ON TOTAL SOJOURN TIME
IN
ACYCLIC JACKSON NETWORKS*

by

Austin J. Lemoine

*This work was supported by the Office of Naval Research under contract No. N00014-78-C-0712 (NR-047-106).

Reproduction in whole or in part is permitted for any purpose of the United States Government.
ABSTRACT

In this paper, we compute the mean of the equilibrium (steady-state) total sojourn time distribution for the important class of infinite capacity acyclic Jackson networks with a single server at each node. In addition, for those acyclic Jackson networks with a 'tree-like' structure, we derive the Laplace transform of the equilibrium total sojourn time distribution and then give a simple recursive procedure for computing the higher moments of the distribution. Such basic results should prove helpful in testing procedures for simulation output analysis of infinite capacity open networks of queues.
0. INTRODUCTION

Queueing network models abound in applications, but despite the immense practical importance of such models the body of available and useful results for networks of queues is far from satisfactory; see Lemoine [9], [10] for a comprehensive review of available equilibrium results and weak convergence results for networks of queues. For studying many queueing network models simulation would appear to be the only practical recourse at the present time. The regenerative method for simulation analysis has been applied to the study of passage time problems in closed networks of queues and finite capacity open networks by Iglehart and Shedler [4], [5], [6]. The report of Lavenberg [8] discusses application of the regenerative method to simulations of closed networks of queues. However, the regenerative method would appear to be inappropriate for the large and important body of infinite capacity open network models; such queuing networks are probably too complicated to return often enough to some "regenerative condition" from which the entire network starts afresh probabilistically. Nevertheless, any candidate procedure for simulation analysis of infinite capacity open networks of queues requires a basic model, and theoretical results for such a model to serve as a testing ground for the procedure. For example, the M/M/1 queue has been invaluable as a testing ground for the development of the various aspects of the regenerative method. For infinite capacity open networks an appropriate test model is the classical Markovian network system of Jackson [7] with a single serve at each node. And, for infinite capacity open networks an important characteristic of system performance is the total
sojourn time (total response time) distribution for typical customers in the network. In this paper, therefore, we compute the mean of the equilibrium (steady-state) total sojourn time distribution for the important class of infinite capacity acyclic Jackson networks with a single server at each node. In addition, for those acyclic Jackson networks with a "tree-like" structure, we derive the Laplace transform of the equilibrium total sojourn time distribution and then give a simple recursive procedure for computing the higher moments of the distribution. Such basic results should prove helpful in testing procedures for simulation output analysis of infinite capacity open networks of queues.
1. THE BASIC MODEL AND STATEMENT OF THE RESULT

The model of interest here is a Markovian network of queues of the type introduced in the classical paper of Jackson [7]. There are \( N \) nodes with node \( i \) having a single-server, a first-come-first-served queue discipline, and a waiting room of unlimited capacity. The external input stream to node \( i \) is Poisson with rate \( \lambda_i \), and these external input streams are assumed to be independent. The service times at node \( i \) are independent and have a common exponential distribution with parameter \( \mu_i \), and are independent of all customer arrivals at node \( i \).

A customer leaving node \( i \) is immediately and independently routed to node \( j \) with probability \( p_{ij} \), and the customer departs the system from node \( i \) with probability \( q_i = 1 - \sum_{j=1}^{N} p_{ij} \).

The state of the network at time \( t \) is taken to be

\[
C(t) = \left( c_1(t), c_2(t), \ldots, c_N(t) \right)
\]

(1)

where \( c_i(t) \) is the number of customers at node \( i \) at time \( t \). Given the independent Poisson external input streams, the exponential service times at the various nodes, and the independent routing scheme, it follows that \( (C(t), t \geq 0) \) is a Markov process with stationary transition probabilities. In this paper we are interested in the total sojourn times of customers in the network when the process \( (C(t), t \geq 0) \) has an equilibrium (or limiting) distribution. In particular, we are interested in the distribution of total sojourn time under the equilibrium Markov
queue-lengths vector process

\[ \{C(t), -\infty < t < +\infty \}. \tag{2} \]

This equilibrium process (when it exists) will henceforth often be denoted by \( C(\cdot) \). And, this equilibrium process exists if (and only if) the "traffic intensity" is less than one at each node in the network. Traffic intensity in this network setting means the following. Let \( \boldsymbol{\mathcal{P}} \) be the \( N \times N \) matrix of the \( p_{ij} \)'s and let \( \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N) \) be the row vector solution of the "traffic equation"

\[ \alpha = \lambda + \alpha \cdot \mathcal{P} \tag{3} \]

where \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_N) \). Since customers eventually leave the system each entry of the matrix \( \mathcal{P}^m \) converges to 0 as \( m \to \infty \), so that the matrix \( \mathbf{I} - \mathcal{P} \) is invertible and (3) has a unique solution given \( \lambda \). In row form (3) is equivalent to

\[ \alpha_i = \lambda_i + \sum_{j=1}^{N} p_{ij} \alpha_j, \quad i = 1,2,\ldots,N. \tag{3a} \]

This is a balance or conservation equation which says that the equilibrium rate of flow through node \( i \), \( \alpha_i \), is the sum of the external input rate, \( \lambda_i \), and the total rate of internal transfers to node \( i \), \( \sum_{j=1}^{N} p_{ji} \alpha_j \). Then, if the "traffic intensity"

\[ \rho_i = \frac{\alpha_i}{\mu_i} < 1 \tag{4} \]
for each $i$, the queue-lengths vector process has a unique equilibrium distribution $\pi$, where for $C = (c_1, c_2, \ldots, c_N)$ a $N$-tuple of non-negative integers

$$\pi(C) = \prod_{i=1}^{N} \psi_i(c_i)$$  \hspace{1cm} (5)

with $\psi_i(c_i) = (1 - \rho_i) c_i^{ci}$. Thus, when (and only when) the condition (4) holds for each node in the network, the random vector $C(t)$, vis-a-vis the equilibrium process $C(\cdot)$, has distribution $\pi$ for each $t$ in $(-\infty, +\infty)$. Another way of saying this is the following. Let $p$ denote the (stationary) transition probability function of the Markov queue-lengths vector process; that is, if $-\infty < s < t < +\infty$ and $C$ and $D$ are possible states then

$$P(C(t) = D | C(s)) = p(D, C, t - s)$$  \hspace{1cm} (6)

on the set $\{C(s) = 0\}$ with probability one. Then $\pi$ is the equilibrium distribution if (and only if)

$$\sum_{D} \pi(D) p(D, C, y) = \pi(C)$$  \hspace{1cm} (7)

for all $y$ in $(0, \infty)$ and all states $C$.

Now, let the random epochs of external customer arrivals in the equilibrium process $C(\cdot)$ be

$$\cdots < t_{-2} < t_{-1} < 0 < t_1 < t_2 \cdots$$  \hspace{1cm} (8)
The points \( \{t_n, n = \pm 1, \pm 2, \ldots \} \) are the superposition of \( N \) independent Poisson processes on \( (-\infty, +\infty) \) with intensities \( \lambda_1, \lambda_2, \ldots, \lambda_N \); and the customer arriving at epoch \( t_n \) enters the network via node \( i \) independently with probability \( \lambda_i/(\lambda_1 + \lambda_2 + \cdots + \lambda_N) \) for all \( n \) and \( i \). Let

\[
t_n + T_n
\]

(9)
de note the random epoch of departure from the network for the customer arriving at epoch \( t_n \), so that \( T_n \) is the total sojourn time in the network for customer \( n \). Since the network is in equilibrium the sojourn times \( \{T_n, n = \pm 1, \pm 2, \ldots \} \) have a common distribution which we denote by \( H \). In this paper we show that if the network is acyclic under \( \mathcal{P} \) then the mean of the distribution \( H \), say \( E[T] \), is given by

\[
E[T] = \left( \sum_{i=1}^{N} \lambda_i \right)^{-1} \sum_{j=1}^{N} \frac{\rho_j}{1 - \rho_j}.
\]

(10a)

Moreover, if the acyclic network has a "tree-like" structure then we show that the Laplace transform, say \( h \), of the equilibrium total sojourn time distribution \( H \) is given by

\[
h(\vartheta) = \tilde{\lambda} [\mathcal{S}(\vartheta) - \mathcal{P}]^{-1} q \]

(10b)

for \( \vartheta > 0 \), where \( \mathcal{S}(\vartheta) \) is a diagonal matrix whose \( i \)th diagonal entry is \( (\mu_i - \alpha_i + \vartheta)/(\mu_i - \alpha_i) \), \( \tilde{\lambda} \) is a row vector whose \( i \)th entry
\[ \lambda_i / (\lambda_1 + \lambda_2 + \ldots + \lambda_N) \], and \( q \) is a column vector whose \( i \)th entry is \( q_i \). We also give a simple recursive procedure for computing all the moments of the distribution with transform given by (10b). The transform result (10b) extends a result of Reich [11] for single-server Markovian queues in tandem.

These results were derived by a heuristic argument in [9]. The argument given here proceeds in stages, each of which comprises a separate section of the paper. The discussion in Sections 2 and 3, and the first part of Section 4 does not require that the network be acyclic. From thereon, however, acyclic structure plays a crucial role.
2. THE PROCESS \( \{C(t), -\infty < t < +\infty\} \)

In this section we observe that the process \( \{C(t), -\infty < t < +\infty\} \)
is a Markov process in equilibrium, with equilibrium distribution \( \pi \) and (stationary) transition probability function \( p \).

Let the process \( \{C(t), -\infty < t < +\infty\} \) be defined on the probability space \((\Omega, \mathcal{F}, P)\). Without loss of generality we take \( C(\cdot) \) to have sample paths which are constant except for isolated jumps, are right continuous, and have left limits, all with probability one. With this setup the process \( C(\cdot) \) has no fixed points of discontinuity with probability one, so that \( P(C(t) = C(t-)) = 1 \) for each \( t \) in \((-\infty, +\infty)\).

Hence, in order to establish that the process \( \{C(t), -\infty < t < +\infty\} \) has the stated properties, it suffices to show that for arbitrary \(-\infty < s < t < +\infty\) we have

\[
P(C(t-) = C; C(u-), u < s) = p(C(s-), t - s, C)
\]

(11)

for all possible states \( C \) with probability one.

Now, given the structure of the sample paths of \( C(\cdot) \) we clearly have

\[
P(C(t-) = C; C(u-), u < s) = P(C(t-) = C; C(u), u < s)
\]

with probability one. Let \( \epsilon_m = 1/2^m \) for \( m = 1, 2, \ldots \). Since \( C(\cdot) \) is a Markov process

\[
P(C(t - \epsilon_m) = C; C(u), u < s - \epsilon_m) = p(C(s - \epsilon_m), C, t - s)
\]
for all $m$, with probability one. By Theorem 9.4.8 in Chung [3]

\[
\lim_{m \to \infty} P(C(t - e_m) \cap \{u < s - e_m\}) = P(C(t) = C(u), u < s) = C\{C(u), u < s\}
\]

with probability one. Moreover, we clearly have $p(C(s - e_m), C, t - s) \longrightarrow p(C(s -), C, t - s)$ as $m \to \infty$, with probability one. The proof of (11) is complete.
3. EXTERNAL ARRIVALS AND EXTERNAL DEPARTURES

As in (8) let \( \{t_n, n=1, 2, \ldots\} \) be the random epochs of external arrivals to the network. Similarly, let

\[ \cdots < d_{-2} < d_{-1} < 0 < d_1 < d_2 < \cdots \]  

be the random epochs of external departures from the network. In this section we observe that the random vectors \( \{C(t_n^-), n=1, 2, \ldots\} \) and \( \{C(d_n), n=1, 2, \ldots\} \) are identically distributed and have \( \pi \) for their common distribution.

Let \( q(C), q(C,D) \) denote the transition rates for the process \( C(*) \). That is, upon entering state \( C \), for example, the process remains there for a random time having an exponential distribution with parameter \( q(C) \), and upon leaving state \( C \) the process goes to some state \( D \neq C \) with probability \( q(C,D)/q(C) \). The equilibrium distribution \( \pi \) satisfies the "balance equation"

\[ \pi(C) q(C) = \sum_{D \neq C} \pi(D) q(D,C) \]  

for all states \( C \). For \( 1 \leq i \leq N \) let \( E_i \) denote the \( N \)-vector with all components zero except for a 1 in component \( i \). If \( C(t) = C \), then the next transition will be either to state \( C + E_i \) (external arrival at node \( i \)), to state \( C - E_i \) (external departure from node \( i \)), or to state \( C - E_i + E_j \) for \( j \neq i \) (transfer from node \( i \) to node \( j \)). Thus for state \( C \)
\[ q(C) = \sum_{i=1}^{N} q(C, C + E_i) + \sum_{i=1}^{N} q(C, C - E_i) \]
\[ + \sum_{i=1}^{N} \sum_{j=1 \atop j \neq i}^{N} q(C, C - E_i + E_j). \]

In addition to satisfying (13), the distribution \( \pi \) also satisfies the "partial balance equation"
\[ \pi(C) \sum_{i=1}^{N} q(C, C + E_i) = \sum_{i=1}^{N} \pi(C + E_i) q(C + E_i, C) \quad (14) \]
for each state \( C \); cf. [19].

Consider the "reversed process" \( \{C(-t), -\infty < t < +\infty\} \). The reversed process is also a Markov process in equilibrium with the same distribution \( \pi \). The transition rates \( q'(C, D) \) for the reversed process are given by
\[ \pi(C) q(C, D) = \pi(D) q'(D, C) \quad (15) \]
and
\[ q(C) = q'(C) \quad (16) \]
for all states \( C \) and \( D \).
If we now apply (15) to each term on the right side of (14), divide both sides of (14) by \( q(C) \), and then invoke (16), we obtain

\[
\pi(C) \sum_{i=1}^{N} \frac{q(C, C+E_i)}{q(C)} = \pi(C) \sum_{i=1}^{N} \frac{q^-(C, C+E_i)}{q^-(C)}
\]

(17)

for each possible state \( C \). In the reversed process the transitions from state \( C \) to state \( C + E_i \), \( 1 \leq i \leq N \), are registered at the epochs \( \{(-d_n^+) +, n = \pm 1, \pm 2, \ldots\} \). Note that the sample paths of the reversed process are left continuous while those of \( C(t) \) are right continuous. Thus, since \( \pi \) is the common distribution of \( C(t) \) and of the reversed process, and since the state space of \( \{C(t_n^-), n = \pm 1, \pm 2, \ldots\} \) and \( \{C((-d_n^+), n = \pm 1, \pm 2, \ldots\} \) coincide with the state space of \( C(t) \) and the reversed process, we conclude from (17) that the random vectors \( \{C(t_n^-), n = \pm 1, \pm 2, \ldots\} \) and \( \{C((-d_n^+), n = \pm 1, \pm 2, \ldots\} \) are identically distributed and have \( \pi \) for their common distribution. Moreover, since \( C(t) \) and the reversed process are equilibrium processes defined over the time interval \( (-\infty, +\infty) \), the vectors \( \{C(d_n), n = \pm 1, \pm 2, \ldots\} \) and \( \{C((-d_n^+), n = \pm 1, \pm 2, \ldots\} \) are identically distributed.
4. EXIT SETS AND NODES

Let $V$ be a non-empty set of nodes, that is, $V$ is a non-empty subset of $\{1, 2, \ldots, N\}$. Let $V^c$ denote the complement of $V$ in $\{0, 1, 2, \ldots, N\}$ where node 0 denotes the network terminus or sink. We say that $V$ is an exit set if $p_{kr} = 0$ for each node $r$ in $V$ and each node $k$ in $V^c$, where $p_{ro} = q_r$ for $r$ in $V$. Equivalently, $V$ is an exit set if upon leaving $V$ there is no path in the network leading back to $V$. Note that $(1, 2, \ldots, N)$ is an exit set.

In the equilibrium process $C(\cdot)$, let $E_{rk}(s,t]$ be the number of customers who depart node $r$ and arrive instantaneously at node $k$ over the time interval $(s,t]$. If $V$ is an exit set then by results of Beutler and Melamed [1] and Walrand and Varaiya [14] the streams $E_{jk}$, $j$ in $V$ and $k$ in $V^c$, are mutually independent Poisson processes with respective intensities $\alpha_j p_{jk}$. In particular, since $(1, 2, \ldots, N)$ is an exit set, the external departure streams $E_{i0}$, $1 \leq i \leq N$, are mutually independent Poisson processes on $(-, +\infty)$ with respective intensities $\alpha_i q_i$.

From here on, suppose that the network is acyclic under $\mathcal{R}$. For node $i$, $1 \leq i \leq N$, let $V_i$ be the set of all nodes $r$ from which $i$ is accessible under $\mathcal{R}$. Observe that node $i$ is not a member of $V_i$ and that

$$\alpha_i = \lambda_i + \sum_{V_i} \alpha_r p_{ri}.$$

Let $V_{(i)}$ consist of node $i$ together with the nodes of $V_i$. Then
both \( V_i \) and \( V(i) \) are exit sets for \( i = 1, 2, \ldots, N \).

Let \( W_i \) denote the complement of \( V_i \) in \( \{1, 2, \ldots, N\} \). Then the streams \( E_{rk} \), \( r \) in \( V_i \) and \( k \) in \( W_i \), are mutually independent Poisson processes with respective intensities \( \alpha_{r}p_{rk} \). Thus, since \( C(\cdot) \) is in equilibrium, the set of nodes \( W_i \) is a Jackson network in equilibrium with Poisson external input intensities

\[
\hat{\lambda}_k = \lambda_k + \sum_{V_i} \alpha_r p_{rk}
\]

for \( k \) in \( W_i \). Also,

\[
\hat{\alpha}_k = \hat{\lambda}_k + \sum_{W_i} \hat{\alpha}_r p_{rk} = \alpha_k
\]

for \( k \) in \( W_i \). In particular, note that \( \hat{\lambda}_i = \alpha_i \), so that if

\[
\cdots < t_{-2i} < t_{-1i} < 0 < t_{1i} < t_{2i} < \cdots
\]  

(18)

are the random epochs of pooled customer arrivals at node \( i \), external arrivals plus internal transfers from other nodes, then the points \( \{t_{n1}, n = \pm 1, \pm 2, \ldots\} \) form a Poisson process on \((-\infty, +\infty)\) with intensity \( \alpha_i \). The equilibrium distribution for the Jackson network \( W_i \), say \( \hat{\pi} \), is given by

- 12 -
\[ \hat{\pi}(\hat{C}) = \prod_{W_i} \psi_k(c_k) \]

where \( \hat{C} = (c_k, k \in W_i) \) is a vector of non-negative integers.

Likewise, if \( W_i^* \) is the complement of \( V(i) \) in \( \{1,2,\ldots,N\} \), then the streams \( E_{rj}, r \in V(i) \) and \( j \) in \( W_i^* \), are mutually independent Poisson processes with respective intensities \( \alpha_r \rho_{rj} \).

The set of nodes \( W_i^* \) is thus a Jackson network in equilibrium with Poisson external input intensities

\[ \lambda_j^* = \lambda_i + \sum_{V(i)} \alpha_r \rho_{rj} \]

for \( j \) in \( W_i^* \), and

\[ \alpha_j^* = \lambda_j^* + \sum_{W_i^*} \alpha_r \rho_{rj} = \alpha_j \]

for \( j \) in \( W_i^* \). The equilibrium distribution for the Jackson network \( W_i^* \), say \( \pi^* \), is given by

\[ \pi^*(\hat{C}^*) = \prod_{W_i^*} \psi_j(c_j) \]

where \( \hat{C}^* = (c_j, j \in W_i^* \) \) is a vector of non-negative integers.

For \(-\infty < t < +\infty\) let

\[ \hat{C}(t) = (c_k(t), k \in W_i) \]
and

\[ C^*(t) = (c_j(t), j \text{ in } W_i^*) \]

Note that \( \hat{C}(t) = (c_i(t), C^*(t)) \) and that if \( \hat{C} = (c, C^*) \) then

\[ \hat{\pi}(C) = \psi_i(c)\pi^*(C^*) \]

Thus, if \( \{\hat{t}_{ni}, n = \pm 1, \pm 2, \ldots\} \) are the points in the Poisson process of external arrivals to the Jackson network \( W_i \), then by Section 3 we have

\[ P(\hat{C}(\hat{t}_{ni}) = (c, C^*)) = \psi_i(c)\pi^*(C^*) \quad (19) \]

for all \( n, c \) and \( C^* \). The customer arriving at epoch \( \hat{t}_{ni} \) enters \( W_i \) via node \( i \) independently with probability \( \frac{\alpha_i}{\sum_k \alpha_k} \) for all \( n \).

Thus, the points \( \{\hat{t}_{ni}, n = \pm 1, \pm 2, \ldots\} \) are independently selected from the points \( \{\hat{t}_{ni}, n = \pm 1, \pm 2, \ldots\} \), and so from (19) we conclude that

\[ P(c_i(\hat{t}_{ni}) = c, C^*(\hat{t}_{ni}) = C^*) = \psi_i(c)\pi^*(C^*) \quad (20) \]

for all \( n, i, c \) and \( C^* \).

For the customer arriving at node \( i \) at epoch \( t_{ni} \), let \( S_{ni} \) be the customer's total sojourn time at node \( i \). Since service times at node \( i \) have an exponential distribution, the distribution of \( S_{ni} \) is completely determined by \( c_i(t_{ni}) = c_i(t_{ni}^*) + 1 \). Thus, it follows that the sojourn times \( \{S_{ni}, n = \pm 1, \pm 2, \ldots\} \) have a common
exponential distribution with parameter $\nu_i = \mu_i - \alpha_i$ for $i = 1, 2, \ldots, N$. Moreover, by virtue of (20),

$$
P\{S_{ni} \leq x, c_i(t_{ni}) = c + 1, C^*(t_{ni}) = C^*\} =
$$

$$
\int_0^x P\{S_{ni} \in dy | c_i(t_{ni}) = c + 1, C^*(t_{ni}) = C^*\} \psi_i(c) \pi^*(c) =
$$

$$
\int_0^x P\{S_{ni} \in dy | c_i(t_{ni}) = c + 1\} \psi_i(c) \pi^*(c) =
$$

$$
P\{S_{ni} \leq x, c_i(t_{ni}) = c + 1\} \pi^*(C^*) ,
$$

where

$$
P\{S_{ni} \leq x, c_i(t_{ni}) = c + 1\} = \psi_i(c) \int_0^x \frac{(\mu_i y)^c}{c!} e^{-\mu_i y} dy .
$$

Hence, summing on $c$ we have that

$$
P\{S_{ni} \leq x, C^*(t_{ni}) = C^*\} = P\{S_{ni} \leq x\} \pi^*(C^*)
$$

(21)

for all $n \, , \, i \, , \, C^*$ and $x$.

Next, let $E_i(s,t]$ be the total number of departures (external plus internal network transfers) from node $i$ over the time interval $(s,t]$. Since $V(i)$ is an exit set we know that the stream $E_i$ is a Poisson process with rate $\alpha_i$. We now observe that, conditioned on
$S_{ni}$, the total flow of customers through node $i$ over the random time interval $(t_{ni}, t_{ni} + S_{ni})$ is also a Poisson process with rate $\alpha_i$. For this it suffices to show that for any $x > 0$ and any non-negative integer $\kappa$

$$P\{E_i(t_{ni}, t_{ni} + y) = \kappa, S_{ni} c \, dx\} =$$

$$\frac{\alpha_i y (\alpha_i y)^\kappa}{\kappa!} e^{-\alpha_i y}, \quad 0 < y < x.$$  

Now, the left side of (22) equals

$$\sum_{c > \kappa} P\{E_i(t_{ni}, t_{ni} + y) = \kappa, S_{ni} c \, dx\} =$$

$$\sum_{c > \kappa} \frac{\mu_i y e^{-\mu_i y}}{\kappa!} \cdot \frac{\mu_i (\mu_i [x-y])^{c-\kappa-1} e^{-\mu_i (x-y)}}{(c - \kappa - 1)!} \cdot \psi(c-1)$$

and this last expression equals the right side of (22).

---

1This same phenomenon has been observed by R. L. Disney and co-workers.
5. THE PROCESS \( \{C^*(t_{ni} + S_{ni} + t), t \geq 0\} \)

In this section we observe that the process \( \{C^*(t_{ni} + S_{ni} + t), t \geq 0\} \) evolves as a Markov process with the same (stationary) transition probability function, say \( p^* \), as the process \( \{C^*(s), -\infty < s < \infty\} \), and with \( C^*(t_{ni} + S_{ni}) \) independent of \( C_i(t_{ni} + S_{ni}) \) and of \( S_{ni} \).

For the customer arriving at node \( i \) at epoch \( t_{ni} \), let \( \delta_{ni} \) be the node visited by this customer upon departing node \( i \). Either \( \delta_{ni} = 0 \), in which case the customer exits the system from node \( i \), or \( \delta_{ni} = j \) for some node \( j \) in \( W_i^* \). For \( j \) in \( W_i^* \), let \( E_j^* = (e_k, k \in W_i^*) \) with \( e_j = 1 \) and \( e_k = 0 \) for \( k \neq j \).

Now let \( Z^*(s) = C^*(s) \). We have

\[
P(c_i(t_{ni} + S_{ni}) = c, Z^*(t_{ni} + S_{ni}) = C^*, S_{ni} \leq x) = \\
\sum_{D^*} \int_0^x P(c_i(t_{ni} + y) = c, Z^*(t_{ni} + y) = C^* | Z^*(t_{ni}) = D^*, S_{ni} \in dy) \\
P(Z^*(t_{ni}) = D^*, S_{ni} \in dy)
\]

Now (22) implies that, conditioned on \( S_{ni} \), node \( i \) contributes to the Poisson external input stream of the Jackson network \( W_i^* \) at rate \( \alpha_i \sum_{W_i^*} p_{ij} \) over the random time interval \( (t_{ni}, t_{ni} + S_{ni}) \). Moreover, by virtue of (21) the sojourn time \( S_{ni} \) and the vector \( Z^*(t_{ni}) \) are independent. And, the total stream of arrivals to node \( i \) is Poisson
with rate $\alpha_i$. Hence, the integral above is equal to

$$
\int_0^X e^{-\alpha_i y} \left( \frac{\alpha_i y}{c!} \right)^i \pi^*(D^*, C^*, y) \pi^*(D^*) \nu_i e^{-\nu_i y} dy.
$$

Thus,

$$
P(c_i(t_{ni} + S_{ni}) = c, Z^*(t_{ni} + S_{ni}) = C^*, S_{ni} \leq x) =
$$

$$
\int_0^X e^{-\alpha_i y} \left( \frac{\alpha_i y}{c!} \right)^i \left[ \sum_{D^*} \pi^*(D^*) \pi^*(D^*, C^*, y) \right] \nu_i e^{-\nu_i y} dy =
$$

$$
\psi_i(c) \pi^*(C^*) \int_0^X \frac{\left( \mu_i y \right)^c}{c!} e^{-\mu_i y} dy,
$$

where we have applied (7) to the transition function $p^*$ and the equilibrium distribution $\pi^*$. Letting $x \to +\infty$ in (23) we see that $c_i(t_{ni} + S_{ni})$ and $Z^*(t_{ni} + S_{ni})$ are independent and have joint distribution $\psi_i(c) \pi^*(C^*)$. Summing on $c$ in (22) we find that $Z^*(t_{ni} + S_{ni})$ and $S_{ni}$ are independent.

We consider next the vector $C^*(t_{ni} + S_{ni})$. Now

$$
P(S_{ni} \leq x, C^*(t_{ni} + S_{ni}) = C^*) = P(S_{ni} \leq x, Z^*(t_{ni} + S_{ni}) = C^*, \delta_{ni} = 0)
$$

$$
+ \sum_{\mathcal{W}_i} P(S_{ni} \leq x, Z^*(t_{ni} + S_{ni}) = C^* - \delta_j^*, \delta_{ni} = j) =
$$

$$
P(S_{ni} \leq x) \cdot \gamma_1(C^*)
$$

(24)
where

$$
\gamma_i(C^*) = q_i \pi^*(C^*) + \sum_{W_i^*} p_{ij} \pi^*(C^* - E_j^*) \quad .
$$

(25)

Now

$$
\sum_{C^*} \pi^*(C^* - E_j^*) = \sum_{C^* = D^* + E_j^*} \pi^*(D^*) = 1
$$

for any $i$ in $W_i^*$. Moreover,

$$
q_i + \sum_{W_i^*} p_{ij} = 1 \quad .
$$

Hence, $\gamma_i$ is a probability distribution on the states of $\{C^*(s), -\infty < s < +\infty\}$, and we conclude from (25) that both $S_{ni}$ and $C_i(t_{ni} + S_{ni})$ are independent of $C^*(t_{ni} + S_{ni})$ and that $\gamma_i$ is the distribution of $C^*(t_{ni} + S_{ni})$.

We now consider the process $\{C^*(t_{ni} + S_{ni} + t), t \geq 0\}$. Let $\hat{p}$ denote the (stationary) transition probability function of the process $\{\hat{C}(s), -\infty < s < +\infty\}$. The epoch of departure $t_{ni} + S_{ni}$ from node $i$ is a stopping time for process $\hat{C}(\cdot)$. And, the pure jump process $\hat{C}(\cdot)$ enjoys the strong Markov property; cf. Breiman [2, pp. 323,328]. Then, for $u, x > 0$
\[ P(C(t_ni + S_{ni}) = (d,D^*), C(t_ni + S_{ni} + u) = (c,C^*)) = \]

\[ P(C(t_ni + S_{ni}) = (d,D^*) \cdot \hat{p}((d,D^*),(c,C^*),u) = \]

\[ \gamma_i(D^*) \left( \psi_i(d) \pi^*((D^*)) \right) p((d,D^*),(c,C^*),u) \]  

(26)

\[ \gamma_i(D^*) \pi^*((D^*)) \cdot P(C(O) = (d,D^*), C(u) = (c,C^*)) , \]

where we have invoked (23) and (24), followed by the (ordinary) Markov property of the process \( \hat{C}(\cdot) \). Summing on \( d \) and on \( c \) in (26), we find that

\[ P(C^*(t_{ni} + S_{ni}) = D^*, C^*(t_{ni} + S_{ni} + u) = C^*) = \]

\[ \gamma_i(D^*) \pi^*((D^*)) \cdot p(C^*(0) = D^*, C^*(u) = C^*) \]

(27)

using the (ordinary) Markov property of the process \( \{C^*(s), -\infty < s < +\infty\} \).

Now suppose \( u,v,x > 0 \). By the conditional independence of the past and the future given the state of the process \( \hat{C}(\cdot) \) at the stopping time \( t_{ni} + S_{ni} + u \), cf. [3, p.316], we have
\[ P(\hat{C}(t_{ni} + S_{ni}) = (d,D^*), \hat{C}(t_{ni} + S_{ni} + u) = (c,C^*), \hat{C}(t_{ni} + S_{ni} + u + b) = (b,B^*) ) = \]

\[ P(\hat{C}(t_{ni} + S_{ni}) = (d,D^*), \hat{C}(t_{ni} + S_{ni} + u) = (c,C^*), \hat{C}(t_{ni} + S_{ni} + u + b) = (b,B^*) ) = \]

\[ \frac{\gamma_i(D^*)}{\pi(D^*)} P(\hat{C}(0) = (d,D^*), \hat{C}(u) = (c,C^*), \hat{C}(u + v) = (b,B^*) ) = \]

\[ \frac{\gamma_i(D^*)}{\pi(D^*)} P(\hat{C}(0) = (d,D^*), \hat{C}(u) = (c,C^*), \hat{C}(u + v) = (b,B^*) ) = \]

also making use of (26) and the Markov property of the process \( \hat{C}(\cdot) \).

Summing on \( d \) and on \( c \) and on \( b \) in (28), we find that

\[ P(C(t_{ni} + S_{ni}) = D^*, C(t_{ni} + S_{ni} + u) = C^*, C(t_{ni} + S_{ni} + u + v) = B^*) ) = \]

\[ \frac{\gamma_i(D^*)}{\pi(D^*)} P(C(0) = D^*, C(u) = C^*, C(u + v) = B^*) = \]

\[ \frac{\gamma_i(D^*)}{\pi(D^*)} p(D^*,C^*,u) p(C^*,B^*,v) . \]

Proceeding by induction, we find that, for all \( 0 = u_0 < u_1 < u_2 < \ldots < u_m < +\infty \) and \( x > 0 \) and all states \( C_0^*, C_1^*, C_2^*, \ldots C_m^* \) of the process \( \{ C(s), -\infty < s < +\infty \} \), we have
\[ p \left\{ \bigcap_{r=0}^{m} \left[ C^* \left( t_{ni} + s_{ni} + u_r \right) = c_r^* \right] \right\} = \]  

\[ \gamma_i(c_0^*) \cdot \prod_{r=1}^{m} p^* \left( c_{r-1}^*, c_r^*, u_r - u_{r-1} \right). \]

And, as a consequence of (30), we conclude that \( \left[ C^* \left( t_{ni} + s_{ni} + t \right), t \geq 0 \right] \) evolves as a Markov process with initial distribution \( \gamma_i^* \) and stationary transition probability function \( p^* \).
6. SOJOURN TIMES \( \{T_{ni}, n = 1, 2, \ldots \} \)

In a manner similar to that of (9) of Section 1, let

\[ t_{ni} + T_{ni} \]

be the epoch of departure from the network for the customer arriving at node \( i \) at epoch \( t_{ni} \) (external arrival to node \( i \) or internal transfer to node \( i \)). Since the set of nodes \( W_i \) (as defined in Section 4) is a Jackson network in equilibrium, the sojourn times \( \{T_{ni}, n = 1, 2, \ldots \} \) are identically distributed; let \( g_i \) denote the Laplace transform of their common distribution. In this section we use the strong Markov property, in conjunction with the developments in Section 5, to derive an expression for the transform \( g_i \) which leads to the results (10a) and (10b).

Let \( Y_{ni} = T_{ni} - S_{ni} \) and \( \theta_1, \theta_2 \geq 0 \). By the conditional independence of the past and the future of the process \( \{\hat{C}(t), -\infty < t < +\infty\} \) at the stopping time \( t_{ni} + S_{ni} \), we have that

\[
\begin{align*}
E & \left\{ e^{-\theta_1 S_{ni}} e^{-\theta_2 Y_{ni}} \right| \left. c_i(t_{ni} + S_{ni}) = c_i \right\} = \\
& \bigg\{ E \left[ e^{-\theta_1 S_{ni}} \hat{C}(t_{ni} + S_{ni}) \right| \left. c_i(t_{ni} + S_{ni}) = c_i \right] \bigg\} \cdot \bigg\{ E \left[ e^{-\theta_2 Y_{ni}} \hat{C}(t_{ni} + S_{ni}) \right| \left. c_i(t_{ni} + S_{ni}) = c_i \right] \bigg\} .
\end{align*}
\]  

(31)

Since \( c_i(t_{ni} + S_{ni}) \) and \( \hat{C}(t_{ni} + S_{ni}) \) are independent,
\[ E \left[ e^{-\theta_1 S_{n_1}} | c(t_{n_1} + S_{n_1}) \right] = E \left[ e^{-\theta_1 S_{n_1}} | c_1(t_{n_1} + S_{n_1}) \right] \]

with probability one; and a version of the conditional expectation on the right side immediately above is clearly

\[ \left( \frac{\mu_i}{\mu_i + \theta_1} \right)^{c_1(t_{n_1} + S_{n_1}) + 1} \]

Hence, the right side of (31) is equal to

\[ \left( \frac{\mu_i}{\mu_i + \theta_1} \right)^{c+1} \left\{ E \left[ e^{-\theta_2 Y_{n_1}} \right] \right\} \left\{ E \left[ e^{-\theta_2 Y_{n_1}} c_i(t_{n_1} + S_{n_1}) = c_i(t_{n_1} + S_{n_1}) \right] \right\} \]

The expectation immediately above is

\[ E \left\{ E \left[ e^{-\theta_2 Y_{n_1}} c_i(t_{n_1} + S_{n_1}) \right] \cdot \mathbbm{1}_{c_i(t_{n_1} + S_{n_1}) = c_i} \right\} \]

which in turn equals

\[ q_i \psi_i(c) + \sum_{W_i^*} E \left\{ E \left[ e^{-\theta_2 Y_{n_1}} c_i(t_{n_1} + S_{n_1}) \right] \cdot \mathbbm{1}_{c_i(t_{n_1} + S_{n_1}) = c_i} \right\} \]

If the customer is routed to node \( j \) in \( W_i^* \) then the customer enters the Jackson network \( W_i^* \) as an external arrival in the sense that

- 24 -
$Z(t_{ni} + S_{ni})$ has distribution $\pi^*$ and $Z^*(t_{ni} + S_{ni})$, $\delta_{ni}$ and $c_i(t_{ni} + S_{ni})$ are all independent of one another. The process 
$(C^*(t_{ni} + S_{ni} + t), t \geq 0)$ evolves as a Markov process with stationary transition probability function $p^*$, and upon entering the set of nodes $W_i^*$ a customer remains within $W_i^*$ until exiting the system. Moreover, the distribution of $c_i(t_{nj})$ is $\psi_i$ for all $n$ and $j$ in $W_i^*$, by virtue of Section 3 applied to the process $\hat{C}(\cdot)$. We conclude that 

\[
E\left\{e^{-\theta_1 S_{ni}} e^{-\theta_2 Y_{ni}} \mid c_i(t_{ni} + S_{ni}) = c, \delta_{ni} = j_i \right\} = \sum_{c} \psi_i(c) \left(\frac{\mu_j}{\mu_j + \theta_1}\right)^{c+1} \left[ q_i \psi_j(c) + \sum_{W_i^*} p_{ij} E\left\{e^{-\theta_2 T_{nj}} \mid c_i(t_{nj}) = c_i \right\} \right] \]

for all $j$ in $W_i^*$. Thus,

\[
E\left\{e^{-\theta_1 S_{ni}} e^{-\theta_2 Y_{ni}} \mid c_i(t_{ni} + S_{ni}) = c_i \right\} = \left(\frac{\mu_j}{\mu_j + \theta_1}\right)^{c+1} \left[ q_i \psi_j(c) + \sum_{W_i^*} p_{ij} E\left\{e^{-\theta_2 T_{nj}} \mid c_i(t_{nj}) = c_i \right\} \right] \]

Summing on $c$ we find that
for all \( n = \pm 1, \pm 2, \ldots \) and all \( i = 1, 2, \ldots, N \). Putting \( \theta_1 = 0 \) and \( \theta_2 = \theta \) in (32) we find that

\[
E\left\{e^{-\theta_1 S_i} e^{-\theta_2 Y_i}\right\} = \frac{v_i}{v_i + \theta_1} + \sum_{j=1}^{N} p_{ij} \sum_{c=0}^{\infty} \left( \frac{\mu_i}{v_i + \theta_1} \right)^{c+1} E\left\{e^{-\theta_2 T_{nj} \mid c_i(t_{nj}) = c}\right\}.
\]

If the acyclic network also has a "tree-like" structure, then \( T_{ni} \) and \( c_i(t_{nj}) \) are independent for \( j \) in \( W_i^* \), whence

\[
E\left\{e^{-\theta T_{nj} \mid c_i(t_{nj}) = c}\right\} = \psi(c) g_j(\theta).
\]

Thus, if the network has a "tree-like" structure, then on putting \( \theta = \theta_1 = \theta_2 \) in (32) we obtain

\[
g_i(\theta) = \frac{v_i}{v_i + \theta} \left[ q_i + \sum_{j=1}^{N} p_{ij} g_j(\theta) \right].
\]

for \( i = 1, 2, \ldots, N \).
7. THE SOJOURN TIMES \( \{T_n, n = 1, 2, \ldots \} \)

For the external network arrival at epoch \( t_n \) let \( \delta_n \) be the node through which the customer enters the network. The sojourn times \( \{T_n, n = 1, 2, \ldots \} \) are identically distributed, and if \( h \) is the Laplace transform of their common distribution then

\[
h(\theta) = \sum_{i=1}^{N} E \left\{ e^{-\theta T_n} \mathbb{1}_{\{\delta_n = i\}} \right\}.
\]

If \( \bar{\lambda}_i = \lambda_i/(\lambda_1 + \lambda_2 + \ldots + \lambda_N) \) then \( P(\delta_n = i) = \bar{\lambda}_i \). By virtue of Section 6 we must have

\[
E \left\{ e^{-\theta T_n} \mathbb{1}_{\{\delta_n = i\}} \right\} = \bar{\lambda}_i g_i(\theta),
\]

and so

\[
h(\theta) = \sum_{i=1}^{N} \bar{\lambda}_i g_i(\theta). \tag{35}
\]

When each transform \( g_i \) satisfies (34), it is a straightforward matter to solve for \( h(\theta) \). Begin by rewriting (34) as

\[
[(v_i + \theta) / v_i] g_i(\theta) = q_i + \sum_{j=1}^{N} p_{ij} g_j(\theta)
\]

for \( i = 1, 2, \ldots, N \). If \( g(\theta) \) is a column vector whose \( i \)th entry
is \( g_i(\theta) \) and if \( \mathcal{A}(\theta) \) and \( q \) are as defined in Section 1, then (34) is equivalent to

\[
(\mathcal{A}(\theta) - \mathcal{R}) g(\theta) = q .
\]

(36)

The matrix \( \mathcal{A}(\theta) \) is clearly invertible for any \( \theta \geq 0 \) and so we can write

\[
\mathcal{A}(\theta) - \mathcal{R} = \left( I - \mathcal{R} [\mathcal{A}(\theta)]^{-1} \right) \mathcal{A}(\theta) .
\]

(37)

Each diagonal element of the diagonal matrix \( [\mathcal{A}(\theta)]^{-1} \) is in \((0,1)\), and so it follows from (37) that \((\mathcal{A}(\theta) - \mathcal{R})\) is invertible for any \( \theta \geq 0 \). With \( \lambda \) a row vector whose ith entry is \( \lambda_i \), we now see that the right side of (35) is equal to

\[
\lambda [\mathcal{A}(\theta) - \mathcal{R}]^{-1} q .
\]

The proof of (10b) is now complete.

Applying Cramer's rule to (36), we see that when (34) holds each transform \( g_i \) is a rational function whose denominator is a polynomial in \( \theta \) of degree at most \( N \). Thus, \( h \) is the transform of a mixture of exponential distributions where (34) holds.
8. THE MOMENTS OF TOTAL SOJOURN TIME

Let \( T_1, T_2, \ldots, T_N, T \) be random variables whose distributions have Laplace transforms \( g_1, g_2, \ldots, g_N, h \), respectively. Any customer never visits a node more than once, and under equilibrium conditions each node is a \( M/M/1 \) queue in equilibrium. Thus, each of the variables \( T_1, T_2, \ldots, T_N, T \) has finite moments of all orders. It follows from (33) that the residual sojourn time \( Y_{ni} \) has mean

\[
\sum_{j=1}^{N} p_{ij} E(T_j) \text{ for all } n \text{ and } i , \text{ and so}
\]

\[
E(T_i) = 1/\nu_i + \sum_{j=1}^{N} p_{ij} E(T_j) \quad (38)
\]

for \( i = 1, 2, \ldots, N \). Then

\[
\sum_{i=1}^{N} \alpha_i E(T_i) = \sum_{i=1}^{N} \frac{\alpha_i}{\nu_i} + \sum_{i=1}^{N} \alpha_i \sum_{j=1}^{N} p_{ij} E(T_j)
\]

\[
= \sum_{i=1}^{N} \frac{\alpha_i}{\nu_i} + \sum_{i=1}^{N} E(T_j) \sum_{i=1}^{N} \alpha_i p_{ij}
\]

\[
= \sum_{i=1}^{N} \frac{\alpha_i}{\nu_i} + \sum_{i=1}^{N} (\alpha_j - \lambda_j) E(T_j) ,
\]

with the last equality holding by virtue of (3a). Hence,
\[
\sum_{i=1}^{N} \lambda_i \ E(T_i) = \sum_{i=1}^{N} \alpha_i / \nu_i .
\] (39)

Noting that \( \nu_i = \mu_i (1 - \rho_i) \), it now follows from (35) and (39) that

\[
E(T_i) = \left( \sum_{i=1}^{N} \lambda_i \right)^{-1} \sum_{i=1}^{N} \frac{\rho_i}{1 - \rho_i} .
\]

Moreover, if \( \nu \) is a column vector whose ith entry is \( 1/\nu_i \), then it follows from (38) that \( E(T_i) \) is the ith entry of the column vector \([I - \nu^T]^{-1} \nu \) for \( i = 1, 2, \ldots, N \).

When each transform \( g_i \) satisfies (34), it is possible to compute the higher moments of total sojourn time by a straightforward recursive procedure. For example, taking the second derivative with respect to \( \theta \) of both sides of (34) and then setting \( \theta = 0 \) gives

\[
E(T_i^2) = \sum_{j=1}^{N} p_{ij} E(T_j^2) + 2 \left( \sum_{j=1}^{N} p_{ij} E(T_j) \right) / \nu_i + 2 / \nu_i^2
\]

for \( i = 1, 2, \ldots, N \), which, in view of (38), is equivalent to

\[
E(T_i^2) = 2E(T_i) / \nu_i + \sum_{j=1}^{N} p_{ij} E(T_j^2)
\] (41)
for \( i = 1, 2, \ldots N \). The same approach used in deriving (39) yields

\[
\sum_{i=1}^{N} \lambda_i E(T_i^2) = 2 \sum_{i=1}^{N} \alpha_i E(T_i)/\nu_i.
\]  

(42)

Thus, we have

\[
E(T^2) = 2 \left( \sum_{i=1}^{N} \lambda_i \right)^{-1} \sum_{i=1}^{N} \frac{\rho_i}{1-\rho_i} E(T_i)
\]  

(43)

when (34) holds. Also, it follows from (41) that if \( \alpha \) is a column vector whose ith entry is \( 2E(T_i)/\nu_i \), then \( E(T_i^2) \) is the ith entry of the column vector \( [I - \Phi]^{-1} \alpha \) for \( i = 1, 2, \ldots, N \). This method of computing \( E(T_i), E(T_i^2) \), for \( i = 1, 2, \ldots, N \), and \( E(T), E(T^2) \) can clearly be continued to obtain further higher moments when (34) holds.

Methods for computing mean sojourn times in some other queueing network models are given in [12] and [13].
REFERENCES


**Title:** On Total Sojourn Time in Acyclic Jackson Networks.

**Author:** Austin J. Lemoine

**Performing Organization Name and Address:**
CONTROL ANALYSIS CORPORATION
800 Welch Road
Palo Alto, CA 94304

**Controlling Office Name and Address:**
Office of Naval Research
Operations Research Program, Code 434
Arlington, Virginia 22217

**Report Date:** January 1979

**Pages:** 38

**Distribution Statement:** Distribution of this document is unlimited.

**Key Words:** Single-Server Queues, Acyclic Jackson Networks, Sojourn Time Distribution, Infinite Capacity Open Network Models

**Abstract:** (Abstract appears on page 1)

**Report Number:** 712-1