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MRC Technical Summary Report #1942

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SOLUTIONS OF $u_t - \Delta \phi(u) = 0$

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Technical Summary Report #1942
March 1979

ABSTRACT

The initial-value problem for equations of the form $u_t - \Delta \varphi(u) = 0$ where $\varphi: \mathbb{R} \to \mathbb{R}$ is nondecreasing arises in many contexts. The main results of this paper concern the continuity of the solutions of this initial-value problem as a function of $\varphi$. Depending on the behavior of $\varphi$ near zero, one finds either that the solutions are continuous into $C([0,\infty);L^1(\mathbb{R}^N))$ as a function of $\varphi$ or into a weaker space in which $L^1(\mathbb{R}^N)$ is replaced by a certain weighted $L^1$ space. A variety of auxiliary results are proved and the sharpness of the condition which distinguishes between the above cases is established.

AMS(MOS) Subject Classification: 35K55, 35K15, 47H15

Key Words: quasilinear parabolic equations
m-accretive operator,
continuous dependence,
porous flow problems.

Work Unit #1 - Applied Analysis

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024. This material is based upon work supported by the National Science Foundation under Grant No. MCS78-01245.
Significance and Explanation

The initial-value problem for equations of the form $u_t - \Delta \psi(u) = 0$
where $\psi: \mathbb{R} \rightarrow \mathbb{R}$ is nondecreasing arises in many contexts. The main
results of this paper concern the continuity of the solutions of this
initial-value problem as a function of $\psi$. This question is of interest
from many points of view. To have a physically meaningful problem, one
wants continuity in $\psi$. Or, for example, one might like to approximate
solutions of the problem $u_t - \Delta (u^3) = 0$ (which is degenerate where the
solution vanishes) by solutions of the nondegenerate problem
$u_t - \Delta (u^3 + \varepsilon u) = 0$ for $\varepsilon > 0$. This is justified by the current
work.
THE CONTINUOUS DEPENDENCE ON $\varphi$ OF SOLUTIONS OF $u_t - \Delta \varphi(u) = 0$

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Introduction.

The initial-value problem for equations of the form $u_t - \Delta \varphi(u) = 0$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is nondecreasing, arises in many contexts. The main results of this paper concern the continuity of solutions of these initial-value problems as functions of the nonlinearity $\varphi$. These results will be obtained via nonlinear semigroup theory, in a generality in which $\varphi$ may be a monotone graph; however we will preview the results here in a more restrictive setting.

If $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous and nondecreasing and $u_0 \in L^1(\mathbb{R}^m) \cap L^m(\mathbb{R}^N)$, then there is a unique $u \in C((0,\infty) : L^1(\mathbb{R}^N)) \cap L^m((0,\infty) \times \mathbb{R}^N)$ which satisfies

$$\begin{cases}
    u_t - \Delta \varphi(u) = 0 & \text{in } D'((0,\infty) \times \mathbb{R}^N), \\
    u(0,x) = u_0(x), & x \in \mathbb{R}^N.
\end{cases}$$

(1)

The existence assertion is explained in Section 1 and the uniqueness is proved in [8].

Assume that continuous nondecreasing functions $\varphi_n : \mathbb{R} \to \mathbb{R}$ with $\varphi_n(0) = 0$ are given for $n = 1,2,\ldots,\infty \in \mathbb{Z}^+ \cup \{\infty\}$ together with initial data $u_{0n} \in L^1(\mathbb{R}^N) \cap L^m(\mathbb{R}^N)$ and let $u_n \in C((0,\infty) : L^1(\mathbb{R}^N))$ be the associated unique solution of

$$\begin{cases}
    u_{nt} - \Delta \varphi_n(u_n) = 0 & \text{in } D'((0,\infty) \times \mathbb{R}^N), \\
    u_n(0,x) = u_{0n}(x), & x \in \mathbb{R}^N.
\end{cases}$$

(1_n)

If $\varphi_n \to \varphi_\infty$ and $u_{0n} \to u_{0\infty}$ in suitable ways, we will prove that $u_n \to u_\infty$. However, the precise situation is rather complicated. If $N \geq 3$, there are (at least) two distinct possibilities depending on the behavior of $\varphi_\infty(r)$ near $r = 0$. Roughly speaking, if the graph of $\varphi_\infty(r)$ is too steep near $r = 0$, the associated diffusion is so strong that the convergence of $u_n$ to $u_\infty$ near $|x| = \infty$ is not as good as it is in the case where $\varphi_\infty(r)$ approaches 0 more rapidly as $r \to 0$. To describe the condition on the behavior

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of \( \varphi_n \) near zero, let \( \beta_n = \varphi_n^{-1} \). Then \( \beta_n \) is a monotone graph which is a function only if \( \varphi_n \) is strictly increasing. Let \( \beta^0_n(r) \) be the element of least modulus of \( \beta_n(r) \) if \( r \in \mathbb{R}(\varphi_n) \) (the range of \( \varphi_n \)). E.g., if \( r > 0 \) and \( r = \varphi_n(s) \) for some \( s \), then

\[
\beta^0_n(r) = \min(t : r = \varphi_n(t)).
\]

If \( r > \varphi_n(s) \), \( r < \varphi_n(s) \) for all \( s \) we set \( \beta^0_n(r) = \infty \) (respectively, \( \beta^0_n(r) = -\infty \)). With these conventions we have:

Theorem: Let \( u_n \in C([0, \omega]) : L^1(\mathbb{R}^N) \cap L^\infty((0, \omega) \times \mathbb{R}^N) \) be the solution of \((1)_n\), \( n = 1, 2, \ldots, \omega \),

\[
\lim_{n \to \infty} \varphi_n(r) = \varphi_\omega(r) \quad \text{for} \quad r \in \mathbb{R},
\]

and

\[
\lim_{n \to \infty} \|u_{n} - u_{\omega}\|_{L^1(\mathbb{R}^N)} = 0.
\]

Then the following assertions hold:

(i) If \( N = 1 \) or \( N = 2 \), then \( u_n \to u_\omega \) in \( C([0, \omega]) : L^1(\mathbb{R}^N) \).

(ii) If \( N \geq 3 \) and

\[
- \int_{a}^{\infty} r^{N-1} \beta^0_\omega \left( \frac{r^{N-2}}{a^{N-2}} \right) dr = \int_{a}^{\infty} r^{N-1} \beta^0_\omega \left( \frac{1}{r^{N-2}} \right) dr = \infty \quad \text{for} \quad a > 0,
\]

then \( u_n \to u_\omega \) in \( C([0, \omega]) : L^1(\mathbb{R}^N) \).

(iii) If \( N \geq 3 \), \( 0 < \alpha \) and \( \rho_\alpha(x) = (1 + |x|^2)^{-\alpha} \) then \( \rho_\alpha u_n \to \rho_\alpha u_\omega \) in \( C([0, \omega]) : L^1(\mathbb{R}^N) \).

To clarify the nature of the condition (4), let \( \sigma_\alpha(r) = |x|^{m} \text{ sign } r \) where \( m > 0 \). Then \( \sigma_\alpha \) is a function and \( \sigma_\alpha = |r|^{1/m} \text{ sign } r \). The condition (4) is fulfilled exactly when \((N-1) - (N-2)/m = -1 \) or \( m \geq (N-2)/N \). It will be shown in Section 3 that if \( 0 < m < (N-2)/N \) the conclusion of (ii) fails in general. However, even if \( 0 < m < (N-2)/N \) we still have (iii) holding. The behavior near \( |x| = \infty \) is adequately damped by the weights \( \rho_\alpha \).

Parts (i) and (ii) of the above theorem are proved in Section 1. The proof of (iii), formulated in an appropriately general way, is given in Section 2. In fact, we associate an \( m \)-accretive operator in the corresponding weighted \( L^1 \) space with each problem \((1)_n \) and show this operator depends continuously on \( \varphi \). Section 3 establishes the necessity of (4) in the class of nonlinearities \( \varphi_\omega(r) = |r|^m \text{ sign } r \), \( 0 < m \).

There appears to be little previous work on continuity of solutions of \((1)_n \) in \( \varphi \).
The problem (1) with $\varphi(u) = |u|^m \sin(u)$ and $u_0 \geq 0$ is discussed in, e.g., [13], [15] via replacing $u_0$ by $u_0 + \varepsilon > 0$ and letting $\varepsilon$ tend to 0. The point of this method is to deal with strictly positive approximations for which the problem is nonsingular (since $\varphi'$ is bounded above and away from zero on the approximations.) In our framework, this corresponds to introducing the approximation $\varphi_\varepsilon(r) = \varphi(r+\varepsilon) - \varphi(\varepsilon)$ of $\varphi$. A simpler (in some sense) approximation is $\varphi_\varepsilon(r) = \varphi(r) + \varepsilon \epsilon$, which yields nonsingular problems if $m \geq 1$. Moreover, it permits $u_0$ to change sign. The question of the dependence on the nonlinearity of solutions of problems related to ours appears in the papers [3], [11], [12], [14], but the results of these works and the current paper have nothing in common. The requirement (4) plays a strong role in [4] in the study of the asymptotic behavior of solutions of an associated time-independent problem.
1. Continuity in \( C([0,\infty); L^1(\mathbb{R}^N)) \).

We begin by a review of the material we will draw upon in the sequel. In order to discuss \( u_t - \Delta w(u) = 0 \) within the nonlinear semigroup theory we first associate an \( m \)-accretive operator \( A_\psi \) in \( L^1(\mathbb{R}^N) \) with the formal expression \( A_\psi u = -\Delta \varphi(u) \). This is done via the results of [5]. Let \( \varphi \) be a maximal monotone graph in \( \mathbb{R} \) (see [7]) and consider the problem

\[
(1.1) \quad u - \Delta w = f \text{ in } D'(\mathbb{R}^N), \quad w(x) \in \varphi(u(x)) \text{ a.e. } x \in \mathbb{R}^N
\]

where \( f \in L^1(\mathbb{R}^N) \). The following theorem holds ([5]):

**Theorem 1** Let \( \varphi \) be a maximal monotone graph in \( \mathbb{R} \), \( 0 \in \varphi(0) \) and \( f \in L^1(\mathbb{R}^N) \). Then:

(i) If \( N \geq 3 \) there exists unique \( u \in L^1(\mathbb{R}^N) \) and \( w \in M^{N/(N-2)}(\mathbb{R}^N) \) for which (1.1) holds.

(ii) If \( N = 2 \) and \( 0 \in \text{int } D(\varphi) \), there is a unique \( u \in L^1(\mathbb{R}^2) \) for which there exists \( w \in L^1_{\text{loc}}(\mathbb{R}^2) \) with \( w(x) \in \varphi(u(x)) \) a.e., \( \text{grad } w \in M^2(\mathbb{R}^2)^2 \) and (1.1) is satisfied.

(iii) If \( N = 1 \) and \( 0 \in \text{int } D(\varphi) \), there is a unique \( u \in L^1(\mathbb{R}) \) for which there exists \( w \in L^1_{\text{loc}}(\mathbb{R}) \) with \( w(x) \in \varphi(u(x)) \) a.e. and (1.1) is satisfied.

**Remarks:** See [5, appendix] concerning the spaces \( M^0(\mathbb{R}^N) \). To relate (1.1) to the problem studied in [5], put \( \beta = \varphi^{-1} \) and rewrite (1.1) as \( \beta(w) = \Delta w + f \).

The operators \( A_\psi \) may now be defined by setting \( A_\psi u = \{ g \in L^1(\mathbb{R}^N) : u \text{ is the solution of (1.1) for } f = g + u \} \) for each \( u \in L^1(\mathbb{R}^N) \). The results of [5] also contain:

**Proposition 2** Under the assumptions of Theorem 1, (i), (ii), (iii), \( A_\psi \) is \( m \)-accretive in \( L^1(\mathbb{R}^N) \). Moreover \( I + A_\psi \) has the following properties:

(i) \( J_\psi \) is a translation and rotation invariant contraction on \( L^1(\mathbb{R}^N) \).

(ii) \( \forall f, g \in L^1(\mathbb{R}^N), \ f \leq g \text{ a.e. } \implies J_\psi f \leq J_\psi g \text{ a.e.} \)

(iii) \( \forall f \in L^1(\mathbb{R}^N), \ -\|f\|_{L^1(\mathbb{R}^N)} \leq J_\psi f \leq \|f\|_{L^1(\mathbb{R}^N)} \text{ a.e.} \)

where \( f^+ = \max(f, 0), f^- = -\min(f, 0) \).

Moreover, if \( f, u = J_\psi f, w \) are related as in Theorem 1, we have:

(iv) If \( N \geq 3 \) there is a constant \( C_N \) depending only on \( N \) such that

\[
\|w\|_{M^{N/(N-2)}(\mathbb{R}^N)} \leq C_N \|f\|_{L^1(\mathbb{R}^N)}.
\]
(v) If $N \geq 2$ there is a constant $D_N$ depending only on $N$ such that

$$
\| \text{grad } w \|_{L^N/(N-2)}(\mathbb{R}^N) \leq D_N \| w \|_{L^1(\mathbb{R}^N)}.
$$

In view of these results, the problem $u_t - \Delta u = 0$, $u(0,x) = u_0(x)$ may be transcribed as

$$
\frac{du}{dt} + A u = 0, \quad u(0) = u_0
$$

and solved by the nonlinear semigroup theory. (See, e.g., [2], [9], [10].) If $\varphi$ is continuous and $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ the resulting solution satisfies

$$
u \in C([0,\infty); L^1(\mathbb{R}^N)) \cap L^\infty([0,\infty) \times \mathbb{R}^N)$$

and (1) of the introduction. (While this is not in print, it is easy to show.) In the general case, we are interested in the continuity of the solution of (1.2) with respect to $\varphi$. For this we will employ the result of nonlinear semigroup theory which states that if each of $A_n$, $n = 1, 2, \ldots, \infty$ is an $m$-accretive operator in a Banach space $X$, $x_n \in D(A_n)$ and $u_n$ is the solution of

$$
\frac{du_n}{dt} + A_n u_n = 0, \quad u_n(0) = x_n
$$

then $A_n \to A$, $x_n \to x$ implies $u_n \to u$ in $C([0,\infty);X)$ where $A_n \to A$ means

$$
\lim_{n \to \infty} (I + A_n)^{-1}x = (I + A)^{-1}x \quad \text{for } x \in X.
$$

(See, e.g., [9], [10] for statements and references.) Thus we are reduced to studying the question of when $\varphi_n \to \varphi$ (as $m$-accretive operators in $\mathbb{R}$) implies $\varphi_n \to \varphi$ as $m$-accretive operators in $L^1(\mathbb{R})$.

Remark: The convergence $\varphi_n \to \varphi$ can be rephrased as $\varphi_n^0(\cdot) \to \varphi^0(\cdot)$ a.e. $r \in \mathbb{R}$, where $\varphi_n^0(\cdot)$ denotes the element of $\varphi_n(\cdot)$ of least modulus and $\varphi^0(\cdot) = \omega(\cdot)$ if $r$ is above (respectively, below) $D(\varphi)$. The main result of this section is:

**Theorem 3.** Let $\varphi_n$, $n = 1, 2, \ldots, \infty$ be maximal monotone graphs in $\mathbb{R}$ with $0 \in \varphi_n(0)$. Let $\varphi_n \to \varphi$ as $n \to \infty$. If

$$
N = 1 \text{ or } N = 2 \text{ and } 0 \in \text{int } D(\varphi),
$$
or
\[ N \geq 3, \quad \beta_\infty = \psi_\infty^{-1} \quad \text{and} \]
\[ \int_a^\infty r^{N-1} \beta_\infty^{\infty} (\frac{1}{r^{N-2}})dr = \int_a^\infty r^{N-1} \beta_\infty^{\infty} (\frac{1}{r^{N-2}})dr = \]
for \( a > 0 \)

then \( A_{\psi_\infty} A_{\psi_\infty} \) as \( m \)-accretive operators in \( L^1(\mathbb{R}^N) \).

If \( \psi_\infty(r) = \|r\|_p \) sign, we saw that (1.7) is equivalent to \( (N-2)/N \leq m \). Another example which helps fix the ideas is \( \psi_\infty(r) = 0 \). Then \( \beta_\infty^{\infty}(r) = 0 \) for \( r > 0 \), \( = \) for \( r < 0 \) and \( 0 \) for \( r = 0 \) and (1.7) holds. Also, if \( \psi_\infty(r) = \mathbb{R} \) for \( r = 0 \) and \( \psi_\infty(r) = \mathbb{S} \) for \( r \neq 0 \), then \( \beta_\infty(r) = 0 \). In this case (1.7) fails and so does the conclusion. If \( \psi_\infty(r) = nr \), then \( \psi_\infty + \psi_\infty \). However, the solution of \( u_n + A_n u_n = u_n - nA_n f = f \) satisfies \( \int_{\mathbb{R}^N} u_n = \int_{\mathbb{R}^N} f \) for \( n = 1, 2, \ldots \) while the solution of \( u_n + A_n u_n = 0 \). Thus \( u_n + u_n \) in \( L^1(\mathbb{R}^N) \) is impossible if \( \int_{\mathbb{R}^N} f \neq 0 \).

Proof of Theorem 3 for \( N \geq 3 \). The principle new step in the proof is:

Proposition 4. Let \( \psi_n, n = 1, 2, \ldots, \infty \) be as in Theorem 3. Let (1.7) hold and \( f \in L^1(\mathbb{R}^N) \). Then \( \{(I + A_n)^{-1}f \} \) \( n = 1, 2, \ldots, \infty \) is precompact in \( L^1(\mathbb{R}^N) \).

Proof: First let \( \varphi \) be an arbitrary maximal monotone graph in \( \mathbb{R} \) with \( 0 \in \varphi(0) \) and \( J_\varphi = \{(I + A_\psi)^{-1} \} \). Let \( \gamma g(x) = g(x+y) \) for \( y \in \mathbb{R}^N \). Since we have that \( J_\varphi 0 = 0 \),

\[ \gamma J_\varphi = J_\varphi \gamma \] and \( J_\varphi \) is a contraction on \( L^1(\mathbb{R}^N) \) (Proposition 2) we also have

\[
\| f \|_{L^1(\mathbb{R}^N)} \leq \| f \|_{L^1(\mathbb{R}^N)}
\]

which shows that \( \{J_\varphi f : \varphi \) a maximal monotone graph in \( \mathbb{R} \) with \( 0 \in \varphi(0) \} \) is a bounded subset of \( L^1(\mathbb{R}^N) \) on which translations are equi-continuous. This implies precompactness in \( L^1_{\text{loc}}(\mathbb{R}^N) \). We need then only show

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |J_\varphi f(x)|dx = 0 \quad \text{uniformly for } n = 1, 2, \ldots, \infty.
\]

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Let \( \epsilon > 0 \) and choose \( N, R_0 > 0 \) such that

\[
\| (f - M_N(\{x \geq R_0\})^\ast - 1_N^N(\mathbb{R}^N) \|_{L^1(\mathbb{R}^N)} \leq \epsilon
\]

where \( t^\ast = \max(t, 0) \) and \( 1_N \) is the characteristic function of \( A \subseteq \mathbb{R}^N \). Set

\[
g = M_N(\{|x| \geq R_0\}) . \quad \text{We have} \quad f \leq t^\ast \leq g + (f-g)^\ast \quad \text{and so}
\]

\[
(\mathcal{J}_n f)^\ast \leq \mathcal{J}_n g \leq \mathcal{J}_n (g + (f-g)^\ast)
\]

because \( \mathcal{J}_n \) is order preserving (Proposition 2). Using this and (1.10),

\[
\int \frac{\mathcal{J}_n f}{|x|^2} \leq \int \frac{\mathcal{J}_n (f^\ast)}{|x|^2} \leq \int \frac{\mathcal{J}_n (g + (f-g)^\ast)}{|x|^2}
\]

\[
\leq \int \frac{\mathcal{J}_n g + \| (f-g)^\ast \|_{L^1(\mathbb{R}^N)}}{|x|^2} \leq \int \frac{\mathcal{J}_n g + \epsilon}{|x|^2}
\]

since \( \mathcal{J}_n \) is an \( L^1(\mathbb{R}^N) \) contraction. Recalling that \( \epsilon > 0 \) is arbitrary, we conclude that it is enough to show

\[
\lim_{R \to \infty} \int \frac{\mathcal{J}_n g}{|x|^2} = 0 \quad \text{uniformly for} \quad n = 1, 2, \ldots, \infty
\]

in order to establish the same for \( (\mathcal{J}_n f)^\ast \). Treating \( (\mathcal{J}_n f)^\ast \) in a similar way, (1.10) holds for \( f \) if it holds for \( g = M_N(\{|x| \geq R_0\}) \), \( N > 0 \), \( R_0 > 0 \). Thus, without loss of generality, we assume

\[
f(x) = \begin{cases} \frac{M}{1 + |x|^2} & \text{if} & |x| \leq R_0 \\ 0 & \text{if} & |x| > R_0 \end{cases}
\]

where \( M > 0 \). The case \( M < 0 \) is entirely similar. We proceed now by giving some estimates on \( \mathcal{J}_n f \) for arbitrary \( \varphi \) and \( f \) given by (1.13). Since \( \mathcal{J}_n \) is rotation invariant there are nonnegative functions \( u(r), w(r) \) for \( r > 0 \) such that

\[
\begin{cases} r^{N-1} u(r) - (r^{N-1} u'(r))^\ast = r^{N-1} \chi_{\{0 < r < R_0\}} \\ w(r) \in \varphi(u(r)) \quad \text{a.e.} \end{cases}
\]
and \( J_r f(x) = u(|x|) \). (Observe that we are abusing notation a bit by using \( u(r), w(r) \) to denote the functions of a single variable corresponding to the rotation invariant functions \( u, w \) of \( N \) variables in Theorem 1 and Proposition 2.) The next task is to estimate \( \int_0^N R^N u(r) \, dr \) (which differs from \( \int \frac{1}{|x| > R} |J_r f| \) by a constant factor). Letting \( \bar{s} = s^{-1} \) we rewrite (1.14) as

\[
(1.15) \quad r^{N-1} \bar{s}(w(r)) - (r^{N-1} w'(r))^\prime = r^{N-1} \mathcal{K}_N(0 < r < R_0)
\]

and hereafter usually ignore the possibility that \( \bar{s} \) is multivalued for notational simplicity. Let

\[
(1.16) \quad H = \frac{1}{N} \int_{\mathbb{R}^N} f = \frac{1}{N} \mu_{N-1} M
\]

where \( \mu_N \) is the area of the unit sphere in \( \mathbb{R}^N \). The relation \( \int_{\mathbb{R}^N} \nabla f \leq \int_{\mathbb{R}^N} f \) is equivalent to

\[
(1.17) \quad \int_0^N r^{N-1} \bar{s}(w(r)) \, dr \leq H.
\]

In addition, the estimate of Proposition 2 (iv) (see [5, Appendix]) implies

\[
(1.18) \quad \int r^{2r} (w(s))^{s-1} ds \leq c_N H (\int r^{N-1} s^{N-1} ds)^{2/n} \leq c_N H r^2
\]

where \( c_N \) will denote various constants depending only on \( N \). Since (1.15) implies

\[
(1.19) \quad (r^{N-1} w'(r))^\prime = r^{N-1} \bar{s}(w(r)) \geq 0 \quad \text{for} \quad r > R_0,
\]

\( r^{N-1} w'(r) \) is nondecreasing on \( (R_0, \infty) \). Thus if \( r_0^{N-1} w'(r_0) > 0 \) for some \( r_0 \in (R_0, \infty) \), \( w' \) will be strictly positive on \( (r_0, \infty) \) and \( w(\infty) > 0 \). However, (1.18) then implies that if \( r > r_0 \)

\[
(1.20) \quad r^{N-1} w'(r) \leq 0 \quad \text{for} \quad r \geq R_0.
\]

Next we integrate the inequality \( t^{N-1} w'(t)/s^{N-1} \geq w'(s) \) (which is valid for \( R_0 \leq s \leq t \)) over \( r \leq s \leq t \) to find

\[
(1.21) \quad w(r) \geq w(t) + \frac{1}{N-1} \left( \frac{1}{r^{N-1}} - \frac{1}{t^{N-1}} \right) (t^{N-1} w'(t)), \quad R_0 \leq r \leq t
\]

and so, choosing \( t = 2r \) and using \( w \geq 0 \),
(1.22) \[ w(r) \geq c_n \frac{1}{r^{N-2}} (-2r)^{N-1} \varphi_N'(2r), \quad R_0 \leq r. \]

Now we set \( \varphi = \varphi_n \) and correspondingly write \( \varphi_n, w_n, \) etc. Let \( \varepsilon > 0 \) be given. It follows from (1.7) that there is an \( R(\varepsilon) > R_0 \) such that

\[
(1.23) \quad \int_{R_0}^{R(\varepsilon)} r^{N-1} \varphi_n \left( \frac{\varepsilon}{r^{N-2}} \right) dr > 2\varepsilon. 
\]

The convergence \( \varphi_n \to \varphi \) implies \( \varphi_n = \varphi_n^{-1} + \bar{\varphi}_n \). By Fatou's lemma we obtain from (1.23) and \( \bar{\varphi}_n \to \bar{\varphi} \) the existence of an integer \( M(\varepsilon) \) such that

\[
(1.24) \quad \int_{R_0}^{R(\varepsilon)} r^{N-1} \varphi_n \left( \frac{\varepsilon}{r^{N-2}} \right) dr > 2\varepsilon \quad \text{for} \quad n \geq M(\varepsilon). 
\]

Using (1.17), (1.20), (1.22), that \( -r^{N-1}w'_n(r) \) is nonincreasing and the monotonicity of \( \bar{\varphi}_n \) we have

\[
\int_{R_0}^{R(\varepsilon)} r^{N-1} \varphi_n \left( \frac{c_n (-2R(\varepsilon))^{N-1} w_n'(2R(\varepsilon))}{r^{N-2}} \right) dr < \varepsilon. 
\]

But then by (1.24) and the monotonicity of \( \bar{\varphi}_n \),

\[
(1.25) \quad 0 \leq r^{N-1} \bar{\varphi}_n(\varepsilon) \leq r^{N-1} \varphi_n(\varepsilon) \leq c_n \varepsilon \quad \text{for} \quad R \geq R(\varepsilon), \quad n \geq M(\varepsilon). 
\]

We are now prepared to demonstrate (1.12), which is equivalent to

\[
(1.26) \quad \lim_{R \to \infty} \int_{R}^{\infty} r^{N-1} \bar{\varphi}_n(w(r)) dr = 0 \quad \text{uniformly for} \quad n = 1,2,\ldots. 
\]

Indeed, by (1.19), if \( k_n(r) = r^{N-1}w_n'(r) \) we have

\[
\int_{R}^{\infty} r^{N-1} \bar{\varphi}_n(w(r)) dr = k_n(\varepsilon) - k_n(\varepsilon). 
\]

However, by (1.25), \( k_n(\varepsilon) - k_n(\varepsilon) \leq c_n \varepsilon \) if \( n \geq M(\varepsilon) \) and \( R \geq R(\varepsilon) \). Thus, if \( \delta > 0 \), we can guarantee

\[
(1.27) \quad \int_{R}^{\infty} r^{N-1} \bar{\varphi}_n(w(r)) dr \leq \delta 
\]

provided only that \( n \) and \( R \) are large enough. For the remaining finite number of indices \( n, (1.27) \) also holds if only \( R \) is sufficiently large and the proof is complete.

End of Proof of Theorem 3 for \( N \geq 3 \): Fix \( f \in L^1(\mathbb{R}^N) \) and let \( \varphi_n, \quad n = 1,2,\ldots. \) satisfy the conditions of Theorem 3. By Proposition 3, \( \{J_n f\} \) is precompact in \( L^1(\mathbb{R}^N) \).

It is enough to show that if \( j_k \), \( k = 1,2,\ldots \) is a subsequence of \( 1,2,\ldots \) then

\[ J_{k_n} f \to u \quad \text{in} \quad L^1(\mathbb{R}^N) \]

implies \( u = J_n f \). Thus we assume \( u_n = J_n f \to u \quad \text{in} \quad L^1(\mathbb{R}^N) \).

Let \( w_n \in H^{N/(N-2)}(\mathbb{R}^N), \quad w_n(x) \to \varphi_n(u_n(x)) \quad \text{a.e.} \) and \( u - \Delta w = f \) (as in Theorem 1).
The estimates (iv) and (v) of Proposition 2 imply \( \{w_n\} \) is bounded in \( L^1, p \) \( (\mathbb{R}^d_+) \) for \( 1 \leq p < N/(N-1) \) (see [5, appendix]) and hence we can assume \( w_n \to w \) in \( L^1, \text{loc} \) \( (\mathbb{R}^N) \).

But then \( \bar{w}(x) = \mathcal{E}_I(u(x)) \) a.e., \( \bar{w} \in H^{N/(N-2)}(\mathbb{R}^N) \) (by Proposition 4 (iv)) and

\[ u = \Delta w = f \text{ in } D'(\mathbb{R}^N). \]

Thus \( u = \mathcal{E}_I f \) by Theorem 1. The proof is complete.

**Proof of Theorem 3 for \( N = 1,2 \).** The proof is analogous to the case \( N \geq 3 \) in many respects and we only sketch it here. The analogue of Proposition 4 is:

**Proposition 5:** Let \( \delta > 0, r_0 > 0 \) and \( \Phi = \{\text{maximal monotone graphs in } \mathbb{R} \text{ with} \}

\( 0 \leq \psi(t) \text{ and } \psi'(t) < r_0 \}. \) Let \( N \in \{1,2\} \) and \( f \in L^1(\mathbb{R}^N) \). Then

\[ \{(I + \Lambda_\psi)^{-1}f; \psi \in \Phi \} \]

is precompact in \( L^1(\mathbb{R}^N) \).

**Proof.** As before we reduce to the case \( f = M_x(\{x \mid x < R_0 \}) \) and the problem of estimating

\[ \int_{R_0}^r s^{N-1} \psi(w(s)) ds = \int_{R_0}^r s^{N-1} u(s) ds \]

where \( \psi(f(x)) = u[|x|], w(r) \in \mathcal{E}(u(r)) \) or \( u(r) \in \mathcal{E}(w(r)) \) and

\[ (1.23) \quad \int_{R_0}^r s^{N-1} \psi(w(r)) - (s^{N-1} \psi'(r)) = M_x(0 < r < R_0). \]

The condition on \( \psi \) corresponds to

\[ (1.24) \quad \delta \leq |\psi(r)| \]

Integrating \( \psi'(r) \geq (s/r)^{N-1} \psi'(s) \) with respect to \( r \) yields

\[ (1.25) \begin{cases} \psi(r) \geq \psi(s) + (r-s) \psi'(s), & R_0 \leq s \leq r, \quad N = 1 \\ \psi(r) \geq \psi(s) + \sin(r/s) \psi'(s), & R_0 \leq s \leq r, \quad N = 2. \end{cases} \]

Thus either \( \psi'(s) \leq 0 \) or \( \psi(w) = \infty \). But \( \psi(w) = \infty \) is inconsistent with (1.24) and

\[ (1.26) \quad \int_{R_0}^r s^{N-1} \psi(w(r)) ds \leq H. \]

Thus \( \psi \leq 0 \) on \( (R_0, \infty) \). Now we integrate \( (r/s)^{N-1} \psi'(r) \geq \psi'(s) \) with respect to \( s \) to find the analogues of (1.20):

\[ (1.27) \begin{cases} \psi(r) \geq \psi(t) + (t-r)(-\psi'(t)), & R_0 \leq r \leq t, \quad N = 1 \\ \psi(r) \geq \psi(t) + \ln(t/r)[-\psi'(t)], & R_0 \leq r \leq t, \quad N = 2. \end{cases} \]

Now the monotonicity of \( \psi(r) \) and hence \( \psi(w(r)) \) implies

\[ \psi(w(2r)) \int_0^{2r} s^{N-1} ds \leq \int_0^r s^{N-1} \psi(w(s)) ds \leq H \text{ for } r \geq 2R_0, \]

so we have
(1.20) \(|\mathbf{w}(r)| < c_n r^{-N} \) for \( r \geq 2R_0 \).

If \( r \) is large enough to guarantee \( c_n r^{-N} < \delta \), we find \( \mathbf{w}(r) \leq r_0 \) from (1.20) and (1.24). Thus (1.27) implies \(-(n-1)\mathbf{w}'(t)) = 0\) as \( t \to \) uniformly for \( \nu \in \Phi \) and the proof is completed as before.

End of proof of Theorem 1 for \( N = 1,2 \). The convergence \( \nu_n \to \nu \) and \( 0 \in \text{int} \, D(\nu_n) \) implies that there are \( \delta, r_0 > 0 \) for which \( \nu_n \in \Phi \) as defined in Proposition 5 if only \( n \) is sufficiently large. Hence for \( f \in L^1(\mathbb{R}^N) \) \( (J_n \, f)n = 1,2,... \) is precompact. The proof is completed as in the case \( N = 3 \). However we must use \( (\mathbf{w}, J_n \, f \, n = 1,2,...) \) is bounded in \( L^1_{\text{loc}}(\mathbb{R}^N) \) in addition to the information in Proposition 2.

See the proofs of [5, Thms 3.1 & 4.1].

We will record one further result here for future reference:

Proposition 6. Let \( \nu \) be a maximal monotone graph in \( \mathbb{R}^N \) with \( 0 \in \nu(0) \). If \( N = 1 \) or \( N = 2 \) assume, in addition, that \( 0 \in \text{int} \, D(\nu) \). Let \( A_\nu \) be the associated \( m \)-accretive operator in \( L^1(\mathbb{R}^N) \). Then

\[
\text{d}(A_\nu) = \{ v \in L^1(\mathbb{R}^N) : v(x) \in \text{d}(\nu) \ \text{a.e.} \}
\]

where \( \text{d}(A_\nu) \) is the closure of \( \text{d}(A_\nu) \) in \( L^1(\mathbb{R}^N) \).

Proof: We must show that if

\[
(1.29) \quad \forall v \in L^1(\mathbb{R}^N) \quad \inf \text{d}(\nu) \leq v(x) \leq \sup \text{d}(\nu) \text{ a.e. } x \in \mathbb{R}^N
\]

then \( v \in \text{d}(A_\nu) \). Let \( \lambda > 0 \) and set \( u_\lambda = (I + \lambda A_\nu)^{-1}v = (I + \lambda A_\nu)^{-1}v - J_\nu v \).

Now \( u_\lambda \in \text{d}(A_\nu) \), and we will establish that \( u_\lambda \to \nu \) in \( L^1(\mathbb{R}^N) \) as \( \lambda \to 0 \), whence the result. One has \( \lambda \nu \to \nu \) as \( \lambda \to 0 \), where

\[
(1.30) \quad \varphi_0(r) = \begin{cases} 
\inf \text{d}(\nu) < r < \sup \text{d}(\nu) \\
-\infty, 0] \\
(0, \infty] \\
\phi
\end{cases}
\]

is the "subdifferential of the indicator function" of \( \text{d}(\nu) \). Since \( (I + \lambda A_\nu)^{-1}v = v \) by (1.29), (1.30), Theorem 1 implies \( u_\lambda \to \nu \) if \( \varphi_0 \) satisfies the hypotheses on \( \varphi_0 \) in Theorem 1. This is clearly the case if \( \inf \text{d}(\nu) < 0 < \sup \text{d}(\nu) \), while the case \( \inf \text{d}(\nu) = \sup \text{d}(\nu) = 0 \) (which is allowed for \( N \geq 3 \)) is trivial. If, for example,

\[
\inf \text{d}(\nu) < \sup \text{d}(\nu) = \max \text{d}(\nu) = 0, \text{ (which is possible for } N \geq 3), \text{ let } \varphi(t) = \varphi(t) \text{ for } t < 0 \text{,}
\]

then
\( \tilde{v}(0) = v(0) \cap (\sim, 0] \), \( \tilde{v}(r) = \emptyset \) for \( r > 0 \). Then, since \( J_{\lambda}^* v \leq J_{\lambda}^* 0 = 0 \) by (1.29) and Proposition 2, and \( \tilde{v}(r) \subseteq v(r) \) for \( r \leq 0 \), \( J_{\lambda}^* v = J_{\lambda}^* v \). The above argument applied to \( \tilde{v} \) in place of \( v \) yields \( J_{\lambda}^* v = J_{\lambda}^* v \) in \( L^1(\mathbb{R}^N) \) as \( \lambda \to 0 \). The remaining case \( \inf D(v) = \min D(v) = 0 \) is exactly the same.
2. Continuity in $L^1(\rho_a)$ for $N \geq 3$.

Throughout this section we assume $N \geq 3$. The weights $\rho_a$ are given by

\begin{equation}
\rho_a(x) = (1 + |x|^2)^{-a}.
\end{equation}

$L^1(\rho_a)$ denotes the weighted $L^1$-space determined by the norm

\begin{equation}
\|u\|_{L^1(\rho_a)} = \int_{\mathbb{R}^N} \rho_a(x)|u(x)|dx.
\end{equation}

The main result of this section is:

**Theorem 7.** Let $\varphi$ be a maximal monotone graph in $\mathbb{R}$ with $0 \leq \varphi(0)$. Let $0 < a \leq (N-2)/2$. Then the operator $A_a$ in $L^1(\rho_a)$ defined by

\begin{equation}
A_a u = \{-\Delta w : w \in L^1(\rho_a+1), -\Delta w \in L^1(\rho_a) \text{ and } w(x) \in \varphi(u(x)) \text{ a.e.}\}
\end{equation}

for $u \in L^1(\rho_a)$ is $m$-accretive in $L^1(\rho_a)$. Moreover, if $\varphi_n, n = 1, 2, \ldots, \infty$ are maximal monotone graphs in $\mathbb{R}$ with $0 \leq \varphi_n(0)$ and $\varphi_n \rightarrow \varphi_m$ (as maximal monotone graphs) as $n \rightarrow \infty$, then $A_{\varphi_n} \rightarrow A_{\varphi_m}$ as $m$-accretive operators in $L^1(\rho_a)$.

As a preliminary to the proof of Theorem 7 we consider the problem $-\Delta w = f \in L^1(\rho_a)$. Let

\begin{equation}
E_N(x) = a_N|x|^{N-2}
\end{equation}

where $a_N$ is that constant for which $-\Delta E_N = \delta_0$, the Dirac mass at the origin. That is, $E_N$ is a fundamental solution of $-\Delta$.

**Proposition 8.** Let $0 < a \leq (N-2)/2$ and $f \in L^1(\rho_a)$. Then there is exactly one solution $w \in L^1(\rho_a+1)$ of $-\Delta w = f$. This solution is given by

\begin{equation}
w(x) = E_N * f(x) = \int_{\mathbb{R}^N} \frac{a_N}{|x-y|^{N-2}} f(y)dy.
\end{equation}

Moreover, grad $w \in L^1(\rho_a+1/2)$ and

\begin{equation}
\|E_N * f\|_{L^1(\rho_a+1)} + \|\text{grad}(E_N * f)\|_{L^1(\rho_a+1/2)} \leq c \|f\|_{L^1(\rho_a)}
\end{equation}

for a constant $c_N$ depending only on $N$.

**Proof:** There are two main points whose proofs we will sketch. First we claim that if $w \in L^1(\rho_a+1)$ and $-\Delta w = 0$, then $w = 0$. This is observed, for example, in the proof of the remark following Theorem 2.1 in [5]. Alternatively, $L^1(\rho_a+1)$ is a subspace of $S'(\mathbb{R}^N)$, the continuous linear functionals on the Schwartz space $S(\mathbb{R}^N)$ of rapidly decreasing $C^\infty$ functions. Hence $w \in L^1(\rho_a+1)$ and $-\Delta w = 0$ implies $w$ is a constant.
But if \( a \leq (N-2)/2 \), the only constant function in \( L^1(\rho_{a+1}) \) is 0. This establishes the uniqueness claim of the proposition. The existence, (2.4) and the estimate (2.5) all follow easily if we establish (2.5). Observe that

\[
\| \nabla N \|_{L^1(\rho_{a+1})} \leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{\rho_N}{|x-y|^{N-2}} (1 + |x|^2)^{-(a+1)} dx |f(y)| dy \right)
\]

and

\[
\| \nabla N \|_{L^1(\rho_{a+1})} \leq \int_{\mathbb{R}^N} \left( \int_{\mathbb{R}^N} \frac{(N-2)a_N}{|x-y|^{N-1}} (1 + |x|^2)^{-(a+1)} dx |f(y)| dy \right)
\]

To estimate these quantities appropriately, we use:

**Lemma 9.** Let \( \beta, \gamma > 0 \), \( \beta + \gamma - N > 0 \) and \( N > \beta \).

Then there is a constant \( C = C(\beta, \gamma, N) \) such that

\[
\int \frac{1}{|x-y|^\beta} \frac{1}{1 + |x|^\gamma} dx \leq \frac{C}{1 + |y|^\delta}
\]

where \( \delta = \min(\beta + \gamma - N, \beta) \).

**Sketch of Proof of Lemma 9.** Using the assumptions that \( \beta + \gamma - N > 0 \) for large \( |x| \) and that \( N > \beta \) for \( x \) near \( y \) one sees

\[
g(y) = \int \frac{1}{|x-y|^\beta} \frac{1}{1 + |x|^\gamma} dx < \infty
\]

and \( g \) is bounded on compact sets. Decomposing the integral defining \( g \) into the sum of three integrals, one over each of the sets \( \Omega_1 = \{ x \in \mathbb{R}^N : |x-y| \leq \frac{1}{2} |y| \} \), \( \Omega_2 = \{ x \in \mathbb{R}^N : |x-y| > \frac{1}{2} |y| \} \) and \( \Omega_3 = \{ x \in \mathbb{R}^N : |x| > 2 |y| \} \) and using that \( |x| \geq \frac{1}{2} |y| \) on \( \Omega_1 \), while \( |x-y| \geq \frac{1}{2} |x| \) on \( \Omega_3 \), one obtains \( g(y) \leq C |y|^{-N-(\beta + \gamma)} + |y|^{-\beta} \)

where the term \( |y|^{-\beta} \) arises from the integral over \( \Omega_2 \). The result follows.

**End of Proof of Proposition 8.** From Lemma 9 we find

\[
\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-2}} \frac{1}{(1 + |x|^2)^{a+1}} dx \leq \frac{\text{cons.}}{1 + |y|^{\delta}} \leq \frac{\text{cons.}}{(1 + |y|^{\gamma/2})^{\delta/2}}
\]

and

\[
\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N-1}} \frac{1}{(1 + |x|^2)^{b+1}} dx \leq \frac{\text{cons.}}{1 + |y|^{\delta}} \leq \frac{\text{cons.}}{(1 + |y|^{\gamma/2})^{\delta/2}}
\]

where \( \delta = \min(2a, N-2) = \min(2a, N-1) = a \) for \( a \leq (N-2)/2 \). Together with (2.6) and (2.7), these estimates imply (2.5).

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The next ingredient in the proof of Theorem 7 is:

Proposition 10: Let \( w \in L^1(\rho_{a+1}) \), \( \Delta w \in L^1(\rho_a) \) and \( 0 < a \leq (N-2)/N \). Let \( p \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \), \( 0 \leq p' \), \( p' \in L^\infty(\mathbb{R}) \) and \( p(0) = 0 \). Then \( p'(w)|\text{grad}w|^2 \in L^1(\rho_a) \) and

\[
\int_N (\Delta w) p(w)\rho_a + \int_N p'(w)|\text{grad}w|^2\rho_a + \int_N j(w)(-\Delta \rho_a) \leq 0
\]

where \( j(\tau) = \int_0^\tau p(s)ds \).

Proof: Let \( \{f_n\} \subset C_0^\infty(\mathbb{R}^N) \), \( f_n \rightarrow -\Delta w \) in \( L^1(\rho_a) \). Then \( w_n = F^{-1}f_n \) satisfies (by Proposition 8)

(i) \( w_n \rightarrow w \) in \( L^1(\rho_{a+1}) \),
(ii) \( \text{grad}w_n \rightarrow \text{grad}w \) in \( (L^1(\rho_{a+1/2}))^N \),
(iii) \( -\Delta w_n = f_n \rightarrow -\Delta w \) in \( L^1(\rho_a) \).

We may integrate the identity

\[
(\Delta w_n) p(w_n)\rho_a = \text{div}(p(w_n)\rho_a \text{grad}w_n - j(w_n)\text{grad}\rho_a)
\]

\[
-|\text{grad}w_n|^2 p'(w_n)\rho_a + j(w_n)\Delta \rho_a
\]

over the ball \( \{ x \in \mathbb{R}^N ; |x| \leq R \} = B_R \) and let \( R \rightarrow \infty \) to conclude that (2.8) holds with \( w \) replaced by \( w_n \). Indeed, since \( f_n \in C_0^\infty(\mathbb{R}^N) \), \( w_n \) and \( \text{grad}w_n \) decay like \( |x|^{-N+2} \) and \( |x|^{-N+1} \), respectively, as \( |x| \rightarrow \infty \) while \( p \in L^\infty \) and \( |j(\cdot)| \leq \|p\|_{L^\infty} \).

Thus

\[
|p(w_n)\rho_a \text{grad}w_n - j(w_n)\text{grad}w_a| \leq \text{const.} \left( |x|^{-2a} |x|^{-N+1} + |x|^{-N+2} |x|^{-2a-1} \right)
\]

\[
\leq \text{const.} \left( |x|^{-(N-1)+2a} \right)
\]

as \( |x| \rightarrow \infty \), and the integral over \( \partial B_R \) arising from the divergence term above tends to zero as \( R \rightarrow \infty \). Now we pass to the limit as \( n \rightarrow \infty \) in the relation

\[
\int_{\partial B_R} (\Delta w_n) p(w_n)\rho_a + \int_{\partial B_R} p'(w_n)|\text{grad}w_n|^2\rho_a + \int_{\partial B_R} j(w_n)(-\Delta \rho_a) = 0
\]

to establish (2.8). The first and third terms above have the desired limiting values by (2.9) (i), (iii) and \( |j(w_n)(-\Delta \rho_a)| \leq \text{const.} \|w_n\|_{\rho_{a+1}} \), while the second term is handled by Fatou's lemma and (2.9) (iii).
Sketch of Proof of Theorem 7. It is now a (somewhat lengthy) exercise in the use of the arguments of [5] to verify that \( A_{\varphi} \) as given by (2.3) is \( m \)-accretive in \( L^1(\rho_a) \). We leave this to the reader. (The third term, \( \int_{\mathbb{R}^N} j(w)(-\Delta \rho_a) \), in (2.8) is nonnegative since \( -\Delta \rho_a \geq 0 \) by direct calculation. It may be dropped while doing the exercise.)

It remains to show that \( \varphi_n * \varphi_m \) implies \( A_n * A_m \) in \( L^1(\rho_a) \). Each \( A_{\varphi} \) has the properties used in the proof of Theorem 3 for \( N \geq 3 \). In particular, \( J_{\varphi} = (I + A_{\varphi})^{-1} \) is order preserving, translation invariant and an \( L^1(\rho_a) \)-contraction. Thus for each \( f \in L^1(\rho_a) \), \( \{ J_{\varphi} f : \varphi \) a maximal monotone graph in \( \mathbb{R} \) with \( 0 \in \varphi(0) \} \) is precompact in \( L^1_{\text{loc}}(\mathbb{R}^N) \). To see this set is precompact in \( L^1(\rho_a) \) we need only show

\[
\lim_{R \to \infty} \sup_{\varphi \in \mathcal{C}} \int_{|x| \geq R} |f(x)| = 0
\]

uniformly in \( \varphi \) for a dense set of \( f \)'s. But if \( f \in L^1(\mathbb{R}^N) \)

\[
\int_{|x| \geq R} |\varphi f(x)| \leq \frac{1}{(1+R)^{2\alpha}} \int_{\mathbb{R}^N} |f(x)| \leq \frac{1}{(1+R)^{2\alpha}} \|f\|_{L^1(\mathbb{R}^N)}
\]

and so (2.10) holds uniformly in \( \varphi \). The convergence \( A_{\varphi_n} * A_{\varphi_m} \) then follows as before.
3. A counterexample for \( N \geq 3 \).

We consider the special case

\[
\begin{align*}
\begin{cases}
    u_t - \Delta(u^n) = 0 \\
    u(0, x) = u_0(x)
\end{cases}
\end{align*}
\]

of (1) where \( \varphi(r) = r^m, \quad 0 < m < (N-2)/N \). (More precisely, we mean \( \varphi(r) = |r|^m \) sign, but will often write \( r^m \) for brevity.) Let \( S_m(t) \) be the semigroup on \( L^1(\mathbb{R}^N) \) associated with (3.1). With \( 2^* = 2N/(N-2) \) we will prove:

**Proposition 10.** Let \( 0 < m < (N-2)/N, \quad \beta = (2-m^+)/2^* \) and \( u_0 \in L^\beta(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \). Then there is an \( t^* > 0 \) such that \( \lim_{t \to t^*} \| S_m(t)u_0 \| = 0 \) for \( t \geq t^* \).

The following results for the preceding results. Set

\[
\varphi_n(r) = \begin{cases}
    \min(nr, |r|^m) & \text{for } r \geq 0, \\
    \max(nr, -|r|^m) & \text{for } r \leq 0
\end{cases}
\]

Then \( \varphi_n(r) \rightarrow |r|^m \) sign as \( n \rightarrow \infty \).

However, one can easily show that if \( u_n \) is the solution of

\[
\begin{align*}
\begin{cases}
    \frac{du_n}{dt} + \Delta u_n = 0 \\
    u_n(0) = u_0
\end{cases}
\end{align*}
\]

then \( \int_{\mathbb{R}^N} u_n(t) = \int_{\mathbb{R}^N} u_0 \) for all \( t \geq 0 \). It follows from Theorem 3.1 that \( u_n(t) \) cannot converge to \( S_m(t)u_0 \) for any \( t \geq t^* \) unless \( \int_{\mathbb{R}^N} u_0 = 0 \).

**Proof of Proposition 10.** The formal idea of the proof runs as follows: Multiply the equation \( u_t - \Delta u^m = 0 \) by \( \frac{1}{s+1} u^s \) and integrate over \( \mathbb{R}^N \) to find

\[
\frac{d}{dt} \int_{\mathbb{R}^N} u^{s+1} + \frac{1}{s+1} \int_{\mathbb{R}^N} \nabla u^m \nabla u^s = 0.
\]

Now the Sobolev inequalities imply

\[
\int_{\mathbb{R}^N} u^m \nabla u^s = \frac{m^s}{(m+\beta)^{1/2}} \int_{\mathbb{R}^N} \nabla u^2 \quad \text{and}
\]

\[
\int_{\mathbb{R}^N} u^2 \nabla u^{m+\beta} = C \left( \int_{\mathbb{R}^N} u^2 \right)^{2/2^*},
\]

where \( 2^* = 2N/(N-2) \). When \( \beta = (2-m^+)/2^* \) we have \( (m+\beta)2^*/2 = \beta+1 \) so, in all,

(3.4), (3.5) imply

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\[(3.6) \quad \frac{d}{dt} \left( \int_{\mathbb{R}^N} u^{s+1} \right) + c \left( \int_{\mathbb{R}^N} u^{s+1} \right)^{2/2} \leq 0. \]

However, every solution of the inequality

\[(3.7) \quad f + C f^\gamma \leq 0, \]

where \( f : [0, \infty) \to [0, \infty) \), \( C > 0 \), \( 0 < \gamma < 1 \) has the property that there is a \( T^* > 0 \) such that \( f(t) = 0 \) for \( t > T^* \). This is proved by comparison. The function

\[ g(t) = a(T-t)^{1-\gamma}, \quad a = \frac{1}{(1-\gamma)c} \]

satisfies

\[(3.8) \quad g' + c g^\gamma = 0 \quad \text{on} \quad [0, T^*] \]

and \( g(0) = aT^{1-\gamma} \geq f(0) \) if \( T^* \) is large enough. Hence \( f \leq g \) on \( [0, T^*] \) and \( f \leq 0 \) for \( t > T^* \).

This formal proof can be made rigorous. We sketch how this is done. Assume that \( u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \). Let \( \lambda > 0 \) and define \( v_0 = u_0 \), and then \( v_{i+1} \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) by

\[(3.9) \quad \frac{v_{i+1} - v_i}{\lambda} = \Delta v_{i+1}^{m} \]

or, more precisely, \( v_{i+1} = (1 + \lambda A)^{-1} v_i \). We multiply (3.9) by \( v_{i+1}^\delta \) and use that \( v_i v_{i+1}^\delta \leq \frac{1}{\beta+1} v_{i+1}'^{\beta+1} + \frac{1}{\beta} v_{i+1}^{\beta} \) to conclude

\[ \frac{1}{\beta+1} (v_{i+1}'^{\beta+1} - v_i^{\beta+1}) - (\Delta v_{i+1}^m) v_{i+1}^\delta \leq 0. \]

Integrating this inequality over \( \mathbb{R}^N \) yields

\[ \frac{1}{\beta+1} \frac{1}{\lambda} \left[ \int_{\mathbb{R}^N} v_{i+1}'^{\beta+1} - \int_{\mathbb{R}^N} v_i^{\beta+1} \right] - \int_{\mathbb{R}^N} v_{i+1}^m v_{i+1}^\delta \leq 0 \]

and so, by Sobolev,

\[(3.10) \quad \frac{1}{\lambda} \int_{\mathbb{R}^N} v_i' v_{i+1}^{\beta+1} - \int_{\mathbb{R}^N} v_i \int_{\mathbb{R}^N} v_{i+1}^m v_{i+1}^\delta \leq 0. \]

The manipulations involving \( \int (\Delta v_{i+1}^m) v_{i+1}^\delta \) can be justified by [5, Appendix]).

Set \( v_\lambda(t) = v_i \) for \( i \lambda \leq t < (i+1)\lambda \). Then the semigroup theory implies \( v_\lambda \to u \), where \( u \) is the semigroup solution of (3.1), in \( L^1(\mathbb{R}^N) \) uniformly on compact subsets of \([0, \infty)\). Since \( \|v_\lambda\|_{L^p} \leq \|u\|_{L^p} \), \( v_\lambda \to u \) in \( L^p(\mathbb{R}^N) \) for \( 1 \leq p < \infty \), uniformly on compact sets. Thus \( f_\lambda(t) = \frac{1}{\beta+1} \int_{\mathbb{R}^N} v_i' v_{i+1}^{\beta+1} \) for \( i \lambda \leq t < (i+1)\lambda \) converges uniformly
on compacts to \( f(t) = \frac{1}{\beta(t)} \int_{\mathbb{R}^N} (u(t))^{\beta+1} \). The relation (3.10) can be rephrased as

\[
\frac{1}{\lambda} \left[ f_\lambda(t+\lambda) - f_\lambda(t) \right] + C \int_0^t f_\lambda(t + \lambda) \frac{2}{\beta+1} \leq 0
\]

for \( 0 \leq t \). Multiply this relation by \( \psi \in C_0^\infty((0,\infty)) \psi \geq 0 \) and integrate to find

\[
\frac{1}{\lambda} \int_0^t \left( f_\lambda(t + \lambda) - f_\lambda(t) \right) \psi(t) \, dt + C \int_0^t f_\lambda(t + \lambda) \frac{2}{\beta+1} \psi(t) \, dt
\]

\[
= \int_0^t f_\lambda(t) \left( \frac{\psi(t+\lambda) - \psi(t)}{\lambda} \right) \, dt + C \int_0^t f_\lambda(t + \lambda) \frac{2}{\beta+1} \psi(t) \, dt \leq 0
\]

provided \( \psi(t) = 0 \) for \( 0 \leq t \leq \lambda \). Letting \( \lambda \rightarrow 0 \) we have

\[
- \int_0^t f(t) \psi'(t) \, dt + C \int_0^t f(t) \frac{2}{\beta+1} \psi(t) \, dt \leq 0
\]

so \( f' + C f(t)^{2/\beta+1} \leq 0 \) in \( D'((0,\infty)) \). However, the comparison argument made above is valid if (3.7) only holds in the sense of distributions. We conclude that \( f(t) \equiv 0 \) for \( t > T^* \), where

\[
T^* = \left( \frac{1}{\beta+1} \int_{\mathbb{R}^N} u_0^m \right)^{1-\gamma} \left( (1-\gamma) \lambda \right)^{-1}.
\]

Now we can eliminate the assumption \( u_0 \in L^\infty(\mathbb{R}^N) \) by approximation (since \( T^* \) depends only on \( \|u_0\| \) ) and \( L^\infty(\mathbb{R}^N) \).

**Remarks:** The solution \( u \) of the problem (which can be generalized)

\[
\begin{cases}
    u_t - \Delta \left| u \right|^m \text{sign} u = 0 & t > 0, \ x \in \Omega \\
    u(0, x) = u_0(x) & x \in \Omega \\
    u^m(t, x) = 0 & \text{for } t > 0, \ x \in \partial \Omega
\end{cases}
\]

where \( \Omega \) is a bounded open subset of \( \mathbb{R}^N \) and \( u_0 \in L^\infty(\Omega) \) has a finite extinction time \( T^* \) such that \( u \equiv 0 \) for \( t > T^* \) if \( 0 < m < 1 \). This may be proved for our notion of solution such as above. The behavior of \( u \) as \( t \uparrow T^* \) is considered in Bensoussan and Fournier [7], who remark in passing on adapting a proof of Sabinina [15] of the existence of \( T^* \) if \( N = 1 \). If \( \Omega = \mathbb{R}^N \), as in our case, then \( T^* \) exists only if \( 0 < m < (N-2)/N \) as our results prove. (There can be no extinction in the case we have continuous dependence in \( L^1(\mathbb{R}^N) \).) The results of Benilan and Aronson [11] establish that there is no extinction for \( (N-2)/N < m < 1 \). Veron [17] exhibits other cases of extinction times.
REFERENCES


[16] Sabinina, E. S., On a class of quasilinear parabolic equations not solvable for the time derivative, Sibirski Mat. Z. 6 (1965), 1074-1100. (Russian)

The initial-value problem for equations of the form \( u_t - \Delta \psi(u) = 0 \), where \( \psi: \mathbb{R} \to \mathbb{R} \) is nondecreasing, arises in many contexts. The main results of this paper concern the continuity of the solutions of this initial-value problem as a function of \( \psi \). Depending on the behavior of \( \psi \) near zero, one finds either that the solutions are continuous into \( C ((0, \infty); L^1 (\mathbb{R}^N)) \) as a function of \( \psi \) or into a weaker space, in which \( L^1 (\mathbb{R}^N) \) is replaced by a certain weighted \( L^1 \) space. A variety of auxiliary results are proved and the sharpness of the condition which distinguishes between the above cases is established.