problem of velocity control will be concerned with all three of these requirements.

The requirement of broader bands complicates the problem of obtaining velocity control by electrical tuning. One can say that an array has complete velocity control if the amplitude and phase relationships between the input voltages are preserved in the transducer face velocities. At some frequencies it is possible to maintain very close velocity control by proper choice of a tuning reactor, but if a wide band of operation is required, one will have to be content with less than perfect velocity control over most of the band, unless the tuning reactor can be changed.

As lower and lower frequencies are required, unless the array designer has very large transducer faces, the separation between adjacent radiators will become small compared to the wavelength in water if a high packing factor is to be maintained. A condition of this sort will tend to increase the mutual impedance effects of the array and this is detrimental to velocity control.

Reference (b) shows that there is a high correlation between lack of velocity control and the occurrence of negative radiation resistances. Negative radiation resistances in an array means that some elements are absorbing power from the rest of the array. Some of this power is dissipated in mechanical and electrical losses in the element, but most of it goes back into the electrical power supply. If many elements are being driven by the same amplifier, then an element returning power to the power supply is merely reapplying electrical power to the other elements. However, the requirement for higher power suggests the use of modular drive, whereby each transducer has its own power supply. In this case the acoustic power absorbed by the element that is not dissipated by internal losses is returned directly to the power supply. If this power becomes excessive, the modular driver may be destroyed.
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<td>Porter, David T.</td>
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<td>800 North Quincy St.</td>
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<td>Arlington, VA 22217</td>
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Various new techniques are being explored in order to maintain velocity control. Some of these are mechnoelectrical feedback, and the use of series or parallel tuning reactors in conjunction with voltage or current drivers. This report will explore the feasibility of maintaining velocity control over a wide band and will not consider feedback.

The problem will be separated into two parts: array factors (factors external to the transducer), and transducer factors (factors internal to the transducer). This report is mainly concerned with the transducer factors. However, a short dissertation on array factors will be contained herein for familiarization purposes. For the idealized case of a planar, broadside, close-packed array of circular pistons in a rigid baffle, some array factors are discussed in reference (b).

ARRAY FACTORS

Reference (b) discusses Thevenin's equivalent circuit for transducers. Figure 1 shows a Thevenin's equivalent circuit for the jth element in an array.

\[ G_j = U_j \left( Z^I + Z_{rj} \right), \]  

and \( Z_{rj} \) is related to the velocities of the other transducers in the array by

\[ Z_{rj} = \frac{N}{i_s} \sum_{i \neq j} \frac{U_i}{U_j}. \]
Reference (b) also introduces a velocity control factor UCF, defined by

\[ UCF = \frac{|Z_{ij}|_{\text{max}}}{|Z^I + Z''|} \]  

(3)

In general, the smaller the UCF, the better velocity control is. Z_{ij} is the self radiation impedance of the elements, which were taken to be uniform over the array. \( |Z_{ij}|_{\text{max}} \) is the maximum absolute value of radiation impedance between pistons occurring in the array. Z_{11} and Z_{12} are factors external to the transducer, which depend on the size, shape, separation, orientation, and frequency of the transducers. The \( Z^I \) is the internal factor, which is taken to be independent of the transducer velocities and radiation impedances.

From equation (3) it is seen that if the real (\( R^I \)) or imaginary (\( X^I \)) part of \( Z^I \) is very large, then UCF will be small, and velocity control will be good. Neither \( R^I \) nor \( R_{11} \) can be small, but they can both be small. \( X_{11} \) cannot be negative, but \( X^I \) can, so that when \( X^I = -X_{11} \), UCF can be very large, and velocity control can be poor. Therefore, the ratio \( X^I/X_{11} \) is more important than the \( X^I \) value itself.

For the external part of the problem, we may find what values of \( Z^I \) are permitted over the frequency band to be used which will not result in negative radiation resistances. In the transducer factors section, it will be shown that \( R^I \) is usually not very frequency dependent, so that in the external part, we may assign the appropriate value of \( R^I \) and find which values of \( X^I \) will result in the occurrence of negative radiation resistances. Figure (2) (taken from reference (b)) shows what values of \( X^I \) result in some negative radiation resistances in a large, planar, broadside, unshaded, close-packed array of circular pistons in a rigid baffle, with \( R^I = 0 \).
COMBINATIONS OF $k\omega$ AND $X^I(R^I=0)$ THAT PRODUCE NEGATIVE RADIATION RESISTANCES IN A LARGE CLOSE—PACKED ARRAY OF CIRCULAR PISTONS

![Graph]

Figure 2.
Occurrence of Negative Radiation Resistances

Had we chosen a larger $R_I$, or looser packing, the undesirable area in Figure (2) would have been smaller. On the other hand, unequal $Z^I$ or unequal $\phi_j$, as might occur due to manufacturing tolerances, would make the undesirable area larger. The undesirable area might also be affected by electrical steering, baffles of different shape (cylindrical, spherical, etc.), and other environmental effects, such as nonrigid baffling and the presence of a dome. The external problem will be considered again at length with more numerical results in a later report.
TRANSUDER FACTORS

Here we will investigate the effects of various transducer parameters on $Z_I$, to find out what values of $Z_I$ would actually be encountered over a frequency band. For any array, a diagram, similar to Figure (2), can be made. If the expected values of $Z_I$ do not fall within the undesirable area of this diagram, then presumably negative radiation resistances will not occur in the frequency band to be used.

The following is a list of important factors affecting $Z_I$:

1. Frequency
2. Power Supply (Voltage or Current Source)
3. Electrical Tuning (Series or Parallel)
4. Coupling Field of the Transducer (Electric Field or Magnetic Field)
5. Effective Coupling Coefficient of the Transducer
6. Front Mass
7. Ratio of Front to Rear Masses
8. Mechanical Resistance Loss (or Mechanical Q in Air)
9. Electrical Resistance in Series with the Power Supply
10. Electrical Conductance in Parallel with the Blocked Reactance

$Z_I$ FOR EIGHT SIMPLE EQUIVALENT CIRCUITS

1. The Results for Eight Cases.

In this section equations will be given for $Z_I$ for the conditions of voltage or current drive, series or parallel tuning, and electric or magnetic field. The circuits considered here consist only of a mechanical resistance, mass and compliance, and electrical blocking and tuning reactances. In the Appendices, the effects on $Z_I$ of adding a rear mass and electrical losses will be considered.

Figure (3) shows the eight equivalent circuits (obtained from reference (c)), equations for their $Z_I$, and sketches of the frequency dependence of $X_I$. Note that in these eight cases $R_I$ is just the mechanical resistance, which is assumed here to be frequency independent. In Figures 4a, b, c, d and e, graphs of the quotient $X_I/X_{ll}$ are presented which illustrate the dependence of $X_I$ on the effective coupling coefficient. Note also that many of the eight expressions for $X_I$ have the same form, so that in Figure (4) only five graphs were needed instead of eight.

2. Derivation of $Z_I$ for the Voltage Source, Series Tuned, Electric Field Case.

The equation for $Z_I$ for the voltage source, series tuned, electric field
(VSE) case will now be derived. To find $Z^I$, we first remove the electromagnetic transformer from circuit 3a by transferring the electrical impedance to the mechanical side (Figure 5a). Then we short circuit the force source, $F = NE$, and examine the impedance looking back into the transducer from the radiating face (Figure 5b).

Here $Z_g$ is the series tuning reactance, $Z_b$ is the blocked reactance, and $Z_m$ is the mechanical impedance. Solving for $Z^I$, we obtain

$$Z^I = Z_m + \frac{1}{Z_s + \frac{1}{N^2Z_gZ_b}}.$$  \hfill (4)

Inserting the proper terms for $Z_m$, $Z_b$, and $Z_g$, we have

$$Z^I = R_m + j\omega M + \frac{1}{j\omega C_m} + \frac{N^2j\omega L_s}{j\omega L_s + \frac{1}{j\omega C_b}}.$$  \hfill (5)

so that

$$R^I = R_m.$$  \hfill (6)
and

\[
\chi^r = \omega M \left( \frac{\omega}{\omega_0} \right)^2 \left[ \frac{\omega}{\omega_0} \right] + \frac{N^2}{\omega_0 C_b (1 - \left( \frac{\omega}{\omega_0} \right)^2)}
\]

Let us define a reference angular frequency \( \omega_0 \), such that

\[
\omega_0 \equiv \frac{1}{\sqrt{M C_m}}.
\]  

This \( \omega_0 \) will correspond to the mechanical resonant frequency in air with the source voltage held constant, and with no tuning reactance in the circuit. Now we choose the tuning inductor such that tuning reactance cancels the blocked reactance at \( \omega_0 \). Then we have

\[
\omega_0 = \frac{1}{\sqrt{L_s C_b}} = \frac{1}{\sqrt{M C_m}}.
\]  

After some algebra,
For constant voltage drive, we may relate $C_b$ and $C_m$ by

$$\frac{N^2 C_m}{C_b} = \frac{k_e^2}{1 - k_e^2}$$

where $k_e$ is the effective coupling coefficient of the transducer. We may eliminate $C_m$ by use of equation (8), so that finally

$$\frac{X^I}{\omega_0 M} = \left(\frac{\omega}{\omega_0}\right)^2 - 1 - \frac{\omega_0}{\omega} \frac{k_e^2}{1 - k_e^2} \frac{1}{(\frac{\omega}{\omega_0})^2 - 1}.$$  

We now express the self radiation reactance $X_{11}$ by

$$X_{11} = \omega M_r,$$

where $M_r$ is the self radiation mass, which can be considered constant when the frequencies are such that the piston face is small relative to a wavelength. For circular pistons of small $k_e$ in a rigid baffle, $X_{11}$ is

$$(\frac{8\pi}{3})k_a \rho c A,$$

so that $M_r$ is

$$\frac{8}{3} \rho a^3.$$

Now the ratio of $X^I$ to $X_{11}$ becomes

$$\frac{X^I}{X_{11}} = \frac{M}{M_r} \left[ \left(\frac{\omega}{\omega_0}\right)^2 - 1 - \frac{k_e^2}{1 - k_e^2} \frac{1}{(\frac{\omega}{\omega_0})^2 - 1} \right].$$

From (13), it is seen that $X^I/X_{11} \to \infty$ at $\omega = \omega_0$. To find the zero crossings of $X^I/X_{11}$, we first set

$$\frac{k_e^2}{1 - k_e^2} = \omega_0 \quad \text{and} \quad \frac{\omega}{\omega_0} = \theta.$$
Then we have

\[
\frac{X_i}{X_{ii}} = \frac{M}{M_i} \left[ \frac{\beta^2 - 1}{\beta^2} - \frac{C_i}{\beta^2 - 1} \right].
\] (14)

For \( X_i = 0 \),

\[
\frac{\beta^2 - 1}{\beta^2} = \frac{C_i}{\beta^2 - 1},
\] (15)

and

\[
(\beta^2 - 1)^2 = C_i \beta^2,
\] (16)

so that

\[
\beta^2 - 1 \pm \sqrt{C_i} = 0,
\] (17)

which has the solution

\[
\beta = \pm \frac{\sqrt{C_i} \pm \sqrt{C_i + 4}}{2},
\] (18)

which can be approximated by

\[
\beta \approx 1 \pm \sqrt{C_i}, \quad \text{for } \sqrt{C_i} \ll 1.
\] (19)
and

\[ \omega_{\text{zero crossing}} = \omega_0 \left(1 \pm \frac{k_e}{2}\right) \]  

(20)

GENERALIZATIONS CONCERNING X$_i$ FOR THE EIGHT SIMPLE EQUIVALENT CIRCUITS

1. Effect of Type of Field

The choice of electric or magnetic field had little effect on the graphs of X$_i$/X$_{11}$ versus \( \omega/\omega_0 \). The only noticeable effect was in the unimportant case of current drive with series tuning (ISE, ISM cases), where the zero crossing is affected differently by coupling coefficient in the ISE case than in the ISM case.

2. Effects of Type of Drive and Tuning

The cases of voltage drive with series tuning and current drive with parallel tuning have X$_i$ curves of the same form, with a pole at \( \omega = \omega_0 \), and zero crossings at about \( \omega/\omega_0 = 1 \pm \frac{k_e}{2} \).

The cases of voltage drive with parallel tuning and current drive with series tuning have X$_i$ curves of the same form, with no poles, and a zero crossing at or near \( \omega = \omega_0 \).

3. Effects of Coupling Coefficient

In the VS and IP cases, increasing the coupling coefficient makes the absolute value of X$_i$ greater in the vicinity of \( \omega = \omega_0 \), and also increases the frequency separation between the zero crossings, effects which are both advantageous for increasing the bandwidth over which velocity control may obtained. The VP cases are independent of coupling coefficient, and the zero crossing is at \( \omega = \omega_0 \). In the ISE case, the frequency of the zero crossing is increased by increasing coupling coefficient, while in the ISM case the frequency of the zero crossing is reduced by increasing coupling coefficient. This results in the ISM case having a positive X$_i$ at \( \omega = \omega_0 \), which is beneficial to velocity control, (see Figure 2), and in the ISE case having a negative X$_i$ at \( \omega = \omega_0 \), which is bad for velocity control. The ISE and ISM cases are mostly of academic interest, as transducers are seldom designed in such a manner. However, these two cases are of interest in considering series tuned transducers which have high source impedances.
4. Effects of Changing the Tuning Reactor

The $X^I$ curves for the VPE and VPM cases are unaffected by the tuning reactor, so that changing the tuning reactor will not affect these cases. In the VSE, VSM, IPE and IPM cases, the pole occurs when the tuning reactance cancels the blocked reactance. Therefore, changing the tuning reactor changes the frequency at which the pole occurs. The $X^I$ curves for the ISE and ISM cases are independent of the series tuning reactor, so that changing the reactor does not affect the ISE or ISM $X^I$ curves.

5. Effects of Changing the Ratio of Front Mass to Self Radiation Mass ($M/M_r$)

For all cases, the ratio of $X^I/X_{11}$ is directly proportional to $M/M_r$, so that increasing $M/M_r$ increases the magnitude of $X^I/X_{11}$, which is generally beneficial to velocity control. However, this is a poor way to obtain velocity control, for two reasons, namely, (a) raising $M$ raises the mechanical $Q$ which forces a redesign of the transducer; and (b) reducing $M_r$ implies using a smaller transducer face, which means a smaller $ka$ of the radiating face, which in turn makes velocity control harder.

A SIMPLE EXAMPLE OF BROADBAND VELOCITY CONTROL

Consider a large, close-packed, broadside array of circular pistons in a rigid baffle. Let $M/M_r = 4$, $R^I = 0$, the coupling coefficient be $44.7\%$, the piston diameters be eight (8) inches, and the resonant frequency in air $f_0 = \frac{440}{D} \text{ be } 1 \text{ k.c.}$ Let the array be series tuned, so that it can be described by circuit 3a. Recalling that the wave number $k$ can be expressed as $\omega/c = 2\pi f/c$, and assuming $c = 5000 \text{ feet/second in water}$, the $ka$ of the pistons will be $0.424$ at $1 \text{ k.c.}$ Now, $ka$ and $\omega/\omega_0$ are related by

$$ka = 0.424 \frac{\omega}{\omega_0} \quad (21)$$

This enables us to plot the undesirable operating area from Figure (2) and the actual ratio of $X^I/X_{11}$ from Figure (4a) on the same graph, as is shown in Figure (6). For series tuning, the ratio $X^I/X_{11}$ will be such as to result in some negative radiation resistances for frequencies between 625 cps and 750 cps and also between 1100 cps and 1225 cps. Therefore, the velocity control obtained between 750 cps and 1100 cps is sufficient to avoid negative radiation resistances, and we may call the velocity control bandwidth of this transducer 350 cycles.
AN EXAMPLE OF BROADBAND VELOCITY CONTROL

Undesirable area in graph is where some pistons absorb power from the array.
Large unsteered, planar array of circular pistons is considered.
Resonant frequency in air: 1 k.c.
Piston diameter: eight (8) inches
Coupling coefficient: 0.447
Electric field
Voltage source
Tuning reactance cancels blocked reactance at 1 k.c.
Piston $ka$ at 1 k.c.: 0.424
Ratio of moving mass to radiation mass: 4:1

The undesirable area is taken from Figure (2).
The actual ratios $X^T/X_{11}$ are taken from Figures (4a) and (4b).
For series tuning the undesirable frequencies are 625 cps to 750 cps
and 1100 cps to 1225 cps.
For parallel tuning undesirable frequencies are from 680 cps to 950 cps.

Figure 6.
We now change the tuning of this array to parallel tuning. Then some elements would have negative radiation resistances for frequencies between 680 cps and 950 cps. However, as we further increase the frequency, there would be no upper frequency at which negative radiation resistances would reoccur, and we cannot assign a velocity control bandwidth to this parallel-tuned array. Although negative radiation resistances do not occur at the resonant frequency in air, they do occur only 50 cps lower, so that the resonant frequency in water of the parallel-tuned array probably does lie in the undesirable area.

DAVID T. PORTER
Mathematician

LIST OF REFERENCES


(c) R. S. Woollett, Lecture Notes on Sonar Transducer Fundamentals.
APPENDIX A

Here we consider the effect on $Z^I$ of adding a rear mass to the circuit in Figure (3a) for the VSE case. In Figure (A-1), the front and rear masses are denoted by $M_1$ and $M_2$.

\[
\omega_0 = \frac{1}{\sqrt{\frac{M_1 M_2}{M_1 + M_2} C_m}}
\]  

(A-1)

This $\omega_0$ will correspond to the mechanical resonant frequency in air with the source voltage held constant, no tuning reactance in the circuit, and no losses in the circuit. The presence of the tuning reactance and some losses will affect the resonant frequency only slightly.

To find $Z^I$ for the circuit of Figure (A-1), we short circuit the voltage source, and look back in from the radiating face, as is shown in Figure (A-2).

\[Z^I\]  

Figure A-2
Finding $Z^I$ for the Circuit of Figure A-1
Setting
\[ Z_c = \frac{Z_1 Z_2}{Z_1 + Z_2} \quad (A-2) \]
\[ Z_7 = Z_3 + Z_6 \quad (A-3) \]
\[ Z_8 = \frac{Z_4 Z_7}{Z_4 + Z_7} \quad (A-4) \]

we find that
\[ Z = Z_c + Z_7 = Z_8 + \frac{1}{\frac{1}{Z_4} + \frac{1}{Z_3 + \frac{1}{Z_2}}} \quad (A-5) \]

Now we define
\[ \omega_1 \equiv \frac{1}{\sqrt{L_c C}} \quad (A-6) \]
\[ x \equiv \frac{\omega}{\omega_1} \]
\[ s \equiv \frac{M_1}{M_2} \]
\[ C_1 \equiv \frac{k_e}{1 - k_e} \]

When we put the appropriate circuit parameters into (A-5) and make use of the definitions in (A-6), we obtain
\[ R = R_m \quad (A-7) \]

and
\[ X = \omega M_1 \left\{ 1 + \frac{1}{s - \left(\frac{\omega}{\omega_0}\right)^2 (s+1) \frac{\sigma^2 - 1}{C_1 + \frac{\sigma^2 - 1}{\alpha^2}}} \right\} \quad (A-8) \]
Suppose now that we want \( X' \) to be very large at \( \omega_0 \), so that velocity control will be obtained for frequencies near mechanical resonance. Assume that we are not free to alter the transducer proper, but that we may adjust the tuning coil. Then we must find the proper value of

\[
\omega_1 = \frac{1}{\sqrt{L_0 C_b}},
\]

such that \( X' \) approaches infinity at \( \omega = \omega_0 \). \( X' \) will be infinite if the denominator in the fraction of (A-8) is zero. As we want this denominator to be zero when \( \omega = \omega_0 \), we set \( \omega_1 = \omega_0 / \omega_1 \), and seek a solution of

\[
S = \frac{1}{\omega_1} (S+1) \frac{\omega^2 - 1}{C_1 + \frac{\omega^2 - 1}{\omega^2}} \tag{A-9}
\]

When the algebra is done, we find simply

\[
\omega_1 = \omega_0 \sqrt{1 - C_1 S} \tag{A-10}
\]

As \( C_1 S \ll 1 \), we can approximate (A-10) by

\[
\omega_1 \approx \omega_0 \left(1 - \frac{C_1 S}{2}\right) \tag{A-11}
\]

Equation (A-11) shows us that the proper tuning coil is almost exactly the same as would be required if the circuit in Figure (A-1) did not include a rear mass, or had an infinite rear mass. Consider a sample case where \( M_1/M_2 = 1/3 = s \), and \( k_e = 1/3 \). Then \( C_1 = 1/8 \), and \( \omega_1 = \omega_0 (1 - 1/48) \).

When the zero crossings of the \( X' \) in (A-8) are found, (having chosen \( \omega_1 \) from (A-10)), it is seen that they lie at almost the same frequencies as for the simpler circuit 3a with no rear mass in it. However, for zero frequency, \( X' \) is now zero.

The most interesting effect of the rear mass on \( X' \) is that there is now another pole, or perhaps we should say that the pole that was at zero frequency has moved to a greater frequency. Upon putting (A-10) into (A-8)
and finding what frequencies give a zero denominator in (A-8), we find that

\[
\frac{\omega}{\omega_0} = \left\{ \frac{1}{\sqrt{1-c_1 s}} \right\} \left\{ \frac{\sqrt{\frac{M_1}{M_1+M_2}}}{\sqrt{M_1/(1+s)}} \right\} \tag{A-12}
\]

The first root in (A-12) is just the already established pole at \( \omega_0 \), given by (A-10), but the second is a low frequency pole. For \( s = M_1/M_2 = 1/3 \), this pole lies at \( \omega/\omega_1 = 1/2 \), and the \( X^I \) curve has the form shown in Figure (A-3).

![Figure A-3](image)

**Figure A-3**

\( X^I \) Curve for a Front to Rear Mass Ratio of 1/3 (VSE Case)

The parallel tuned case (VPE) also has a low frequency pole, which is determined by the parallel resonance of \( C_m \) and \( M_2 \). It can be shown that \( X^I \) approaches infinity for \( \omega = \omega_0 \sqrt{\delta/(1+s)} \). The \( X^I \) curve for the VPE case is shown in Figure (A-4).
APPENDIX B

ADDITION OF AN ELECTRICAL LOSS, $G_b$, IN PARALLEL WITH THE BLOCKED REACTANCE

Here we consider the effect on $Z^I$ of adding an internal loss $G_b$ to the basic circuit of Figure (3a) for the VSE case. This loss is shown in Figure B-1.

\[
Z^I = R_m + j\omega M \left(1-\left(\frac{\omega}{\omega_c}\right)^2\right) + \frac{N^2}{G_b + j\omega C_b \left(1-\left(\frac{\omega}{\omega_c}\right)^2\right)} \tag{B-1}
\]
which at \( \omega = \omega_0 \) reduces to

\[
R^I = R_m + \frac{N^2}{G_b}, \quad X^I = 0. \tag{B-2}
\]

Note that we have lost the pole at \( \omega = \omega_0 \), but \( R^I \) may still be very large if \( N^2/G_b \) is large. Indeed, as \( G_b \) approaches 0, \( R^I \) approaches infinity, and complete velocity control is established, just as in the case when we did not take \( G_b \) into account. It will be convenient now to assume that \( G_b \) is linearly frequency dependent, such that \( \tan \delta \) remains fairly constant over a wide frequency range. \( \tan \delta \) is given by

\[
\tan \delta = \frac{G_b}{\omega \epsilon_b} = \frac{G_b}{\omega \epsilon_b} \tag{B-3}
\]

Then it can be shown that

\[
R^I = R_m + \frac{\omega M}{\omega_0} \frac{k_e^2}{\omega / \omega_0} \frac{\tan \delta}{1 - k_e^2 (1 - (\omega / \omega_0)^2)^2 + \tan^2 \delta} \tag{B-4}
\]

and

\[
X^I = \omega M (1 - (\omega / \omega_0)^2) - \frac{\omega M}{\omega / \omega_0} \frac{k_e^2}{1 - k_e^2 (1 - (\omega / \omega_0)^2)^2 + \tan^2 \delta}. \tag{B-5}
\]

Sketches of \( R^I \) and \( X^I \) versus frequency are shown in Figure B-2.
Effect on $R^I$ and $X^I$ of $\mathcal{G}_b$ for VSE Case

Then at $\omega = \omega_0$, $R^I$ becomes

$$R^I = R_m + \omega_m M \frac{k^2 \omega^2}{1 - k^2} \frac{1}{\tan \delta} = R_m \left(1 + \frac{\omega_m}{\tan \delta} \frac{k^2 \omega^2}{1 - k^2}\right)$$ (B-6)

or,

$$\frac{R^I}{X_m} = \frac{M}{M_r} \left[ \frac{R_m}{\omega_0 M} + \frac{k^2 \omega^2}{(1 - k^2) \tan \delta} \right]$$ (B-7)

Just as in the case when we did not consider $\mathcal{G}_b$ ($\tan \delta = 0$), the magnitude of $Z^I$ and the resulting degree of velocity control are very dependent upon the coupling coefficient. If the electrical part of the transducer circuit is weakly coupled to the mechanical part, then the high impedance of the tank circuit made by $L_b$ and $C_b$ will only give a small effect to $Z^I$.

The analysis contained in equations (B-1) through (B-7) is also pertinent to the case of a current source with parallel tuning. $Z^I$ for the cases of voltage source and parallel tuning (VPE, VPM) is not affected by $\mathcal{G}_b$. 

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APPENDIX C

ADDITION OF AN ELECTRICAL LOSS, $R_1$, IN SERIES WITH THE TRANSDUCER

1. VSE Case

Now we alter circuit 3a by adding a resistance $R_1$ in series with the tuning coil. This resistance may be considered as either a source impedance or an electrical loss internal to the coil or transducer.

![Diagram](image)

**Figure C-1**

Addition of Series Resistance to Circuit 3a (VSE Case)

Now $Z^I$ can be shown to be

$$Z^I = R_m + j\omega M \frac{\left(\frac{\omega}{\omega_0}\right)^2 - 1}{\omega/\omega_0}$$

(C-1)

$$+ \frac{k_e^2}{1-k_e^2} \omega M \frac{\omega_0 L_s - j \frac{\omega_0}{\omega}}{1 - j \omega_0 L_s R_1 \left(1 - \frac{(\omega/\omega_0)^2}{\omega/\omega_0}\right)}$$

Defining

$$\Theta_1 \equiv \frac{\omega_0 L_s}{R_1}$$

we get

$$Z^I = R_m + j\omega M \frac{\left(\frac{\omega}{\omega_0}\right)^2 - 1}{\omega/\omega_0}$$

(C-3)

$$+ \frac{k_e^2}{1-k_e^2} \omega M \frac{\Theta_1 - j \frac{\omega_0}{\omega}}{1 - j \Theta_1 \left(1 - \frac{(\omega/\omega_0)^2}{\omega/\omega_0}\right)}$$
If the $Z^I$ of (C-3) were separated into $R^I$ and $X^I$, then the $R^I$ and $X^I$ would appear as in the sketch in Figure (C-2), which is similar to B-2, which accounted for the $Q_b$ loss.

![Figure C-2](image)

**Figure C-2**

Effect on $R^I$ and $X^I$ of Series Loss $R_1$ for VSE Case

When $\omega = \omega_o$, then $R^I$ and $X^I$ simplify to

$$R^I = R_m + \frac{k_e^2}{1 - k_e^2} \omega_o M Q_1$$

(C-4)

and

$$X^I = -\frac{k_e^2}{1 - k_e^2} \omega_o M$$

(C-5)

Note that as $R_1$ becomes larger, $Q_1$ becomes smaller, and $R^I$ for $\omega = \omega_o$ also becomes smaller, and the velocity control becomes poorer. On the other hand, as $R_1$ becomes smaller, $R^I$ approaches infinity, and complete velocity control is established.

2. VPE Case

If we add a series resistance $R_1$ to the original VPE circuit, Figure (3b),
we now obtain Figure (C-3).

\[ Z^I = R_m + j\omega M + \frac{1}{j-\omega_m} \]

\[ + N^2 \frac{jR_i \omega L_p}{1 - \left(\frac{\omega}{\omega_v}\right)^2} \]

\[ \frac{1}{R_i + j\omega L_p} \frac{1 - \left(\frac{\omega}{\omega_v}\right)^2}{1 - \left(\frac{\omega}{\omega_v}\right)^2} \]

(C-6)

If \( R_1 \) becomes very large, the voltage source in conjunction with the large \( R_1 \) acts like a current source, and we have the IFE Case again. From Figure (4-a) we see that the IFE case has a pole at \( \omega = \omega_0 \), which makes the velocity control good. Therefore, addition of a series loss to the VPE case is beneficial to velocity control at \( \omega = \omega_0 \). Usually, internal electrical losses hurt velocity control, but this case is an exception.

When \( \omega = \omega_0 \), we have

\[ Z^I = R_m + N^2 R_1 \]

(C-7)

and

\[ \chi^I = 0 \]

(C-8)

After some more algebra it can be shown that at \( \omega = \omega_0 \)

\[ R^I = R_m + \frac{k_v^2}{1 - k_v^2} \frac{M}{\omega L_p} = R_m + \frac{k_v^2}{1 - k_v^2} \frac{\omega M}{Q_1} \]

\[ = R_m \left(1 + \frac{Q_m}{Q_1} \frac{k_v^2}{1 - k_v^2}\right) \]

(C-9)
where

\[ G_1 = \frac{\omega_0 L_p}{R_1} = \frac{1}{\omega C_b R_1} \]  

(C-10)

Now let us compare \( R^1 \) at \( \omega = \omega_0 \) for the series tuned case (C-4) and the parallel case (C-9). From these two equations we see that velocity control at \( \omega = \omega_0 \) is better in the series tuned case if \( Q_1 \) is greater than 1.0, but is better in the parallel tuned case if \( Q_1 \) is less than 1.0. Therefore, if the series resistance \( R_1 \) is less than the reactance of the tuning coil (or the blocked reactance of \( C_b \)), then at \( \omega = \omega_0 \) series tuning gives better velocity control, while if \( R_1 \) is greater than the tuning reactance at \( \omega = \omega_0 \), then parallel tuning yields better velocity control.

APPENDIX D

DERIVATION OF \( X^1 \) EQUATIONS FOR MAGNETIC FIELD CASES

Figure (D-1) shows a simplified two-port circuit for a magnetic field transducer with no tuning reactor (Figure (3d), page 52, Reference (c)).

\[ F = \frac{1}{j\omega L_{em}} E + Z_m U \]  

(D-1)

and

\[ I = \gamma_b E + \frac{1}{j\omega L_{em}} U. \]  

(D-2)
To find $Z^r$ we look back into the transducer from the radiating face, so that

$$Z^r = \frac{E}{I},$$

(D-3)

with $E = 0$ for the voltage source case, and $I = 0$ for the current source case. And so, for the voltage source case, $Z^I = Z_m$. For the current source case, we first use (D-2) to find

$$E = -\frac{Z_b}{j\omega L_m},$$

(D-4)

and put this result into (D-1) to obtain

$$Z^I = \frac{E}{I} = \frac{Z_b}{\omega L_m} + Z_m,$$

(D-5)

If a parallel tuning capacitor $C_p$ is used, then $Z_b$ will be the parallel combination of $C_p$ and $L_b$. If a series tuning capacitor $C_s$ is used, then we must replace $E$ by $E - I/j\omega C_s$ in (D-1) and (D-2).

When we define the coupling coefficient by

$$k_c^2 = \frac{L_b C_m}{L_m},$$

(D-6)

then the $X^I$ functions for Figures 3e, f, g, and h may be derived by algebra similar to the derivation of 3a.
<table>
<thead>
<tr>
<th>Case</th>
<th>Designation</th>
<th>Power Source</th>
<th>Tuning Field</th>
<th>Equivalent Circuit</th>
<th>$I^2$ Graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>VSE</td>
<td>Voltage</td>
<td>Series Electric</td>
<td>![Equation](1/\beta - \frac{C_1}{\beta^2})</td>
<td><img src="4a" alt="Graph" /></td>
</tr>
<tr>
<td>b</td>
<td>VPE</td>
<td>Voltage</td>
<td>Parallel Electric</td>
<td>![Equation](1/\beta - \frac{C_2}{\beta^2})</td>
<td><img src="4b" alt="Graph" /></td>
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<tr>
<td>c</td>
<td>ISM</td>
<td>Current</td>
<td>Series Electric</td>
<td>![Equation](1/\beta - \frac{C_1}{\beta^2})</td>
<td><img src="4c" alt="Graph" /></td>
</tr>
<tr>
<td>d</td>
<td>IPE</td>
<td>Current</td>
<td>Parallel Electric</td>
<td>![Equation](1/\beta - \frac{C_2}{\beta^2})</td>
<td><img src="4a" alt="Graph" /></td>
</tr>
<tr>
<td>e</td>
<td>VSM</td>
<td>Voltage</td>
<td>Series Magnetic</td>
<td>![Equation](1/\beta - \frac{C_2}{(\beta^2)} R_m)</td>
<td><img src="4d" alt="Graph" /></td>
</tr>
<tr>
<td>f</td>
<td>VPM</td>
<td>Voltage</td>
<td>Parallel Magnetic</td>
<td><img src="1/%5Cbeta" alt="Equation" /></td>
<td><img src="4b" alt="Graph" /></td>
</tr>
<tr>
<td>g</td>
<td>ISM</td>
<td>Current</td>
<td>Series Magnetic</td>
<td>![Equation](1/\beta + \frac{C_1}{\beta})</td>
<td><img src="4e" alt="Graph" /></td>
</tr>
<tr>
<td>h</td>
<td>IPM</td>
<td>Current</td>
<td>Parallel Magnetic</td>
<td>![Equation](1/\beta - \frac{C_1}{(\beta^2)} R_m)</td>
<td><img src="4d" alt="Graph" /></td>
</tr>
</tbody>
</table>

Note: $\beta = \omega_0 R_m$ and the tuning resonant current $\omega_0 = \omega_0$.  

Note: $R_m = R_m$ for the 8 cases (a) through (d).  

$C_1 = \frac{k_0^2}{1-k_0^2}$  

$C_2 = k_0^2$
Figure 4c  $\frac{x}{H} \text{ for ESE Case}$
Figure 4d  $\frac{X}{X_1}$ for VSM, IPM Cases

$$\frac{X}{X_1} = \frac{M_0 (a^2/2)^{1/3}}{M_1 \alpha (a^2/2)}$$

VSM Case

- $C = 0.01$, $k = 10.0\%$, $R = 9.95 \%

- $C = 0.25$, $k = 50.0\%$, $R = 44.7\%$

IPM Case

$\gamma = \frac{\alpha}{\omega_0}$