COMPUTATION OF THE
BIVARIATE NORMAL DISTRIBUTION
OVER CONVEX POLYGONS

by
A. R. DONATO
M. F. JARNAKIN, Jr.
R. K. HAGEMAN
Strategic Systems Department

SEPTEMBER 1978

Approved for public release; distribution unlimited

NAVAL SURFACE WEAPONS CENTER
Bolling, Virginia 22448 Silver Spring, Maryland 20910

79 06 04 050
NAVAL SURFACE WEAPONS CENTER
Dahlgren, Virginia 22448

Paul L. Anderson, Capt., USN
Commander
**Title:** Computation of the Bivariate Normal Distribution Over Convex Polygons

**Authors:** A. R. Di Donato, M. P. Barnagin, Jr., R. K. Hageman

**Performing Organization Name and Address:** Naval Surface Weapons Center (KOS)  Dahlgren, Virginia 22448

**Controlling Office Name and Address:** Naval Surface Weapons Center (KOS)  Dahlgren, Virginia 22448

**Abstract:**
A procedure is given for computing the bivariate normal probability over an angular region or a convex polygon. The procedure is implemented into a Fortran IV computer program which is designed to yield 3, 6, or 9 decimal digits of accuracy. Comparisons with two other published methods, for the same achievable accuracy, show our program to be much faster.
FOREWORD

The work described in this report was done in the Science and Mathematics Research Group of the Strategic Systems Department. Requests for such work over the last fifteen years were instrumental in its initiation. We acknowledge Ann Howes' contributions to the editorial aspects of this report.

R. A. NIEMANN, Head
Strategic Systems Department
ABSTRACT

A procedure is given for computing the bivariate normal probability over an angular region or a convex polygon. The procedure is implemented into a Fortran IV computer program which is designed to yield 3, 6, or 9 decimal digits of accuracy. Comparisons with two other published methods, for the same achievable accuracy, show our program to be much faster.
CONTENTS

Foreword ................................................................. iii
Abstract ................................................................. iv
1. Introduction ......................................................... 1
2. Algorithm for P[A(R, \(\theta_1, \theta_2\))] ................................. 2
3. Use of P(A) to Compute P(H) ......................................... 8
4. Computer Program for P(H) (and P(A)) ............................. 9
5. Comparison with Gideon and Gurland Method ..................... 17
6. Comments on Drezner's Method ..................................... 18
7. Some Numerical Results ........................................... 22
References ............................................................. 23

Appendices

A. Program Parameters. Chebyshev Coefficients for erfc (x)/z(x), x \(> 0\) ........................................ 25
B. Listing of Drezner Program .......................................... 27
C. Listing of Test Program with Some Numerical Results .......... 35
D. Fortran Listing of the Program ...................................... 41

Distribution List
1. INTRODUCTION

In this report we give a numerical procedure for integrating the bivariate normal density function over a convex polygon. The Fortran IV computer program* developed from it is fast and is designed to yield the output probability to 3, 6, or 9 decimal digit accuracy. As far as we know, it is the fastest and most versatile program of its kind — most versatile in the sense that it handles, with three prespecified levels of accuracy in the output, arbitrary convex polygons** rather than just triangles and quadrilaterals. We make note at this time that the program serves as a basic subroutine for the automatic computation of the bivariate normal over an arbitrary polygon. A complete program for this much more general case has been written, checked out, and is operational. Its description is deferred to a later report.

Our procedure for the convex polygon case depends on a fast method, with prespecified accuracy, to evaluate the bivariate normal distribution over an angular region, Λ. In particular, we wish to evaluate

\[
\Phi(\Lambda) = \frac{(1 - \rho^2)^{-\frac{1}{2}}}{2\pi a_1 a_2} \int_{\Lambda} \exp \left\{ -\frac{(w - \mu_1)^2}{a_1^2} - 2\rho \frac{(w - \mu_1)(z - \mu_2)}{a_1 a_2} \right\} \frac{1}{2(1 - \rho^2)} \text{d}w \text{d}z,
\]

where \((\mu_1, \mu_2)\) is the mean and \(\begin{bmatrix} a_1^2 & \rho a_1 a_2 \\ \rho a_1 a_2 & a_2^2 \end{bmatrix}\) the covariance matrix of the normal random variable \((w, z)\) with correlation coefficient \(\rho\). The angular region \(\Lambda\) is defined as the semi-infinite part of the plane bounded by two intersecting directed straight lines. Of course by this definition there are four such regions, and therefore it is necessary to always state which of them is involved.

The well-known linear transformation

\[
\begin{align*}
x &= \left[ \frac{w - \mu_1}{a_1} - \rho \frac{z - \mu_2}{a_2} \right] \sqrt{1 - \rho^2}, \\
y &= \frac{z - \mu_2}{a_2},
\end{align*}
\]

reduces the integrand of (1) to one with circular symmetry, namely

\[
\Phi(\Lambda) = \frac{1}{2\pi} \int_{\Lambda} \exp \left\{ -\frac{x^2 + y^2}{2} \right\} \text{d}x \text{d}y.
\]

*The program is coded for the CDC-6700, a large-scale binary computer capable of one million operations per second. It has a 60 bit binary word length of which 48 are used to express the mantissa of a number.

**The term convex polygon will always mean a closed convex polygon.
where $A$, like $\Lambda$, is an angular region, since (2) takes straight lines into straight lines. Thus we deal only with (3) hereafter unless noted otherwise.

An extensive literature exists on methods for integrating the bivariate normal variate over various simple geometries, where the ultimate objective is to appropriately utilize these integrations to evaluate the distribution over a polygon, [2,3,4,5,6,7]. One such case is where $\Lambda$ in (1) forms a right angle at $(h,k)$ with the sides of the angle directed parallel to the $w$ and $z$ directions. When the mean is zero and the variances are equal to one, (1) for the angular region just described is denoted by $\Phi(h,k,\rho)$ and is called the bivariate normal integral [3] or the bivariate normal probability function [9, p. 936]. We shall make reference to $\Phi$ in Section 6. We show it is equivalent to (3) where the given right angle is transformed to an angular region $A$ and then show that a recent method for computing $\Phi$, [3], is slower than our procedure for obtaining the same result from (3).

The idea of integrating over an angular region seems to have originated with Gideon and Gurland (hereafter G & G), [4] * [5]. As observed by them, the idea of integrating over an angular region, as expressed by (3), is a natural and easily visualized way to obtain the probability over a polygon. In Section 5 we shall discuss and compare their computing method with ours.

In Section 3 we show how, by utilizing (3) over a set of angular regions, we obtain the probability over a convex polygon. Our approach differs here also from what G & G advocate. In Section 7, we give some numerical results. The computer program is described in Section 4 with its Fortran listing given in Appendix D. In the next section we give some analysis and also the algorithm for evaluating (3). Its implementation into a computer program is not straightforward since certain precautions are necessary as will be explained in Section 4.

2. ALGORITHM FOR $P(A(R, \theta_1, \theta_2))$

In this section we derive the algorithm by which we evaluate (3); i.e., we obtain that part of the circular normal distribution over the angular region $A(R, \theta_1, \theta_2)$ as the shaded region shown in Figure 1. Lines $\mathcal{L}_1$ and $\mathcal{L}_2$ form the boundaries of this region. $R$ denotes the distance from the origin to the vertex of $A(R, \theta_1, \theta_2)$.

It is convenient because of circular symmetry in the integrand of (3), to perform a rotation of axes such that the line $L$ and the $x$ axis coincide with $A$ rigidly rotated as shown in Figure 2. Hereafter we shall always assume such a rotation, through the angle $\psi$, has been carried out.

The coordinate transformation

\begin{equation}
  x = R + r \cos \theta, \quad y = r \sin \theta, \quad 0 \leq \theta < \pi,
\end{equation}

is used in (3) to obtain

*We are grateful to Pete Shugart at White Sands Missile Range, New Mexico for bringing their Wisconsin report [4] to our attention.
Figure 1. Angular Region, \(A(R, \theta_1, \theta_2)\), (shaded region)

Figure 2. Angular Region \(A(R, \theta_1, \theta_2)\) After Rotation
\begin{align*}
(5) \quad P(A) &= \frac{1}{2\pi} \int_{\phi_1}^{\theta_2} \int_{0}^{\infty} \exp \left[ -\frac{1}{2} \left( R^2 + 2rR \cos \theta + r^2 \right) \right] r \, dr \, d\theta \\
&= \frac{1}{2\pi} e^{-R^2/2} \int_{\phi_1}^{\theta_2} \int_{0}^{\infty} e^{-r^2/2} e^{-pr} \, dr \, d\theta,
\end{align*}

where
\begin{align*}
(6) \quad p &\equiv R \cos \theta.
\end{align*}

An integration by parts on the integral in $r$ yields
\begin{align*}
(7) \quad \int_{0}^{\infty} re^{-r^2/2} e^{-pr} \, dr &= 1 - p \int_{0}^{\infty} e^{-r^2/2} e^{-pr} \, dr \\
&= 1 - po^{3/2} \int_{0}^{\infty} e^{-\frac{1}{4}(1+p)^2} \, dr \\
&= 1 - p\sqrt{\pi} \left( \text{erfc}(p/\sqrt{2}) \right) / z(p/\sqrt{2}),
\end{align*}

where
\begin{align*}
(8) \quad \text{erfc}(x) &= 1 - \text{erf}(x) = \int_{x}^{\infty} z(t) \, dt, \quad z(x) \equiv \sqrt{\frac{2}{\pi}} e^{-x^2}.
\end{align*}

Using (7) in (5), carrying out the obvious part of the $\theta$ integration, (5) becomes
\begin{align*}
(9) \quad P(A) &= e^{-R^2/2} \left[ \frac{\theta_2 - \theta_1}{2\pi} - \frac{1}{\pi} \int_{\phi_1}^{\theta_2} u[\text{erfc}(u)/z(u)] \, du \right],
\end{align*}

where
\begin{align*}
(10) \quad u &\equiv p\sqrt{2} = (R\sqrt{2}) \cos \theta.
\end{align*}

We note for $R = 0$, (9) gives the exact result directly,
\begin{align*}
(11) \quad P[A(0, \Delta \theta)] &= \Delta \theta / 2\pi. \quad \Delta \theta \equiv \theta_2 - \theta_1.
\end{align*}

Equation (9) gives the relation for $P(A)$ upon which our program is based. A similar relation was originally derived by Amos, [11], in an entirely different way.

The difficulty in evaluating the integral in (9) is resolved by obtaining, for a given $\delta > 0$, the minimax polynomial fit to $\text{erfc}(u)/z(u)$ for $0 < u < C(\delta)$. Namely, a set of constants $a_k$ and a least positive integer $K$ are found such that
The constant $C$ is chosen, once $\delta$ is specified, such that

$$
\frac{1}{2\pi} \int \exp \left[ -\frac{1}{2}(x^2 + y^2) \right] dx dy = \frac{1}{2} \text{erfc} \left( \frac{R}{\sqrt{2}} \right) = \varepsilon \equiv \delta/\sqrt{\pi},
$$

with

$$
C = \frac{R}{\sqrt{2}} , \quad \widetilde{A} \equiv A \left( R, -\frac{\pi}{2}, \frac{\pi}{2} \right).
$$

For $\delta \approx 5(-4), 5(-7), 5(-10), 2(-13)$ the $a_k$ and $C$ are given in Appendix A. For example for $\delta \approx 5(-13), (=5 \times 10^{-7})$, we give $C = 3.5505$, $K = 9$. The way in which $\varepsilon$ was chosen in (13) is explained below.

The integration in (9) can now be carried out numerically by recurrence relations. Indeed, from (12) and (9) we have

$$
P(A) \sim \frac{e^{-R^2/2}}{\pi} \left[ \frac{\theta_2 - \theta_1}{2} - \sum_{k=0}^{\infty} a_k \left( \frac{R}{\sqrt{2}} \right)^{k+1} \int_{\theta_1}^{\theta_2} \cos^{k+1} \theta \, d\theta \right], \quad 10^{-1} < \frac{\pi}{2},
$$

where

$$
J_k \equiv \left( \frac{R}{\sqrt{2}} \right)^k \int_{\theta_1}^{\theta_2} \cos^k \theta \, d\theta,
$$

so that

$$
J_{k+1} = \frac{1}{k+1} \left[ \left( \frac{R}{\sqrt{2}} \cos \theta \right)^k \frac{R}{\sqrt{2}} \sin \theta \right|_{\theta_1}^{\theta_2} + \frac{R}{\sqrt{2}} J_k J_{k+1}.
$$

with

$$
J_0 = \theta_2 - \theta_1, \quad J_1 = \frac{R}{\sqrt{2}} \sin \theta_2 - \frac{R}{\sqrt{2}} \sin \theta_1.
$$

Hence

$$
P(A) \sim \frac{e^{-R^2/2}}{\pi} \left[ \frac{\theta_2 - \theta_1}{2} - \sum_{k=0}^{\infty} a_k J_{k+1} \right], \quad 10^{-1} < \frac{\pi}{2},
$$

where it is emphasized that (15) and (19) hold only when $10^{-1} \ll \frac{\pi}{2}, i = 1, 2$. This follows from (10), because $u > 0$ in (12) and $R > 0$ in (10) imply $\cos \theta > 0$. For cases outside this range, we make use of the fact that
\[
\text{P}[A(R, 0, \theta)] = \frac{1}{2} \text{erfc} \left( \frac{R}{\sqrt{2}} \sin \theta \right) - \text{P}[A(R, 0, \pi - \theta)], \quad \frac{\pi}{2} \leq \theta \leq \pi,
\]

where we prefer to work with the coerror function, \(\text{erfc}\) (see (8)), instead of the univariate cumulative distribution function of a normal variable. They are related by

\[
\frac{1}{2} \text{erfc} \left( x/\sqrt{2} \right) = 1 - \frac{1}{2\sqrt{2}} \int_{-\infty}^{x} z(t/\sqrt{2}) \, dt.
\]

The implementation of (19) and (20) is discussed in Section 4, which deals with the computer program.

We now show that if a maximum error of \(2/\sqrt{\pi} \delta\) is made in approximating

\[
f(u) = \text{erfc} (u), \quad 0 < u < C(\delta),
\]

as noted in (12), then the truncation error in computing \(P(A)\), using (19), can be no larger than \(\delta/\sqrt{\pi}\). Indeed, from (12)

\[
|F(u) - \sum_0^K a_k u^k| \leq \delta e^u^2, \quad u \geq 0, \quad F(u) = \frac{f(u)}{\vartheta(u)},
\]

and since \(u \geq 0\), we have

\[
\left| \int_{\theta_1}^{\theta_2} u \vartheta^2 \, d\theta \right| \leq \int_{\theta_1}^{\theta_2} u \vartheta^2 \, d\theta < \delta e^{-R^2/2} \int_{\theta_1}^{\theta_2} u \vartheta^2 \, d\theta.
\]

But, with (10),

\[
e^{-R^2/2} \int_{\theta_1}^{\theta_2} u \vartheta^2 \, d\theta = \int_{\theta_1}^{\theta_2} \left( \frac{R}{\sqrt{2}} \cos \theta \right) \exp \left[ -\frac{R^2}{2} \sin^2 \theta \right] \, d\theta
\]

\[
= \frac{\sqrt{\pi}}{2} \left[ \text{erf} \left( \frac{R}{\sqrt{2}} \sin \theta_2 \right) - \text{erf} \left( \frac{R}{\sqrt{2}} \sin \theta_1 \right) \right] \leq \sqrt{\pi},
\]

and (24) then implies

\[
\left| \int_{\theta_1}^{\theta_2} u \vartheta^2 \, d\theta \right| \leq \delta \sqrt{\pi} (= \varepsilon).
\]

This accounts for the way \(\varepsilon\) was chosen in (13).

The dominant part of the computation in evaluating \(P(A)\) from (19) is the generation of the sum of terms \(\{a_k, k \geq 1\}\). Two situations can occur for which this sum does not contribute to the value of \(P(A)\) to within the accuracy specified, namely when \(R\) is "small" and when \(R\) is "large." In the first case, we have
where with \( uF(u) \equiv g(u) \),

\[
g(u) = \frac{R}{\sqrt{2}} \cos \theta (1 + \frac{R^2}{2} \cos^2 \theta) \sqrt{\frac{\pi}{2}} \left[ 1 - \frac{\cos \theta}{\sqrt{2} \cos \theta} + O(R^3) \right]
\]

Carrying out the \( \theta \) integration in (27), we obtain

(28) \[ P(A) = \frac{\Delta \theta}{2\pi} - \frac{1}{3\sqrt{2}} \left( \frac{R}{\sqrt{2}} \sin \theta_2 - \frac{R}{\sqrt{2}} \sin \theta_1 \right) + \frac{1}{4\pi} \left( \frac{R^2}{2} \sin 2\theta_2 - \frac{R^2}{2} \sin 2\theta_1 \right) + O(R^3). \]

Thus, when

(29) \[ \frac{1}{2\sqrt{\pi}} \left| \frac{R}{\sqrt{2}} \sin \theta_2 - \frac{R}{\sqrt{2}} \sin \theta_1 \right| \leq \frac{1}{\sqrt{2} \sqrt{\pi}} \leq \epsilon, \quad (\epsilon \equiv \delta/\sqrt{\pi}, \text{ see (13)}), \]

then

(30) \[ P(A) \equiv \Delta \theta/2\pi. \]

Extending the above analysis one can show that the \( R^3 \) term in (28) is given by

\[
\frac{1}{2\sqrt{\pi}} \left( \frac{R}{\sqrt{2}} \right)^3 \sin^2 \theta \cos \theta,
\]

and upon integration yields

(31) \[ E \equiv + \frac{1}{6\sqrt{\pi}} \left( \frac{R}{\sqrt{2}} \right)^3 (\sin^3 \theta_2 - \sin^3 \theta_1). \]

Hence \( P(A) \) is approximated to within \( \epsilon \) by (28), without the \( O(R^3) \) term, when

(32) \[ |E| \leq \left( \frac{R}{\sqrt{2}} \right)^3 \frac{1}{3\sqrt{\pi}} \leq \epsilon. \]

In the other circumstance, when \( R \) is sufficiently large, a parameter \( \overline{R} \) can be determined, depending on \( \epsilon \), such that if

(33) \[ R \gg \overline{R} \quad (\text{or } R^2/2 \gg \overline{R}^2/2). \]
then \( P[A(R, \theta_1, \theta_2)] < \epsilon \) for \( |\theta_1|, |\theta_2| \leq \pi/2 \). So in this case that part of the computation for 
\( P(A) \) which uses (19) can be omitted, but one erfc function is still required for each \( |\theta_i| > \pi/2 \) 
\( (i = 1, 2) \) (see (20)).

We note from (13) and the fact that \( P[A(R, -\pi/2, \theta)] \) is an increasing function of \( \theta \), that for 
\( R > R \)

\[
P[A(R, \theta_1, \theta_2)] \leq P[A(\overline{R}, -\pi/2, \pi/2)] = \frac{1}{2} \text{erfc} \left( \frac{\overline{R} \sqrt{2}}{2} \right), \quad |\theta_1|, |\theta_2| \leq \pi/2, \quad \theta_1 \leq \theta_2.
\]

Consequently, we choose \( \overline{R} \) such that

\[
\frac{1}{2} \text{erfc} \left( \frac{\overline{R} \sqrt{2}}{2} \right) = \epsilon = \delta \sqrt{\pi},
\]

and observe that \( C \) from (14) and \( \overline{R} \sqrt{2} \) are the same for a given \( \epsilon \). Geometrically it means that the 
region to the right of the vertical line \( x = \overline{R} \sqrt{2} \) does not contribute to \( P(A) \) to within the specified 
accuracy.

### 3. USE OF \( P(A) \) TO COMPUTE \( P(H) \)

In this section we show how, using probabilities over angular regions, the probability, \( P(H) \), 
over a convex polygon \( H \) is obtained. In [4] they propose using probabilities over triangles and 
quadilaterals to obtain the same. Our procedure, however, is, in general more efficient. As shown 
below, we require only \( N \) angular regions for an \( N \)-sided convex polygon, whereas they need at 
least \( 3(N - 2) \) regions if \( H \) is decomposed into triangles. If, for \( N \) even, \( H \) is decomposed into quadri-
laterals, or quadilaterals plus one triangle for \( N \) odd, then one needs \( 2N - 4 \) or \( 2N - 3 \) angular regions, 
respectively. We remind the reader that our ultimate purpose in developing a program for computing 
\( P(H) \) is to use it as a subroutine to evaluate the probability over an arbitrary polygon. As stated 
earlier this has been done and will be discussed together with a computing program in a later report.

Let \( H(N, t_1, \ldots, t_N) \) denote a convex polygon of \( N \) sides with vertices at coordinate points 
\( t_1, \ldots, t_N \), where \( t_k \equiv (x_k, y_k) \) and the points \( \{t_i\} \) are given in counterclockwise order; i.e., so 
that the area of \( H \) is on the left as one traverses the boundary continuously. Then

\[
P(H) = P(A_1) - \sum_{i=2}^{N-1} P(A_i) + P(A_N),
\]

where using Figure 3 with \( N = 6 \), \( A_1 \) is the angular region determined by any interior angle of 
\( H \) with its vertex assigned as \( t_1 \), \( A_i, i = 2, \ldots, N - 1 \), are angular regions determined by the exterior 
angles of \( H \) at vertices \( t_2, \ldots, t_{N-1} \), respectively, as shown in Figure 3 and \( A_N \) is the angular region 
obtained from the vertical angle of the interior angle of \( H \) at \( t_N \). It is easy to argue the validity of 
(36) by noting, e.g. in Figure 3 (\( N = 6 \)), that the probabilities over the disjoint shaded regions \( E_i, 
\( i = 2, 3, \ldots, N - 1 \), excessively diminish the result for \( P(H) \) by an amount exactly compensated 
for by the addition of \( P(A_N) \). A formal proof of (36) is not given in this report.

*Note that (2) maps convex polygons into convex polygons.*
In keeping with our efforts to maintain an efficient program, as described in the previous section, we make use of the following result. Let $\theta(i)$ denote the quantity $\theta_2 - \theta_1$, which appears in (9), for the $i^{th}$ angular region probability, $\mathcal{P}(A_i)$. Then, since the interior angles of an $N$-convex polygon add up to $(N-2)\pi$ radians, we have

$$\theta(N) = -\theta(1) + \sum_{i=2}^{N-1} \theta(i).$$

Thus our program generally will require only $N-1$ calls to the $\tan^{-1}$ routine instead of $N$ by using (37). The accumulation of the right-hand side is denoted in the flow charts as $\phi$; e.g., see boxes [2,8,11,24,33] in flow chart II, page 11

4. COMPUTER PROGRAM FOR $P(H)$ (AND $P(A)$)

In this section we discuss the Fortran IV program for the evaluation of $P(H)$ or $P(A)$, the normal probability distribution over a convex polygon $H$ of $N$ sides, or an angular region $A$, respectively. Two flow charts I, II are given on the next two pages which show the flow and major steps of the program. It will be helpful to refer to these charts during the discussion. A location in the flow charts will be identified by chart number and box number, e.g. [1,10] refers to chart I box 10.

One input to the program is the sequence $\{t_k\}_1^N$, where $t_k$ denotes the $(x,y)$ coordinates of the $k^{th}$ vertex of $H$ with the vertices ordered in a counterclockwise direction. The value of $N$ is
FLOW CHART i

Subroutine (VALR16)

Input: $N$, $\{\{x_k, y_k\}_k\}$

Output: $P$

$\begin{align*}
\sum_k &= (x_k, y_k) \\
\text{Input: } &N, \{\{x_k, y_k\}_k\} \quad \text{Output: } P
\end{align*}$

$\begin{align*}
\tan^{-1}\left(\frac{\sum w - \sum z}{\sum u + \sum v}\right) &\rightarrow \theta \\
\theta &= \frac{2\pi}{\sum w} \\
\sum w - \sum z &\Rightarrow \theta \\
\sum u + \sum v &\Rightarrow \theta
\end{align*}$

$\begin{align*}
x_kw + y_kz &\Rightarrow s_1 \\
x_kz - y_kw &\Rightarrow h_1 \\
x_ku + y_kv &\Rightarrow s_2 \\
x_kv - y_ku &\Rightarrow h_2
\end{align*}$

$\begin{align*}
\frac{1}{2}E(h_1) - L &\Rightarrow \frac{1}{2}E(h_2) \\
\frac{1}{2}E(-h_1) &\Rightarrow \frac{1}{2}E(-h_2)
\end{align*}$

$\begin{align*}
\frac{1}{2}E(h_1) - L &\Rightarrow \frac{1}{2}E(h_2) \\
\frac{1}{2}E(-h_1) &\Rightarrow \frac{1}{2}E(-h_2)
\end{align*}$
FLOW CHART II
Subroutine (VALR16) (Con't)

*Use 4-quadrant \( \tan^{-1} \) routine \((\pi, \pi)\)

\[ S = g_1h_2 - g_2h_1 \quad C = g_1g_2 + h_1h_2 \]

I - 7 = chart I, box 7.
specified with \( N \) set to one if \( P(A) \) is desired. In this case 3 points are required, as in the case of a triangle where \( N = 3 \), in counterclockwise order, i.e., so that the region \( A \) is to the left as one transverses the boundary lines with the only vertex at \( t_1 \). A parameter is set specifying whether 3, 6, or 9 decimal digits are desired in the output \( P(H) \) or \( P(A) \).* The associated values of various parameters are given in Appendix A, namely \( a_1, a_2, a_3, R/\sqrt{2} \). Also listed in that Appendix are values of these parameters for \( P(H) \) or \( P(A) \) computable to twelve decimal digits. These however are not incorporated into the program but could be with no difficulty if desired.

It is imperative for the program to operate properly that the \( t_k \) be given in counterclockwise order; i.e., with the area on the left as one travels along the boundary of \( H \) or \( A \). Two typical examples are shown in Figure 4, where \( P \) is wanted over the shaded or hatched regions.

![Figure 4. Typical Regions for H and A](image)

Point \( t_1 \) for \( H \) can be taken initially as any vertex point, however when this program is used for arbitrary polygons, it will cycle the points and renumber them so that the new \( t_1 \) is the point with the property

\[
(38) \quad t_1 = \left[ (x_j, y_j) \right], \quad y_j < y_k \text{ with } x_j < x_k \text{ if } y_j = y_k, \quad k = 1, \ldots, N
\]

(This feature is not shown on the flow charts.).

For \( A \), \( t_1 \) must be specified as the vertex point, as shown in Figure 4 above.

In order to evaluate \( P(H) \), \( N \) angular regions \( \{A_k\} \) are treated, one at each vertex of \( H \), and their probabilities \( \{ P(A_k) \} \) combined appropriately as explained in Section 3 (see (36)). For a particular \( A_k = A(R, \theta_1, \theta_2) \), the inequality below

*We make note of the fact here that the specified number of correct decimal digits in computing \( P(H) \) may not be achieved in the unlikely case that the errors associated with a majority of the angular regions have the same sign and thus add to a total error of as much as \( N \delta \).
is tested where $\alpha_1$ is taken from (29). If it holds then [I, 5] is used to evaluate $P(A)$, which is then stored in $I$,

$$P(A) = (\theta_2 - \theta_1)/2\pi = \frac{1}{2\pi} \tan^{-1} \left( \frac{u_z - v_z}{u_w + v_z} \right),$$

where the $\tan^{-1}$ is obtained from a four quadrant subroutine which gives its output in $(-\pi, \pi]$. The quantities $u, v, w, z$ are initially defined in [I, 1] and subsequently in [II, 20, 25, 37] depending on which angular region is involved. The angles $\theta_2$ and $\theta_1$ are measured in radians and are as shown in Figures 1 or 2, page 3, with $\Delta \theta$ always positive from $\theta_1$ counterclockwise to $\theta_2$.

If the inequality in (39) is not true, then a rotation of axes is carried out, [I, 9], as indicated in Figures 1 and 2. Quantities $g_1/D_1, h_1/D_1, g_2/D_2, h_2/D_2$ are computed, [I, 10], where

$$\begin{align*}
\frac{g_1}{D_1} &= \frac{R}{\sqrt{2}} \cos \theta_1 \rightarrow g_1, \quad \frac{h_1}{D_1} = \frac{R}{\sqrt{2}} \sin \theta_1 \rightarrow h_1, \\
\frac{g_2}{D_2} &= \frac{R}{\sqrt{2}} \cos \theta_2 \rightarrow g_2, \quad \frac{h_2}{D_2} = \frac{R}{\sqrt{2}} \sin \theta_2 \rightarrow h_2,
\end{align*}$$

with

$$D_1 = (2(w^2 + z^2))^{1/4}, \quad D_2 = (2(u^2 + v^2))^{1/4}. $$

We have for the first of (40)

$$\frac{R}{\sqrt{2}} \cos \theta_1 = \left( \frac{x_k^2 + y_k^2}{2} \right)^{1/2} \frac{x_k w + y_k z}{(x_k w + y_k z)^2 + (x_k z - y_k w)^2}^{1/4}
= (x_k w + y_k z)/(2(w^2 + z^2))^{1/4} = g_1/D_1.$$

The other relations in (40) are found in the same way. The location denoted in the charts as $P$ contains the output $P(A)$ if $N = 1$, or $P(H)$ if $N \geq 3$. Location $\phi$ contains 0 if $N = 1$. If $N \geq 3$ and (33) does not hold, then $\phi$ contains $\theta(N)$, (See (37)), at exit.

In [I, 28], the inequality

$$B < \alpha_2 \equiv (9\pi e^2)^{1/3}, \quad (B = R^2/2),$$

is tested. If it holds then $P(A_k)$ is given by (28), [I, 5, 32], where $\alpha_2$ is taken from (32).

In general, the program distinguishes 12 different types of angular regions which are exhaustive and are characterized by the signs of the numbers $g_1, g_2, h_1, h_2$ as computed in [I, 10]. Examples of each of the 12 regions are shown below in Figure 5 with the terms used to evaluate $P$. 13
\[ P = P(A) = P(\bar{A}) \]

\[ P = \frac{1}{2} \text{erfc}(h_2) - P(\bar{A}) \]

\[ P = \frac{1}{2} \text{erfc}(-h_1) - P(\bar{A}) \]

\[ P = \frac{1}{2} \text{erfc}(h_2) - \frac{1}{2} \text{erfc}(h_1) + P(\bar{A}) \]

\[ \bar{A} \equiv A(R, \theta_3, \theta_4) \]

Figure 5. Various Cases for \( A \)

Note \( h_1, g_1, h_2, g_2 \) here refer to [1, 10].
It is assumed the rotation, \( [1,9] \) has been carried out as described above, so that the vertex of \( A \) is on the positive x-axis (not at the origin). The angle between the directed lines labelled (1) and (2) is always measured from (1) to (2) in the counterclockwise direction and it is non-negative and always no larger than \( \pi \) (\( \sin (\theta_2 - \theta_1) > 0 \)), since we are dealing with convex polygons. We allow \( \pi < \Delta \theta < 2\pi \) only if \( N = 1 \). In this case we evaluate \( \text{P}(E-A) = \bar{P} \), where \( E \) denotes the entire plane, and find \( \text{P}(A) = 1 - \bar{P} \). The boxes that apply for \( N = 1 \) only, showing the details just mentioned, are (1, 36, 37, 38). In the situations shown in Figure 5, we denote the probability over the angular region between (1) and (2) by \( P \), and note in the expressions for \( P \) below each diagram, that if \( g_i < 0 \) (\( i = 1, 2 \)) then \( \text{erfc} (h_i) \) is required, where \( |h_i| \) is the normal distance from line (1) to the origin (See (20)). The lines (3) and (4) shown in the diagrams bound the angular region denoted by \( A \). In diagrams (1), (2), (3), \( A \) and \( \bar{A} \) coincide.

If \( |h| (|h_1| \text{ or } |h_2|) \) is sufficiently small, \( \text{erfc} (h) \) can be replaced by one and a call to the \text{erfc} routine avoided. This feature appears in the program through (1, 12, 13, 16, 20, 34) where the inequality

\[
|h| < \alpha_3
\]

is tested. We have if (44) holds

\[
\frac{1}{2} \left| \text{erfc}(h) - 1 \right| < \frac{1}{2} \frac{2}{\sqrt{\pi}} |h| \leq \frac{1}{\sqrt{\pi}} \alpha_3 = \varepsilon/2 \quad (\varepsilon \text{ defined in } (13))
\]

so that \( \alpha_3 \) is taken as

\[
\alpha_3 = \sqrt{\pi} \varepsilon/2.
\]

Box (11, 7) is used to check if \( \bar{R} \) is sufficiently large for the computation of (19) to be bypassed. The choice for \( \bar{R}/2 \), which has already been discussed on page 8, is made so that with \( \bar{R} \geq \bar{R} \),

\[
\text{P}(A(R, -\pi/2, \pi/2)) \leq \varepsilon.
\]

The program for \( \text{P}(\bar{A}) \) by (19) is displayed in (11, 12, 23) and (11, 27), with (11, 4) showing the computation for \( J_0 \) which denotes \( \pm \) the angle of \( \bar{A} \) where the sign agrees with the sign preceding \( \text{P}(\bar{A}) \) in the relations given for \( \text{P} \) in the diagrams of Figure 5 (See (18), also).

The program is designed to recognize and avoid a subtle situation that can occur due to round-off error that leads to a catastrophic erroneous result. As an example suppose we are dealing with a polygon where one of the exterior angular regions, say \( A_k \), \( k \neq 1, N \), as shown by the solid lines in Figure 6, subtends an angle \( \theta \) of nearly \( \pi \) radians with sides of \( A \) at large perpendicular distances from the origin, so that \( \text{P}(A) \sim 1 \).
Figure 6. Shows A Singular Case Situation

Suppose, however, by rounding error line 1 is actually given by the computer as line 3 so that the angular region $\tilde{A}$ subtends an angle $\tilde{\theta}$ near $(-\pi)$ radians. Thus the program in this case would yield a value $[-P(\tilde{A})]$; i.e., a small value. Moreover it would be negative since $\tilde{\theta}$ is measured from 3 to 2 which is clockwise rather than counterclockwise. This singular case situation and others that can occur are handled in the program through boxes 11,221, 11,231, 11,241, and 11,291. When $N \geq 3$, a singular case occurs for the $k^{th}$ angular region of $\mathbf{H}$ ($\Delta \theta \not \in [0,\pi]$) when:

\begin{equation}
S = -\frac{R}{\sqrt{2}} \sin (\theta_2 - \theta_1) < 0.
\end{equation}

If this is the case, a second inequality is tested, namely,

\begin{equation}
C = -\frac{R}{\sqrt{2}} \cos (\theta_2 - \theta_1) < 0.
\end{equation}

If (47) and (48) are satisfied, as in Figure 6, we set $P(A_k) = \frac{1}{2}\text{erfc}(t)$, where $t = -h_1$ if $g_1 < 0$, or $t = h_2$ if $g_1 > 0$. If (47) holds and (48) does not, then we set $P(A_k) = 0$ since $|\Delta \theta| \sim 10^{-14}$. A singular situation that cannot be resolved occurs in the unlikely case that (47) holds, (48) does not, $R \gg R$, and $g_1, g_2$ are negative. When all of these conditions are true, $A_k$ may contain the origin so that for sufficiently large $R (> 10^4)$, $P(A_k)$ is not close to zero. However $\Delta \theta (\sim 10^{-14})$ should always be in $[0,\pi]$ for a convex polygon, but it is not since (47) holds. Hence we cannot find, within the single precision capabilities of the CDC6700, the value of $P(A_k)$, because the value of $\Delta \theta$ cannot be resolved.

In the next section, we discuss the Gideon-Gurand method (G & G) for evaluating $P(A)$. In their report and published paper however they do not consider the programming aspects of their method, which must also deal with the singular case problem just mentioned.

Extensive checking of our program was carried out. Comparisons of results were made with a program of the G & G method that we developed. Also comparisons were made with two other independent programs for computing $P(H)$ for the special case of triangles. These programs also allowed independent checking for convex polygons other than triangles, since a convex polygon can always be decomposed into a set of triangles.
Our computing program is designated as VALR16. In Appendix C a Fortran IV listing is given of a test program which generates coordinates representing the vertices of a set of triangles such that all phases of the VALR16 program are tested by evaluating the probability over these triangles. Some numerical results are also given there.

5. COMPARISON WITH GIDEON AND GURLAND METHOD

The work of Gideon and Gurland (G & G) [41, 51] gives a set of relations by which \( P(A) \) can be evaluated. Their unique procedure is very efficient and though limited to 5 decimal digit accuracy appears to be one of the best of the methods we reviewed in the literature [1, 2, 3, 6, 7]. Essentially they assume the angular region \( A \) has been rotated as shown in Figure 2 such that if \( |\theta_i| \leq \frac{\pi}{4} \), \((i = 1, 2)\) then

\[
P[A(R, 0, \theta_i)] = \frac{1}{4} \text{erfc}(R/\sqrt{2}) \left( b_0 + b_1 R + b_2 R^2 \theta_i + (b_3 R + b_4 R^2)\theta_i^3 + (b_5 R + b_6 R^2)\theta_i^5 \right).
\]

The coefficients \( b_j \) were determined by least squares for each of 15 subintervals in \( R, [0, 1/2), [1/2, .75), \ldots, [j/4, (j + 1)/4), \ldots, [3.75, 4). \) In order to evaluate \( P(A) \), they need to use (49) twice, with the same value of \( R \), once for \( \theta_1 \) and once for \( \theta_2 \). Because the use of (49) is constrained to \( \theta \leq \frac{\pi}{4} \), G & G require in addition to (20) the relation

\[
P[A(R, 0, \theta)] = \frac{1}{4} \text{erfc}\left(\frac{R}{\sqrt{2}} \sin \theta \right) \text{erfc}\left(\frac{R}{\sqrt{2}} \cos \theta \right) - P[A(R, 0, \frac{\pi}{2} - \theta)], \quad \frac{\pi}{4} < \theta \leq \frac{\pi}{2}.
\]

We have programmed the (G & G) method and found the average computing time per angular region to be about 20% longer than ours at the 6 decimal digits accuracy level. We estimate a 25% to 30% difference if we modified our method for 5 instead of 6 decimal digit accuracy.

Although it takes less time to evaluate the righthand side of (49) twice, without \( \text{erfc}\left(\frac{R}{\sqrt{2}}\right) \), than it does to evaluate the recursive procedure given by (19), our method has significantly less calls to the various special function routines, except for the exponential. In particular since the minimax approximation for \( \text{erfc}\left(\frac{u}{\sqrt{2}}\right) \) holds for \( u \gg 0 \) (\( |\theta_i| \leq \pi/2 \)), we do not need (50). Moreover, in evaluating the number of \( \text{erfc} \) functions required by G & G it is recalled that we need an \( \text{erfc} \) function when \( R/2 < \theta < 3\pi/4 \) or when \( 3\pi/4 < \theta < \pi \). In the second case they also need one \( \text{erfc} \), however for the first inequality they need two. Consequently, for each \( A \), counting the \( \text{erfc} \) function needed in (49) once and using (20) and (50) it is easy to show by enumeration of cases (for example, in [3] of Figure 5, page 14, they could need 5 while we would need none) that their method takes on the average 3½ times as many \( \text{erfc} \) functions as ours. In addition, they treat \( \theta_1 \) and \( \theta_2 \) separately while we treat the difference \( \theta_2 - \theta_1 \) (except for the functions \( g_1 \), \( h_1 \), \( g_2 \), \( h_2 \) which are expressed as algebraic functions of the coordinates of \( A \)). Thus, they need two separate calls to the arctangent routine for \( P(A) \) whereas we require one, and for \( A \) a triangle they need 5 arctangents (taking advantage of (37)) while we need only 2. They also need \( R \) which requires a square root while we need an exponential. The average number of calls to special functions for a convex polygon of \( N \) sides is summarized in Table 1.

*We note a serious omission in [5] where it is not explicitly stated that (49) only holds for \( |\theta| \leq \pi/4 \).
Table 1. Average No. of Calls to Special Function Routines for N-Angular Regions

<table>
<thead>
<tr>
<th>Function</th>
<th>Us</th>
<th>G &amp; G</th>
</tr>
</thead>
<tbody>
<tr>
<td>erfc</td>
<td>N</td>
<td>3.5N</td>
</tr>
<tr>
<td>tan⁻¹</td>
<td>N-1</td>
<td>2N-1</td>
</tr>
<tr>
<td>square root</td>
<td>N+1</td>
<td>2N+1</td>
</tr>
<tr>
<td>exponential</td>
<td>N</td>
<td>0</td>
</tr>
</tbody>
</table>

We also note that in [4] they advocate treating N sided polygons by decomposing them into sets of quadrilaterals and triangles. In the case of N-convex polygons, this would mean treating 2N-3 angular regions for N odd, and 2N-4 for N even, whereas we would only require N angular regions as explained in Section 3. Also in the case of an arbitrary polygon it will be more efficient in general to decompose it into as few convex polygons as possible rather than, as G & G propose, into triangles and quadrilaterals.

6. COMMENTS ON DREZNER'S METHOD

In a recent paper by Drezner, [3], a method is given for computing the bivariate normal integral, \( \Phi(m, k, \rho) \); i.e.,

\[
\Phi(m, k, \rho) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left( -\frac{w^2 - 2\rho wz + z^2}{2(1 - \rho^2)} \right) \, dw \, dz.
\]

By letting

\[
\begin{align*}
&u = (m - w)\sqrt{2(1 - \rho^2)}, \\
v = (k - z)\sqrt{2(1 - \rho^2)}, \\
&M = m\sqrt{2(1 - \rho^2)}, \\
&K = k\sqrt{2(1 - \rho^2)},
\end{align*}
\]

he obtains

\[
\Phi(m, k, \rho) = \frac{1}{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-u^2 - v^2} f(u, v) \, du \, dv,
\]

where

\[
f(u, v) = \exp[M(2u - M) + K(2v - K) + 2\rho(u - M)(v - K)].
\]

Drezner then uses Gaussian integration, when \( m < 0, k < 0, \rho < 0 \), so that

\[
\Phi(m, k, \rho) \approx \frac{1}{\pi} \frac{\rho^2}{\pi} \sum \sum A_i A_j f(u_i, v_j),
\]

18
where the weights $A_i$ and abscissae $u_i$ (or $v_j$) are given in (8). He makes the significant observation that if

$$m \leq 0, \quad k \leq 0, \quad \rho \leq 0,$$

then $f(u,v) \leq 1$ and he can use (55) directly to evaluate $F$ within a given error for relatively small values of $J$. For example, the maximum observed error for $J = 5$, is reported to be $5.5(-7)$. He also takes advantage of the fact that if the argument of the exponent in (54) is sufficiently less than zero, $f$ can be replaced by zero. For $J = 5$, his cutoff value is stated as $-12$.

In cases where one or more of the three inequalities in (56) does not hold, one or two erfc functions must also be computed. In case $mk \rho > 0$, then two sums such as appear in (55) are needed in addition to possibly one or two erfc functions. The necessary relations are all given in (3). Two typographical errors are noted there. In (10) $1/\sqrt{2\pi}$ should replace $1/2\pi$ and in (12)

$$e^{\frac{1}{4} \frac{\text{Sgn}(m) \text{Sgn}(k)}}{4},$$

where the minus sign replaces an incorrect plus sign.

Clearly (1) with $\mu_i = 0, \sigma_i = 1, \quad i = 1, 2$, reduces to (51) where the angular region $\Lambda$ has a right angle at $(m,k)$. Applying the transformation in (2) which reduces to

$$x = (w - \rho z)\sqrt{1 - \rho^2}, \quad y = z,$$

the $90^\circ$ angular region $\Lambda$ in the $w - z$ plane is transformed into an angular region $\Lambda$ in the $x - y$ plane with vertex at $(x_0, y_0)$, where

$$x_0 = (m - \rho k)\sqrt{1 - \rho^2}, \quad y_0 = k,$$

with a subtended angle $\theta_0$.

$$\theta_0 = \tan^{-1} \left(\sqrt{1 - \rho^2} \left(\frac{m}{-\rho}\right)\right),$$

where $\theta_0$ is measured counterclockwise from the negative side of the line $y = k$. The angular region $\Lambda$ therefore is always below the line $y = k$.

In particular, given a set of values $(m,k,\rho)$ there exists a corresponding angular region in the $x - y$ plane specified by $R, \theta_1, \theta_2$ (see Figures 1 and 2). The connection between these sets of variables can be shown to be
\[ R = \{(m^2 - 2pmk \mp k^2)/(1 - \rho^2)\}^{1/2} \]

\[
\begin{align*}
\theta &= \tan^{-1}\{k\sqrt{1 - \rho^2}/(\rho k - m)\}, \\
\theta &= \tan^{-1}\{-k\sqrt{1 - \rho^2}/(\rho m - k)\}, \\
g_1 &= (\rho k - m)\sqrt{2(1 - \rho^2)}, \\
g_2 &= (\rho m - k)\sqrt{2(1 - \rho^2)}, \\
h_1 &= k\sqrt{2}, \\
h_2 &= -m\sqrt{2}, \\
or \\
m &= -\sqrt{2} - \frac{R}{\sqrt{2}} \sin \theta_2 = -\sqrt{2} h_2, \\
k &= \sqrt{2} \frac{R}{\sqrt{2}} \sin \theta_1 = \sqrt{2} h_1 \quad \text{(see (40))}, \\
\rho &= \frac{2}{R^2}(g_1 g_2 + h_1 h_2) = -\cos(\theta_2 - \theta_1), \\
\sqrt{1 - \rho^2} &= \sin(\theta_2 - \theta_1) \geq 0.
\end{align*}
\]

We have programmed the Drezner procedure and compared it to our method. A Fortran IV listing is given in Appendix B. We did not expect it to be as efficient because of the large number of exponentials required. For \( J = 5 \), 25 exponentials are required when \( mkp < 0 \), and 50 are needed when \( mkp > 0 \). However neither method suffers in comparison to the other in computing additional \( \text{erfc} \) functions (or equivalently normal probability integrals in one dimension) since it can be shown both require the same number (none, one, or two) in any particular case.

Timing runs for the two programs showed that the Drezner method is 4 times slower on the average than ours for 6 decimal digit accuracy and 8 times as slow for 9 decimal digits of accuracy.

We also note that Drezner's procedure is incomplete for programming because he does not state how to treat the cases \( \rho = 1, \rho = -1 \). These values can occur through numerical rounding and must be dealt with before a working program can be obtained. This problem is resolved by noting that if \( \rho = 1 - \epsilon, \epsilon > 0 \), then

\[
\lim_{\epsilon \to 0} \Phi(m,k, 1 - \epsilon) = \frac{1}{2} \text{erfc}(-T/\sqrt{2}),
\]

where \( T = \text{minimum of } m \) and \( k \);

if \( \rho = -1 + \epsilon, \epsilon > 0 \), then

\[
\lim_{\epsilon \to 0} \Phi(m,k,-1 + \epsilon) = \begin{cases} \\
\frac{1}{2}[\text{erfc}(-k/\sqrt{2}) - \text{erfc}(m/\sqrt{2})], & \text{if } k > -m \\
0 & \text{otherwise}.
\end{cases}
\]

These formulas are easily seen to be true by noting that the line \( w = m \) transforms by \((58)\) to the line, call it \( L \).
The line $L$ clearly has the property that, whether $m$ is positive or negative, it is tangent to the circle $(D)$

$$(64) \quad x^2 + y^2 = m^2.$$  

The following additional facts are easily proved from (63) and will help the reader in following Figures 7 and 8 and similar situations: (1) The slope of line $L$, or $dy/dx$, is $+\sqrt{1 - \rho^2}/(-\rho)$, and hence the slope has the opposite sign to that of $\rho$, so that $\rho$ must be negative in Figure 7 and positive in Figure 8; (2) The $x$-intercept of $L$ is $m\sqrt{1 - \rho^2}$, so that $m$ is positive in Figure 7 and negative in Figure 8, $m$ having the same sign as the $x$-intercept of $L$.

Hence if $\rho = -1 + \epsilon$, $\epsilon > 0$ and small, $k > -m$, we have the situation shown in Figure 7. Now as $\epsilon \to 0$, the point (C) approaches $+\infty$ along $y = k$, and $L$ approaches tangency to the circle $(D)$ at $(0,-m)$ (but note that $0 < k < m$ in this case since the $x$-intercept of line $L$ is positive). Consequently, in the limit as $\epsilon \to 0$, $\Phi(m,k,-1)$ is given by (62). Similar heuristic arguments, which can be made rigorous, can be given for any other situation with $\rho \to 1$ as well as $\rho \to -1$. For example, with $\rho = 1 - \epsilon$, $\epsilon > 0$, $m < k$ as in Figure 8.
In this figure, \( \rho = 1 - \epsilon, m < k < 0 \) (x-intercept negative), and we observe that as \( \epsilon \to 0 \), the shaded area in Figure 8 approaches the region below and including the line \( y = m(< k < 0) \) as required by (61), or the limit is \( (1/2) \text{erfc}(-m/\sqrt{2}) \).

7. SOME NUMERICAL RESULTS

In this section we give the numerical results for \( \text{P}(H_1) \) and \( \text{P}(H_2) \), using our program VALR-16, where \( H_1 \) is a 6 sided convex polygon containing the origin and \( H_2 \) is an 8 sided convex polygon not containing the origin. The \( x, y \) columns of Table 2 below contain the \( x, y \) coordinates of the vertices; the three columns that follow list the values of \( \text{P}(A_k) \) for each angular region (See Figure 3) for \( \epsilon_1 \equiv 2.5(-4), \epsilon_2 \equiv 2.6(-7), \epsilon_3 \equiv 2.9(-10) \). The last row headed \( \text{P}(H) \) contains the value of \( \text{P}(H) \) for \( \epsilon_1, \epsilon_2, \epsilon_3 \). All the \( \text{P} \) values have been truncated from 14 digit CDC 6700 output.

Table 2

<table>
<thead>
<tr>
<th>( k )</th>
<th>( x )</th>
<th>( y )</th>
<th>( \text{P}(A_k), \epsilon_1 )</th>
<th>( \text{P}(A_k), \epsilon_2 )</th>
<th>( \text{P}(A_k), \epsilon_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5</td>
<td>-2.0</td>
<td>.911227</td>
<td>.911064879</td>
<td>.91106477067</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>0.0</td>
<td>.046858</td>
<td>.046998988</td>
<td>.04699911886</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>2.0</td>
<td>.052500</td>
<td>.052666886</td>
<td>.05266699792</td>
</tr>
<tr>
<td>4</td>
<td>-0.5</td>
<td>1.5</td>
<td>.059487</td>
<td>.059482771</td>
<td>.05948276788</td>
</tr>
<tr>
<td>5</td>
<td>-1.5</td>
<td>0.0</td>
<td>.042640</td>
<td>.042515227</td>
<td>.04251511748</td>
</tr>
<tr>
<td>6</td>
<td>-1.0</td>
<td>-1.5</td>
<td>.017780</td>
<td>.017747368</td>
<td>.01774728061</td>
</tr>
<tr>
<td>( \text{P}(H) = \text{P}(H_1) \rightarrow )</td>
<td>.727521</td>
<td>.727148375</td>
<td>.72714804914</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1.5</td>
<td>-1.5</td>
<td>.851151</td>
<td>.850975856</td>
<td>.85097578896</td>
</tr>
<tr>
<td>2</td>
<td>2.0</td>
<td>-0.75</td>
<td>.018352</td>
<td>.018726585</td>
<td>.01872664573</td>
</tr>
<tr>
<td>3</td>
<td>1.75</td>
<td>1.75</td>
<td>.038192</td>
<td>.038305815</td>
<td>.03830565474</td>
</tr>
<tr>
<td>4</td>
<td>1.25</td>
<td>1.75</td>
<td>.064039</td>
<td>.064042498</td>
<td>.06404250011</td>
</tr>
<tr>
<td>5</td>
<td>0.50</td>
<td>1.5</td>
<td>.486789</td>
<td>.486841686</td>
<td>.48684172637</td>
</tr>
<tr>
<td>6</td>
<td>0.25</td>
<td>0.25</td>
<td>.042253</td>
<td>.042256194</td>
<td>.04225618944</td>
</tr>
<tr>
<td>7</td>
<td>0.25</td>
<td>-0.25</td>
<td>.026270</td>
<td>.026273494</td>
<td>.02627348809</td>
</tr>
<tr>
<td>8</td>
<td>0.50</td>
<td>-1.25</td>
<td>.116365</td>
<td>.116775266</td>
<td>.11677520430</td>
</tr>
<tr>
<td>( \text{P}(H) = \text{P}(H_2) \rightarrow )</td>
<td>.291919</td>
<td>.291304849</td>
<td>.29130478878</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(See (36))
REFERENCES


APPENDIX A
PROGRAM PARAMETERS. CHEBYSHEV COEFFICIENTS FOR
\( \text{erfc}(x)/z(x), \ x > 0 \)

In this appendix we list the pertinent constants that appear in the program for three levels of accuracy (3, 6, 9 decimal digits), and an additional set which is designed to yield 12 correct decimal digits for the probability over an angular region.

<table>
<thead>
<tr>
<th>Acc.</th>
<th>( \delta )</th>
<th>( C(\delta) = \frac{R}{\sqrt{2}} )</th>
<th>( \epsilon )</th>
<th>( \alpha_1 )</th>
<th>( \alpha_2 )</th>
<th>( \alpha_3 )</th>
<th>( \frac{1}{2} E(R/\sqrt{2}) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>4.50(-4)</td>
<td>2.46</td>
<td>2.54(-4)</td>
<td>2.02(-7)</td>
<td>1.22(-2)</td>
<td>2.25(-4)</td>
<td>2.52(-4)</td>
</tr>
<tr>
<td>B</td>
<td>4.56(-7)</td>
<td>3.5505</td>
<td>2.57(-7)</td>
<td>2.08(-13)</td>
<td>1.23(-4)</td>
<td>2.28(-7)</td>
<td>2.57(-7)</td>
</tr>
<tr>
<td>C</td>
<td>5.21(-10)</td>
<td>4.382</td>
<td>2.94(-10)</td>
<td>2.72(-19)</td>
<td>1.35(-6)</td>
<td>2.61(-10)</td>
<td>2.88(-10)</td>
</tr>
<tr>
<td>D</td>
<td>1.78(-13)</td>
<td>5.1092</td>
<td>1.00(-13)</td>
<td>3.17(-26)</td>
<td>6.58(-9)</td>
<td>8.90(-14)</td>
<td>2.50(-13)</td>
</tr>
</tbody>
</table>

\( \epsilon = \delta/\sqrt{\pi} \) \hspace{2cm} C(\delta) \hspace{2cm} \text{See page 6.} \hspace{2cm} \alpha_2 = (9\pi\epsilon^2)^{1/3} \hspace{2cm} \text{See pages 7, 13.}

\( R/\sqrt{2} \) \hspace{2cm} \text{See page 8.} \hspace{2cm} \frac{1}{2} E(R/\sqrt{2}) = \frac{1}{2} \text{erfc}(R/\sqrt{2}) \hspace{2cm} \text{See page 8.}

\( \alpha_1 = \pi\epsilon^2 \) \hspace{2cm} \text{See pages 7, 13.} \hspace{2cm} \alpha_3 = \delta/2 \hspace{2cm} \text{See page 15.}

\( \frac{1}{2} E(R/\sqrt{2}) = 2.5\epsilon \) \hspace{2cm} \text{for } \text{(D)} \)

The first column of the table labeled Acc. (for accuracy) lists \( \text{A, B, C, D} \) referring to 3, 6, 9, 12 decimal digits of accuracy, respectively, for the probability over an angular region. Pages are given above where the parameters are defined in the report.

The minimax coefficients, \( a_k \), for approximating \( \text{erfc}(x) \) on \( C(\delta) \) (See (12), (15)) are given below for four accuracy levels as indicated in the tables below by \( \text{A, B, C, D} \). They were computed by a double precision minimax subroutine utilizing values of \( \text{erfc}(x) \) accurate to 18 significant digits on \( [\frac{1}{2}, 1] \) and \( \text{erf}(x) \) accurate to 25 digits on \( [0, \frac{1}{2}] \).

For \( \text{A} \) (Average time per angular region = \( 2.2 \times 10^{-4} \) sec)

\[
\begin{align*}
a_0 &= .885777518572895D + 00 \\
a_1 &= -.981151952778050D + 00 \\
a_2 &= .759305502082485D + 00 \\
a_3 &= -.353644980686977D + 00 \\
a_4 &= .695232092435207D - 01
\end{align*}
\]
For (B) (Average time per angular region = 4.6 x 10^{-4} sec)

\[ a_0 = 0.886226470016632 \times 10^0 \]  
\[ a_1 = -0.999950714561036 \times 10^0 \]  
\[ a_2 = 0.88534882003892 \times 10^0 \]  
\[ a_3 = -0.660611239043357 \times 10^0 \]  
\[ a_4 = 0.421821197160099 \times 10^0 \]  
\[ a_5 = -0.22289805667208 \times 10^0 \]  
\[ a_6 = 0.905057384150449 \times 10^{-1} \]  
\[ a_7 = -0.25490611884287 \times 10^0 \]  
\[ a_8 = 0.430895168984138 \times 10^{-2} \]  
\[ a_9 = -0.323377239693247 \times 10^0 \]

For (C) (Average time per angular region = 6.5 x 10^{-4} sec)

\[ a_0 = 0.886226924931465 \times 10^0 \]  
\[ a_1 = -0.999999899776252 \times 10^0 \]  
\[ a_2 = 0.88622373186722 \times 10^0 \]  
\[ a_3 = -0.666626670510907 \times 10^0 \]  
\[ a_4 = 0.442851899328568 \times 10^0 \]  
\[ a_5 = -0.265638206366025 \times 10^0 \]  
\[ a_6 = 0.14506043403012 \times 10^0 \]  
\[ a_7 = -0.71490927799889 \times 10^{-1} \]  
\[ a_8 = 0.3091929552120 \times 10^{-1} \]  
\[ a_9 = -0.11232532148441 \times 10^{-1} \]  
\[ a_{10} = 0.324944543171185 \times 10^{-2} \]  
\[ a_{11} = -0.704260243309096 \times 10^{-3} \]  
\[ a_{12} = 0.10578757480633 \times 10^{-3} \]  
\[ a_{13} = -0.97184686416046 \times 10^{-5} \]  
\[ a_{14} = 0.408335517232165 \times 10^{-6} \]

For (D) (Average time per angular region = 9.1 x 10^{-4} sec)

\[ a_0 = 0.886226925452593 \times 10^0 \]  
\[ a_1 = -0.99999989776252 \times 10^0 \]  
\[ a_2 = 0.886226922786746 \times 10^0 \]  
\[ a_3 = -0.66666611866661 \times 10^0 \]  
\[ a_4 = 0.443112868048919 \times 10^0 \]  
\[ a_5 = -0.26662729014111 \times 10^0 \]  
\[ a_6 = 0.147687136321938 \times 10^0 \]  
\[ a_7 = -0.761365853850292 \times 10^{-1} \]  
\[ a_8 = 0.368032849350860 \times 10^{-1} \]  
\[ a_9 = -0.16719050688183 \times 10^{-1} \]  
\[ a_{10} = 0.71029625734052 \times 10^{-2} \]  
\[ a_{11} = -0.278170932906224 \times 10^{-2} \]  
\[ a_{12} = 0.981112629090333 \times 10^{-3} \]  
\[ a_{13} = -0.302588640752108 \times 10^{-3} \]  
\[ a_{14} = 0.789960968802448 \times 10^{-4} \]  
\[ a_{15} = -0.16865181767046 \times 10^{-4} \]  
\[ a_{16} = 0.283646635409322 \times 10^{-5} \]  
\[ a_{17} = -0.358314466908290 \times 10^{-6} \]  
\[ a_{18} = 0.317679497040006 \times 10^{-7} \]  
\[ a_{19} = -0.175440651940430 \times 10^{-8} \]  
\[ a_{20} = 0.452534347337305 \times 10^{-10} \]

Average time per angular region refers to the average computing time on the CDC-6700 to obtain \( P(A) \).
APPENDIX B.
LISTING OF DREZNER PROGRAM

This appendix contains a listing of the program for computing $P(A)$ or $\Phi(H)$ by Drezner's procedure. It is designed to use $J = 3, 5, 8$ where $J$ is defined by equation (5) in [3]. Thus, referring to Table 1 in [3], $P$ will be computed correctly to at least 3, 6, or 9 digits, respectively by this program.

Call line to Z. Drezner Subroutine

CALL DREZNR (x, y, N, P_k, IOP) where

- $x$ is the input array of abscissas of the vertices of the polygon
- $y$ is the input array of ordinates of the vertices of the polygon
- $N$ is the number of sides of the polygon.
- $P_k$ is the location of the answer as computed by the Drezner method
- $IOP = 1$ specifies the Drezner subroutine to use a table of $J = 3$ weights in computing $P_k$ (See (55)).
- $IOP = 2$ specifies the Drezner routine to use a table of $J = 5$ weights in computing $P_k$.
- $IOP = 3$ specifies a table of $J = 8$ weights in computing $P_k$.

*N = 1 for an angular region $A$ with 3 points given in counterclockwise order with first point at vertex of $A$, (See page 12). Note $0 < \Delta \theta < 2\pi$ for $N = 1$, but $0 < \Delta \theta < \pi$ for $N > 3$. 

27
SUBROUTINE OREZNR( X, Y, N, ANS, IOP )
DIMENSION X(N), Y(N), U(2), V(2), G(2), H(2)
DIMENSION AM(51), AK(51), RHO(51)
REAL L
DATA RT2 / 1.4142 13562 373 /
NM1=N-1
K=1
ANS=0.
NBAR=N
IF ( N.EQ.1 ) NBAR=3
U(2)=X(NBAR)-X(1)
V(2)=Y(NBAR)-Y(1)
KP1=K+1
U(1)=X(KP1)-X(K)
V(1)=Y(KP1)-Y(K)
IF ( N.GT.1 ) GO TO 3141
SGN=1
SN=V(2)*U(1)-U(2)*V(1)
IF ( SN.GE.0. ) GO TO 3141
SGN=-1.
T1=U(1)
U(1)=U(2)
U(2)=T1
T1=V(2)
V(2)=V(1)
V(1)=T1
3141 CONTINUE
BGD1=SQRT( 2.*U(1)*U(1)+V(1)*V(1))
BGD2=SQRT( 2.*U(2)*U(2)+V(2)*V(2))
3151 CONTINUE
L=0.
B=5.*(X(K)*X(K)+Y(K)*Y(K))
G(1)=U(1)*X(K)+V(1)*Y(K)
G(2)=U(2)*X(K)+V(2)*Y(K)
H(1)=-Y(K)*U(1)+X(K)*V(1)
H(2)=-Y(K)*U(2)+X(K)*V(2)
G(1)=G(1)/BGD1
G(2)=G(2)/BGD2
H(1)=H(1)/BGD1
H(2)=H(2)/BGD2
AM(K)=-RT2*H(2)
AK(K)=-RT2*H(1)
IF ( P.NE.0. ) GO TO 3181
RHO(K)=-(2.*U(2)*U(1)+V(2)*V(1))/BGD1*BGD2
GO TO 3191
3181 CONTINUE
RHO(K) = -(\text{G}(1) + \text{G}(2) + \text{H}(1) \cdot \text{H}(2)) / \text{B}

3191 CONTINUE
IF ( K.LT.NM1 ) GO TO 3631
IF ( K.EQ.NM1 ) GO TO 3661
CALL FLAN ( AM(K), AK(K), RHO(K), ANS1, IOP )
ANS=ANS+ANS1
IF ( N.EQ.1 ) RETURN
IF ( SGN.EQ.1. ) RETURN
ANS=1+ANS
RETURN

3631 CONTINUE
K=K+1
KP1=K+1
IF ( K.NE.2 ) GO TO 3651
KM1=K-1
CALL FLAN ( AM(KM1), AK(KM1), RHO(KM1), ANS1, IOP )
ANS=ANS1
U(2)=X(KP1)-X(K)
V(2)=Y(KP1)-Y(K)
BG02=SQRT( 2.0*(U(2)*U(2)+V(2)*V(2)) )
GO TO 3151

3651 CONTINUE
U(1)=U(2)
V(1)=V(2)
U(2)=X(KP1)-X(K)
V(2)=Y(KP1)-Y(K)
BG01=EG02
BG02=SQRT( 2.0*(U(2)*U(2)+V(2)*V(2)) )
GO TO 3671

3661 CONTINUE
X=N
U(1)=X(N)-X(1)
V(1)=Y(N)-Y(1)
BG01=SQRT( 2.0*(U(1)*U(1)+V(1)*V(1)) )

3671 CONTINUE
KM1=K-1
CALL FLAN ( AM(KM1), AK(KM1), RHO(KM1), ANS1, IOP )
ANS=ANS-ANS1
GO TO 3151
END
SUBROUTINE PLAN ( H, AK, R, ANS, IOP )
DIMENSION EPS3(11)
DATA ( EPS3(I),I=1,3 ) / 2.E-5, 2.E-7, 2.E-10 / 
DATA RT2/1.4142 13562 373 / 
OM=1.0-EPS3(IOP)
ANS=0.0
IF ( R.LE.-OM ) GO TO 3171
IF ( (H*AK*R).GT.0. ) GO TO 3155
IF ( H.GT.0. ) GO TO 2031
IF ( AK.GT.0. ) GO TO 2021
IF ( F.GT.0. ) GO TO 2011
ANS=BFHI(H,AK,R,IOP )
GO TO 3161
2011 CONTINUE
IF ( AK.NE.0. ) GO TO 2061
GO TO 2023
2021 CONTINUE
IF ( F.LT.0. ) GO TO 2041
2023 CONTINUE
ANS=EC9(H,AK,R,IOP )
GO TO 3161
2031 CONTINUE
IF ( AK.EQ.0. ) GO TO 2051
2035 CONTINUE
IF ( AK.LT.0. ) GO TO 2061
2041 CONTINUE
ANS=EC7(H,AK,R,IOP )
GO TO 3161
2051 CONTINUE
IF ( F.GT.0. ) GO TO 2061
GO TO 2041
2061 CONTINUE
ANS=EC8(H,AK,R,IOP )
GO TO 3161
3155 CONTINUE
ANS=EC1I(H,AK,R,IOP )
3161 CONTINUE
RETURN
3171 CONTINUE
IF ( AK.LE.(-H*EPS3(IOP))) GO TO 3161
T1=A/R/RT2
T2=H/RT2
ANS=5*(ERFC(0,T1)-ERFC(0,T2))
GO TO 3161
END
FUNCTION EQ7 (H, AK, R, IOP)
DATA RT2/1.4142 13562 373 /
T=-H/RT2
T1=-AK/RT2
EQ7=4PHI(-H, -AK, R, IOP) + 5*(ERFC(0, T) + ERFC(0, T1))-1.
RETURN
END

FUNCTION EQ8 (H, AK, R, IOP)
DATA RT2/1.4142 13562 373 /
T=-AK/RT2
EQ8=4PHI(-H, -AK, -R, IOP) + 5*ERFC(0, T)
RETURN
END

FUNCTION EQ9 (H, AK, R, IOP)
DATA RT2/1.4142 13562 373 /
T=-H/RT2
EQ9=4PHI(-H, -AK, -R, IOP) + 5*ERFC(0, T)
RETURN
END
FUNCTION EQ11(H,AK,R,IOP)
DIMENSION EPS3(11)
DATA (EPS3(I),I=1,3) / 2.E-5,2.E-7,2.E-10 /
DATA FT2/1.4142 13562 373 /
91 FORMAT (1H0,3E22.15)
OM=1.0-EPS3(IOP)
IF (F,LT,OM) GO TO 2001
T=H
IF (AK.LE.H) T=AK
T1=-T/RT2
EQ11=0.5*ERFC(0,T1)
GO TO 1991
1991 CONTINUE
RETURN
2001 CONTINUE
CST=SORT(H*H-2.0*R*H*AK+AK*AK)
T1=R*H-AK
C1=1.
T2=SIGN(C1,H)
T1=(T1*T2)/CST
T4=1.
T3=H*AK
T5=SIGN(T4,T3)
TDEL=(1.0-T5)*.25
T3=R*AK-H
C1=1.
T2=SIGN(C1,AK)
T3=(T3*T2)/CST
IF (P.GT.0.) GO TO 2031
IF (T1.GT.0.) GO TO 2023
T4=phi(H,0.,T1,IOP)
GO TO 2051
2023 CONTINUE
T4=EQ2(H,0.,T1,IOP)
GO TO 2051
2031 CONTINUE
IF (T1.LT.0.) GO TO 2041
T4=EQ8(H,0.,T1,IOP)
GO TO 2051
2041 CONTINUE
T4=EQ7(H,0.,T1,IOP)
2051 CONTINUE
IF (AK.GT.0.) GO TO 3031
IF (T3.GT.0.) GO TO 3023
T6=phi(AK,0.,T3,IOP)
GO TO 3351
3023 CONTINUE
T6=EQ6(AK,0.,T3,IOP)
GO TO 3051
3031 CONTINUE
IF ( T3.LT.0. ) GO TO 3041
T6=EQ7(AK,0.,T3,IOP )
GO TO 3051
3041 CONTINUE
T6=EQ7(AK,0.,T3,IOP )
3051 CONTINUE
EQ11=T4+T6=TDOL
RETURN
FUNCTION $PHI ( M,AK,R ,IOP )
DIMENSION A(21),X(21),LL0(6),LHI(6)
DIMENSION EPS1(11)
DIMENSION EPS3(11)
DATA ( A(I) ,I=1,18 ) / 
  1 4.4602 97704 66658E-1, 3.9646 82669 98335E-1, 
  2 4.3726 86798 77644E-2, 2.4640 61520 28443E-1, 
  3 3.9233 1.666 52399E-1, 2.1141 81930 76057E-1, 
  4 3.3246 66035 13439E-2, 5.2485 33445 15628E-4 / 
DATA ( X(I) ,I=1,18 ) / 
  1 1.9055 41497 98192E-1, 0.4825 16675 44577E-1, 
  2 1.7997 76578 41573E+0, 1.0024 21519 68216E-1, 
  3 4.0281 39660 46201E-1, 1.0609 49821 52572E+0, 
  4 1.7797 29418 52026E+0, 2.6697 60356 38766E+0 / 
DATA ( A(I) ,I=9,16 ) / 
  1 1.3410 91884 53360E-1, 2.6833 07544 72640E-1, 
  2 2.7595 33973 86422E-1, 1.5744 62826 18790E-1, 
  3 4.0841 1.991 74625E-2, 5.3679 35756 (2526E-3, 
  4 2.0206 36491 32437E-4, 1.1925 96926 59532E-6 / 
DATA ( X(I) ,I=9,16 ) / 
  1 5.2978 64393 18516E-2, 2.6739 03721 67767E-1, 
  2 6.1630 28841 82402 E-1, 1.0642 46312 11623E+0, 
  3 1.5886 55862 27006E+0, 2.1839 21153 39860E+0, 
  4 2.8631 33903 70000E+0, 3.6600 07162 72440E+0 / 
DATA ( EPS1(I) ,I=1,3 ) / -8.,-12.,-20. / 
DATA PI / 3.1415 92653 58979 / 
DATA ( LL0(I) ,I=1,3 ) / 1,4.5 / 
DATA ( LHI(I) ,I=1,3 ) / 3,6,16 / 
DATA RT2 / 1,4142 13562 373 / 
DATA ( EPS3(I) ,I=1,3 ) / 2.E-5,2.E-7,2.E-10 / 
QM=1.-EPS3(IOP)
ILO=LL0(IOP)
IHI=LHI(IOP)
EPS=EPS1(IOP)
RSQ=R*Q
IF ( RSQ.LT.1.) GC TO 2991
T3=1.
GOTO 3001
33
2991 CONTINUE
T3=SQRT(1.-RSQ)
CST=RT2*T3
3001 CONTINUE
BPHI=0.
IF ( R.LE.-OM ) GO TO 3011
IF ( R.LT.0.04 ) GO TO 3331
T=H
IF ( AK.LE.H ) T=AK
T1=-T/RT2
BPHI=.5*ERF3(0,T1)
GO TO 3371
3011 CONTINUE
IF ( AK.LE.(-H*EPS3(IOP) ) ) GO TO 3371
T1=-AK/RT2
T2=H/RT2
ANS=.5*(ERF3(0,T1)-ERFC(0,T2))
GO TO 3371
3331 CONTINUE
H1=H/CST
AK1=AK/CST
SUM=0.
DO 3361 I=IL0,INI
SUM1=0.
DO 3351 J=IL0,INI
T1=H1*(2.*X(I)-H1)+AK1*(2.*X(J)-AK1)
1 +2.*R*(X(I)-H1)*(X(J)-AK1)
IF ( T1.LT.EPS ) GO TO 3351
SUM1=SUM1+EXP(T1)*A(J)
3351 CONTINUE
SUM=SUM+A(I)*SUM1
3361 CONTINUE
BPHI=(SUM*T3)/PI
3371 CONTINUE
RETURN
END
APPENDIX C.

LISTING OF TEST PROGRAM WITH SOME NUMERICAL RESULTS

The test program listed in this appendix is designed to "see" all paths of the VALR16 subroutine, the basic routine of this report. The test program treats three different sets of triangles for each $\epsilon$, i.e., $\epsilon_1$, $\epsilon_2$, $\epsilon_3$. A total of 351 triangles are treated. Our subroutine VALR16 is used to obtain the probability $P(H)$ over each triangle and the result is compared with the result obtained by the routine based on Drezner's method. The numerical results below state the case number, $(x, y)$ vertices, VALR16 result, Drezner result, and absolute value of the difference, corresponding to that case for which the absolute value of the difference in $P(H)$ for the two methods was a maximum for each set and for each $\epsilon$. Thus there are nine cases given below.

| Case No. | $x$ | $y$ | $P(H)$ and $|\Delta P|$ |
|---------|-----|-----|-------------------------|
| 3       | 2.0000 000000 0000 | 1.0000 000000 0000 | .01116 23895 4828 |
|         | 1.0000 000000 0000 | 0.0000 000000 0000 | .01144 55124 4546 |
|         | 3.0000 000000 0000 | 1.0000 000000 0000 | 2.83(−4) |
| 116     | 3.0000 000000 0000 | 1.5000 000000 0000 | .07276 76379 5214 |
|         | 0.0000 000000 0000 | −0.0006 66881 7000 | .07312 88147 1695 |
|         | 3.0000 000000 0000 | 0.0000 000000 0000 | 3.61(−4) |
| 76      | .11048 34376 7180 | .11048 34376 7180 | .07464 96837 3470 |
|         | −1.8895 16562 3282 | .11048 34376 7180 | .07443 77215 8773 |
|         | −1.8895 16562 3282 | −.88951 65623 2820 | 2.12(−4) |
|         | $\epsilon_1 = 2.54(\times 10^{−4})$ |
| 15      | 0.0000 000000 0000 | −2.0000 000000 0000 | .17865 07387 5631 |
|         | 1.0000 000000 0000 | 0.0000 000000 0000 | .17864 99501 5890 |
|         | 0.0000 000000 0000 | 1.0000 000000 0000 | 7.89(−7) |
| 90      | 3.0000 000000 0000 | 3.0000 000000 0000 | .00059 65636 6379 |
|         | 3.0000 000000 0000 | 0.0000 000000 0000 | .00059 79525 2985 |
|         | 6.0000 000000 0000 | 1.5000 000000 0000 | 9.29(−7) |
| 79      | .01109 91882 5761 | .01109 91882 5761 | .11200 59368 5515 |
|         | −1.9889 00811 7424 | .01109 91882 5761 | .11200 54478 9902 |
|         | .01109 91882 5761 | −.98890 08117 4239 | 4.89(−7) |
|         | $\epsilon_2 = 2.57(\times 10^{−7})$ |

*See Appendix A.
<table>
<thead>
<tr>
<th>Case No.</th>
<th>x</th>
<th>y</th>
<th>P(H) and</th>
<th>ΔP</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>e₃ = 2.94(-10)*</td>
<td></td>
</tr>
<tr>
<td>96</td>
<td>4.0000 00000 0000</td>
<td>0.0000 00000 0000</td>
<td>.12383 33015 1674</td>
<td></td>
</tr>
<tr>
<td></td>
<td>2.0000 00000 0000</td>
<td>2.0000 00000 0000</td>
<td>.12383 33012 5254</td>
<td>2.64(-10)</td>
</tr>
<tr>
<td></td>
<td>0.0000 00000 0000</td>
<td>7.8256 90500 (-10)</td>
<td>4.7645 25380 3718</td>
<td>4.85(-10)</td>
</tr>
<tr>
<td>65</td>
<td>1.9999 80000 0000</td>
<td>4.5000 00000 0000</td>
<td>.47645 25375 5211</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-3.0000 00000 0000</td>
<td>0.0000 00000 0000</td>
<td>.47645 25375 5211</td>
<td>4.85(-10)</td>
</tr>
<tr>
<td></td>
<td>3.0000 00000 0000</td>
<td>0.0000 00000 0000</td>
<td>5.19(-10)</td>
<td></td>
</tr>
<tr>
<td>94</td>
<td>.00116 10499 1180</td>
<td>.00116 10499 1180</td>
<td>.22803 35273 2106</td>
<td></td>
</tr>
<tr>
<td></td>
<td>-1.9988 38950 0882</td>
<td>-1.9988 38949 3056</td>
<td>.22803 35268 0156</td>
<td>5.19(-10)</td>
</tr>
<tr>
<td></td>
<td>2.0011 61049 9118</td>
<td>-1.9988 38950 0882</td>
<td>5.19(-10)</td>
<td></td>
</tr>
</tbody>
</table>

*See Appendix A.
PROGRAM DRET ( OUTPUT )
COMMON IOP
DIMENSION X(201), Y(201), X1(201), Y1(201)
DIMENSION EPSI(4), X3(3), Y3(3)
DIMENSION APH2I(3), APH3I(3)
DIMENSION IRAY(21)
DIMENSION DEL1(3), DEL31(3), ALPHA(3)

SET A FOR PJ */ HULL DECK
DATA ( X(I), I=1,48 )/
1 1.0, 3.0, 2.0, 1.0, 2.0, -1.0, 1.0, 0.0, 0.0,
2 1.0, 0.0, 0.0, 1.0, -1.0, 2.0, 1.0, 0.0, 0.0,
3 1.0, 0.0, 3.0, 1.0, 2.0, 2.0, 1.0, 2.0, 2.0,
4 0.0, 2.0, 1.0, 0.0, 1.0, -1.0, 0.0, -1.0, 1.0,
DATA ( Y(I), I=1,48 )/
1 0.0, 1.0, 1.0, 0.0, 1.0, 0.0, 1.0, 0.0, 2.0,
2 0.0, 1.0, 1.0, 0.0, 1.0, -2.0, 0.0, -1.0, 2.0,
3 0.0, -2.0, 1.0, 0.0, -2.0, 1.0, 0.0, -1.0, 2.0,
4 0.0, 1.0, 1.0, 0.0, -2.0, 1.0, 0.0, -1.0, 1.0,

SET B FOR PJ */ HULL DECK
DATA ( X(I), I=49,90 )/
1 1.0, 5.0, 2.0, 1.0, 1.0, 5.0, 2.0, 1.0, 5.0,
2 1.0, 1.0, 1.0, 1.0, 6.66666, 1.0, 1.0, 2.0, 1.0,
3 1.0, 2.0, 1.0, 1.0, -1.0, 1.0, 1.0, 1.0, 1.0,
4 1.0, 1.0, 2.0, 1.0, 2.0, 1.0,
DATA ( Y(I), I=49,90 )/
1 0.0, 1.0, 1.0, 0.0, 1.0, 0.0, 0.0, -5.0, 0.0,
2 0.0, 1.0, 0.0, 0.0, 1.0, 0.0, 0.0, 0.0, 1.0,
3 0.0, 0.0, 1.0, 0.0, 0.0, -1.0, 0.0, 0.0, -1.0,
4 0.0, -2.0, 1.0, 0.0, -5.0, 1.0,
DATA ( X(I), I=91,102 )/
1 5.0, 1.0, 5.0, 2.0, 0.0, 4.0, 2.0, 0.0, 4.0,
DATA ( Y(I), I=91,102 )/
1 5.0, 1.0, 2.0, 2.0, 1.0, 0.0, 2.0, 1.0, 0.0,
DATA ( X(I), I=103,117 )/
1 1.0, 0.0, 0.0, 1.0, 0.0, 0.0, 1.0, 0.0, 1.0,
2 1.0, 1.0, 0.0,
DATA ( Y(I), I=103,117 )/
1 0.0, 1.0, 1.0, 0.0, 1.0, -5.0, 0.0, 5.0, 1.0,
2 0.0, 1.0, -5.0, 0.0, 5.0, 1.0,
DATA ( EPS1(I), I=1,3 )/
1 2.53623 66450E-4,
DATA ( DEL1(I), I=1,3 )/
1 4.49542E-4, 4.55777E-7, 5.21712E-10,
DATA ( B11(I), I=1,3 )/
1 2.463, 3.5505, 4.382,
DATA ( APH21(I), I=1,3 )/
1 2.24771E-3, 2.27885E-6, 2.60856E-5,
DATA ( APH31(I), I=1,3 )/
1 1.220659E-1, 1.231919E-3, 1.3480369E-5,
90 FORMAT ( 1H0,60X,I10 )
91 FORMAT ( 1H-,8E16.9 )
92 FORMAT ( 1H- )
93 FORMAT ( 1H0,I10 )
97 FORMAT ( 1H1 )
95 FORMAT ( 1H ,44X,2E12.4 )

N=3
PRINT 97
DO 3021 I=1,151

X(I)=X(I)
Y(I)=Y(I)
3021 CONTINUE

DO 3071 I3=1,3
3071 PRINT 97
PRINT 90;I3
IFNT=0
DO 3061 I5=1,3
PRINT 92
TMAX=0.
M=0
DO 3025 I=1,151
3025 CONTINUE

APH3=APH31(I3)
Y(92)=1.5-3.*APH3
Y(95)=3.*APH3
Y(98)=-3.*APH3
Y(101)=1.5+3.*APH3
Y(104)=9.*APH3
Y(105)=-9.*APH3
Y(107)=Y(104)
Y(111)=Y(105)
Y(113)=Y(104)
Y(117)=Y(105)
DO 3027 I=1,131
Y(I)=Y(I)
3027 CONTINUE
IF ( I5.EQ.1 ) GO TO 3035
IF ( I5.EQ.3 ) GO TO 3035
DO 3031 I=1,151
X(I)=3.*X1(I)
Y(I)=3.*Y1(I)
3031 CONTINUE
3035 CONTINUE
DO 3051 I=1,115,3
3037 CONTINUE
IP2=I+2
DO 3049 I1=1,3
M=M+1
IF ( IPNT.NE.0 ) PRINT 93,M
3041 CONTINUE
IF ( I1.EQ.1 ) GO TO 3047
T1=X(I)
T2=Y(I)
X(I)=X(I+1)
Y(I)=Y(I+1)
X(I1)=X(IP2)
Y(I1)=Y(IP2)
X(IP2)=T1
Y(IP2)=T2
3047 CONTINUE
IF ( I5.NE.3 ) GO TO 3045
IF ( I1.GT.1 ) GO TO 3045
T1=APH21(I3)
T2=SQRT(T1)-1.5-12
SX=X(I1)-T2
SY=Y(I1)-T2
DO 3043 I7=1,12
X(I7)=X(I7)-SX
Y(I7)=Y(I7)-SY
3043  CONTINUE
3045  CONTINUE
3048  CONTINUE
   Ti=(X(I)-X(I+1))*(Y(I)-Y(I+2))
   T2=(X(I)-X(I+2))*(Y(I)-Y(I+1))
   IF ( Ti.GE.TZ ) GO TO 304
   T1=X(I+2)
   X(I+2)=Y(I+1)
   X(I+1)=T1
   T2=Y(I+2)
   Y(I+2)=Y(I+1)
   Y(I+1)=T2
304  CONTINUE
   IF ( IFNT.NE.0 )
      PRINT 95,X(J),Y(J),J=I,IP2
      IOP1=I3
      CALL VALR16( X(I),Y(I),N,ANS1,IOP1 )
      ANS=ANS1
      IOP=I3
      CALL DREZNR( X(I),Y(I),3,ANS3,I3 )
      DEL=ABS(ANS-ANS3)
      IF ( DEL.GE.TMAX ) GO TO 3049
      MSAV=MS
      SADEL=DEL
      TMAX=DEL
      X3(1)=X(I)
      X3(2)=X(I+1)
      X3(3)=X(I+2)
      Y3(1)=Y(I)
      Y3(2)=Y(I+1)
      Y3(3)=Y(I+2)
      SAVOR=ANS3
      SAVPJ=ANS
3049  CONTINUE
3051  CONTINUE
   PRINT 93,MSAV
   PRINT 96,X3(1),Y3(1)
   PRINT 96,X3(2),Y3(2)
   PRINT 96,X3(3),Y3(3)
   PRINT 94,SAVPJ,SAVDR,SAVDEL
3061  CONTINUE
3071  CONTINUE
4011  CONTINUE
   94 FORMAT ( 1H0,4E15.8 )
   96 FORMAT ( 1H0,6E22.15 )
9011  CONTINUE
   STOP
   FND
APPENDIX D.

FORTRAN LISTING OF THE PROGRAM

This appendix contains the basic subroutine of this report which calculates $P(A)$ or $P(H)$ to 3, 6, or 9 correct decimal digits.

CALL VALR16 (x, y, N, ans, IOP)

-where:

- $x, y$ are input arrays of the coordinates of the vertices. Vertices must be listed in counterclockwise order. See pp. 9, 10.

- $N$ is the number of sides of the polygon.*

- ans identifies the location where the $P_k$ is returned.

- $IOP = 1, 2$ or $3$ for $3, 6$ or $9$ decimal digits of accuracy, respectively.

* $N = 1$ for an angular region $A$ specified by three points given in counterclockwise order with the first point at the vertex of $A$, (See page 12). Note $0 < \Delta \theta < \frac{2\pi}{N}$ for $N = 1$, but $0 < \Delta \theta < \pi$ for $N \geq 3$. 
SUBROUTINE VALR16( X, Y, N, ANS, IO )
DIMENSION RSQ(4)
DIMENSION X(1), Y(1), U(2), V(2), G(2), h(2)
DIMENSION E(5), E(10), E(15)
DIMENSION APH1(3), APH2(3), APH3(3)
REAL L
DATA PI/3.1415 92653 58979 /
DATA T/6.2831 5307 17958 /
DATA ALNP/1.447 2986 84940 /
DATA C1/2.28209 47917 73877 /
DATA C2/3.15915 49430 91895 /
DATA ( E(L), I*10 )
1 .885777518572895E+00 , -.981151952776058E+00 ,
2 .759385502082405E+00 , -.353649830686977E+00 ,
3 .695232092435207E-01 /
DATA ( E2(I), I=1, 10 )/
1 .886226497816632E+00 , -.999950714561836E+00 ,
2 .885438828003392E+00 , -.66661239043357E+00 ,
3 .218211971600999E+00 , -.228896895667206E+00 ,
4 .90507384150449E-01 , -.25491061184287E-01 ,
5 .430951698413E-02 , -.323377239693247E-03 /
DATA ( E3(I), I=1, 15 ) /
1 .88625692931465E+00 , -.999999897726252E+00 ,
2 .8662373318722E+00 , -.66662667051097E+00 ,
3 .4425169938569E+00 , -.26563826366825E+00 ,
4 .1458343430314E+00 , -.7149038795889E-01 ,
5 .3091929352121E-01 , -.1232353214841E-01 ,
6 .32494543171185E-02 , -.704260243309096E-03 ,
7 .105787574480335E-03 , -.971864864160461E-05 ,
8 .408355172316E-06 /
DATA ( APH1(I), I=1, 3 ) /
1 2*02E-7, 2*08E-13, 2*72E-19 /
DATA ( APH2(I), I=1, 3 ) /
1 1*22E-2, 1*23E-4, 1*35E-6 /
DATA ( APH3(I), I=1, 3 ) /
1 2*28E-4, 2*28E-7, 2*61E-10 /
DATA ( RSQ(I), I=1, 3 ) /
1 6*0516, 12*60605, 19*201924 /
NH=H-1
K=1
PHIN=0.
PHIK=0.
ANS=0.
U(1)=X(1)-X(K)
V(1)=Y(1)-Y(K)
IF ( N*NE=1 ) GO TO 3131
U(2)=X(3)-X(1)
V(2)=Y(3)-Y(1)
SN=V(2)*U(1)-U(2)*V(1)
IF ( SN.GE.0. ) GO TO 3141
ANS=-1.
T1=U(1)
U(1)=U(2)
U(2)=T1
T1=V(2)
V(2)=V(1)
V(1)=T1
GO TO 3141

3131 CONTINUE
U(2)=X(N)-X(1)
V(2)=Y(N)-Y(1)

3141 CONTINUE
BGD1=SQRT( 2.*(U(1)*U(1)+V(1)*V(1)))
BGD2=SQRT( 2.*(U(2)*U(2)+V(2)*V(2)))

3151 CONTINUE
L=0.
ALAM=0.
B=+5.*(X(K)*X(K)+Y(K)*Y(K))
IF ( B.GT.ATOM(IOP) ) GO TO 3171
CAPG=0.

3161 CONTINUE
T1=ABS(V(2)*U(1)-U(2)*V(1))
T2=U(2)*U(1)+V(2)*V(1)
PHIK=ATAN2(T1,T2)
ANS1=PHIK/TMOP=CAPG
GO TO 3621

3171 CONTINUE
G(1)=U(1)*X(K)+V(1)*Y(K)
G(2)=U(2)*X(K)+V(2)*Y(K)
H(1)=-Y(K)*U(1)+X(K)*V(1)
H(2)=-Y(K)*U(2)+X(K)*V(2)
G(1)=U(1)/BGD1
G(2)=U(2)/BGD2
H(1)=H(1)/BGD1
H(2)=H(2)/BGD2
SN=(2.*(V(2)*U(1)-U(2)*V(1)))/(BGD1*BGD2)
IF ( SN.GT.0. ) GO TO 3185
CN=G(1)*G(2)+H(1)*H(2)
IF ( CN.GE.0. ) GO TO 3183
PHIK=PI
IF ( G(1).LT.0. ) GO TO 3141
ANS1=5*ERFC(0,H(2))
GO TO 3621

3181 CONTINUE
ANS1=5*ERFC(0,-H(1))
GO TO 3621
3183 CONTINUE
   PHIK=0.
   ANSI=0.
   GO TO 3621
3185 IF ( B*LE.*APH2(IOP) ) GO TO 3381
   IF ( G(1)*LT.0. ) GO TO 3261
   IF ( G(2)*GE.0. ) GO TO 3471
   G(2)=-G(2)
   H(2)=-H(2)
   ALAM=PI
   IF ( ABS(H(2))*.LE.*APH3(IOP) ) GO TO 3251
   L=5*ERFC(0,-H(2))
   GO TO 3471
3251 CONTINUE
   L=5
   GO TO 3471
3261 CONTINUE
   G(1)=-G(1)
   H(1)=-H(1)
   IF ( G(2)*LT.0. ) GO TO 3271
   ALAM=PI
   IF ( ABS(H(1))*.LE.*APH3(IOP) ) GO TO 3251
   L=5*ERFC(0,H(1))
   GO TO 3471
3271 CONTINUE
   G(2)=-G(2)
   H(2)=-H(2)
   IF ( ABS(H(1))*.LE.*APH3(IOP) ) GO TO 3291
   IF ( ABS(H(2))*.LE.*APH3(IOP) ) GO TO 3281
   L=5*(ERFC(0,H(1))-ERFC(0,H(2)))
   GO TO 3471
3281 CONTINUE
   L=5*(ERFC(0,H(1))-1.)
   GO TO 3471
3291 CONTINUE
   IF ( ABS(H(2))*.LE.*APH3(IOP) ) GO TO 3471
   L=5*(1.-ERFC(0,H(2)))
   GO TO 3471
3301 CONTINUE
   CAPG=G1*(H(2)-H(1))-G2*(1.)*H(2)-G1*H(1))
   GO TO 3161
3471 CONTINUE
   IF ( B*LT.*RSQ(IOP) ) GO TO 3479
   PHIN=-8.
   GO TO 3495
3479 CONTINUE
   IF ( K*NE.*a ) GO TO 3480
   IF ( FFIN*LE.*0. ) GO TO 3480
AJO=PHIN=ALAN
GO TO 3481

3480 CONTINUE
SN=G(1)*H(2)-G(2)*H(1)
CN=G(1)*G(2)+H(1)*H(2)
AJO=ATAN2(SN,CN)
PHIK=AJO
IF ( AJO.LT.0. ) PHIK=PI*AJO

3481 CONTINUE
CAPG=AJO
CAPH=AJO
N=1
F=0.
AJO=H(2)-H(1)
CIRCH=AJO
IF ( IOP.EQ.3 ) GO TO 3691
IF ( IOP.EQ.2 ) GO TO 3741
SUM=NEW(N)*AJO

3482 CONTINUE
M=M+1
H(2)=H(2)*G(2)
H(1)=H(1)*G(1)
T=H(2)-H(1)
F=F+8
CAPV=(F*CAPG+T)/M
SUM=SUM+E(M)*CAPV
IF ( N.LE.5 ) GO TO 3491
CAPH=CIRCH
CIRCH=CAPV
GO TO 3492

3491 CONTINUE
ANS1=L*EXP(-(B+ALNP))*(CAPH-SUM)
GO TO 3621

3495 CONTINUE
ANS1=L

3621 CONTINUE
IF ( (K-NH1) ) 3631, 3661, 3623

3623 CONTINUE
ANS=ABS(ANS+AES(ANS1))
RETURN

3631 CONTINUE
K=K+1
KP1=K+1
IF ( K.LE.2 ) GO TO 3651
ANS=ABS(ANS1)
U(2)=X(KP1)-X(K)
V(2)=Y(KP1)-Y(K)
PHIN=PHIN-PHIK

45
BGDZ=SQRT(  2.*(U(2)*U(2)+V(2)*V(2)) )
GO TO 3151

3651 CONTINUE
U(1)=U(2)
V(1)=V(2)
U(2)=X(KP1)-X(K)
V(2)=Y(KP1)-Y(K)
BGD1=BGDZ
BGDZ=SQRT(  2.*(U(2)*U(2)+V(2)*V(2)) )
GO TO 3671

3661 CONTINUE
K=N
U(1)=X(N)-X(1)
V(1)=Y(N)-Y(1)
BGD1=SQRT(  2.*(U(1)*U(1)+V(1)*V(1)) )

3671 CONTINUE
PHIN=PHIN+PHIK
ANS=ANS-ABS(ANS1)
GO TO 3151

3681 CONTINUE
SUM=E3(QH)*AJ1
3691 CONTINUE
M=M+1
H(2)=H(2)*G(2)
H(1)=H(1)*G(1)
F=F+F
CAPV=(F*CAPG+T)/M
SUM=SUM+E2(M)*CAPV
IF ( M.GE.15 ) GO TO 3491
CAPG=CIRCH
CIRCH=CAPV
GO TO 3691

3701 SUM=E2(QH)*AJ1
3711 M=M+1
H(2)=H(2)*G(2)
H(1)=H(1)*G(1)
F=F+F
CAPV=(F*CAPG+T)/M
SUM=SUM+E2(M)*CAPV
IF ( M.GE.10 ) GO TO 3491
CAPG=CIRCH
CIRCH=CAPV
GO TO 3711
END
DISTRIBUTION LIST

Chief of Naval Operations
Department of the Navy
Washington, D.C. 20350
Attn: OP-980
OP-982
OP-982E
OP-982F
OP-983
OP-987
OP-961

Commander, Naval Air Systems Command
Department of the Navy
Washington, D.C. 20360
Attn: Code NAIR-03
Code NAIR-03D
Code NAIk-5034

Commander, Naval Electronics Systems Command
Department of the Navy
Washington, D.C. 20360
Attn: Code NELEX-03A

Commanding Officer and Director
Naval Ship Research and Development Center
Washington, D.C. 20034
Attn: Code 18
Code 154
Code 184
Code 1541
Code 1802
Code 1805
Library

Commander, Naval Sea Systems Command
Department of the Navy
Washington, D.C. 20362
Attn: SEA 03A
SEA 034E
SEA 035
SEA 035B

Fleet Analysis Center
Naval Weapons Center
Seal Beach
Corona, California 91720
Attn: Library

ECOM Office Building
U.S. Army Electronic Command
Fort Monmouth, New Jersey 07703
Attn: Technical Library

Director, Naval Research Laboratory
Washington, D.C. 20390
Attn: Code 7800
Library

U.S. Army Electronic Command
China Lake, California 93555
Attn: Code 6073
Code 60704
Library

Naval Sea Systems Command
Department of the Navy
Washington, D.C. 20362

Director, Naval Research Laboratory
Washington, D.C. 20390
Attn: Code 7800
Library

Office of Naval Research
Washington, D.C. 20360
Attn: Math and Information Sciences Division
Library

U.S. Naval Observatory
34th Street and Massachusetts Avenue, N.W.
Washington, D.C. 20390
Attn: Library
U.S. Naval Oceanographic Office
Washington, D.C. 20390
Attn: Code 0814
Library

Navy Publications and Printing Service Office
Naval District of Washington
Washington, D.C. 20390

Commanding Officer
U.S. Army Harry Diamond Laboratories
Washington, D.C. 20438

AFADSB Headquarters, U.S. Air Force
Washington, D.C. 20330

Director
Defense Research and Engineering
Washington, D.C. 20390
Attn: WSEG
Deputy Director, Tactical Warfare Programs
Deputy Director, Test and Evaluation

The Library of Congress
Washington, D.C. 20540
Attn: Exchange and Gift Division (4)

Director
Defense Intelligence Agency
Washington, D.C. 20301

Lawrence Radiation Laboratory
Technical Information Department
P.O. Box 808
Livermore, California 94550

Sandia Corporation
Livermore Branch
P.O. Box 969
Livermore, California 94550
Attn: Technical Library

Numerical Analysis Research Library
University of California
405 Hilgard Avenue
Los Angeles, California 90024

Superintendent
U.S. Naval Postgraduate School
Monterey, California 93940
Attn: Library, Tech Reports Section

Commander
Naval Undersea Research and Development Center
3202 East Foothill Boulevard
Pasadena, California 91107
Attn: Code 254
Code 2501
Code 25403
Code 25406

Director
Office of Naval Research Branch Office
1030 East Green Street
Pasadena, California 91101

Commanding Officer
Marine Air Detachment
Naval Missile Center
Point Mugu, California 93041

Office of Naval Research
Branch Office, Chicago
219 South Dearborn Street
Chicago, Illinois 60604

Superintendent
U.S. Naval Academy
Annapolis, Maryland 21402
Attn: Library, Serials Division

Commanding Officer
U.S. Army Aberdeen R&D Center
Aberdeen, Maryland 21005
Attn: Dr. F. E. Grubbs
Library

Director
U.S. Army Munitions Command
Edgewood Arsenal, Maryland 21010
Attn: Operations Research Group
President
Naval War College
Newport, Rhode Island 02840

6151 West Century Boulevard
Los Angeles, California 90045

Prof. M. Albertson
Department of Civil Engineering
Colorado State University
Fort Collins, Colorado 80521

California Institute of Technology
Pasadena, California 91109
Attn: Prof. T. Y. Wu

Rutgers University
Statistics Center
New Brunswick, New Jersey 08903
Attn: Prof. M. F. Shakun

Dr. George Ioup, Physics Department
Louisiana State University
Lake Front
New Orleans, Louisiana 70122

Burroughs Corporation Research Center
Paoli, Pennsylvania 19301
Attn: R. Mirsky, Advanced Systems

Pennsylvania State University
University Park, Pennsylvania 16802
Attn: Prof. P. C. Hammer, Computer Science Department

Defense Documentation Center
Cameron Station
Alexandria, Virginia 22314

Langley Aeronautical Laboratory
National Aeronautics and Space Administration
Langley Field, Virginia 23365
Attn: Mr. J. B. Parkinson

Prof. Gary Makowski
Department of Math and Statistics
Marquette University
Milwaukee, Wisconsin 53233

University of Wyoming
Statistics Department
Box 3275, University Station
Laramie, Wyoming 82070
Attn: Dr. W. C. Guenther
Mr. J. Terragno

Mrs. Pamela M. Morse
Canada Department of Agriculture
Sir John Carling Building, Room E265
Statistical Research Service
C.E.F. Ottawa, Ontario
Canada

National Research Council
Montreal Road
Ottawa 2, Canada
Attn: Mr. E. S. Turner

Prof. H. Primas
Swiss Federal Institute of Technology
Physical Chemistry Laboratory
Universitatsstrasse 22
8006 Zurich
Switzerland

V.P.I. and State University
Blacksburg, Virginia 24060
Attn: Dr. J. Arnold, Statistics Dept.
Dr. R. H. Myers, Statistics Dept.

Dr. Donald Amos
Division 5122
Sandia Laboratories
Albuquerque, New Mexico 87115

Alan B. Bligh
Code 7810
Naval Research Laboratory
Washington, D.C. 20375
N
N-10
N-10(S. Vittoria)
N-20
N-30
N-40
X-21(2)

Math Department, WOL
Dr. J. W. Enig, R-10
Dr. A. H. VanTuyl, A-43
Technical Library, X-21 (4)