ABSTRACT. Two new combinatorial inequalities are presented. The main result states that if \( \gamma_j, 1 \leq j \leq n \), are fixed complex scalars with \( \sigma = |\sum \gamma_j| > 0 \) and \( \delta = \max_{1 \leq i, j} |\gamma_i - \gamma_j| > 0 \), and if \( \mathcal{V} \) is a normed vector space over the complex field, then
\[
\max_{\pi} \left| \sum_{j=1}^{n} \gamma_j a_{\pi(j)} \right| \geq \left[ \frac{\sigma}{(2\sigma + \delta)} \right] \max_j |a_j|, \quad \forall a_1, \ldots, a_n \in \mathcal{V},
\]
\( \pi \) varying over permutations of \( n \) letters. Next, we consider an arbitrary generalized matrix norm \( N \) and discuss methods to obtain multiplicativity factors for \( N \), i.e., constants \( v > 0 \) such that \( vN \) is submultiplicative. Using our combinatorial inequalities, we obtain multiplicativity factors for certain \( C \)-numerical radii which are generalizations of the classical numerical radius of an operator.

1. SOME NEW COMBINATORIAL INEQUALITIES

In a recent paper [5] we studied a somewhat less general version of the following problem: Given fixed complex scalars \( \gamma_1, \ldots, \gamma_n \), and a normed vector space \( \mathcal{V} \) over the complex field \( \mathbb{C} \), can we find a constant \( K > 0 \) such that the inequality
\[
\max_{\pi \in S_n} \left| \sum_{j=1}^{n} \gamma_j a_{\pi(j)} \right| \geq K \cdot \max_j |a_j|, \quad \forall a_1, \ldots, a_n \in \mathcal{V},
\]

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is satisfied? Here \( S_n \) is the symmetric group of \( n \) letters, and \(|a_j|\) is the norm of the vector \( a_j \).

We start with the following lemma.

**Lemma 1.1.** For any \( \gamma_1, \ldots, \gamma_n \in \mathbb{C} \) and \( a_1, \ldots, a_n \in \mathbb{V} \),

\[
\max_{\pi} \left| \sum_j \gamma_j a_{\pi(j)} \right| \geq \frac{1}{2} \max_{i,j} |\gamma_i - \gamma_j| \cdot \max_{i,j} |a_i - a_j|.
\]

**Proof.** We may rearrange the \( \gamma_j \) and the \( a_j \) so that

\[
|\gamma_1 - \gamma_n| = \max_{i,j} |\gamma_i - \gamma_j|, \quad |a_1 - a_n| = \max_{i,j} |a_i - a_j|.
\]

Now, consider the vectors

\[
b_1 = \gamma_1 a_1 + \gamma_2 a_2 + \cdots + \gamma_{n-1} a_{n-1} + \gamma_n a_n,
b_2 = \gamma_1 a_n + \gamma_2 a_2 + \cdots + \gamma_{n-1} a_{n-1} + \gamma_n a_1.
\]

We have

\[
\max_{\pi} \left| \sum_j \gamma_j a_{\pi(j)} \right| \geq \max \{|b_1|, |b_2|\} \geq \frac{1}{2} |b_1 - b_2| \\
= \frac{1}{2} |\gamma_1 a_1 + \gamma_n a_n - \gamma_1 a_n - \gamma_n a_1| \\
= \frac{1}{2} |\gamma_1 - \gamma_n| \cdot |a_1 - a_n|,
\]

and the proof is complete. \( \square \)

Denoting

\[
(1.2) \quad \sigma = \left| \sum_j \gamma_j \right|, \quad \delta = \max_{i,j} |\gamma_i - \gamma_j|,
\]

we prove the following result.

**Theorem 1.2.** There exists a constant \( K > 0 \) that satisfies (1.1) if and only if \( c\delta > 0 \). If \( c\delta > 0 \) then (1.1) holds with \( K = c\delta/(2\sigma + \delta) \).

**Proof.** Suppose \( c\delta = 0 \). If \( \sigma = 0 \), take \( a_j = a \), \( 1 \leq j \leq n \), for some \( a \neq 0 \); if \( \delta = 0 \), then the \( \gamma_j \) are equal, so choose \( a_j \) not all zero with \( \sum a_j = 0 \). In both cases,
\[ \max \left| \sum_j \gamma_j a_{\pi(j)} \right| = 0 \quad \text{but} \quad \max_j |a_j| > 0; \]
hence no \( K > 0 \) satisfies (1.1).

Conversely, suppose \( \sigma^5 > 0 \) and let us show that \( K = \sigma^5/(2\sigma + 5) \) satisfies (1.1). The following proof, which is shorter than the original one in [5], is due to Redheffer and Smith [8].

Order the \( a_j \) so that
\[ a_1 = \max_j |a_j|, \quad |a_1 - a_n| = \max_j |a_1 - a_j| = \varepsilon|a_1| \quad (0 \leq \varepsilon \leq 2). \]

Thus, by Lemma 1.1,
\[ (1.3) \quad \max \left| \sum_j \gamma_j a_{\pi(j)} \right| \geq \frac{\sigma^5}{2} \max_j |a_j|. \]

Next, consider the vectors
\[ c_j = \gamma_1 a_{k+j} + \gamma_2 a_{2+j} + \ldots + \gamma_n a_{n+j}, \quad J = 1, \ldots, n, \]
where \( k + j = (k + j) \mod n \). We have
\[ (1.4) \quad \max \left| \sum_j \gamma_j a_{\pi(j)} \right| \geq \max_j |c_j| \geq \frac{1}{n} |c_1 + \ldots + c_n| = \frac{\sigma}{n} |a_1 + \ldots + a_n| = \frac{\sigma}{n} |na_1 - (a_1 - a_2) - (a_1 - a_3) - \ldots - (a_1 - a_{n-1})| \geq \frac{\sigma}{n} \{n|a_1| - (n-1)|a_1 - a_n| \} = \sigma(1 - \frac{n-1}{n} \varepsilon) \max_j |a_j|. \]

By (1.3) and (1.4), therefore,
\[ (1.5) \quad \max \left| \sum_j \gamma_j a_{\pi(j)} \right| \geq \max \left\{ \frac{\sigma^5}{2}, \sigma(1 - \frac{n-1}{n} \varepsilon) \right\} \max_j |a_j|. \]

The expressions in the braces are functions of \( \varepsilon \) describing straight lines with opposite slopes and intersecting value \( \sigma^5/(2\sigma + 5 - 2\sigma/n) \). Thus, for any \( \varepsilon \),
\[ (1.6) \quad \max \left\{ \frac{\sigma^5}{2}, \sigma(1 - \frac{n-1}{n} \varepsilon) \right\} \geq \frac{\sigma^5}{2\sigma + 5 - 2\sigma/n} > \frac{\sigma^5}{2\sigma + 5}. \]

By (1.5) and (1.6), the theorem follows. \( \Box \)
What is the best (greatest) possible $K$ which satisfies (1.1)? In answer to that question, Redheffer and Smith proved the following [8].

**THEOREM 1.3.** If $\sigma \delta > 0$, then the best $K$ for (1.1) satisfies

$$
\frac{\sigma \delta}{2\sigma + 3 - 2\sigma/n} \leq K \leq \min \left\{ \sigma, \frac{\sigma \delta}{2\sigma + 3 - 2\sigma/n - 2\delta/n} \right\},
$$

and the inequality on the right becomes an equality when the $\gamma_j$ and $a_j$ are real numbers.

We note that the left-hand inequality in (1.7) was established already in the proof of Theorem 1.2. For the complete proof of Theorem 1.3, see [2].

From Theorem 1.3, Redheffer and Smith immediately conclude that while the Goldberg-Straus constant in Theorem 1.2 is not optimal for any $n$, it is the best that can be chosen independently of $n$, even if the $\gamma_j$ and $a_j$ are real.

Under certain restrictions on the $\gamma_j$, we can improve the constant obtained in Theorem 1.2.

**THEOREM 1.4.** If $\gamma_1, \ldots, \gamma_n$ are of the same argument, then (1.1) holds with $K = 3/2$.

**Proof.** We may assume that

$$
\gamma_1 \geq \cdots \geq \gamma_n.
$$

Arrange the $a_j$ so that

$$
|a_1| = \max_j |a_j|,
$$

and let $P$ be a projection of $V$ in the direction of $a_1$. We write

$$
P a_j = \lambda_j a_j, \quad j = 1, \ldots, n,
$$

and set

$$
\rho_j = \text{Re } \lambda_j, \quad j = 1, \ldots, n.
$$

Since
\[ \lambda_1 = 1 \geq |\lambda_j|, \quad j = 2, ..., n, \]

it follows that

\[ \rho_1 = 1 \geq |\rho_j|, \quad j = 2, ..., n. \]

So we may order \( a_2, ..., a_n \) to satisfy

\[ 1 = \rho_1 \geq \rho_2 \geq \cdots \geq \rho_n. \]

We have

\[
\max_{\pi} \left| \sum_j \gamma_j a_{\pi(j)} \right| \geq \max_{\pi} \left| p \left( \sum_j \gamma_j a_{\pi(j)} \right) \right|
= \max_{\pi} \left| \sum_j \gamma_j \lambda_j \cdot |a_j| \right| \geq \max_{\pi} \left| \text{Re} \left( \sum_j \gamma_j c_{\pi(j)} \right) \right| \cdot |a_1|
= \max_{\pi} \left| \sum_j \gamma_j \rho_{\pi(j)} \right| \cdot \max_j |\lambda_j|.
\]

Now, if \( \rho_n > 0 \), then

\[
\max_{\pi} \left| \sum_j \gamma_j \rho_{\pi(j)} \right| = \sum_j \gamma_j \rho_j \geq \gamma_1 \rho_1 \geq \frac{1}{2} \left( \gamma_1 - \gamma_n \right) = \frac{5}{2};
\]

and if \( \rho_n < 0 \), then, by Lemma 1.1,

\[
\max_{\pi} \left| \sum_j \gamma_j \rho_{\pi(j)} \right| \geq \frac{5}{2} \max_{i,j} |\rho_i - \rho_j| = \frac{5}{2} \left( \rho_1 - \rho_n \right) \geq \frac{5}{2}.
\]

This together with (1.8) completes the proof. \( \square \)

Note that when the \( \gamma_j \) are of the same argument, then \( \sigma > 0 \) implies \( \sigma > 0 \), in which case

\[
\frac{5}{2} > \frac{\sigma \gamma}{2\sigma + \gamma}.
\]

That is, the constant of Theorem 1.4 is indeed an improvement over the \( K \) of Theorem 1.2.

2. MATRIX NORMS AND GENERALIZED NUMERICAL RADII

In this section we review (mainly without proof) some of the results in [5] which lead to applications of our combinatorial inequalities.

We start with the following definitions [7]: let \( C_{n \times n} \) denote the algebra of \( n \times n \) complex matrices. A mapping
$N : \mathbb{C}^{n \times n} \rightarrow \mathbb{R}$
is a seminorm if for all $A, B \in \mathbb{C}^{n \times n}$ and $\alpha \in \mathbb{C}$,

\begin{align*}
N(A) &\geq 0, \\
N(\alpha A) &= |\alpha| N(A), \\
N(A + B) &\leq N(A) + N(B).
\end{align*}

If in addition

\begin{align*}
N(A) > 0, \quad \forall A \neq 0,
\end{align*}

then $N$ is a generalized matrix norm. Finally, if $N$ is also (sub) multiplicative, i.e.,

\begin{align*}
N(AB) \leq N(A)N(B),
\end{align*}

we say that $N$ is a matrix norm.

**EXAMPLES.** (i) If $\cdot \cdot$ is any norm on $\mathbb{C}^n$, then

\begin{align*}
\|A\| = \max\{|Ax| : |x| = 1\}
\end{align*}

is a matrix norm on $\mathbb{C}^{n \times n}$. In particular, we recall the spectral norm

\begin{align*}
\|A\|_2 = \max\{(x^*Ax)^{1/2} : x^*x = 1\}.
\end{align*}

(ii) The numerical radius,

\begin{align*}
r(A) = \max\{|x^*Ax| : x^*x = 1\},
\end{align*}

is a nonmultiplicative generalized matrix norm (e.g., [6, §173, 176], [3]).

In [5] we introduced the following generalization of the numerical radius: Given matrices $A, C \in \mathbb{C}^{n \times n}$, the $C$-numerical radius of $A$ is the nonnegative quantity

\begin{align*}
r_C(A) = \max\{|\text{tr}(CU^*AU)| : U \ n \times n \text{ unitary}\}.
\end{align*}

It is not hard to see that
\[ r(A) = r_C(A) \quad \text{with} \quad C = \text{diag}(1,0,\ldots,0); \]

thus \( R(A) \) is a special case of \( r_C(A) \).

It follows from the definition that for each \( C \), \( r_C \) is a seminorm on \( \mathbb{C}_{n \times n} \). We may then ask whether \( r_C \) is a generalized matrix norm. Since the situation is trivial for \( n = 1 \), we hereafter assume that \( n \geq 2 \).

**THEOREM 2.1 ([5]).** \( r_C \) is a generalized matrix norm on \( \mathbb{C}_{n \times n} \) if and only if \( C \) is a nonscalar matrix and \( \text{tr} \, C \neq 0 \).

Next, we consider multiplicativity, which seems to be a complicated question.

For a given seminorm \( N \) and a constant \( \nu > 0 \), evidently

\[ N_\nu = \nu N \]

is a seminorm, too. Similarly, if \( N \) is a generalized matrix norm, then so is \( N_\nu \). In each case the new norm may or may not be multiplicative. If it is, we call \( \nu \) a multiplicativity factor for \( N \).

It is an interesting fact that seminorms do not have multiplicativity factors, while generalized matrix norms always do. More precisely, we have the following result.

**THEOREM 2.2 ([5]).** (i) A nontrivial seminorm has multiplicativity factors if and only if it is a generalized matrix norm.

(ii) If \( N \) is a generalized matrix norm, then \( \nu \) is a multiplicativity factor if and only if

\[ \nu \geq \nu_N = \max_{A,B \neq 0} \frac{N(AB)}{N(A)N(B)}. \]

Theorems 2.1 and 2.2 guarantee that \( r_C \) has multiplicativity factors if and only if \( C \) is nonscalar and \( \text{tr} \, C \neq 0 \). In practice, however, Theorem 2.2 was of no help to us since we were unable to apply it to C-numerical radii.

An alternative way of obtaining multiplicativity factors is suggested by the following theorem of Gastinel [2] (originally in [1]).
THEOREM 2.3. Let $N$ be a generalized matrix norm, $M$ a matrix norm, and $\eta \geq \xi > 0$ constants such that

$$\xi M(A) \leq N(A) \leq \eta M(A), \quad \forall A \in \mathbb{C}_{n\times n}.$$ 

Then any $\nu \geq \eta/\xi^2$ is a multiplicativity factor for $N$.

Proof. For $\nu \geq \eta/\xi^2$, we have

$$N_{\nu}(AB) \leq \nu N(AB) \leq \nu \eta M(AB) \leq \nu \eta M(A)M(B) \leq \frac{\nu}{\xi^2} N(A)N(B) \leq \nu^2 N(A)N(B) = N_{\nu}(A)N_{\nu}(B),$$

and the proof is complete. $\square$

Since any two generalized matrix norms on $\mathbb{C}_{n\times n}$ are equivalent, constants $\xi \geq \eta > 0$ as required in Theorem 2.3 always exist.

Having Gastinel's theorem and the inequalities of Section 1, we are now ready to obtain multiplicativity factors for $C$-numerical radii with Hermitian $C$.

Combining Lemmas 9 and 10 of [5], we state:

LEMMA 2.3. If $C$ is Hermitian with eigenvalues $\gamma_j$, and if $K$ satisfies (1.1), then

$$\left[\frac{K}{2}\right] \|A\|_2 \leq r_C(A) \leq \left[\sum_j |\gamma_j|\right] \|A\|_2, \quad \forall A \in \mathbb{C}_{n\times n}.$$ 

Using the notation of (1.2), we prove:

THEOREM 2.4. Let $C$ be Hermitian, nonscalar, with $\text{tr} C \neq 0$ and eigenvalues $\gamma_j$. Then any $\nu$ with

$$\nu \geq 4 \sum_j |\gamma_j| \left(\frac{2\sigma + 5}{\sigma^2}\right)^2$$

is a multiplicativity factor for $r_C$; i.e., $\nu r_C = r_{\nu C}$ is a matrix norm.

Proof. Since $C$ is nonscalar, the $\gamma_j$ are not all equal; and since $\text{tr} C \neq 0$, $\sum_\gamma_j \neq 0$. Thus $\sigma^2 > 0$, so inequality (1.1) is satisfied by the positive constant $K$ of Theorem 1.2. By Lemma 2.3, therefore,
\[
\frac{1}{2} \cdot \frac{\sigma_6}{2\sigma_6 + 8} \|A\|_2 \leq r_\sigma(A) \leq \sum \gamma_j \|A\|_2, \quad \forall A \in \mathbb{C}_{n \times n},
\]
and Gastinel's theorem completes the proof. \(\square\)

For Hermitian definite \(C\), we improve Theorem 2.4 as follows.

**THEOREM 2.5.** Let \(C\) be Hermitian nonnegative (nonpositive) definite. If \(C\) is nonscalar with eigenvalues \(\gamma_j\), then any \(\nu\) with \(\nu \geq 16\sigma/5^2\) is a multiplicativity factor for \(r_C\).

**Proof.** Since \(C\) is Hermitian definite, the \(\gamma_j\) are of the same sign. So (1.1) holds with \(K\) of Theorem 1.4, and Lemma 2.3 implies that

\[
\frac{5}{4} \|A\|_2 \leq r_C(A) \leq \sum \gamma_j \|A\|_2 = 5 \|A\|_2, \quad \forall A.
\]

Since \(C\) is nonscalar, the \(\gamma_j\) are not all equal; so \(\delta > 0\), and Theorem 2.3 completes the proof. \(\square\)

The optimal (least) multiplicativity factor for \(r, \nu_r\), is the subject of our last result.

**THEOREM 2.6.** \(\nu_r\) is a matrix norm if and only if \(\nu \geq 4\). That is, \(\nu_r = 4\).

**Proof.** It is well known (e.g., [6, §173]) that

\[
\frac{1}{2} \|A\|_2 \leq r(A) \leq \|A\|_2, \quad \forall A \in \mathbb{C}_{n \times n}.
\]

Thus, by Gastinel's theorem, \(\nu \geq 4\) is a multiplicativity factor for \(r\), and by Theorem 2.2, \(\nu_r \leq 4\).

To show that \(\nu_r \geq 4\), consider the \(n \times n\) matrices

\[
A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2}, \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \oplus 0_{n-2}.
\]

A simple calculation shows that \(r(A) = r(B) = 1/2\) and \(r(AB) = 1\). Hence \(\nu_r = \nu_{AB}\) satisfies

\[
r_r(AB) \leq r_r(A)r_r(B)
\]
if and only if $v \geq 4$, and the theorem follows. □

Note that the results of Theorems 2.4 - 2.6 depend neither on the dimension $n$ nor on the space $V$.

REFERENCES


Combining combinatorial inequalities, matrix norms, and generalized numerical radii.

The main result states that for $1 \leq j \leq n$, are fixed complex scalars with $\sigma = \sum \gamma_j > 0$ and $\delta = \max |\gamma_i - \gamma_j| > 0$, and if $y$ is a normed vector space over the complex field, then...
20. Abstract continued.

\[
\max_{\pi} \left| \sum_j \gamma_j a_{\pi(j)} \right| \geq \frac{\sigma}{2^d} \max_j |a_j| \quad \forall a_1, \ldots, a_n \in \mathbb{C},
\]

\(\pi\) varying over permutations of \(n\) letters. Next, we consider an arbitrary generalized matrix norm \(N\) and discuss methods to obtain multiplicativity factors for \(N\), i.e., constants \(\nu > 0\) such that \(\nu N\) is submultiplicative. Using our combinatorial inequalities, we obtain multiplicativity factors for certain \(C\)-numerical radii which are generalizations of the classical numerical radius of an operator.