A Finite-Element Model For Plane-Strain Plasticity

Philip G. Hodge, Jr. Professor of Mechanics

Hendrik M. Van Rij Research Assistant

Department of Aerospace Engineering and Mechanics
University of Minnesota
Minneapolis, Minnesota 55455

Prepared for

OFFICE OF NAVAL RESEARCH
Arlington, VA 22217

OFFICE OF NAVAL RESEARCH
Chicago Branch Office
536 South Clark St.
Chicago, IL 60605
Plasticity, perfectly-plastic, finite-elements, plane strain, Prandtl punch, slip, velocity discontinuities, combined-finite-element-model.

A finite-element model is proposed which allows for both straining within each element and slip between two elements. Basic equations are derived and are shown to almost completely uncouple into two constituent components: the conventional finite-element equations for continuous displacement fields and the "slip" equations which were recently derived for a model based on slipping of rigid triangles. The model is applied to the Prandtl punch problem and is shown to combine the best features of its two constituents.
Two models, the canonical and the intermediate, are considered in this paper. These models are based on a combination of linear and non-linear dynamics.

In the canonical model, the dynamics are described by a set of differential equations. The motion of the system is determined by the forces and torques acting on it. However, in the intermediate model, a portion of the motion is treated as a constraint, while the other part is described by the canonical equations.

The paper introduces a new approach to modeling the dynamics of complex systems, which combines the advantages of both models. This approach allows for a more accurate and efficient simulation of the system's behavior.

The results obtained from this model are compared with experimental data, and the agreement is found to be satisfactory. The model can be applied to a wide range of applications, including robotics, aerospace engineering, and biomechanics.

In conclusion, the proposed approach provides a powerful tool for modeling the dynamics of complex systems. The model's versatility and accuracy make it a valuable addition to the field of dynamical systems.
The statics of each model can be obtained from Figs. 2 and 3 and the Principle of Virtual Work. If there are no body forces, then it is evident that the resulting static equations for each interior node will be homogeneous and linear, and that the combined model is simply the sum of the other two.

It remains then, to consider the constitutive equations and the boundary conditions, and we will do that in Secs. 2 and 3, respectively. Section 2 will also list generic kinematic and static equations. Then in Sec. 4 we will examine in detail a specific boundary-value problem in order to clearly indicate the character of the proposed model, with particular reference to the close relation between it and the classical and slip models for the same problem. Section 5 will apply the model to an approximation to the Prandtl punch problem [3, 4]. The paper will conclude with a discussion of the merits of the model.

2. Basic equations. We begin by reviewing the well-known equations for the classical model, follow with a summary of the slip-model equations from [1], and conclude by demonstrating that the slip-model constitutive equations are the only ones which must be modified before these equations can be applied to the combined model.

For the classical model the generalized displacements are the dimensionless nodal displacements

\[ u_k = \frac{U_k}{I} \quad v_k = \frac{V_k}{I} \quad (1) \]

where \( I \) is the length of the triangle hypotenuse. These displacements determine a unique continuous piecewise-linear displacement field which leads to piecewise-constant strains.

Taking these as generalized strains \( \epsilon_x^a, \epsilon_y^a, \gamma_{xy}^a \), we obtain for triangle ADB (triangle 1) in Fig. 2a,

\[ \begin{align*}
\epsilon_x &= u_B - u_A \quad \epsilon_y = v_B - v_A \quad \gamma_{xy} = u_B - 2u_A + v_B - v_A \\
&= (1/k)(\sigma_x^a, \sigma_y^a, \tau_{xy}^a) \quad (2)
\end{align*} \]

with similar expressions for the other elements.

Generalized stresses will be defined by

\[ \sigma_x^a = \frac{4}{k^2} \int_{\Delta^a} \sigma_x(x, y) \, dA \quad (3) \]

etc. where \( k \) is the yield stress in shear. For any reasonable homogeneous material, constant strains will produce constant stresses so that (3) reduces to

\[ (\sigma_x^a, \sigma_y^a, \tau_{xy}^a) = (1/k)(\sigma_x^a, \sigma_y^a, \tau_{xy}^a) \]

If point D is the only node with a non-zero displacement in Fig. 2a, then the internal work done in triangle ABD is

\[ W_{int} = \int_{\Delta} (\sigma_x^a \epsilon_x + \sigma_y^a \epsilon_y + \tau_{xy}^a \gamma_{xy}) \, dA = (k/t^2/2)(\epsilon_x^a - \epsilon_y^a) (4) \]

The total internal work done by a motion of point D is

\[ W_{int} = (kt^2/2) \int \left[u_D - (\sigma_x^a \epsilon_x + \sigma_y^a \epsilon_y + \tau_{xy}^a \gamma_{xy}) + (\sigma_y^a \epsilon_y + \sigma_x^a \epsilon_x + \tau_{xy}^a \gamma_{xy})\right] \]

If there is no force applied to node D, the external work must vanish for all choices of \( u_D \) and \( v_D \), hence we obtain the linear homogeneous static equations

\[ (\sigma_x^a \epsilon_x + \sigma_y^a \epsilon_y + \tau_{xy}^a \gamma_{xy}) = 0 \quad (7a) \]

\[ (\tau_{xy}^a \gamma_{xy}) = 0 \quad (7b) \]

associated with a generic small node. Similarly, at a large node, Fig. 2b, we are led to
(15)
\[ 0 = \theta H_1 - 2H_1 + H_1 + L_1 + 2L_1 \]

(16)
\[ 0 = \theta z - 2V z + V z + \theta H_1 + 2\theta G_1 + \theta D_1 \]

The present section is devoted to the presentation of
the determinant's equation. In this section, we consider
the equation's determinant in detail. The determinant's
determinants are derived for each point of the intersection
and the determinant of the two equations along".

(17)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(18)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(19)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(20)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(21)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(22)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(23)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(24)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(25)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(26)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(27)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(28)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(29)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(30)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(31)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(32)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(33)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(34)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(35)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(36)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(37)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(38)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(39)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(40)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(41)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(42)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(43)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(44)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(45)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(46)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(47)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(48)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(49)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(50)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(51)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(52)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(53)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(54)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(55)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(56)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(57)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(58)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(59)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic

(60)
\[ \frac{\theta D_1}{\theta z} = \frac{\theta D_1}{\theta z} \]

The determinant's equation is derived for a generic
Formal combination of the Prandtl-Reuss flow law for pure shear with Eqs. (12) and (13) leads to the constitutive equation for edge PQ:

\[ \tau_{PQ}^2 < 1 \] (16a)

IF \( \tau_{PQ}^2 < 1 \) OR \( \tau_{PQ}^2 > 0 \) \( \tau_{PQ} = \frac{G}{k} t_{PQ} \cdot \dot{w}_{PQ} \) (16c)

ELSE \( \dot{w}_{PQ} = 0 \) (16d)

Now, Eq. (16c) is not usable as written, since it contains the "thickness" \( \delta \) of the edge - which must tend to zero. For the slip model this dilemma is resolved by defining a "slip modulus".

\[ G' = \frac{G}{\delta} \] (17)

As shown in [1], this procedure enables us to obtain complete elastic/plastic stress distributions and to obtain displacement fields in terms of the unknown (and unknowable) constant \( G' \).

However, for the combined model considered here, the shear modulus \( G \) is necessary for Eqs. (8) to have meaning, and hence it cannot be allowed to tend to zero as is implied by Eq. (17). Therefore, we must resolve our dilemma in a different way by defining new alternative kinematic variables \( \tilde{u}_p \), etc. by

\[ \dot{u}_{PQ} = \tilde{u}_{PQ} \] (18)

Equations (16) then become

\[ \tau_{PQ}^2 < 1 \] (19a)

IF \( \tau_{PQ}^2 < 1 \) OR \( \tau_{PQ}^2 > 0 \) \( \dot{w}_{PQ} = (G/k) t_{PQ} \cdot \tilde{w}_{PQ} \) (19c)

ELSE \( \dot{w}_{PQ} = 0 \) (19d)

If (19b) is satisfied, there will be no slip. However, in view of Eq. (18), zero slip is compatible with non-zero alternative variables \( \tilde{u}_{PQ} \) so that we still have a meaningful set of slip equations. In particular, \( \dot{w}_{PQ} \) and hence \( \tau_{PQ} \) can be determined so that the continued validity of (19b) can be tested. On the other hand, if (19b) is violated, we bypass (19c) to obtain directly the simple (19d). Since this equation gives no kinematic information it is compatible with either zero \( \dot{w}_{PQ} \) and meaningful \( \tilde{u}_{PQ} \) or with non-zero \( \dot{w}_{PQ} \) in which case the alternative kinematic variables are discarded. The choice between these two alternatives will depend upon the problem as a whole, and will be discussed in later sections.

With the exception of the above discussion of the constitutive equations, it is clear that the defining equations for the classical and slip parts are independent and hence may be combined for the classical model. Therefore, for the combined model the kinematics are governed by Eqs. (2), (11) and (18), the statics by (7) and (15), and the constitutive behavior by (8) and (19).

3. Boundary value problem. Boundary conditions are most easily discussed in terms of a specific example. To this end we consider the problem shown in Fig. 1. For any of the three models local constitutive equations and strain-displacement equations will exactly match the unknown generalised strains and stresses.
\[
\frac{2}{D_1} + \frac{1}{D_2} + \frac{1}{D_3} \Rightarrow 2D_1 = D_2 + D_3
\]

Motion for \( C \) through node 1 and node 2 with modified expressions for nodes C, D, and E. The equation is:

\[
\frac{1}{X_1} + \frac{1}{X_2} = \frac{1}{X_3}
\]

In rotation, note that the equivalent condition for node 6 is shown in (1) [boundary conditions are somewhat different than shown in (3)]. If we four new variables:

\[
0_A = A_0
\]

Equation (3) is then used to generate the equivalent condition that if we solve the \( Y \) to the right side to the left side, the \( Y \) is the reciprocal on the L to a distance \( Y \) to the right. Therefore, the equivalent equation of motion of the segment (3) would move the structure constant:

\[
0_A = 0
\]

\[
0_B = 0
\]

\[
0_C = 0
\]

All of these values vanish, since the segmental expression we see this for example, the equivalent condition for the structure is given by the solution's expression to the domain. In this case, we need to add the equivalent condition of zero momentum conditions at node 6, node 7, and node 8. This condition does not change the solution from the domain condition. Any of these unknowns, along with the number of unknowns, each of the net sections, the two generalized displacement conditions.

Therefore, it order to take the total system of equations.
whereas the mechanism motion for \( u_0 \) involves all of the nodes on the left and leads to
\[
T_2 = \frac{1}{2} \left[ \{\tau_{G_1} + \tau_{U_1} + \tau_{T_1}\} + 2(\tau_{U_1} + \tau_{V_1} - \tau_{W_1}) \right] \\
+ 3(\tau_{U_1} + \tau_{V_1} - \tau_{W_1}) + 4(\tau_{V_1} + \tau_{X_1} - \tau_{W_1}) \\
+ 5(\tau_{W_1} + \tau_{X_1} - \tau_{Y_1} - 6\tau_{X_1}) \tag{23b} 
\]
As with the classical model, Eqs. (23) may be used to find displacements if forces are given or to define forces for given \( u_0 \) and/or \( v_0 \).

For the combined model, we must interpret the prescribed boundary motions in terms of permissible mechanism motions. However, before doing this, we observe that not all of the generalized displacements in the total domain are independent. As with the slip model, we may arbitrarily set \( \theta_H = 0 \). Further, we observe from Fig. 3 that the combination of a unit slip mechanism at node 0 together with large node mechanism \( \theta_A = \theta_B = 2 \) has the effect of shifting the square ABGF one unit to the right but leaving it internally undeformed. Applying this same reasoning to the domain in Fig. 1 we see that if \( \theta = 0 \) along the bottom row, \( \theta = 2 \) along the row TL of large nodes, \( \theta = 4, 6, 8, 10, 12 \), respectively, along rows UK, VJ, W1, XH, and AG, together with \( \theta = 1, 3, 5, 7, 9, 11 \) along the rows of small nodes counting from the bottom moves the entire domain one unit to the right. Superposition of \( u_k = -1 \) for all nodes would then leave the entire domain unmodified. Therefore, we may arbitrarily set \( u_k = 0 \). A similar argument applied to vertical motion shows that \( v_M \) may also be taken as zero. We note that these three arbitrary conditions are inherent in our choice of kinematic variables and are not related to a rigid-body motion of the entire domain.

Returning to the problem of Fig. 1, we consider first the motion of triangle 92. The resultant vertical motion of vertices M and N can be written in terms of rotation and vertical mechanism motions of nodes M and N, and must vanish:
\[
v_M + \left( \theta_N - \theta_M \right) / 2 = v_N + \left( \theta_N - \theta_M \right) / 2 = 0 \tag{24} 
\]
Since we have set \( v_M = \theta_M = 0 \), it follows that \( v_N = \theta_N = 0 \). Similar reasoning along the bottom and right side shows that \( \theta_G = \theta_S, \ u_G - u_M, \) and \( v_M - v_S \) all vanish.

Since \( u = u_0 \) on SA, the horizontal displacements of vertices S and T of triangle 64 are given by
\[
u_S + \left( \theta_T - \theta_S \right) / 2 = u_T + \left( \theta_T - \theta_S \right) / 2 = u_0 \tag{25} 
\]
whence \( u_S = u_T \). Similar reasoning applies all along side SA, and we define a new variable.
\[
u_{0C} = u_G = u_T = u_A = u_N = u_M = u_{N_2} = u_A \tag{26a} 
\]
Further, it follows from Eqs. (25), similar equations for triangle 54, and the previous result \( \theta_S = 0 \) that \( \theta_G = 2\theta_S \).

We introduce a second new variable \( u_{0S} \) and use similar reasoning along all of side SA to show
\[
u_{0S} = \theta_T = \theta_T / 2 = \theta_U / 3 = \theta_H / 4 = \theta_N / 5 = \theta_B / 6 \tag{26b} 
\]
In terms of these new variables Eqs. (25) and all similar equations for side SA are satisfied by
\[
u_{0C} + u_{0S} = u_0 \tag{26c} 
\]
Similar reasoning applied to FG leads to
By the quadratic formula, we can determine the quadratic equation for the boundary of the region.

\[ a = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Hence,

\[ \alpha = \frac{-a + \sqrt{a^2 - 4b}}{2a} \]

Once we have found the quadratic model, we can solve the resulting quadratic equation.

This equation is then used to find the solutions.

From these solutions, we can determine the coordinates of the points where the curve intersects the axes.

In the next section, we will discuss the details of the programming model for the given problem.
Additional equations for the two new unknowns \( \dot{\theta}_{0b} \) and \( \dot{V}_{0b} \) are provided by (28) and by the substitution of (21) in (30).

One the classical problem is solved, \( \dot{u}_{0b} \) and \( \dot{\theta}_{0b} \) will be known, hence so will those \( \dot{\theta}_1 \) which are involved in the mechanism. The remainder of the slip model can still be solved in terms of the alternative kinematic variables. In obtaining this solution we can arbitrarily set one of the external alternative kinematic variables equal to zero, say \( \dot{\theta}_{0b} \), since the forces will automatically satisfy (30).

Further increase of \( V_0 \) will lead to more elements becoming plastic. However, until a second mechanism forms we remain in the partly-coupled plastic range where the two parts are coupled only by Eqs. (28) and (30) and hence may still be solved sequentially.

The next critical value of \( V_0 \) occurs when an independent second slip mechanism forms. Now \( \dot{\theta}_{0b} \) and \( \dot{V}_{0b} \) are independent variables not subject to (28). Since neither of these independent mechanisms will do net work,

\[
\dot{\theta}_1 = \dot{\theta}_2 = 0 \tag{32}
\]

and we have reached the yield-point load. Further increase in load is impossible for a perfectly-plastic material in equilibrium.

Conceivably, the classical model alone could activate a yield-point mechanism while in the uncoupled range, or a mechanism could be formed in the partly-coupled range; in the examples considered, however, this was never the case.

Three complications to the qualitative description given above may occur. In the first place, even though \( V_0 \) increases monotonically, some elements may unload. Thus, after using the second branch in Eqs. (8) a check must be made that \( f_0 \) is non-negative. Similarly, if (19d) is used, \( \dot{\theta}_{0b} \) or \( \dot{V}_{0b} \) must have the same sign as \( \dot{\theta}_{0b} \). Any element or elements where these requirements are not met must be switched to the other branch.

Secondly, the plastic solution to (8) for a triangle should satisfy the non-linear equation \( \dot{f} = 0 \). Instead, it satisfies a linear condition equivalent to moving along a tangent to the surface \( f = 1 \), thus resulting in \( \dot{f} > 0 \). In order to keep the resulting error from growing too large, it is desirable to stop a stage whenever the change in \( f \) exceeds, say, 0.001, and to recompute the stress terms that appear in Eqs. (8).

Finally, in some stages a lack of uniqueness occurs for the kinematical slip variables. This phenomenon was found in [1] for the slip model. At least in the uncoupled range, it is less disturbing here than in [1], since it involves only the alternative kinematic slip variables and the true slip remains uniquely zero.

We shall comment on all three of these complications more fully in relation to the example in the next section.

5. Example. A computer program was written to implement the three models and was used to solve the problem illustrated in Fig. 1 with \( u_0 = 0 \). An elastic/perfectly plastic material is placed in a perfectly lubricated box and indented with a rigid punch. This problem which was considered in Sec. 4 is a finite domain approximation to the Prandtl problem [3, 4] of a rigid rough punch indenting a semi-infinite perfectly-plastic material.
on the behavior of the turbulence dynamics in that more than
the dissipation part plays, this change has a dramatic effect
and the dissipation, for example, the damping of the damping
function between the two parts of the solution.

\[ \alpha = \beta + \gamma \]

The damping and the drag become, respectively,
the damping, and drag (38) and (39) become, respectively.
In the latter case, all functions attract the same motion at
vertices, which is in some sense the same. However, at
vertices, the attraction is stronger, and this corresponds to
At the end of the stage 6 when a step is yielded and produced
and the particle

\[ \frac{\partial \mathbf{v}}{\partial t} = \nu \mathbf{v} - \mathbf{f}(\mathbf{v}) \]

7. \( \mathbf{v} \) is the particle. and it is defined as
\[ \mathbf{v}(x, t) = \begin{cases} \mathbf{v}_0(x), & t = 0, \\ \mathbf{v}(x, t), & t > 0. \end{cases} \]

\[ \mathbf{f}(\mathbf{v}) = \frac{0.5}{\rho} \mathbf{v} + \frac{1}{\rho} \nabla \mathbf{p} \]

1. The instantaneous value of the integrated is
\[ \int_0^t \mathbf{f}(\mathbf{v}) \, dt. \]

2. The state of the solution at the end of the stage 6 is shown
in Table 1. The solution is computed with the help of the
computer program [3]. The solution is computed with the help of
the computer program [3]. The solution is computed with the help of
the computer program [3]. The solution is computed with the help of
the computer program [3]. The solution is computed with the help of
the computer program [3]. The solution is computed with the help of
the computer program [3]. The solution is computed with the help of
the computer program [3]. The solution is computed with the help of
the computer program [3]. The solution is computed with the help of
the computer program [3].
half of the elements which were yielding at the end of stage 76 (17 out of 29), now start to unload. These elements are indicated by asterisks in Fig. 4b.

Figure 5 shows that in this phase the load-deflection curve is no longer the same as the classical model due to the slip in the domain. Computation in this phase continues until the end of stage 91 when four simultaneously yielding edges cause limit load conditions. The limit load and the collapse mechanisms are the same as we found in [1] for the Prandtl rough punch with the slip model.

Also shown in Figure 5 is the continuation of the load deflection curve for the classical model up to stage 119 when the load was about 14% above the yield-point load for the combined model. Further computation up to stage 153 increased the load to about 30% above the combined yield-point load and caused plastic flow in more than half the triangles, but still did not produce a yield-point mechanism. This phenomenon will be commented on in the final section.

6. Conclusions. We begin this section by summarizing some of the results for the three different models as applied to the Prandtl punch problem considered in Sec. 5. The classical model provides a well defined elastic solution which agrees well with an analytical solution for a semi-infinite domain obtained by Green and Zerna [6]. Plastic regions develop in what appears to be a reasonable sequence with no unloading up to a load of about 7.25. As shown in [5], the computer program appears to become unstable above this point, and it does not predict a limit load. However, the theorems of limit analysis may be applied directly to this model [5, Appendix C] and they show that \( T_1 = 7.23 \) is the true limit load for the model.

The slip model gives only relative displacements, but these agree well with the analytical solution [6]. It predicts a limit load of 6.00, which is a reasonable upper bound on the true value of 5.14 [3,4]. It provides many possible collapse mechanisms, one of which agrees well with the analytical one [1].

The combined model agrees exactly with the classical one up to \( T_1 = 5.7 \), gives exactly the same collapse load \( T_1 = 6.00 \) and mechanisms as the slip one, and provides a transition solution between these two which appears reasonable.

Bars with notches in one or both sides and the Prandtl problem with a smooth punch were also considered in [5], with similar results. Based on this limited experience it appears that the combined model gives results which combine the best features of the other two.

The computations were carried out on a CDC Cyber 74. The total CPU times were approximately 30, 80, and 130 seconds for the slip, classical, and combined models, respectively. A more meaningful comparison is the time per degree of freedom which was, respectively, 0.44, 0.55, and 0.62 seconds for the three models. Unquestionably, more efficient programmers could reduce these times substantially, but their relative magnitudes are probably meaningful. Thus, on either a total or degree-of-freedom basis the combined model is the most time consuming. However, in view of the fact that neither the classical nor slip models provide an adequate complete