1. Introduction

In this paper I would like to give a very partial account of integration theory in Hilbert space and related questions of absolute continuity which may be important in problems of stochastic realization theory, linear and non-linear filtering, detection theory and quantum communication theory. This theory is largely the creation of I.E. Segal and his former students, notably Gross and Nelson. The need for such a theory arose for the purpose of putting quantum field theory on a rigorous mathematical basis. The theory has a distinct algebraic character and I believe is particularly suited to the needs of stochastic system theory. An account of this algebraic approach may be found in SEGAL-KUNZE [1], SEGAL ([1] and the bibliography cited therein) where a countably additive measure which is translation and notation invariant can be given mathematical meaning.

The other approach to some of the ideas of the Russian school (cf. GELFAND-VILEN'KIN) in the sense that essentially Hilbert space techniques are used and in general one works with "weak" processes as opposed to "strict" processes. In this theory non-linear functions of processes can be handled and in particular certain non-linear functionals of white noise can be given mathematical meaning. The other approach to some of these questions is due to GROSS (cf. GROSS [1], [2] and the bibliography cited therein) where a countably additive "extension" on a separable Banach space of the finitely-additive Gaussian measure on a Hilbert space is obtained. These ideas have recently been modified and developed by Balakrishnan (cf. for example BALAKRISHNAN [1], [2]) in a series of papers related to detection and filtering theory.

2. Segal-Gross Theory of Weak Processes

It is a known fact that there is no analog of Lebesgue measure (i.e. a countably additive measure which is translation and notation invariant) on an infinite dimensional Hilbert space. In fact no such measure exists even when invariance is relaxed to quasi-invariance. Such an "invariant" measure however exists if we do not insist on countable additivity.

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we denote by $\mathcal{E}(f)$. A is a ($C^*$-algebra with unit in the supnorm and hence $A = C(X)$ for some compact Hausdorff space $X$. Let $f \mapsto \hat{f}$ be this isomorphism. Moreover $E$ is a continuous positive linear function on $A$ and hence by the Riesz representation theorem

$$E(f) = \int f \, dm,$$

where $m$ in this case is probability measure. The isomorphism $f \mapsto \hat{f}$ can be extended to tame functions in $E(H,H)$ by density such that $(f\tilde{g})' = \hat{f}(\tilde{g})$. Hence if $f = \text{char-fin}(A)$, $A$ a tame set, $\hat{f}$ is characteristic function of some measurable set $A$ and $\mu(A) = m(A)$. Using the Gelfand Transform, we can see that $\hat{f}$ is an extension of $f$ from $A$ to all of $X$. Now $A$ and in fact $H$ is such that $m(H) = 0$.

Let $H$ be identified with $\mathbb{C}$. The continuous linear functionals on $H$ are in $L^*\overline{H}$ $\overline{H}$. To the linear functional determined by $\gamma$ there corresponds a measurable $f(y)(\cdot)$ on $X$. $F : H \rightarrow L^*\overline{H}$ is norm-preserving. It can be shown (1) that the map $F$ completely determines the extension $f \mapsto \hat{f}$. (2) the map $f \mapsto \hat{f}$ can be extended to tame functions and (3) $\hat{f} = \phi(f(y_1), \ldots, f(y_n))$ where $(x,y \in \mathbb{C})$.

The functions $F(y)(\cdot)$ on $X$ are normally distributed with variance $||y||^2$ and $\hat{f}$, $\ldots, \hat{f}$ are orthogonal, then $F(y_1), \ldots, F(y_n)$ are independent. More concrete realizations of $H$ and the measure space $(X,\mu)$ can be obtained, for example

- a) where $\mu = \mathbb{R}^2$, $X = \mathbb{R}$, $m$ the product measure corresponding to Gaussian measure on each coordinate,
- b) $H = L^2(0,1)$, $X = L^2(0,1)$ with $m$ Wiener measure. However it can be easily proved that these various extensions are all measure theoretically isomorphic.

3. Abstract Wiener Spaces and Absolute Continuity

The discussion above could be formalized using the ideas of Abstract Wiener Space due to Gross.

Let $H$ be a real Hilbert space and let $\mu$ be the tame measure given by (2.3). A measurable norm on $H$ is a norm $||\cdot||$ such that $\forall \epsilon > 0$, \exists finite dimensional projection $F$ such that $\forall$ finite dimensional $P$, $\mu(\{x \in H | ||P(x)|| > \epsilon\}) < \epsilon$.

Let $E = \text{completion of H with respect to } ||\cdot||$. $E$ is a Banach space. The canonical embedding $i : H \rightarrow E$ is compact. Identifying $H$ and $H^*$, we obtain by duality the embeddings

$$\begin{array}{ccc}
\mathbb{C} & \rightarrow & \mathbb{C}^* \rightarrow & E^* \\
\phi & \mapsto & \phi & \mapsto \left( f \mapsto \phi(f) \right)
\end{array}$$

$E^*$ can be identified with its image in $H$ and $H$ with its image in $E$. The measure $\mu$ has a countably additive extension $\mu'$ on the Borel fields of $E$. The triple $(E,\mathbb{B},\mu)$ is called a Wiener space and $\mu$ Wiener measure on $E$.

Now, $E^*$ could be interpreted as functions on $E$ belonging to $L^1(E,\mu)$ and their $L^2$-norm equals the $H^*$-norm. Hence the closure of $E^2$ in $L^2(E,\mu)$ can be identified with $H^*$. If $\alpha \in \mathbb{C}^*$, we denote by $\alpha$ the corresponding random variable on $E$. Let $\mathbb{P}$ be a finite dimensional orthogonal projection on $E$ such that

$$\mathbb{P} = \sum_{a} a \otimes a,$$

where $a$ is an orthonormal in $E$, then $\mathbb{P} = \sum_{a} a \otimes a$ defines a random variable on $E$ with values in $H$.

A function $f$ on $H$ with values in a Banach space $F$ determines a random variable $\hat{f}$ on $E$ with values in $F$. If for any sequence of finite dimensional projection $P_n = \mathbb{P}$ strongly in $H$, the sequence of random variables $f \mathbb{P}_n$ converge in measure on $E$.

Some Preliminaries

Let $H$ be a real Hilbert space and let $L(\mathcal{H})$ denote the space of bounded linear operators on $H$. Let $I$, denote the Banach space of nuclear operators of $H$ under the norm $||K|| = tr(\mathfrak{K}K)^2$. $I$, is a $\mathcal{F}$-ideal in $L(H)$. Let $\mathcal{H}$ denote the Banach space of Hilbert Schmidt operators on $H$ with norm $||K|| = tr(\mathfrak{K}K)^1$. $I_2$ is also a $\mathcal{F}$-ideal in $L(H)$.

If $K \in I$, the Fredholm determinant of $(1+K)$ is defined by $det(1+K) = \prod_{\lambda > 0}(1+\lambda_j)$ where $\lambda_j$ are the eigenvalues of $K$ counted with their multiplicities. If $K \in L$, the Carleman Fredholm determinant of $1+K$ is defined by $det(1+K) = \prod_{\lambda < 0}(1+\lambda_j)$. $det(1+K)$ is an analytic function on $I$ and $\prod_{\lambda < 0}(1+\lambda_j)$ is an analytic function on $I_2$.

Preliminary Lemmas

The following lemmas follow from the work of Gross (cf. Gross (192)).

**Lemma 3.2** Let $K \in L(\mathcal{H})$. Then $K$ determines a random variable $K$ on $E$ with values in $E$.

**Lemma 3.2** Let $P$, be a family of orthogonal projections converging to $I$ strongly. Let $K \in I$. Then $(K^2P)$ is a Cauchy sequence in $L^2(E;P;H)$ and $\text{Prob}(K \in I) = 1$.

Suppose $K \in I_2$. Then in general $<Kx,x>$ and $\text{Tr}(K)$ need not exist. However if $<Kx,x> = \text{Tr}(K)$ can be given a meaning as a real random variable on $E$ via stochastic extension.

In fact, for certain non-linear operators $K : E \rightarrow E$ if $K$ is continuous and its $H$-derivative (defined below) $K_x$ is Hilbert-Schmidt.

Let $U \subset E$ be open. A function $f : U \rightarrow F$, $F$ Banach is $\mathcal{B}$-continuous at $x \in U$ if the function $g(x) = f(x)\mathbb{P}$ defined on $(0 \setminus \{x\}) \cap H$ is continuous at the origin in the induced (Hilbert) toplogy, if $f$ is $H$-differentiable at $x$ if $f$ is Frechet differentiable at the origin in $H$. If it can be shown (cf. Reiter).

**Proposition:** Let $U \subset E$ be open and let $K : U \rightarrow E$ be such that (i) $K(U) \subset H$, (ii) the $H$-derivative at $x$, $K_x : U \rightarrow L(H,E)$ is continuous and Hilbert-Schmidt. Let $(e_n)$ be an orthonormal basis in $H$ such that $e_n \in \mathcal{E}(E,\mathcal{B})$, $\forall i$. Let $P_n = \sum_{i=1}^{n} e_i \otimes e_i : E \rightarrow E$. Then
Theorem 3.3: ably edit ion on the borel
measure on \( E \) with

\[
\text{such that}
\]

\( a_n(x) \) converges almost everywhere on \( E \) to

a random variable on \( U \). Denoting this random variable by \( \langle K_x,x \rangle - \text{tr(K}_x \rangle \) if \( a_n(x) \) is any other sequence for which \( (a_n(x)) \) a.e., then \( a_n(x) + \langle K_x,x \rangle - \text{tr(K}_x \rangle \) a.e.

(11) \( \langle K_x,x \rangle - \text{tr(K}_x \rangle \) does not depend on the choice of basis \( \{e^i\}_{i \in \mathbb{N}} \) in \( E \).

Absolute Continuity and Computation of the Radon–Nikodym Derivative

Case I (Translation)

Theorem 3.1 (Segal)

Let \((I,H,E)\) be an abstract Wiener Space and let \( p \) be standard Wiener measure on \( E \). Let \( e \in E \) and let \( T : E \to E : x \mapsto x + e \). Then the transformed measure \( p_T \) and \( p \) are mutually absolutely continuous if and only if \( e \in H \). The R–N derivative of \( p_T \) with respect to \( p \) is the random variable \( \exp(-|e|^2/2) \).

Remark:

If \( E = C(0,1,\mu) \) where \( \mu \) denotes Wiener measure then \( H = H^{1/2}(0,1,\mu) \), the Sobolev space with Gaussian measure.

Case II (Linear Transformation)

Theorem 3.2 (Segal-Feldman)

Let \((I,H,E)\) be an abstract Wiener Space and \( p \) standard Wiener measure. Let \( q \) be a Gaussian measure on \( E \) with covariance \( Q \). Then \( p \) and \( q \) are either mutually singular or mutually absolutely continuous. They are mutually absolutely continuous if and only if there exists a \( K \in \mathcal{L}_Q \), symmetric, such that the quadratic form \( Q(x) \) on \( E \) is of the form \( Q(x) = \langle (I+K)x,x \rangle \).

The R–N derivative of \( q \) with respect to \( p \) is the random variable on \( E \) given by

\[
\frac{\lambda_0}{\lambda_1} (\lambda_1^{1/2} \exp(h_0(1\lambda_1^{1/2} - 1)^2). \]  

It is possible to use Theorem 3.2 to prove Theorem 3.3

Let \((I,H,E)\) be abstract Wiener Space and let \( p \) be the standard Wiener measure on \( E \). Let \( T = I + K \) be an invertible linear transformation on \( E \) with \( K \in \mathcal{L}(E,H) \). Then \( K \) is invertible and \( \text{tr(K}_x \rangle \) exists. Then the R–N derivative of the transformed measure \( p_T \) with respect to \( p \) is given by

\[
|\delta(T)| \exp[-\langle Kx,x \rangle - \text{tr(K}_x \rangle] = |\delta| |Kx|^2 \text{ a.e.} \]  

The affine case could now be proved using Theorem 3.1. There is a non-linear version of Theorem 3.3.

Theorem 3.4 (Sacer)

Let \((I,H,E)\) be an abstract Wiener Space and \( p \) be standard Wiener measure on \( E \). Let \( U \subset E \) be open and let \( T : I + K : U \to E \) be a continuous non-linear transformation such that

(i) \( T \) is a homeomorphism of \( U \) onto an open subset of \( E \).

(ii) \( K(U) \subset H \) and \( K : U \to H \) is continuous.

(iii) \( K \) is a standard Wiener process assumed to be independent.

Then \( p \) and the transformed measure \( p_T \) are mutually absolutely continuous as measures on \( U \). The R–N derivative of \( p_T \) with respect to \( p \) is given by

\[
|\delta(T)| \exp[-\langle Kx,x \rangle - \text{tr(K}_x \rangle] = |\delta| |Kx|^2 \text{ a.e.} \]  

Remarks:

(i) As mentioned earlier \( \langle Kx,x \rangle - \text{tr(K}_x \rangle \) is a random variable. It is intriguing to see the resemblance to the Wong-Zakai correction term relating the Ito and Stratanovich integrals.

(ii) Consider the Kalman Filtering problem

\[
dx_t = Fx_t dt + GdW_t \]

\[
dy_t = E x_t dt + d\eta_t \]  

where \( \eta_t \) and \( \eta_t \) are standard Wiener processes assumed to be independent.

Then by passing to the Innovations Representation

\[
dy_t = Ey_t dt + d\nu_t \]  

where \( E = E|x_t|^2 \) and \( ey_t \) is the Innovations process (which is a standard Wiener process) and noting that

\[
\tilde{e}_t = \int_0^t K(t,s) d\nu_s \]  

with \( K(t,s) \in L^2([0,t]; x(0,t); L(H;\mathbb{R}^2)) \), we are in the situation of Theorem 3.3. A "causal" representation for the R–N derivative could be obtained by invoking the Krein Factorization Theorem in conjunction with Theorem 3.3 (cf. HITSUDA where the reverse process is followed).

4. The Free Quantum Field and Kalman Filtering

In the previous section we have indicated how starting from a Hilbert space \( H \) with Gauss measure of unit variance \( \mu \) on it we can construct a Banach space \( E \) and a measure \( \mu \) which is countably additive on the borel sigma algebra of \( \mathbb{E} \). It is also an extension of the Shirai measure of \( \mathbb{E} \) in a certain precise sense. Integration of functions on \( H \) and questions of absolute continuity can be answered by passing to the Banach space by an appropriate stochastic extension. There is a purely Hilbert space point of view due to Segal which may turn out to be more important for the needs of System Theory. Due to lack of space we do not give a detailed exposition of this theory here. This theory is
The particle representation which involves the symmetric tensor products of a complex Hilbert space \( H \) with itself, (ii) the wave representation (functional integration) in the space \( L^2(H^\ast) \) of a real paraboloid \( H \) and (iii) the complex-wave representation in which a space \( K \) of entire anti-holomorphic functions on \( H \) are involved.

The intertwining operators between the various representations requires absolute continuity consideration of the Fourier Transform. Mathematically, the field is diagonalized in the functional integration representation whereas the particle numbers are diagonalized in the tensor product representation. In the complex-wave representation the creation operators achieve a kind of diagonalization.

Brockett (cf. BROCKETT) has recently shown that the group with 4 generators \( H, P, Q, E \) with the commutation relations

\[
\begin{align*}
[H, P] &= -Q, \quad [H, Q] = P, \quad [P, Q] = E
\end{align*}
\]

with the rest zero plays an important role in Kalman filtering theory. This group has been called the Harmonic Oscillator Group (cf. STREETER). The group generated by \( P, Q, E, \) the Heisenberg group, is a subgroup of the oscillator group. The oscillator group is not nilpotent but solvable.

Barger has used all the continuous unitary irreducible representations of the Harmonic Oscillator group. He shows that if complex Lie algebras are allowed then one can obtain the Bargmann-Segal representation of the harmonic oscillator by holomorphic functions using the Fourier transform technique of Kirillov. In this representation the creation operator \( C(z) \) is multiplication by \( z \) and the annihilation operator \( C^\ast(z) \) is \( \overline{z} \).

Segal (cf. SEGAL [2]) has explicitly given the intertwining operators between the holomorphic and real representations. It would thus appear that the Zakai equations for the unnormalized conditional density corresponding to the Kalman filtering problem defines a "field" which is analogous to the "free quantum field".

**Definition:**
A concrete free boson field over a given complex Hilbert space \( H \), denoted as \( \Phi(H) \) is a quadruple \( (K, U, \Gamma, V) \) consisting of

1. A complex Hilbert space \( K \),
2. A continuous mapping \( \Gamma : W(a) : H \to U(K) \), the space of unitary operators on \( K \) satisfying the Weyl relations

\[
W(z)W(z') = \exp(\frac{1}{2} \text{Im} z, z' W(z, z'), \forall z, z' \in H,
\]
3. A continuous representation \( \Gamma \) from \( U(K) \) satisfying

\[
\Gamma(UH)\Gamma(UH^{-1}) = W(Uz), \forall U \in U(H), z \in H,
\]
4. A unit vector \( v \in K \) having the properties that \( \Gamma(U)v = v \forall U \in U(H) \) and \( W(z)v, z \in H \) span \( K \) topologically
5. \( \Gamma \) is positive in the sense that if \( A \) is any positive self-adjoint operator on \( H \), then \( \text{tr}(A) \) is positive where for any positive self-adjoint \( A \) in \( H \) \( \text{tr}(A) \) is the self-adjoint generator of the one-parameter unitary group \( \{ e^{itA} \} | t \in \mathbb{R} \).

Let \( H' \) be a real Hilbert space and let \( g \) denote the centred Gaussian weak distribution on \( H' \) with variance 1. We define a positive linear functional \( \tilde{Z} \) (expectation) on the algebra \( \mathcal{A}(H') \) of all bounded tame functions on \( H' \). Let \( L^2(H', g) \) be the completion of \( \mathcal{A}(H') \) with respect to the inner product \( \langle f, f' \rangle = E(\text{tr}(ff')) \). Let \( \hat{\Theta} \) denote the complexification of \( \mathcal{A}(H', g) \).

If \( H \) is a complex Hilbert space, it has also the structure of a real Hilbert space with inner product equal to the real part of the complex inner product in \( H \). In this way, we can define \( g \) on \( H \) and hence \( L^2(H, g) \). From the work of Segal, we know \( L^2(H, g) \) can be regarded as the completion of the algebra \( F'(H) \) of functions of the form

\[
f(x) = p(x, e_1, \ldots, x, e_n, \ldots), \quad p \text{ a real polynomial and the } e_i \text{ orthonormal.}
\]

In addition to \( F' \), one can also consider the algebra of functions

\[
f(x) = p(x, e_1, \ldots, x, e_n, \ldots), \quad p \text{ a polynomial function on } C^\infty \text{ and complex conjugates of the above. Let } F(H) \text{ denote the last mentioned algebra.}
\]

In the complex-wave representation the representation space \( K \) is the closure of \( F(H) \) in \( L^2(H, g) \). Segal has shown that the elements of \( K \) can be identified as functions well defined at every point of \( H \) and which satisfy an \( L^2 \)-boundedness condition. We do not go into the details of the construction of \( K \) and \( \Gamma \) of the Weyl System here. It can be shown that there exists a unique (up to unitary equivalence) Weyl System. It is however worthwhile stating explicitly the form of the "creation" and "annihilation" operators.

**Definition:**
For any representation \( \Phi = (K, U, \Gamma, V) \) of the free boson field over the given Hilbert space \( H \) and for given vector \( z \in H \), the creation operator for \( z \) denoted by \( C(z) \) is defined as the operator

\[
\Phi(z) = \delta W(z) - \delta_{W(z)}, \quad \delta_{W(z)} \text{ denotes the self-adjoint generator of the one-parameter group } (W(z)| z \in \mathbb{R}).
\]

**Theorem (Segal)**
The operators \( C(z) \) and \( C^\ast(z) \) are closed, densely defined and mutually adjoint. In the complex-wave (anti-holomorphic) representation, \( C(z) \) has domain consisting of all \( F \) such that \( \langle z, \cdot \rangle F(z) \in X \). \( C(z) \) is the mapping

\[
\Phi(z) = \delta W(z) - \delta_{W(z), \Phi(z) \in X}.
\]

\( C^\ast(z) \) has domain consisting of all \( F \in X \) such that \( F(z) \) is the mapping

\[
\Phi(z) = \delta W(z) - \delta_{W(z), \Phi(z) \in X}.
\]
$$F = -\sqrt{2}\mathbf{i} \mathbf{F}_x$$

The Particle Representation

Let \( H' \) be a real Hilbert space and let \( H^{\otimes n} \) be its complexification. Let \( H^0 \) be the \( n \)-fold symmetric tensor product of \( H \) with itself. We give \( H^0 \) the inner product

$$\langle \mathbf{f} \otimes \mathbf{g}, \mathbf{f}' \otimes \mathbf{g}' \rangle = \sum_{\pi} \mathbf{f}(\pi(1)) \mathbf{g}'(\pi(n))$$

where \( \pi \) is a permutation of \((1,2,\ldots,n)\) and \( \mathbf{f} \) and \( \mathbf{g} \) are symmetric tensors. For brevity we denote by \( \otimes \) the symmetric product.

Let \( F \) be the weak centred Gaussian distribution of unit variance on \( H' \). Associated with \( H' \) is a probability space \((\Omega, \mathcal{F}, \mu)\), where \( \mathcal{F} \) is generated by \( F(f), f \in H \) and if \( f_1, \ldots, f_n \) are orthonormal in \( H' \) and \( \phi \) is a Radon-Nikodym derivative on \( R^2 \), then

$$\mathbb{E}(e^{i\langle \phi, \mathbf{f} \rangle}) = \prod_{k=1}^{n} \mathbb{E}(e^{i\langle \phi, f_k \rangle}) = \prod_{k=1}^{n} \mathbb{E}(e^{i\langle \phi, f_k \rangle})$$

Let \( \mathcal{F} \) be the closed linear span in \( L^2(\mu) \) of all elements of the form \( F(f_1) \cdots F(f_n) \) \( n < \infty \) and let \( S(H') \) be the orthogonal completion of \( \mathcal{F} \). For \( f_1, \ldots, f_n \) in \( H' \), define \( \mathbf{f}(f_1) \cdots \mathbf{f}(f_n) \) to be the orthogonal projection of \( \mathbf{f}(f_1) \cdots \mathbf{f}(f_n) \) on \( S(H') \).

Then it is easy to see that

$$\mathbf{f}(f_1) \cdots \mathbf{f}(f_n) = \sum_{\pi}\mathbf{f}(\pi(1)) \cdots \mathbf{f}(\pi(n))$$

extends uniquely to a unitary mapping from \( S(H') \) onto \( H^0 \). We identify \( S(H') \) with \( H^0 \) via this mapping. Segal showed that \( S(H') \) span \( L^2(\mu) \).

Hence \( L^2(\mu) = \overline{\mathcal{F} S(H')} \). This is Fock space.

Let \( \Gamma(H) \) denote \( \sum_{n=0}^{\infty} H^0 \). \( \Gamma(H) \) is intrinsically attached to the structure of \( H \) as a real Hilbert space. Hence if \( U : H' \rightarrow H' \) is an orthogonal mapping of one real Hilbert space into another it induces a unitary mapping \( \Gamma(U) : \Gamma(H) \rightarrow \Gamma(K) \). Of \( S(H) \), \( \Gamma(U) \) is \( U \otimes \mathbf{U} \) (n-factors). The ideas of Fock space are important in filtering theory and related to Wiener's homogeneous chaos. For a recent application see MARCUS-MITTER-OONE.

5. Conclusions

The mathematics used in quantum field theory may have applications to modelling of stochastic systems and filtering theory. In this paper I have concentrated on ideas surrounding the free quantum field. I believe ideas of non-linear quantum field theory, for example, those developed in SEGAL [3] have applications in non-linear filtering theory. But this we have to leave for the future.

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