NORTHWESTERN UNIVERSITY
The Technological Institute
Department of Civil Engineering
Evanston, Illinois 60201

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THE STRESS FIELD CREATED BY A CIRCULAR SLIDING CONTACT ON TRANSVERSELY ISOTROPIC SPHERES

A THESIS

SUBMITTED TO THE GRADUATE SCHOOL
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

for the degree

MASTER OF SCIENCE

Field of Civil Engineering

by

Dale Bruce Mowry

Evanston, Illinois

June 1978

NORTHWESTERN UNIVERSITY

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Abstract

The field equations for a transversely isotropic half-space are defined in terms of potential functions. A form of the potential functions satisfying equilibrium is assumed and the boundary value problem of a tangentially loaded spherical indentor on the half-space is solved.

Expressions are obtained defining the radius of no-slip for the static case, and the relationship between the horizontal surface displacement under the indentor and the horizontally applied force. Stresses for tangential loading are superposed with those previously obtained for normal loading of the indentor, and the stress field is defined in the half-space and on the surface for both static and sliding cases.

Von Mises' criteria for the sliding case is calculated and plotted in the half-space and on the surface for two transversely isotropic metals using two separate coefficients of limiting friction, and on the surface for the static case for magnesium with a coefficient of limiting friction of .5.

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1. Introduction

The stress field created by a circular sliding contact on an isotopic half-space has been solved by Hamilton and Goodman [1]. Motivated by consideration of mechanical failure they examined constant lines of von Mises' yield criteria in the half-space and on the surface. Dahan and Zarka [2] have recently solved the stress field in a transversely isotropic half-space in contact with a spherical indentor under normal loading. They also plotted von Mises' criteria for several transversely isotropic metals to show the effect of the anisotropy for the indentation of an elastic half-space. Both solutions give results in the half-space in a closed form.

The purpose of the present analysis is to derive the stress expressions for identical transversely isotropic spheres in contact under both normal and tangential loading, and to obtain expressions for the stresses as the spheres slide relative to each other as a result of the tangential load. Von Mises' yield criteria is then examined to compare the anisotropic case to the isotropic case in [1].

2. Formulation

(a) Basic Equations

For a transversely isotropic body the stress-strain relations are given by Green and Zerna [3] as

$$(\sigma_{xx},\sigma_{yy},\sigma_{zz},\sigma_{yz},\sigma_{zx},\sigma_{xy}) = \begin{bmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ & c_{11} & c_{13} & 0 & 0 & 0 \\ & & c_{33} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{44} & 0 \\ & & & & & \frac{1}{2}(c_{11}-c_{12}) \end{bmatrix} \begin{bmatrix} e_{xx} \\ e_{yy} \\ e_{zz} \\ e_{yz} \\ e_{zx} \\ e_{xy} \end{bmatrix} \qquad (1)$$

The relations between the strains and displacements are

$$e_{xx} = \frac{\partial u_x}{\partial x}, \quad e_{yy} = \frac{\partial u_y}{\partial y}, \quad e_{zz} = \frac{\partial u_z}{\partial z}$$

$$e_{yz} = \left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right), \quad e_{zx} = \left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right), \quad e_{xy} = \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right) \qquad (2)$$

and the stress-strain relations become

$$\sigma_{xx} = c_{11}\frac{\partial u_x}{\partial x} + c_{12}\frac{\partial u_y}{\partial y} + c_{13}\frac{\partial u_z}{\partial z}$$

$$\sigma_{yy} = c_{12}\frac{\partial u_x}{\partial x} + c_{11}\frac{\partial u_y}{\partial y} + c_{13}\frac{\partial u_z}{\partial z}$$

$$\sigma_{zz} = c_{13}\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) + c_{33}\,\frac{\partial u_z}{\partial z}$$

$$\sigma_{yz} = c_{44}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right)\,,\quad \sigma_{zx} = c_{44}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right)\,,\quad \sigma_{xy} = \tfrac{1}{2}(c_{11}-c_{12})\cdot \left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right). \tag{3}$$

Substitution of the above stress equations into the equilibrium equations, $\partial\sigma_{ij}/\partial x_j = 0$, gives the following expressions for equilibrium in terms of displacement:

$$c_{11}\,\frac{\partial^2 u_x}{\partial x^2} + \tfrac{1}{2}(c_{11}-c_{12})\,\frac{\partial^2 u_x}{\partial y^2} + c_{44}\,\frac{\partial^2 u_x}{\partial z^2} + \partial/\partial x\left[\tfrac{1}{2}(c_{11}+c_{12})\,\frac{\partial u_y}{\partial y} + (c_{13}+c_{14})\,\frac{\partial u_z}{\partial z}\right] = 0 \tag{4a}$$

$$\tfrac{1}{2}(c_{11}-c_{12})\,\frac{\partial^2 u_y}{\partial x^2} + c_{11}\,\frac{\partial^2 u_y}{\partial y^2} + c_{44}\,\frac{\partial^2 u_y}{\partial z^2} + \partial/\partial y\left[\tfrac{1}{2}(c_{11}+c_{12})\,\frac{\partial u_x}{\partial x} + (c_{13}+c_{44})\,\frac{\partial u_z}{\partial z}\right] = 0 \tag{4b}$$

$$c_{44}\left(\frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2}\right) + c_{33}\,\frac{\partial^2 u_z}{\partial z^2} + (c_{13}+c_{44})\,\partial/\partial z\left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}\right) = 0 \tag{4c}$$

The displacements are defined next by potential functions as in [3]:

$$u_x = \frac{\partial \phi}{\partial x} + \frac{\partial \psi}{\partial y}\ , \quad u_y = \frac{\partial \phi}{\partial y} - \frac{\partial \psi}{\partial x}\ , \quad u_z = k\frac{\partial \phi}{\partial z} \tag{5}$$

where k is a constant to be determined later.

Substituting these displacement expressions into the equilibrium equations (4) leads to

$$c_{11}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \left[c_{44} + k(c_{13}+c_{44})\right]\frac{\partial^2\phi}{\partial z^2}$$

$$+ \tfrac{1}{2}(c_{11}-c_{12})\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] + c_{44}\frac{\partial^2\psi}{\partial z^2} = 0 \tag{6a}$$

$$c_{11}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \left[c_{44} + k(c_{13}+c_{44})\right]\frac{\partial^2\phi}{\partial z^2}$$

$$- \tfrac{1}{2}(c_{11}-c_{12})\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right] - c_{44}\frac{\partial^2\psi}{\partial z^2} = 0 \tag{6b}$$

$$(c_{13} + c_{44} + kc_{44})\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + kc_{33}\frac{\partial^2\phi}{\partial z^2} = 0 \tag{6c}$$

Clearly these equations are satisfied if

$$c_{11}\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \left[c_{44} + k(c_{13}+c_{44})\right]\frac{\partial^2\phi}{\partial z^2} = 0 \tag{7a}$$

$$(c_{13} + c_{44} + kc_{44})\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + kc_{33}\frac{\partial^2\phi}{\partial z^2} = 0 \tag{7b}$$

and

$$-\tfrac{1}{2}(c_{11} - c_{12})\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) - c_{44}\frac{\partial^2\psi}{\partial z^2} = 0 \tag{7c}$$

Equations (7a) and (7b) can be written

$$\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{c_{44} + k(c_{13} + c_{44})}{c_{11}}\frac{\partial^2\phi}{\partial z^2} = 0 \tag{8}$$

and

$$\left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right) + \frac{kc_{33}}{(c_{13} + c_{44} + kc_{44})}\frac{\partial^2\phi}{\partial z^2} = 0 \tag{9}$$

As pointed out in [3], a suitable, non-zero ϕ can be found if (8) and (9) are identical, which leads to

$$\frac{k(c_{13} + c_{44}) + c_{44}}{c_{11}} = \frac{kc_{33}}{kc_{44} + c_{13} + c_{44}} = \nu \ . \tag{10}$$

Equation (10) generates a quadratic equation for the solution of ν,

$$c_{11}c_{44}\nu^2 + [c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}]\,\nu + c_{33}c_{44} = 0 \tag{11}$$

from which it is found that

$$\nu_1 = \frac{-[c_{13}(2c_{44}+c_{13}) - c_{11}c_{33}] - \{[c_{13}(2c_{44}+c_{13}) - c_{11}c_{33}]^2 - 4c_{11}c_{33}c_{44}^2\}^{\frac{1}{2}}}{2c_{11}c_{44}} \tag{12a}$$

and

$$\nu_2 = \frac{-[c_{13}(2c_{44}+c_{13}) - c_{11}c_{33}] + \{[c_{13}(2c_{44}+c_{13}) - c_{11}c_{33}]^2 - 4c_{11}c_{33}c_{44}^2\}^{\frac{1}{2}}}{2c_{11}c_{44}} \tag{12b}$$

with the functions ϕ_1 and ϕ_2 applicable to ν_1 and ν_2. Hence the equations that determine ϕ_1 and ϕ_2 are

$$\left(\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2}\right) + \nu_1 \frac{\partial^2\phi_1}{\partial z^2} = 0 \tag{13}$$

and

$$\left(\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial y^2}\right) + \nu_2 \frac{\partial^2\phi_2}{\partial z^2} = 0 \tag{14}$$

Defining $z_1 = \frac{1}{\sqrt{\nu_1}} z$ and $z_2 = \frac{1}{\sqrt{\nu_2}} z$

$$\frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} + \frac{\partial^2\phi_1}{\partial z_1^2} = 0 \tag{15}$$

$$\frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial^2\phi_2}{\partial y^2} + \frac{\partial^2\phi_2}{\partial z_2^2} = 0 \tag{16}$$

From (15) and (16) a form of the potential functions is assumed as

$$\phi_1 = \sum_{n=0}^{\infty} \int_0^{\infty} A_{1n}(\xi)\, e^{-\xi z/\sqrt{\nu_1}}\, J_n(\xi r)\, d\xi \cos n\theta \tag{17}$$

$$\phi_2 = \sum_{n=0}^{\infty} \int_0^{\infty} A_{2n}(\xi)\, e^{-\xi z/\sqrt{\nu_2}}\, J_n(\xi r)\, d\xi \cos n\theta \tag{18}$$

To determine Ψ, equation (7c) is written in the form

$$\left(\frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2}\right) + \frac{2c_{44}}{(c_{11}-c_{12})} \frac{\partial^2 \Psi}{\partial z^2} = 0 \tag{19}$$

Let $\Psi = \phi_3$, $\dfrac{2c_{44}}{(c_{11}-c_{12})} = \nu_3$, and $z_3 = \dfrac{1}{\sqrt{\nu_3}} \; z$.

Then,

$$\frac{\partial^2 \phi_3}{\partial x^2} + \frac{\partial^2 \phi_3}{\partial y^2} + \frac{\partial^2 \phi_3}{\partial z_3^{\,2}} = 0 \tag{20}$$

and ϕ_3 has the form

$$\phi_3 = \sum_{n=0}^{\infty} \int_0^{\infty} A_{3n}(\xi)\, e^{-\xi z/\sqrt{\nu_3}}\, J_n(\xi r)\, d\xi \sin n\theta \tag{21}$$

The form of the displacement is

$$u_x = \partial/\partial x(\phi_1+\phi_2) + \frac{\partial \phi_3}{\partial y}, \quad u_y = \partial/\partial y(\phi_1+\phi_2) - \frac{\partial \phi_3}{\partial x}, \quad u_z = k\, \partial/\partial_z(\phi_1+\phi_2) \tag{22}$$

With (17), (18), and (21) representing the functions ϕ_1, ϕ_2 and ϕ_3 the application of the boundary conditions determines $A_{in}(\xi)$, ϕ_i, and finally, from the stress-strain relations (1), the stress field.

The stress field for the case of two identical transversely isotropic spheres in contact is obtained by first solving the problem of normally loaded spheres. Then the case of tangential load in the positive x-direction insufficient for sliding to occur is considered, where sliding refers to the condition of total slip between the two spheres. The stress field obtained is then specialized to give results for the case where sliding of the sphere occurs. The problem of normally loaded transversely isotropic spheres as previously noted, has been solved in [2] so that the following solution is for tangential loading. The two solutions will then be superposed.

With the orientation of the axes as shown in figure 2.1, the boundary conditions for the application of a tangential force in the positive x-direction, P_x, are (when appropriate symmetry conditions are considered for the two spheres)

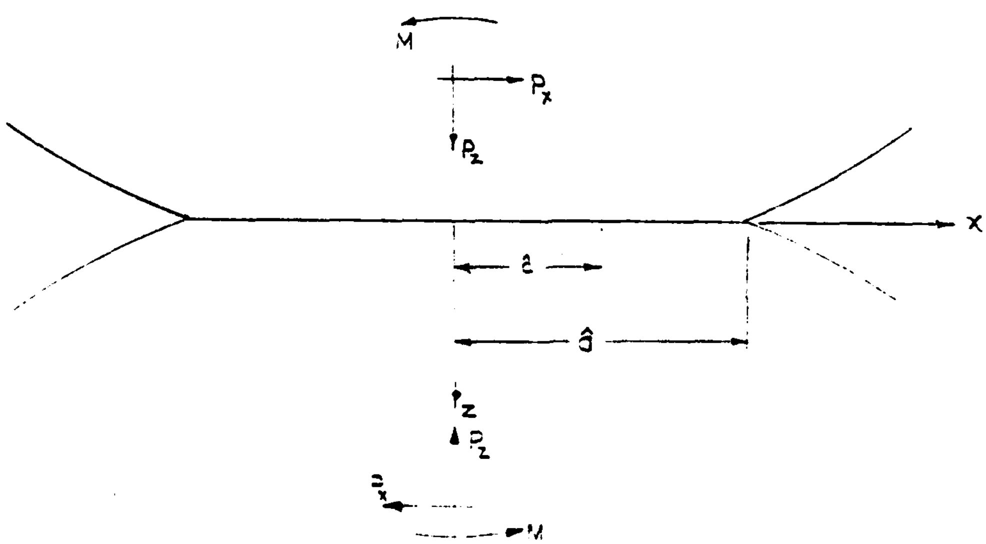

Figure 2.1 Physical Orientation of the Problem

on $z = 0$:

$$
\begin{aligned}
u_y &= 0 && 0 < r < \hat{c} \\
u_x &= \alpha && 0 < r < \hat{c} \\
\sigma_{zx} &= g(r) && \hat{c} < r < \hat{a} \\
\sigma_{zx} &= 0 && \hat{a} < r < \infty \\
\sigma_{zy} &= 0 && \hat{c} < r < \infty
\end{aligned}
\tag{23}
$$

where $0 < r < \hat{c}$ is the region of no slip, $\hat{c} < r < \hat{a}$ is the region that experiences slip, α is constant, and $g(r)$ is a function of r to be determined later.

Since the forms of the potentials are in cylindrical polar coordinates, the boundary conditions are put in that coordinate system as follows:

$$
\begin{aligned}
u_r &= u_x \cos\theta + u_y \sin\theta = \alpha \cos\theta && 0 < r < \hat{c} \\
u_\theta &= u_y \cos\theta - u_x \sin\theta = -\alpha \sin\theta && 0 < r < \hat{c}
\end{aligned}
\tag{24}
$$

If u_r and u_θ are expressed in Fourier cosine and sine series,

$$
\begin{aligned}
u_r &= \sum_{n=0}^{\infty} \hat{u}_{rn} \cos n\theta = \alpha \cos\theta && 0 < r < \hat{c} \\
u_\theta &= \sum_{n=0}^{\infty} \hat{u}_{\theta n} \sin n\theta = -\alpha \sin\theta && 0 < r < \hat{c}
\end{aligned}
\tag{25}
$$

From symmetry considerations and equation (25), n = 1 and

$$u_r = \hat{u}_r \cos\theta$$
$$u_\theta = \hat{u}_\theta \sin\theta \qquad (26)$$

Then in the region $0 < r < \hat{c}$,

$$u_x = u_r \cos\theta - u_\theta \sin\theta = \tfrac{1}{2}(\hat{u}_r - \hat{u}_\theta) + \tfrac{1}{2}(\hat{u}_r + \hat{u}_\theta)\cos 2\theta \qquad (27a)$$

$$u_y = u_r \sin\theta + u_\theta \cos\theta = \tfrac{1}{2}(\hat{u}_r + \hat{u}_\theta)\sin 2\theta \qquad (27b)$$

From (27a) and (23), with $\Delta = 2\alpha$,

$$\hat{u}_r - \hat{u}_\theta = \Delta , \qquad 0 < r < \hat{c} \qquad (28a)$$

$$\hat{u}_r + \hat{u}_\theta = 0 , \qquad 0 < r < \hat{c} \qquad (28b)$$

In the same manner, with $\sigma_{zr} = \hat{\sigma}_{zr} \cos\theta$, $\sigma_{z\theta} = \hat{\sigma}_{z\theta} \sin\theta$, (29)
and $f(r) = 2g(r)$,

$$\sigma_{zx} = \tfrac{1}{2}(\hat{\sigma}_{zr} - \hat{\sigma}_{z\theta}) + \tfrac{1}{2}(\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta})\cos 2\theta = g(r), \quad \hat{c} < r < \hat{a} \qquad (30)$$

yields

$$\hat{\sigma}_{zr} - \hat{\sigma}_{z\theta} = f(r) \quad , \qquad \hat{c} < r < \hat{a} \quad . \qquad (31a)$$

$$\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta} = 0 \quad , \qquad \hat{c} < r < \hat{a} \quad . \qquad (31b)$$

Since

$$\sigma_{zx} = \tfrac{1}{2}(\hat{\sigma}_{zr} - \hat{\sigma}_{z\theta}) + \tfrac{1}{2}(\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta}) \cos 2\theta \qquad (32a)$$

and $$\sigma_{zy} = \tfrac{1}{2}(\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta}) \sin 2\theta, \qquad (32b)$$

then $$\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta} = 0, \qquad \hat{a} < r < \infty \qquad (33a)$$

$$\hat{\sigma}_{zr} - \hat{\sigma}_{z\theta} = 0, \qquad \hat{a} < r < \infty \qquad (33b)$$

The boundary conditions in polar form are

$$\hat{u}_r - \hat{u}_\theta = \Delta \qquad 0 < r < \hat{c} \qquad (34a)$$

$$\hat{u}_r + \hat{u}_\theta = 0 \qquad 0 < r < \hat{c} \qquad (34b)$$

$$\hat{\sigma}_{zr} - \hat{\sigma}_{z\theta} = f(r) \qquad \hat{c} < r < \hat{a} \qquad (34c)$$

$$\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta} = 0 \qquad \hat{a} < r < \infty \qquad (34d)$$

$$\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta} = 0 \qquad \hat{c} < r < \infty \qquad (34e)$$

on $z = 0$.

Next, the stresses and displacements are defined in the form of the potentials from equations (17), (18), and (21).

Equations for the polar displacements and stress field due to ϕ_1 and ϕ_2 are given in [3]. Those for ϕ_3 are derived by coordinate transformation. As previously noted, $n = 1$ so that

$$\phi_1 = \int_0^\infty A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}}\, J_1(\xi r)\, d\xi \cos\theta \tag{35a}$$

$$\phi_2 = \int_0^\infty A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}}\, J_1(\xi r)\, d\xi \cos\theta \tag{35b}$$

$$\phi_3 = \int_0^\infty A_3(\xi)\, e^{-\xi z/\sqrt{\nu_3}}\, J_1(\xi r)\, d\xi \sin\theta \tag{35c}$$

Making use of the equations given in [3] and noting that

$$\frac{d^2}{dr^2}\left[J_1(\xi r)\right] = \left[\xi^2 J_{-1}(\xi r) + \frac{2}{r^2} J_1(\xi r) - \frac{\xi}{r} J_o(\xi r)\right]$$

$$= \left[-\xi^2 J_1(\xi r) + \frac{1}{r^2} J_1(\xi r) - \frac{1}{r}\frac{d}{dr} J_1(\xi r)\right]$$

the expressions for displacements and stresses become

$$u_r = \int_0^\infty \left\{\left[A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}}\right]\frac{d}{dr} J_1(\xi r)\right.$$

$$\left. + A_3(\xi)\, e^{-\xi z/\sqrt{\nu_3}} \left[\frac{1}{r} J_1(\xi r)\right]\right\} d\xi \cos\theta \tag{36a}$$

$$u_\theta = \int_0^\infty \left\{\left[-A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}} - A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}}\right]\frac{1}{r} J_1(\xi r)\right.$$

$$\left. - A_3(\xi)\, e^{-\xi z/\sqrt{\nu_3}} \left[\frac{d}{dr} J_1(\xi r)\right]\right\} d\xi \sin\theta \tag{36b}$$

$$u_z = \int_0^\infty \left\{\left[k_1 \left(\frac{\xi}{\sqrt{\nu_1}}\right) A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}}\right.\right.$$

$$\left.\left. - k_2 \left(\frac{\xi}{\sqrt{\nu_2}}\right) A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}}\right] J_1(\xi r)\right\} d\xi \cos\theta \qquad (36c)$$

and, with some regrouping after differentiation

$$\sigma_{rr} = \int_0^\infty \left\{\left[\left(\frac{c_{13}k_1}{\nu_1} - c_{11}\right) \xi^2 A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}}\right.\right.$$

$$+ \left(\frac{c_{13}k_2}{\nu_2} - c_{11}\right) \xi^2 A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}}$$

$$\left. + \left(\frac{c_{11}-c_{12}}{r^2}\right) \left(A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}} - A_3(\xi)\, e^{-\xi z/\sqrt{\nu_3}}\right)\right] J_1(\xi r)$$

$$+ \left(\frac{c_{12}-c_{11}}{r}\right) \frac{d}{dr}\left[\left(A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}} - A_3(\xi)\, e^{-\xi z/\sqrt{\nu_3}}\right) \cdot\right.$$

$$\left.\left. J_1(\xi r)\right]\right\} d\xi \cos\theta \qquad (37a)$$

$$\sigma_{\theta\theta} = \int_0^\infty \left\{\left[\left(\frac{c_{13}k_1}{\nu_1} - c_{12}\right) \xi^2 A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}}\right.\right.$$

$$+ \left(\frac{c_{13}k_2}{\nu_2} - c_{12}\right) \xi^2 A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}}$$

$$\left. + \left(\frac{c_{12}-c_{11}}{r^2}\right) \left(A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}} - A_3(\xi)\, e^{-\xi z/\sqrt{\nu_3}}\right)\right] J_1(\xi r)$$

$$+ \left(\frac{c_{11}-c_{12}}{r}\right) \frac{d}{dr} \left[\left(A_1(\xi)\ e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)\ e^{-\xi z/\sqrt{\nu_2}} - A_3(\xi)\ e^{-\xi z/\sqrt{\nu_3}}\right) \cdot\right.$$

$$\left.\left. J_1(\xi r)\right]\right\} d\xi \cos\theta \qquad (37b)$$

$$\sigma_{r\theta} = \int_0^\infty \left\{\left[\left(\frac{c_{11}-c_{12}}{2}\right) \xi^2 A_3(\xi)\ e^{-\xi z/\sqrt{\nu_3}}\right.\right.$$

$$\left. + \left(\frac{c_{11}-c_{12}}{r^2}\right) \left(A_1(\xi)\ e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)\ e^{-\xi z/\sqrt{\nu_2}} - A_3(\xi)\ e^{-\xi z/\sqrt{\nu_3}}\right)\right] J_1(\xi r)$$

$$+ \left(\frac{c_{12}-c_{11}}{r}\right) \frac{d}{dr} \left[\left(A_1(\xi)\ e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)\ e^{-\xi z/\sqrt{\nu_2}} - A_3(\xi)\ e^{-\xi z/\sqrt{\nu_3}}\right) \cdot\right.$$

$$\left.\left. J_1(\xi r)\right]\right\} d\xi \sin\theta \qquad (37c)$$

$$\sigma_{z\theta} = \int_0^\infty \left\{ \frac{1}{r}\left[\frac{c_{44}(1+k_1)}{\sqrt{\nu_1}}\ \xi A_1(\xi)\ e^{-\xi z/\sqrt{\nu_1}} + \frac{c_{44}(1+k_2)}{\sqrt{\nu_2}}\ \xi A_2(\xi)\ e^{-\xi z/\sqrt{\nu_2}}\right] J_1(\xi r)\right.$$

$$\left. + \frac{d}{dr}\left[\frac{c_{44}}{\sqrt{\nu_3}}\ \xi A_3(\xi)\ e^{-\xi z/\sqrt{\nu_3}} J_1(\xi r)\right]\right\} d\xi \sin\theta \qquad (37d)$$

$$\sigma_{rz} = \int_0^\infty \left\{\frac{d}{dr}\left[\left(\frac{-c_{44}(1+k_1)}{\sqrt{\nu_1}}\ \xi A_1(\xi)\ e^{-\xi z/\sqrt{\nu_1}} - \frac{c_{44}(1+k_2)}{\sqrt{\nu_2}}\ \xi A_2(\xi)\ e^{-\xi z/\sqrt{\nu_2}}\right) J_1(\xi r)\right]\right.$$

$$\left. - \frac{1}{r}\left[\frac{c_{44}}{\sqrt{\nu_3}}\ \xi A_3(\xi)\ e^{-\xi z/\sqrt{\nu_3}} J_1(\xi r)\right]\right\} d\xi \cos\theta \qquad (37e)$$

$$\sigma_{zz} = \int_0^\infty \left\{ \left[\left(\frac{k_1 c_{33}}{\nu_1} - c_{13}\right) \xi^2 A_1(\xi)\, e^{-\xi z/\sqrt{\nu_1}} + \left(\frac{k_2 c_{33}}{\nu_2} - c_{13}\right)\xi^2 A_2(\xi)\, e^{-\xi z/\sqrt{\nu_2}}\right] \cdot J_1(\xi r)\right\} d\xi \cos\theta \tag{37f}$$

The above stresses and displacements are now used with the boundary conditions to solve for the $A_i(\xi)$, and subsequently, the stress field.

(b) Boundary Value Problem Solution

From the form of the boundary conditions (34) it becomes necessary to define $(\hat{u}_r - \hat{u}_\theta)$, $(\hat{u}_r + \hat{u}_\theta)$, $(\hat{\sigma}_{zr} - \hat{\sigma}_{z\theta})$, and $(\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta})$ in terms of the $A_i(\xi)$.

Comparing (26) and (36) after differentiation it is seen that

$$\hat{u}_r - \hat{u}_\theta = \int_0^\infty \left\{A_1(\xi)e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)e^{-\xi z/\sqrt{\nu_2}} + A_3(\xi)e^{-\xi z/\sqrt{\nu_3}}\right\}\xi J_o(\xi r)\, d\xi \tag{38}$$

Making use of the identity [4] that

$$\left[\xi J_o(\xi r) - \frac{2}{r} J_1(\xi r)\right] = \xi\left[J_o(\xi r) - \frac{2}{\xi r} J_1(\xi r)\right] = -\xi J_2(\xi r)$$

it follows that

$$\hat{u}_r + \hat{u}_\theta = \int_0^\infty \left\{A_1(\xi)e^{-\xi z/\sqrt{\nu_1}} + A_2(\xi)e^{-\xi z/\sqrt{\nu_2}} - A_3(\xi)e^{-\xi z/\sqrt{\nu_3}}\right\}\left[-\xi J_2(\xi r)\right] d\xi \tag{39}$$

For $(\hat{\sigma}_{zr} - \hat{\sigma}_{z\theta})$ and $(\hat{\sigma}_{zr} + \hat{\sigma}_{z\theta})$ comparing (29) and (37) after differentiation

$$\hat{\sigma}_{rz} + \hat{\sigma}_{z\theta} = c_{44} \int_0^\infty \Big\{(1+k_1)\ A_1(\xi)\Big(-\frac{\xi}{\sqrt{\nu_1}}\Big)e^{-\xi z/\sqrt{\nu_1}}$$

$$+ (1+k_2)\ A_2(\xi)\Big(-\frac{\xi}{\sqrt{\nu_2}}\Big)e^{-\xi z/\sqrt{\nu_2}}$$

$$- A_3(\xi)\Big(-\frac{\xi}{\sqrt{\nu_3}}\Big)e^{-\xi z/\sqrt{\nu_3}}\Big\} \Big[-\xi J_2(\xi r)\Big]\, d\xi \tag{40}$$

and

$$\hat{\sigma}_{rz} - \hat{\sigma}_{z\theta} = c_{44} \int_0^\infty \Big\{(1+k_1)\ A_1(\xi)\Big(-\frac{\xi}{\sqrt{\nu_1}}\Big)e^{-\xi z/\sqrt{\nu_1}}$$

$$+ (1+k_2)\ A_2(\xi)\Big(-\frac{\xi}{\sqrt{\nu_2}}\Big)e^{-\xi z/\sqrt{\nu_2}}$$

$$+ A_3(\xi)\Big(-\frac{\xi}{\sqrt{\nu_3}}\Big)e^{-\xi a/\sqrt{\nu_3}}\Big\} \Big[\xi J_o(\xi r)\Big]\, d\xi \tag{41}$$

It can be noted at this point that from symmetry considerations $\sigma_{zz} = 0$ on the entire half-space surface, and a relationship exists between $A_1(\xi)$ and $A_2(\xi)$, simplifying the above expressions. From (37f), on $z = 0$, equating σ_{zz} to zero gives rise to

$$\Big(\frac{k_1 c_{33}}{\nu_1} - c_{13}\Big) A_1(\xi) = \Big(c_{13} - \frac{k_2 c_{33}}{\nu_2}\Big) A_2(\xi) \quad . \tag{42}$$

Let

$$K = \frac{\left(\frac{k_1 c_{33}}{\nu_1} - c_{13}\right)}{\left(c_{13} - \frac{k_2 c_{33}}{\nu_2}\right)} \quad ,$$

then $A_2(\xi) = K\, A_1(\xi)$. (43)

This simplifies the expressions at hand to

$$\hat{u}_r + \hat{u}_\theta = \int_0^\infty [(-1-K)\ A_1(\xi) + A_3(\xi)]\ \xi J_2(\xi r)\ d\xi \tag{44a}$$

$$\hat{u}_r - \hat{u}_\theta = \int_0^\infty [(1+K)\ A_1(\xi) + A_3(\xi)]\ \xi J_o(\xi r)\ d\xi \tag{44b}$$

$$\hat{\sigma}_{rz} + \hat{\sigma}_{z\theta} = c_{44} \int_0^\infty \left[\left(\frac{1+k_1}{\sqrt{\nu_1}} + K\,\frac{1+k_2}{\sqrt{\nu_2}}\right) A_1(\xi) - \frac{1}{\sqrt{\nu_3}}\ A_3(\xi)\right] \xi^2 J_2(\xi r)\ d\xi \tag{44c}$$

$$\hat{\sigma}_{rz} - \hat{\sigma}_{z\theta} = c_{44} \int_0^\infty \left[\left(-\frac{1+k_1}{\sqrt{\nu_1}} - K\,\frac{1+k_2}{\sqrt{\nu_2}}\right) A_1(\xi) - \frac{1}{\sqrt{\nu_3}}\ A_3(\xi)\right] \xi^2 J_o(\xi r)\ d\xi \tag{44d}$$

Define $a = 1 + K$, $b = c_{44}\left[\frac{(1+k_1)}{\sqrt{\nu_1}} + K\,\frac{(1+k_2)}{\sqrt{\nu_2}}\right]$, $c = \frac{c_{44}}{\sqrt{\nu_3}}$ (45)

Then

$$\hat{u}_r + \hat{u}_\theta = \int_0^\infty [eC(\xi) + fD(\xi)]\ \xi J_2(\xi r)\ d\xi \tag{46a}$$

$$\hat{u}_r - \hat{u}_\theta = \int_0^\infty [-fC(\xi) - eD(\xi)]\, \xi J_o(\xi r)\, d\xi \tag{46b}$$

$$\hat{\sigma}_{rz} + \hat{\sigma}_{z\theta} = \int_0^\infty -cC(\xi)\, \xi^2 J_2(\xi r)\, d\xi \tag{46c}$$

$$\hat{\sigma}_{rz} - \hat{\sigma}_{z\theta} = \int_0^\infty cD(\xi)\, \xi^2 J_o(\xi r)\, d\xi \tag{46d}$$

where $C(\xi) = -\frac{b}{c} A_1(\xi) + A_3(\xi)$, $D(\xi) = -\frac{b}{c} A_1(\xi) - A_3(\xi)$ (47)

and $d = \frac{b}{ac}$, $e = \frac{1}{2}(d^{-1}+1)$, $f = \frac{1}{2}(d^{-1}-1)$. (48)

The boundary conditions (34) can now be written

$$\int_0^\infty [eC(\xi) + fD(\xi)]\, \xi J_2(\xi r)\, d\xi = 0 \qquad 0 < r < \hat{c} \tag{49a}$$

$$-\int_0^\infty [fC(\xi) + eD(\xi)]\, \xi J_o(\xi r)\, d\xi = \Delta \qquad 0 < r < \hat{c} \tag{49b}$$

$$c\int_0^\infty D(\xi)\, \xi^2 J_o(\xi r)\, d\xi = f(r) \qquad \hat{c} < r < \hat{a} \tag{49c}$$

$$c\int_0^\infty D(\xi)\, \xi^2 J_o(\xi r)\, d\xi = 0 \qquad \hat{a} < r < \infty \tag{49d}$$

$$- c \int_0^\infty C(\xi)\, \xi^2 J_2(\xi r)\, d\xi = 0 \qquad \hat{c} < r < \infty \tag{49e}$$

Motivated by the work of Westmann [5] and Goodman and Keer [6] a solution is assumed in the form

$$\xi\, C(\xi) = 0 \tag{50a}$$

$$\xi\, D(\xi) = \xi^{\frac{1}{2}} \int_0^{\hat{c}} t^{\frac{1}{2}} \chi_1(t)\, J_{-\frac{1}{2}}(\xi t)\, dt$$

$$- 2f'\, \xi^{\frac{1}{2}} \int_{\hat{c}}^{\hat{a}} t^{\frac{1}{2}} \phi_o(t)\, J_{-\frac{1}{2}}(\xi t)\, dt \tag{50b}$$

where f' is the coefficient of limiting friction to be subsequently introduced.

From (50a) it is seen that (49e) is satisfied identically. Substituting (50b) into (49d) leads to

$$\frac{d}{dr} \int_0^{\hat{c}} \chi_1(t)\, dt - 2f' \frac{d}{dr} \int_{\hat{c}}^{\hat{a}} \phi_o(t)\, dt = 0 \tag{51}$$

which is satisfied automatically.

Substituting (50b) into (49c) leads to

$$\sqrt{\frac{\pi}{2}}\, \frac{f(r)}{2f'c} = \frac{1}{r} \frac{d}{dr} \int_r^{\hat{a}} \phi_o(t)\, [t(t^2-r^2)^{-\frac{1}{2}}]\, dt, \qquad \hat{c} < r < \hat{a} \tag{52}$$

At this point the nature of $f(r)$ is examined. As noted in [2] the pressure distribution under the area of contact, $0 < r < \hat{a}$, for normal loading, P_z, has been determined as

$$\rho(r) = \frac{\rho_o}{\hat{a}} (\hat{a}^2 - r^2)^{\frac{1}{2}} \qquad (53)$$

where ρ_o, the pressure under the center of the contact region, is

$$\rho_o = \frac{3P_z}{2\pi\hat{a}^2} \qquad (54)$$

Upon application of a tangential force, P_x, sufficient to create a slip region, $\hat{c} < r < \hat{a}$, the surface traction in the slip region may be expressed as

$$\sigma_{zx} = f' \rho(r) \qquad \hat{c} < r < \hat{a} \qquad (55)$$

Then by (23) and the fact that $f(r) = 2g(r)$,

$$f(r) = \frac{2f'\rho_o}{\hat{a}} \ (\hat{a}^2 - r^2)^{\frac{1}{2}} \qquad (56)$$

and (51) may be written

$$B(\hat{a}^2 - r^2)^{\frac{1}{2}} = \frac{1}{r} \frac{d}{dr} \int_r^{\hat{a}} \phi_o(t) \ [t(t^2 - r^2)^{-\frac{1}{2}}]dt, \qquad \hat{c} < r < \hat{a} \qquad (57)$$

where $B = \sqrt{\frac{\pi}{2}}\ \frac{\rho_o}{\hat{a}\,c}$. Knowing that $\phi_o(t)$ must satisfy the form (57), by examining p. 60 of [4] an expression for $\phi_o(t)$ is found to be

$$\phi_o(t) = G(t^2 - \hat{a}^2) \tag{58}$$

where, defining c by (45),

$$G = \sqrt{\frac{\pi}{2}}\ \frac{\rho_o}{\hat{a}} \left(\frac{\sqrt{\nu_3}}{2c_{44}}\right) \tag{59}$$

Substituting equations (50) into (49b), and dividing through by e gives the form

$$-\sqrt{\frac{\pi}{2}}\ \frac{\Delta}{e} = \int_o^r \frac{\chi_1(t)}{(r^2-t^2)^{\frac{1}{2}}}dt \qquad 0 < r < \hat{c} \tag{60}$$

Examining [4], p. 67, it becomes apparent that if $\chi_1(t)$ is constant, (60) will be satisfied. With $\chi_1(t) = H$, integrating,

$$-\sqrt{\frac{\pi}{2}}\ \frac{\Delta}{e} = H(\pi/2) \tag{61}$$

and $H = -\frac{\Delta}{e}\sqrt{\frac{2}{\pi}}$. Therefore

$$\chi_1(t) = -\frac{\Delta}{e}\sqrt{\frac{2}{\pi}} \tag{62}$$

satisfies (49b). It is easily verified that (67a) is automatically satisfied by $\chi_1(t)$ and $\phi_o(t)$ as defined above. The boundary value problem is thereby satisfied by

$$\phi_o(t) = G(t^2-\hat{a}^2) \qquad \chi_1(t) = -\frac{\Delta}{e}\sqrt{\frac{2}{\pi}} \tag{63}$$

Utilizing the identity $J_{-\frac{1}{2}}(\xi t) = \sqrt{\frac{2}{\pi\xi t}}\cos(\xi t)$ and integrating (50b), $D(\xi)$ is found to be

$$D(\xi) = \left(-\frac{\Delta}{e}\ \frac{2}{\pi}\right)\left[\frac{\sin \hat{c}\xi}{\xi^2}\right] - 2f'G\sqrt{\frac{2}{\pi}}\left[2\hat{a}\ \frac{\cos(\hat{a}\xi)}{\xi^3} - 2\ \frac{\sin(\hat{a}\xi)}{\xi^4}\right.$$

$$\left. - 2\hat{c}\ \frac{\cos(\hat{c}\xi)}{\xi^3} + 2\ \frac{\sin(\hat{c}\xi)}{\xi^4} + (\hat{a}^2-\hat{c}^2)\ \frac{\sin(\hat{c}\xi)}{\xi^2}\right] \tag{64}$$

From (47) it is seen that

$$A_1(\xi) = -\frac{c}{2b}D(\xi) \tag{65a}$$

$$A_3(\xi) = -\tfrac{1}{2}D(\xi) \tag{65b}$$

and by (43)

$$A_2(\xi) = -\frac{Kc}{2b}\ D(\xi) \tag{65c}$$

(c) Auxiliary Relationships

Before proceeding to the evaluation of the stresses it is of interest to examine two conditions key to the determination of $\hat{c}$, the radius of the no-slip region, and the relationship between P_x, the tangentially applied force and α, the displacement in the x-direction on the surface under the area of contact.

In the work of Goodman and Keer [6], upon tangential loading and creation of the slip region, $\hat{c} < r < \hat{a}$, a singularity arises in σ_{zx} which can be eliminated by an appropriate choice of $\hat{c}$. To examine that phenomenon in this analysis attention is turned to the expression for σ_{zx} given in (32a). Using (46c) and (50a) it is seen that the sum of the Fourier coefficients vanishes. Therefore any singularity in σ_{zx} is reflected in $(\hat{\sigma}_{zr} - \hat{\sigma}_{z\theta})$.

Performing the indicated integration and differentiation it is found that

$$\frac{\hat{\sigma}_{zr}-\hat{\sigma}_{z\theta}}{c} = \left(-\frac{\Delta}{e}\frac{2}{\pi}\right)(\hat{c}^2-r^2)^{-\frac{1}{2}} + 2f'G\sqrt{\frac{2}{\pi}}\left[2(\hat{a}^2-r^2)^{\frac{1}{2}} - (\hat{c}^2-r^2)^{\frac{1}{2}}\right]$$

$$- 2f'G\sqrt{\frac{2}{\pi}}(\hat{a}^2-r^2)(\hat{c}^2-r^2)^{-\frac{1}{2}} \qquad 0 < r < \hat{c} \tag{66}$$

As the radius r goes to $\hat{c}$ the singularity generated in (66) can be eliminated if

$$\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) = 2f'G\sqrt{\frac{2}{\pi}}(\hat{a}^2-\hat{c}^2) \tag{67}$$

or

$$\Delta = -2f'Ge\sqrt{\frac{\pi}{2}}(\hat{a}^2-\hat{c}^2) \tag{68}$$

An expression for the radius of contact under normal load, P_z, has been determined (similiar to that of [2]) as

$$\hat{a} = \left[\frac{3P_zR}{4}\left(\frac{\delta_1-\delta_2}{2}\right)\right]^{1/3} \tag{69}$$

where R is the radius of the sphere, and δ_1 and δ_2 are quantities to be defined later which are dependent only on the elastic constants of the half-space. Therefore with (69) to determine $\hat{a}$, (68) can be solved for $\hat{c}$ if Δ is known.

In order to determine the relationship between the displacement, α, and the tangential force, P_x, (and therefore between Δ and P_x) equilibrium must be considered. On the surface z = 0, summing forces in the x-direction gives that

$$-P_x = \int_0^{\hat{a}} \int_0^{2\pi} \sigma_{zx}(r,\theta,o)\ r\ d\ \theta\ d\ r \tag{70}$$

Again using (32a) to define σ_{zx} on the surface, with the sum of the Fourier stress coefficients equal to zero as noted previously, performing the necessary differentiation and integration on the difference of the Fourier stress coefficients it is found that

$$-P_x = 2c\left[-\frac{\Delta}{e}\hat{c} - f'G/2\pi\left(\hat{a}^2\hat{c} - \frac{2\hat{a}^3}{3} - \frac{\hat{c}^3}{3}\right)\right] \tag{71}$$

By means of (68) and (71), Δ (and therefore α) and $\hat{c}$ can be calculated given $\hat{a}$ and P_x.

3. Stress Expressions

(a) Preliminary Integration

Before proceeding it is necessary to comment on notation. Since the stresses from (37) are for the tangentially loaded spheres and will be superposed with the solution of Dahan and Zarka [2] for the normally loaded case, the notation to follow will, as much as possible, be consistent with [2]. In the present analysis for tangential loading, the elastic constants, c_{ij}, are the moduli for a given transversely isotropic metal. In [2] analysis is carried out using the compliance, a_{ij}, as the elastic constant in the basic equations. The relationship between the two is herewith stated as:

$$c_{11} = \frac{a_{11}a_{33}-a_{13}^2}{\ell} , \quad c_{12} = \frac{a_{13}^2-a_{12}a_{33}}{\ell}$$

$$c_{13} = \frac{a_{13}a_{12}-a_{11}a_{13}}{\ell} , \quad c_{33} = \frac{a_{11}^2-a_{12}^2}{\ell} , \quad c_{44} = \frac{1}{a_{44}} . \tag{72}$$

$$\text{where } \ell = a_{11}^2 a_{33} - 2a_{13}^2 a_{11} - a_{12}^2 a_{33} + 2a_{13}^2 a_{12} . \tag{73}$$

Further relations are given in Appendix A.

It is apparent from equations (37) that several key integrals will be involved in the stress expressions. From equations (65) by letting

$$M_1 = -\frac{c}{2b} , \quad M_2 = -K\frac{c}{2b} , \quad M_3 = -\tfrac{1}{2} . \tag{74}$$

it can be seen that $A_i(\xi) = M_i\ D(\xi)$ where $i = 1,2$ or 3. All Bessel functions in (37) are in some form of $J_1(\xi r)$, so that the key integrals are

$$\int_0^\infty D(\xi) e^{-\xi z/\sqrt{\nu_i}} J_1(\xi r)\, d\xi \quad , \tag{75a}$$

$$\int_0^\infty \xi\, D(\xi) e^{-\xi z/\sqrt{\nu_i}} J_1(\xi r)\, d\xi \quad , \tag{75b}$$

$$\int_0^\infty \xi^2 D(\xi) e^{-\xi z/\sqrt{\nu_i}} J_1(\xi r)\, d\xi \quad , \tag{75c}$$

By (64)

$$\int_0^\infty \xi^2 D(\xi) e^{-\xi z/\sqrt{\nu_i}} J_1(\xi r)\, d\xi$$

$$= \left(-\frac{\Delta}{e}\frac{2}{\pi}\right) \int_0^\infty \sin(\hat{c}\xi) e^{-\xi z/\sqrt{\nu_i}} J_1(\xi r)\, d\xi$$

$$-2f'G\sqrt{\frac{2}{\pi}} \left\{-2\hat{a} \int_0^\infty \left[\frac{\sin(\hat{a}\xi)}{\hat{a}\xi^2} - \frac{\cos(\hat{a}\xi)}{\xi}\right] e^{-\xi z/\sqrt{\nu_i}}\ J_1(\xi r)\, d\xi\right.$$

$$+2\hat{c} \int_0^\infty \left[\frac{\sin(\hat{c}\xi)}{\hat{c}\xi^2} - \frac{\cos(\hat{c}\xi)}{\xi}\right] e^{-\xi z/\sqrt{\nu_i}}\ J_1(\xi r)\, d\xi$$

$$\left. + (\hat{a}^2-\hat{c}^2) \int_0^\infty \sin(\hat{c}\xi) e^{-\xi z/\sqrt{\nu_i}}\ J_1(\xi r)\, d\xi\right\} \tag{76}$$

From [2] define

$$S_{i,1}(\hat{m}) = \int_0^\infty \frac{\sin(\hat{m}\xi)}{\xi} J_o(\xi r) e^{-\xi z s_i} d\xi \tag{77a}$$

$$S_{i,1}^{\;1}(\hat{m}) = \int_0^\infty \frac{\sin(\hat{m}\xi)}{\xi} J_1(\xi r) e^{-\xi z s_i} d\xi \tag{77b}$$

$$T_{i,1}^{\;1}(\hat{m}) = \int_0^\infty \frac{\cos(\hat{m}\xi)}{\xi} J_1(\xi r) e^{-\xi z s_i} d\xi \tag{77c}$$

and

$$C_{ij}(\hat{m}) = \int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\,\xi^j} - \frac{\cos(\hat{m}\xi)}{\xi^{j-1}}\right] J_o(\xi r) e^{-\xi z s_i} d\xi \tag{78a}$$

$$D_{ij}(\hat{m}) = \int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\xi^j} - \frac{\cos(\hat{m}\xi)}{\xi^{j-1}}\right] J_1(\xi r) e^{-\xi z s_i} d\xi \tag{78b}$$

where $\hat{m} = \hat{a}$ or $\hat{c}$, and $s_i = 1/\sqrt{\nu_i}$.

It can be seen that

$$\int_0^\infty \sin(\hat{c}\xi)\, J_1(\xi r) e^{-\xi z/\sqrt{\nu_i}}\, d\xi = -\frac{d}{d\hat{c}}\, T_{i,1}^{\;1}(\hat{c}) \tag{79}$$

so that equation (76) becomes

$$\int_0^\infty \xi^2 D(\xi)\ J_1(\xi r) e^{-\xi z/\sqrt{\nu_i}} d\xi$$

$$= \left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[-\frac{d}{d\hat{c}}\ T_{i,1}^{\ 1}(\hat{c})\right]$$

$$-2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}D_{i2}(\hat{a}) + 2\hat{c}D_{i2}(\hat{c}) + (\hat{a}^2-\hat{c}^2)\left(-\frac{d}{d\hat{c}}\ T_{i,1}^{\ 1}\ (\hat{c})\right)\right]$$

(80)

By the same manner

$$\int_0^\infty \xi\ D(\xi)\ J_1(\xi r) e^{-\xi z/\sqrt{\nu_i}} d\xi$$

$$= \left(-\frac{\Delta}{e}\frac{2}{\pi}\right) S_{i,1}^{\ 1}\ (\hat{c})\ -\ 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\ D_{i3}(\hat{a}) + 2\hat{c}\ D_{i3}(\hat{c})\right.$$

$$\left. + (\hat{a}^2-\hat{c}^2)\ S_{i,1}^{\ 1}\ (\hat{c})\right] \tag{81}$$

and

$$\int_0^\infty D(\xi)\ J_1(\xi r) e^{-\xi z/\sqrt{\nu_i}} d\xi = \left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\ \hat{c}\ \left(D_{i2}(\hat{c}) + T_{i,1}^{\ 1}(\hat{c})\right)$$

$$-2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\ D_{i4}(\hat{a}) + 2\hat{c}\ D_{i4}(\hat{c})\right.$$

$$\left. + (\hat{a}^2-\hat{c}^2)\ \hat{c}\ \left(D_{i2}(\hat{c}) + T_{i,1}^{\ 1}\ (\hat{c})\right)\right] \tag{82}$$

To evaluate the integrals defined in (77) and (78), from [2] let

$$\gamma_i^4 = \frac{1}{\hat{m}^4}(r^2 - \hat{m}^2 + s_i^2 z^2)^2 + 4\frac{s_i^2 z^2}{\hat{m}^2} \tag{83a}$$

$$\alpha_i^2 = \tfrac{1}{2}(r^2 - \hat{m}^2 + s_i^2 z^2 + \hat{m}^2 \gamma_i^2) \tag{83b}$$

$$\beta_i = -\frac{s_i z \hat{m}}{\alpha_i} \tag{83c}$$

then

$$S_{i,1}(\hat{m}) = \arctan\left(\frac{\hat{m}-\beta_i}{\alpha_i + s_i z}\right) \tag{84a}$$

$$S_{i,1}^{1}(\hat{m}) = \frac{\beta_i + \hat{m}}{r} \tag{84b}$$

$$T_{i,1}^{1}(\hat{m}) = \frac{\alpha_i - s_i z}{r} \tag{84c}$$

and

$$C_{i2}(\hat{m}) = 1 - \frac{1}{\hat{m}}\left(r\, S_{i,1}^{1}(\hat{m}) + s_i z\, S_{i,1}(\hat{m})\right) \tag{85a}$$

$$C_{i3}(\hat{m}) = \frac{1}{2\hat{m}}\left(\hat{m}^2 - \frac{r^2}{2} + s_i^2 z^2\right) S_{i,1}(\hat{m}) + \frac{3 s_i z r}{4\hat{m}} S_{i,1}^{1}(\hat{m})$$

$$+ \frac{r}{4} T_{i,1}^{1}(\hat{m}) - \frac{s_i z}{2} \tag{85b}$$

$$D_{i2}(\hat{m}) = \frac{1}{2\hat{m}}\left(r\, S_{i,1}(\hat{m}) - s_i z S_{i,1}^{\;1}(\hat{m}) - \hat{m}\, T_{i,1}^{\;1}(\hat{m})\right) \tag{85c}$$

$$D_{i3}(\hat{m}) = \frac{1}{3\hat{m}}\left(\hat{m}^2 - r^2 + \frac{s_i^2 z^2}{2}\right) S_{i,1}^{\;1}(\hat{m}) + \frac{s_i z}{6} T_{i,1}^{\;1}(\hat{m})$$

$$- \frac{r s_i z}{2\hat{m}} S_{i,1}(\hat{m}) + r/3 \tag{85d}$$

$D_{i4}(\hat{m})$ is not given in [2] and is evaluated by definition (78b) and integration by parts. The result is

$$D_{i4}(\hat{m}) = \frac{\hat{m}}{4}\int_0^\infty \frac{\sin(\hat{m}\xi)}{\xi^2} J_1(\xi r) e^{-\xi z s_i} d\xi$$

$$+ \frac{r}{4}\int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\xi^3} - \frac{\cos(\hat{m}\xi)}{\xi^2}\right] J_o(\xi r) e^{-\xi z s_i} d\xi$$

$$- \frac{s_i z}{4}\int_0^\infty \left[\frac{\sin(\hat{m}\xi)}{\hat{m}\xi^3} - \frac{\cos(\hat{m}\xi)}{\xi^2}\right] J_1(\xi r) e^{-\xi z s_i} d\xi \tag{86a}$$

or, by the previous definitions

$$D_{i4}(\hat{m}) = \frac{\hat{m}^2}{4}\left(D_{i2}(\hat{m}) + T_{i,1}^{\;1}(\hat{m})\right) + \frac{r}{4} C_{i3}(\hat{m}) - \frac{z s_i}{4} D_{i3}(\hat{m}) \tag{86b}$$

(b) Stresses in the Half-Space

(i) Static Case

By differentiation and substitution of the appropriate integrals (80)-(82) into (37) the stresses in the half-space become

$$\frac{\sigma_{rr}}{\cos\theta} = \left(\frac{c_{13}k_j}{\nu_j} - c_{11}\right) L_j \left\{ \left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left(-\frac{d}{d\hat{c}} T_{j,1}^{\ 1}(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{j2}(\hat{a})\right.\right.$$

$$\left.\left. + 2\hat{c}\, D_{j2}(\hat{c}) + (\hat{a}^2 - \hat{c}^2)\left(-\frac{d}{d\hat{c}} T_{j,1}^{\ 1}(\hat{c})\right)\right]\right\}$$

$$+ \left(\frac{c_{11}-c_{12}}{r^2}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\hat{c}\left(D_{i2}(\hat{c}) + T_{i,1}^{\ 1}(\hat{c})\right) - 2fG\sqrt{\frac{2}{\pi}}\ \cdot\right.$$

$$\left.\left[-2\hat{a}D_{i4}(\hat{a}) + 2\hat{c}D_{i4}(\hat{c}) + (\hat{a}^2-\hat{c}^2)\hat{c}\left(D_{i2}(\hat{c}) + T_{i,1}^{\ 1}(\hat{c})\right)\right]\right\}$$

$$+ \left(\frac{c_{12}-c_{11}}{r}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\hat{c}\left(D_{i2}'(\hat{c}) + T_{i,1}^{\ 1}{}'(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\ \cdot\right.$$

$$\left.\left[-2\hat{a}\, D_{i4}'(\hat{a}) + 2\hat{c}\, D_{i4}'(\hat{c}) + (\hat{a}^2-\hat{c}^2)\hat{c}\left(D_{i2}'(\hat{c}) + T_{i,1}^{\ 1}{}'(\hat{c})\right)\right]\right\}$$

(87a)

where $j = 1,2$ summed, $i = 1,2,3$ summed, primed notation (with the exception of f') denotes d/dr, and

$$L_1 = M_1, \quad L_2 = M_2, \quad L_3 = -M_3 \ .$$

With the same notation,

$$\frac{\sigma_{\theta\theta}}{\cos\theta} = \left(\frac{c_{13}k_j}{\nu_j} - c_{12}\right) L_j \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left(-\frac{d}{d\hat{c}} T_{j,1}^{\;1}(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{j2}(\hat{a})\right.\right.$$

$$\left.\left. + 2\hat{c}\, D_{j2}(\hat{c}) + (\hat{a}^2-\hat{c}^2)\left(-\frac{d}{d\hat{c}} T_{j,1}^{\;1}(\hat{c})\right)\right]\right\}$$

$$+ \left(\frac{c_{12}-c_{11}}{r^2}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) \hat{c}\left(D_{i2}(\hat{c}) + T_{i,1}^{\;1}(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\;\cdot\right.$$

$$\left.\left[-2\hat{a}\, D_{i4}(\hat{a}) + 2\hat{c}\, D_{i4}(\hat{c}) + (\hat{a}^2-\hat{c}^2)\hat{c}\left(D_{i2}(\hat{c}) + T_{i,1}^{\;1}(\hat{c})\right)\right]\right\}$$

$$+ \left(\frac{c_{11}-c_{12}}{r}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) \hat{c}\left(D_{i2}'(\hat{c}) + T_{i,1}^{\;1\prime}(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\;\cdot\right.$$

$$\left.\left[-2\hat{a}\, D_{i4}'(\hat{a}) + 2\hat{c}\, D_{i4}'(\hat{c}) + (\hat{a}^2-\hat{c}^2)\hat{c}\left(D_{i2}'(\hat{c}) + T_{i,1}^{\;1}{}'(\hat{c})\right)\right]\right\} \tag{87b}$$

$$\frac{\sigma_{r\theta}}{\sin\theta} = \left(\frac{c_{11}-c_{12}}{2}\right) L_3 \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left(-\frac{d}{d\hat{c}} T_{3,1}^{\;1}(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{32}(\hat{a})\right.\right.$$

$$\left.\left. + 2\hat{c}\, D_{32}(\hat{c}) + (\hat{a}^2-\hat{c}^2)\left(-\frac{d}{d\hat{c}} T_{3,1}^{\;1}(\hat{c})\right)\right]\right\}$$

$$+ \left(\frac{c_{11}-c_{12}}{r^2}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) \hat{c}\left(D_{i2}(\hat{c}) + T_{i,1}^{\;1}(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{i4}(\hat{a})\right.\right.$$

$$\left.\left. + 2\hat{c}\, D_{i4}(\hat{c}) + (\hat{a}^2-\hat{c}^2)\hat{c}\left(D_{i2}(\hat{c}) + T_{i,1}^{\;1}(\hat{c})\right)\right]\right\}$$

$$+ \left(\frac{c_{12}-c_{11}}{r}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) \hat{c} \left(D_{i2}'(\hat{c}) + T_{i,1}^{1}{}'(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{i4}'(\hat{a})\right.\right.$$

$$\left.\left. + 2\hat{c}\, D_{i4}'(\hat{c}) + (\hat{a}^2-\hat{c}^2)\hat{c}\left(D_{i2}'(\hat{c}) + T_{i,1}^{1}{}'(\hat{c})\right)\right]\right\} \qquad (87c)$$

$$\frac{\sigma_{z\theta}}{\sin\theta} = \frac{c_{44}(1+k_j)}{\sqrt{\nu_j}} L_j \frac{1}{r} \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) S_{j,1}^{1}(\hat{c}) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{j3}(\hat{a}) + 2\hat{c}\, D_{j3}(\hat{c})\right.\right.$$

$$\left.\left. + (\hat{a}^2-\hat{c}^2)\, S_{j,1}^{1}(\hat{c})\right]\right\}$$

$$- \frac{c_{44}}{\sqrt{\nu_3}} L_3 \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) S_{3,1}^{1}{}'(\hat{c}) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{33}'(\hat{a}) + 2\hat{c}\, D_{33}'(\hat{c})\right.\right.$$

$$\left.\left. + (\hat{a}^2-\hat{c}^2)\, S_{3,1}^{1}{}'(\hat{c})\right]\right\} \qquad (87d)$$

$$\frac{\sigma_{rz}}{\cos\theta} = \frac{-c_{44}(1+k_j)}{\sqrt{\nu_j}} L_j \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) S_{j,1}^{1}{}'(\hat{c}) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{j3}'(\hat{a}) + 2\hat{c}\, D_{j3}'(\hat{c})\right.\right.$$

$$\left.\left. + (\hat{a}^2-\hat{c}^2)\, S_{j,1}^{1}{}'(\hat{c})\right]\right\}$$

$$+ \frac{c_{44}}{\sqrt{\nu_3}} \frac{1}{r} L_3 \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right) S_{3,1}^{1}(\hat{c}) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{33}(\hat{a}) + 2\hat{c}\, D_{33}(\hat{c})\right.\right.$$

$$\left.\left. + (\hat{a}^2-\hat{c}^2)\, S_{3,1}^{1}(\hat{c})\right]\right\} \qquad (87e)$$

$$\frac{\sigma_{zz}}{\cos\theta} = \left(\frac{c_{33}k_j}{\nu_j} - c_{13}\right) L_j \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left(-\frac{d}{d\hat{c}} T_{j,1}^{1}(\hat{c})\right) - 2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\, D_{j2}(\hat{a})\right.\right.$$

$$\left.\left. + 2\hat{c}\, D_{j2}(\hat{c}) + (\hat{a}^2-\hat{c}^2)\left(-\frac{d}{d\hat{c}} T_{j,1}^{1}(\hat{c})\right)\right]\right\} \tag{87f}$$

Equations (87) represent expressions for the stress components due to a tangential loading, P_x, insufficient to cause overall sliding of the spheres. The results of Dahan and Zarka [2] for the stresses due to the normal load, P_z, are

$$\sigma_{rr} = \rho_o\left[\frac{-1}{(s_1-s_2)\sqrt{d'}}\left(s_1C_{12}(\hat{a}) - s_2C_{22}(\hat{a})\right)\right.$$

$$\left. + \frac{\nu}{r(s_1-s_2)}\left(s_1\rho_2D_{13}(\hat{a}) - s_2\rho_1D_{23}(\hat{a})\right)\right] \tag{88a}$$

$$\sigma_{\theta\theta} = \rho_o\left[\frac{\sqrt{d'}}{(s_1-s_2)(a'c'-d')}\left(s_1q_2C_{12}(\hat{a}) - s_2q_1C_{22}(\hat{a})\right)\right.$$

$$\left. - \frac{\nu}{r(s_1-s_2)}\left(s_1\rho_2D_{13}(\hat{a}) - s_2\rho_1D_{23}(\hat{a})\right)\right] \tag{88b}$$

$$\sigma_{zz} = \rho_o\,\frac{s_2C_{12}(\hat{a}) - s_1C_{22}(\hat{a})}{(s_1-s_2)} \tag{88c}$$

$$\sigma_{rz} = \rho_o\,\frac{D_{12}(\hat{a}) - D_{22}(\hat{a})}{(s_1-s_2)\sqrt{d'}} \tag{88d}$$

With equations (88) superposed with equations (87) the solution for the stress field of the static case is complete.

(ii) Stresses for the Sliding Spheres

From the static case, if the force P_x is gradually increased, the size of the no-slip region, $0 < r < \hat{c}$, decreases until P_x reaches a value of $f'P_z$, at which point $\hat{c}$ becomes zero and the spheres slide over the entire surface of the contact. The effect of $\hat{c}\to 0$ is analyzed and generates the stress expressions in the half-space due to sliding.

As $\hat{c}\to 0$ it can be seen that

$$S^{1}_{i,1}(\hat{c})\to 0, \quad S^{1\prime}_{i,1}(\hat{c})\to 0, \text{ and } -\frac{d}{d\hat{c}}\,T^{1}_{i,1}(\hat{c})\to 0,$$

so that the complete stress expressions for sliding, including contributions from both normal and tangential loading, can be written as

$$\begin{aligned}\sigma_{rr} = {} & \left[\left(\frac{c_{13}k_j}{\nu_j} - c_{11}\right)L_j D_{j2}(\hat{a}) + \left(\frac{c_{11}-c_{12}}{r^2}\right) L_i D_{i4}(\hat{a}) \right. \\ & \left. + \left(\frac{c_{12}-c_{11}}{r}\right) L_i\, D'_{i4}(\hat{a})\right] \left(4f'G\sqrt{\frac{2}{\pi}}\,\hat{a}\right) \cos\theta \\ & + \rho_o\left[\frac{-1}{(s_1-s_2)\sqrt{d'}} \left(s_1 C_{12}(\hat{a}) - s_2 C_{22}(\hat{a})\right)\right. \\ & \left. + \frac{\nu}{r(s_1-s_2)} \left(s_1\rho_2 D_{13}(\hat{a}) - s_2\rho_1 D_{23}(\hat{a})\right)\right]\end{aligned} \tag{89a}$$

$$\sigma_{\theta\theta} = \Big[\Big(\frac{c_{13}k_j}{\nu_j} - c_{12}\Big) L_j D_{j2}(\hat{a}) + \Big(\frac{c_{12}-c_{11}}{r^2}\Big) L_i D_{i4}(\hat{a})$$

$$+ \Big(\frac{c_{11}-c_{12}}{r}\Big) L_i D_{i4}'(\hat{a})\Big] \Big(4f'G\sqrt{\frac{2}{\pi}}\,\hat{a}\Big) \cos\theta$$

$$+ \rho_o\Big[\frac{\sqrt{d'}}{(s_1-s_2)(a'c'-d')} \Big(s_1q_2C_{12}(\hat{a}) - s_2q_1C_{22}(\hat{a})\Big)$$

$$- \frac{\nu}{r(s_1-s_2)} \Big(s_1\rho_2D_{13}(\hat{a}) - s_2\rho_1D_{23}(\hat{a})\Big)\Big] \qquad (89b)$$

$$\sigma_{r\theta} = \Big[\frac{(c_{11}-c_{12})}{2} L_3D_{32}(\hat{a}) + \Big(\frac{c_{11}-c_{12}}{r^2}\Big) L_iD_{i4}(\hat{a})$$

$$+ \Big(\frac{c_{12}-c_{11}}{r}\Big) L_iD_{i4}'(\hat{a})\Big] \Big(4f'G\sqrt{\frac{2}{\pi}}\,\hat{a}\Big) \sin\theta \qquad (89c)$$

$$\sigma_{z\theta} = \Big[\frac{c_{44}(1+k_j)}{\sqrt{\nu_j}} L_j\Big(\frac{1}{r}\Big)D_{j3}(\hat{a}) - \frac{c_{44}}{\sqrt{\nu_3}} L_3D_{33}'(\hat{a})\Big] \Big(4f'G\sqrt{\frac{2}{\pi}}\,\hat{a}\Big) \sin\theta$$

$$(89d)$$

$$\sigma_{rz} = \Big[\frac{-c_{44}(1+k_j)}{\sqrt{\nu_j}} L_jD_{j3}'(\hat{a}) + \frac{c_{44}}{\sqrt{\nu_3}} \Big(\frac{1}{r}\Big) L_3D_{33}(\hat{a})\Big] \Big(4f'G\sqrt{\frac{2}{\pi}}\,\hat{a}\Big) \cos\theta$$

$$+ \rho_o \frac{D_{12}(\hat{a})-D_{22}(\hat{a})}{(s_1-s_2)\sqrt{d'}} \qquad (89e)$$

$$\sigma_{zz} = \left(\frac{k_j c_{33}}{\nu_j} - c_{13}\right) L_j \left(4f'G\sqrt{\frac{2}{\pi}}\,\hat{a}\,D_{j2}(\hat{a})\right)\cos\theta$$

$$+ \rho_o \frac{s_2 C_{12}(\hat{a}) - s_1 C_{22}(\hat{a})}{(s_1 - s_2)} \tag{89f}$$

where, as before, $j = 1,2$ summed, $i = 1,2,3$ summed, $L_1 = M_1$, $L_2 = M_2$, $L_3 = -M_3$.

(iii) Stresses as $r \to 0$

By substituting the leading terms of the series expansions of the Bessel functions into equations (37) the stresses can be examined as the radius r approaches zero for tangential loading. The analysis for normal loading has been performed in [2]. For the tangential loading σ_{rr}, $\sigma_{\theta\theta}$, σ_{zz}, and $\sigma_{r\theta}$ vanish as $r \to 0$. In the normal case σ_{rz} vanishes and $\sigma_{r\theta}$ and $\sigma_{z\theta}$ do not arise under normal loading.

For the case where tangential loading is not producing sliding of the spheres, the superposition of the stresses for $r \to 0$ is

$$\sigma_{rr} = \sigma_{\theta\theta} = \rho_o\left\{\frac{\mu}{2} - \frac{1}{\sqrt{d}} - \frac{z}{\hat{a}}\left[\left(m_1 - \frac{\ell_1}{2}\right)\arctan\left(\frac{\hat{a}}{s_1 z}\right)\right.\right.$$

$$\left.\left. - \left(m_2 - \frac{\ell_2}{2}\right)\arctan\left(\frac{\hat{a}}{s_2 z}\right)\right]\right\} \tag{90a}$$

$$\sigma_{r\theta} = 0 \tag{90b}$$

$$\sigma_{z\theta} = \frac{c_{44}(1+k_i)}{2\sqrt{\nu_i}} M_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left(\frac{\hat{c}}{\hat{c}^2+s_i^2 z^2}\right) -2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\left(1-\frac{s_i z}{\hat{a}}S_{i,1}(\hat{a})\right)\right.\right.$$

$$\left.\left. + 2\hat{c}\left(1-\frac{s_i z}{\hat{c}}S_{i,1}(\hat{c})\right) + (\hat{a}^2-\hat{c}^2)\left(\frac{\hat{c}}{\hat{c}^2+s_i^2 z^2}\right)\right]\right\} \sin\theta$$

(90c)

$$\sigma_{rz} = \frac{-c_{44}(1+k_i)}{2\sqrt{\nu_i}} M_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left(\frac{\hat{c}}{\hat{c}^2+s_i^2 z^2}\right) -2f'G\sqrt{\frac{2}{\pi}}\left[-2\hat{a}\left(1-\frac{s_i z}{\hat{a}}S_{i,1}(\hat{a})\right)\right.\right.$$

$$\left.\left. + 2\hat{c}\left(1-\frac{s_i z}{\hat{c}}S_{i,1}(\hat{c})\right) + (\hat{a}^2-\hat{c}^2)\left(\frac{\hat{c}}{\hat{c}^2+s_i^2 z^2}\right)\right]\right\} \cos\theta$$

(90d)

$$\sigma_{zz} = \rho_o\left\{-1 - \frac{s_1 s_2}{s_1-s_2}\cdot\frac{z}{\hat{a}}\left[\arctan\left(\frac{\hat{a}}{s_1 z}\right) - \arctan\left(\frac{\hat{a}}{s_2 z}\right)\right]\right\}$$

(90e)

where $i = 1,2,3$ summed and $k_3 = 0$.

For the case of the sliding spheres $\hat{c} \to 0$ and equations (90a,b,e) remain the same. Equations (90c,d) become

$$\sigma_{rz} = \frac{-c_{44}(1+k_i)}{\sqrt{\nu_i}} M_i \left(2f'G\sqrt{\frac{2}{\pi}}\right)\hat{a}\left(1-\frac{s_i z}{\hat{a}}S_{i,1}(\hat{a})\right)\cos\theta \tag{91a}$$

$$\sigma_{z\theta} = \frac{c_{44}(1+k_i)}{\sqrt{\nu_i}} M_i \left(2f'G\sqrt{\frac{2}{\pi}}\right) \hat{a} \left(1 - \frac{s_i z}{\hat{a}} S_{i,1}(\hat{a})\right) \cos\theta \tag{91b}$$

with the same summation convention as (90).

(c) Stresses on the Surface

On the surface $z = 0$ the exponential term in (37) goes to unity. The integral expressions (75) are likewise simplified and evaluated.

Performing the necessary differentiation and substitution in (37), the stress expressions on the surface for tangential loading are obtained. The stresses on the surface for the normal loading are given in [2]. The superposed results give the stress state for the static case on $z = 0$ as

on

$0 < r < \hat{c}$

$$\sigma_{rr} = \rho_o \left\{ -\frac{1}{\sqrt{d'}} \left(1 - \frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} + \frac{\mu \hat{a}^2}{3r^2} \left[1 - \left(1 - \frac{r^2}{\hat{a}^2}\right)^{3/2}\right] \right\} \tag{92a}$$

$$\sigma_{\theta\theta} = \rho_o \left\{ \left(\mu - \frac{1}{\sqrt{d'}}\right)\left(1 - \frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} - \frac{\mu \hat{a}^2}{3r^2} \left[1 - \left(1 - \frac{r^2}{\hat{a}^2}\right)^{3/2}\right] \right\} \tag{92b}$$

$$\sigma_{r\theta} = 0 \tag{92c}$$

$$\sigma_{rz} = \frac{c}{2} \left\{ \left(-\frac{\Delta}{e}\frac{2}{\pi}\right) [(\hat{c}^2 - r^2)^{-\frac{1}{2}}] - 2f'G\sqrt{\frac{2}{\pi}} [-2(\hat{a}^2 - r^2)^{\frac{1}{2}} \right.$$

$$\left. + (\hat{a}^2 - r^2)(\hat{c}^2 - r^2)^{-\frac{1}{2}} + (\hat{c}^2 - r^2)^{\frac{1}{2}}] \right\} \cos\theta \tag{92d}$$

$$\sigma_{z\theta} = -\frac{c}{2}\left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)(\hat{c}^2-r^2)^{-\frac{1}{2}} - 2f'G\sqrt{\frac{2}{\pi}}\left[-2(\hat{a}^2-r^2)^{\frac{1}{2}} + (\hat{c}^2-r^2)^{\frac{1}{2}}\right.\right.$$

$$\left.\left. + (\hat{a}^2-r^2)(\hat{c}^2-r^2)^{-\frac{1}{2}}\right]\right\}\sin\theta \qquad (92e)$$

$$\sigma_{zz} = -\rho_o\left(1-\frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} \qquad (92f)$$

on $\hat{c} < r < \hat{a}$

$$\sigma_{rr} = \left(\frac{c_{13}k_j}{\nu_j} - c_{11}\right) L_j \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{-\frac{1}{2}}}{r}\right]\right.$$

$$- 2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\pi r}{2} - \frac{\hat{c}}{4r}(r^2-\hat{c}^2)^{\frac{1}{2}} + r\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right.$$

$$\left.\left. + \left(\frac{\hat{a}^2\hat{c}}{r} - \frac{\hat{c}^3}{4r} - \frac{3r\hat{c}}{4}\right)(r^2-\hat{c}^2)^{-\frac{1}{2}}\right]\right\}\cos\theta$$

$$+ \left(\frac{c_{11}-c_{12}}{r}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{\frac{1}{2}}}{r^2}\right]\right.$$

$$- 2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\pi r^2}{8} + \frac{\hat{c}(r^2-\hat{c}^2)^{3/2}}{2r^2} + \left(\frac{\hat{a}^2\hat{c}}{r^2} - \frac{3\hat{c}}{4}\right)(r^2-\hat{c}^2)^{\frac{1}{2}}\right.$$

$$\left.\left. + \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right]\right\}\cos\theta$$

$$+ \rho_o\left\{-\frac{1}{\sqrt{d'}}\left(1-\frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} + \frac{\mu\hat{a}^2}{3r^2}\left[1-\left(1-\frac{r^2}{\hat{a}^2}\right)^{3/2}\right]\right\} \qquad (93a)$$

$$\sigma_{\theta\theta} = \left(\frac{c_{13}k_j}{\nu_j} - c_{12}\right) L_j \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{-\frac{1}{2}}}{r}\right]\right.$$

$$- 2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\pi r}{2} - \frac{\hat{c}}{4r}(r^2-\hat{c}^2)^{\frac{1}{2}} + r\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right.$$

$$\left.\left. + \left(\frac{\hat{a}^2\hat{c}}{r} - \frac{\hat{c}^3}{4r} - \frac{3r\hat{c}}{4}\right)(r^2-\hat{c}^2)^{-\frac{1}{2}}\right]\right\}\cos\theta$$

$$+ \left(\frac{c_{12}-c_{11}}{r}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{\frac{1}{2}}}{r^2}\right] -2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\pi r^2}{8} + \frac{\hat{c}(r^2-\hat{c}^2)^{3/2}}{2r^2}\right.\right.$$

$$\left.\left. + \left(\frac{\hat{a}^2\hat{c}}{r^2} - \frac{3\hat{c}}{4}\right)(r^2-\hat{c}^2)^{\frac{1}{2}} + \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right]\right\}\cos\theta$$

$$+ \rho_o\left\{\left(\mu - \frac{1}{\sqrt{d'}}\right)\left(1 - \frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} - \frac{\mu\hat{a}^2}{3r^2}\left[1 - \left(1 - \frac{r^2}{\hat{a}^2}\right)^{3/2}\right]\right\} \tag{93b}$$

$$\sigma_{r\theta} = \left(\frac{c_{11}-c_{12}}{2}\right) M_3\left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{-\frac{1}{2}}}{r}\right] -2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\pi r}{2} - \frac{\hat{c}}{4r}(r^2-\hat{c}^2)^{\frac{1}{2}}\right.\right.$$

$$\left.\left. + r\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \left(\frac{\hat{a}^2\hat{c}}{r} - \frac{\hat{c}^3}{4r} - \frac{3r\hat{c}}{4}\right)(r^2-\hat{c}^2)^{-\frac{1}{2}}\right]\right\}\sin\theta$$

$$+ \left(\frac{c_{11}-c_{12}}{r}\right) L_i\left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{\frac{1}{2}}}{r^2}\right] -2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\pi r^2}{8} + \frac{\hat{c}(r^2-\hat{c}^2)^{3/2}}{2r^2}\right.\right.$$

$$\left.\left. + \left(\frac{\hat{a}^2\hat{c}}{r^2} - \frac{3\hat{c}}{4}\right)(r^2-\hat{c}^2)^{\frac{1}{2}} + \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right]\right\}\sin\theta \tag{93c}$$

$$\sigma_{rz} = \frac{c}{2}\left\{-2f'G\sqrt{\frac{2}{\pi}}\,[-2(\hat{a}^2-r^2)^{\frac{1}{2}}]\right\}\cos\theta \tag{93d}$$

$$\sigma_{z\theta} = -\frac{c}{2}\left\{-2f'G\sqrt{\frac{2}{\pi}}\,[-2(\hat{a}^2-r^2)^{\frac{1}{2}}]\right\}\sin\theta \tag{93e}$$

$$\sigma_{zz} = -\rho_o\left(1-\frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} \tag{93f}$$

and on $\hat{a} < r < \infty$

$$\sigma_{rr} = \left(\frac{c_{13}k_j}{\nu_j} - c_{11}\right) L_j \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{-\frac{1}{2}}}{r}\right] - 2f'G\sqrt{\frac{2}{\pi}}\left[\frac{\hat{a}}{4r}(r^2-\hat{a}^2)^{\frac{1}{2}}\right.\right.$$

$$- r\sin^{-1}\left(\frac{\hat{a}}{r}\right) - \frac{\hat{c}}{4r}(r^2-\hat{c}^2)^{\frac{1}{2}} + r\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \left(\frac{4\hat{a}^2\hat{c}}{4r} - \frac{3\hat{c}r}{4}\right)\cdot$$

$$\left.\left.(r^2-\hat{c}^2)^{-\frac{1}{2}}\right]\right\}\cos\theta + \left(\frac{c_{11}-c_{12}}{r}\right)L_i\left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{\frac{1}{2}}}{r^2}\right]\right.$$

$$- 2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\hat{a}(r^2-\hat{a}^2)^{3/2}}{2r^2} + \left(\frac{3\hat{a}}{4} - \frac{\hat{a}^3}{r^2}\right)(r^2-\hat{a}^2)^{\frac{1}{2}} - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right.$$

$$\left.\left.+ \frac{\hat{c}(r^2-\hat{c}^2)^{3/2}}{2r^2} + \left(\frac{\hat{a}^2\hat{c}}{r^2} - \frac{3\hat{c}}{4}\right)(r^2-\hat{c}^2)^{\frac{1}{2}} + \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right]\right\}\cos\theta$$

$$+ \rho_o\frac{\mu}{3}\cdot\frac{\hat{a}^2}{r^2} \tag{94a}$$

$$\sigma_{\theta\theta} = \left(\frac{c_{13}k_1}{\nu_j} - c_{12}\right) L_j \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{-\frac{1}{2}}}{r}\right]\right.$$

$$- 2f'G\sqrt{\frac{2}{\pi}}\left[\frac{\hat{a}}{4r}(r^2-\hat{a}^2)^{\frac{1}{2}} - r\sin^{-1}\left(\frac{\hat{a}}{r}\right) - \frac{\hat{c}}{4r}(r^2-\hat{c}^2)^{\frac{1}{2}}\right.$$

$$\left.\left. + r\sin^{-1}\left(\frac{\hat{c}}{r}\right) + \left(\frac{4\hat{a}^2\hat{c}}{4r} - \frac{3\hat{c}r}{4}\right)(r^2-\hat{c}^2)^{-\frac{1}{2}}\right]\right\}\cos\theta$$

$$+ \left(\frac{c_{12}-c_{11}}{r}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{\frac{1}{2}}}{r^2}\right] - 2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\hat{a}(r^2-\hat{a}^2)^{3/2}}{2r^2}\right.\right.$$

$$+ \left(\frac{3\hat{a}}{4} - \frac{\hat{a}^3}{r^2}\right)(r^2-\hat{a}^2)^{\frac{1}{2}} - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right) + \frac{\hat{c}(r^2-\hat{c}^2)^{3/2}}{2r^2}$$

$$\left.\left. + \left(\frac{\hat{a}^2\hat{c}}{r^2} - \frac{3\hat{c}}{4}\right)(r^2-\hat{c}^2)^{\frac{1}{2}} + \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right]\right\}\cos\theta$$

$$- \rho_o\frac{\mu}{3}\cdot\frac{\hat{a}^2}{r^2} \tag{94b}$$

$$\sigma_{r\theta} = \left(\frac{c_{11}-c_{12}}{2}\right) M_3 \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{-\frac{1}{2}}}{r}\right] - 2f'G\sqrt{\frac{2}{\pi}}\left[\frac{\hat{a}}{4r}(r^2-\hat{a}^2)^{\frac{1}{2}}\right.\right.$$

$$- r\sin^{-1}\left(\frac{\hat{a}}{r}\right) - \frac{\hat{c}}{4r}(r^2-\hat{c}^2)^{\frac{1}{2}} + r\sin^{-1}\left(\frac{\hat{c}}{r}\right)$$

$$\left.\left. + \left(\frac{4\hat{a}^2\hat{c}}{4r} - \frac{3\hat{c}r}{4}\right)(r^2-\hat{c}^2)^{-\frac{1}{2}}\right]\right\}\sin\theta$$

$$+ \left(\frac{c_{11}-c_{12}}{r}\right) L_i \left\{\left(-\frac{\Delta}{e}\frac{2}{\pi}\right)\left[\frac{\hat{c}(r^2-\hat{c}^2)^{\frac{1}{2}}}{r^2}\right] -2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\hat{a}(r^2-\hat{a}^2)^{3/2}}{2r^2}\right.\right.$$

$$+ \left(\frac{3\hat{a}}{4} - \frac{\hat{a}^3}{r^2}\right)(r^2-\hat{a}^2)^{\frac{1}{2}} - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right) + \frac{\hat{c}(r^2-\hat{c}^2)^{3/2}}{2r^2}$$

$$\left.\left. + \left(\frac{\hat{a}^2\hat{c}}{r^2} - \frac{3\hat{c}}{4}\right)(r^2-\hat{c}^2)^{\frac{1}{2}} + \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{c}}{r}\right)\right]\right\} \sin\theta \tag{94c}$$

$$\sigma_{rz} = \sigma_{z\theta} = \sigma_{zz} = 0 \tag{94d}$$

where $i = 1,2,3$ summed, $j = 1,2$ summed, $L_1 = M_1$, $L_2 = M_2$, $L_3 = -M_3$ before, and c is defined by (45).

The above results are easily specialized for sliding as follows:

on

$0 < r < \hat{a}$

$$\sigma_{rr} = \left[\left(\frac{c_{13}k_j}{\nu_j} - c_{11}\right) L_j \left(f'G\sqrt{\frac{2}{\pi}}\,\pi r\right) + \left(\frac{c_{11}-c_{12}}{r}\right) L_i \left(f'G\sqrt{\frac{2}{\pi}}\frac{\pi r^2}{4}\right)\right]\cos\theta$$

$$+ p_o\left\{-\frac{1}{\sqrt{d'}}\left(1 - \frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} + \frac{\mu\hat{a}^2}{3r^2}\left[1 - \left(1 - \frac{r^2}{\hat{a}^2}\right)^{3/2}\right]\right\} \tag{95a}$$

$$\sigma_{\theta\theta} = \left[\left(\frac{c_{13}k_j}{\nu_j} - c_{12}\right) L_j \left(f'G\sqrt{\frac{2}{\pi}}\,\pi r\right) + \left(\frac{c_{12}-c_{11}}{r}\right) L_i \left(f'G\sqrt{\frac{2}{\pi}}\frac{\pi r^2}{4}\right)\right]\cos\theta$$

$$+ p_o\left\{\left(\mu - \frac{1}{\sqrt{d'}}\right)\left(1 - \frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} - \frac{\mu\hat{a}^2}{3r^2}\left[1 - \left(1 - \frac{r^2}{\hat{a}^2}\right)^{3/2}\right]\right\} \tag{95b}$$

$$\sigma_{r\theta} = \left[\left(\frac{c_{11}-c_{12}}{2}\right) M_3 \left(f'G\sqrt{\frac{2}{\pi}}\,\pi\, r\right) + \left(\frac{c_{11}-c_{12}}{r}\right) L_i \left(f'G\sqrt{\frac{2}{\pi}}\,\frac{\pi r^2}{4}\right)\right] \sin\theta \tag{95c}$$

$$\sigma_{rz} = c\left[2f'G\sqrt{\frac{2}{\pi}},(\hat{a}^2-r^2)^{\frac{1}{2}}\right] \cos\theta \tag{95d}$$

$$\sigma_{z\theta} = -c\left[2f'G\sqrt{\frac{2}{\pi}}\,(\hat{a}^2-r^2)^{\frac{1}{2}}\right] \sin\theta \tag{95e}$$

$$\sigma_{zz} = -\,\rho_o\left(1 - \frac{r^2}{\hat{a}^2}\right)^{\frac{1}{2}} \tag{95f}$$

and on $\hat{a} < r < \infty$

$$\sigma_{rr} = \left(\frac{c_{13}k_j}{\nu_j} - c_{11}\right) L_j \left\{-2f'G\sqrt{\frac{2}{\pi}}\left[\frac{\hat{a}}{4r}\,(r^2-\hat{a}^2)^{\frac{1}{2}} - r\,\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right]\right\} \cos\theta$$

$$+ \left(\frac{c_{11}-c_{12}}{r}\right) L_i \left\{-2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\hat{a}(r^2-\hat{a}^2)^{3/2}}{2r^2} + \left(\frac{3\hat{a}}{4} - \frac{\hat{a}^3}{r^2}\right)(r^2-\hat{a}^2)^{\frac{1}{2}}\right.\right.$$

$$\left.\left. - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right]\right\} \cos\theta$$

$$+ \rho_o\,\frac{\mu}{3}\cdot\frac{\hat{a}^2}{r^2} \tag{96a}$$

$$\sigma_{\theta\theta} = \left(\frac{c_{13}k_j}{\nu_j} - c_{12}\right) L_j \left\{-2f'G\sqrt{\frac{2}{\pi}}\left[\frac{\hat{a}}{4r}\,(r^2-\hat{a}^2)^{\frac{1}{2}} - r\,\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right]\right\} \cos\theta$$

$$+ \left(\frac{c_{12}-c_{11}}{r}\right) L_1 \left\{-2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\hat{a}(r^2-\hat{a}^2)^{3/2}}{2r^2} + \left(\frac{3\hat{a}}{4} - \frac{\hat{a}^3}{r^2}\right)(r^2-\hat{a}^2)^{\frac{1}{2}}\right.\right.$$

$$\left.\left. - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right]\right\} \cos\theta$$

$$- \rho_o \frac{\mu}{3} \cdot \frac{\hat{a}^2}{r^2} \tag{96b}$$

$$\sigma_{r\theta} = \left(\frac{c_{11}-c_{12}}{2}\right) M_3 \left\{-2f'G\sqrt{\frac{2}{\pi}}\left[\frac{\hat{a}}{4r}\,(r^2-\hat{a}^2)^{\frac{1}{2}} - r\,\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right]\right\} \sin\theta$$

$$+ \left(\frac{c_{11}-c_{12}}{r}\right) L_1 \left\{-2f'G\sqrt{\frac{2}{\pi}}\left[-\frac{\hat{a}(r^2-\hat{a}^2)^{3/2}}{2r^2} + \left(\frac{3\hat{a}}{4} - \frac{\hat{a}^3}{r^2}\right)(r^2-\hat{a}^2)^{\frac{1}{2}}\right.\right.$$

$$\left.\left. - \frac{r^2}{4}\sin^{-1}\left(\frac{\hat{a}}{r}\right)\right]\right\} \sin\theta \tag{96c}$$

$$\sigma_{rz} = \sigma_{\theta z} = \sigma_{zz} = 0\ . \tag{96d}$$

4. Numerical Results

In order to show the effect of this kind of anisotropy on the stress state, the stresses in the half-space and on the contact were examined numerically for magnesium and cadmium, two metals studied in [2]. For these calculations the coefficients of limiting friction $f_1' = .25$ and $f_2' = .50$ were used. As in [1] and [2], von Mises' criteria for plastic yielding,

$$J = \left\{\sigma_{zy}^2+\sigma_{xz}^2+\sigma_{xy}^2+ 1/6\left[(\sigma_{yy}-\sigma_{zz})^2+(\sigma_{zz}-\sigma_{xx})^2+(\sigma_{xx}-\sigma_{yy})^2\right]\right\}^{\frac{1}{2}} \tag{97}$$

was calculated and lines of constant J/ρ_o plotted in the sphere on the $y = 0$ plane, and on the $z = 0$ surface inside the contact zone. Figures 4.1-4.4 illustrate the results in the half-space, while figures 4.5-4.8 illustrate the results on the surface for $r \le \hat{a}$. Figure 4.9 represents the static case on the surface for magnesium with $f' = .5$, and $\hat{c} = .5\hat{a}$, and is included for contrast with the sliding cases.

It is noted in [2] that it is not generally possible to use von Mises' criteria to predict yielding in anisotropic metals. It is plotted herein as a means of contrasting the stress disturbance in the transversely isotropic metals to that in isotropic metals as displayed in [1]. It can be seen that magnesium behaves in much the same manner as the isotropic case in [1], while cadmium's elastic properties create a more unusual situation. Except for the case of magnesium with $f' = .5$ (which parallels the isotropic case of $f' = .5$) the maximum value of J/ρ_o is found on $z = 0$ at the leading edge of the contact, as is the case in [1].

The elastic moduli of magnesium and cadmium used in the numerical analysis are given in Table 4.1. Also listed are the moduli of steel for $\nu = .3$ illustrating values for an isotropic case. All units are N/m^2.

TABLE 4.1 ELASTIC MODULI FOR MAGNESIUM, CADMIUM, AND STEEL

	Magnesium	Cadmium	Steel ($c_{11}=c_{33}, c_{12}=c_{13}$, $c_{44}=\frac{1}{2}(c_{11}-c_{12})$)
c_{11}	5.857×10^{10}	1.092×10^{11}	2.691×10^{11}
c_{12}	2.501×10^{10}	3.976×10^{10}	1.153×10^{11}
c_{13}	2.079×10^{10}	3.754×10^{10}	1.153×10^{11}
c_{33}	6.110×10^{10}	4.602×10^{10}	2.691×10^{11}
c_{44}	1.658×10^{10}	1.562×10^{10}	7.690×10^{10}
$\frac{1}{2}(c_{11}-c_{12})$	1.678×10^{10}	3.472×10^{10}	7.690×10^{10}

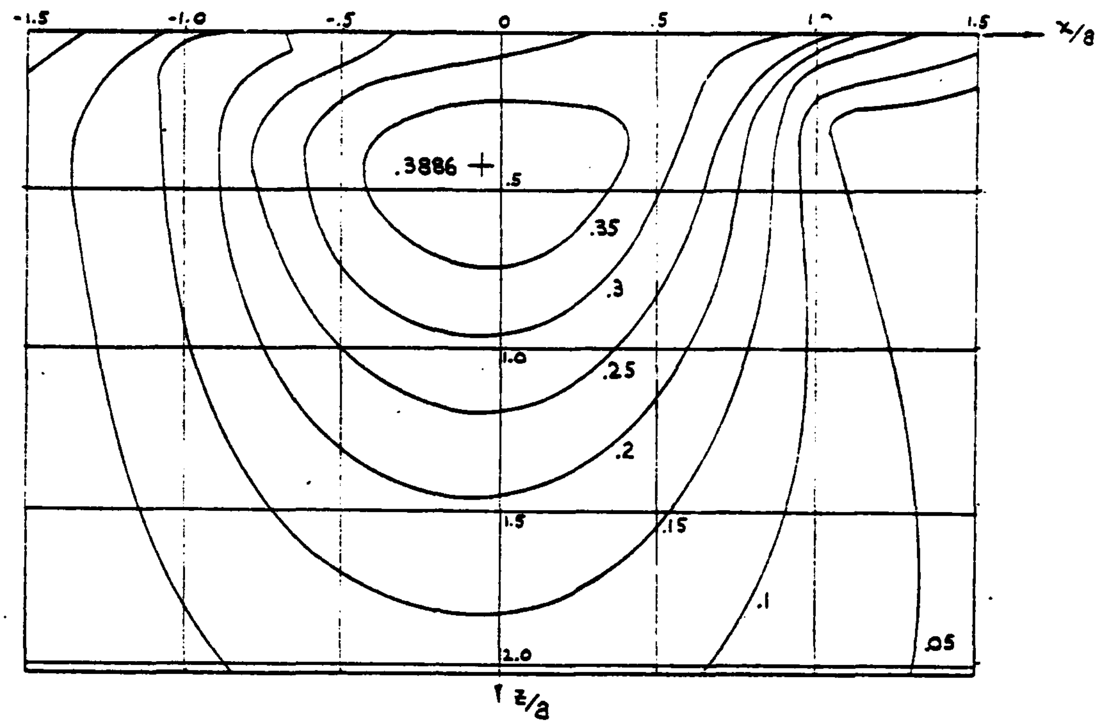

Figure 4.1: Magnesium, f'=.25

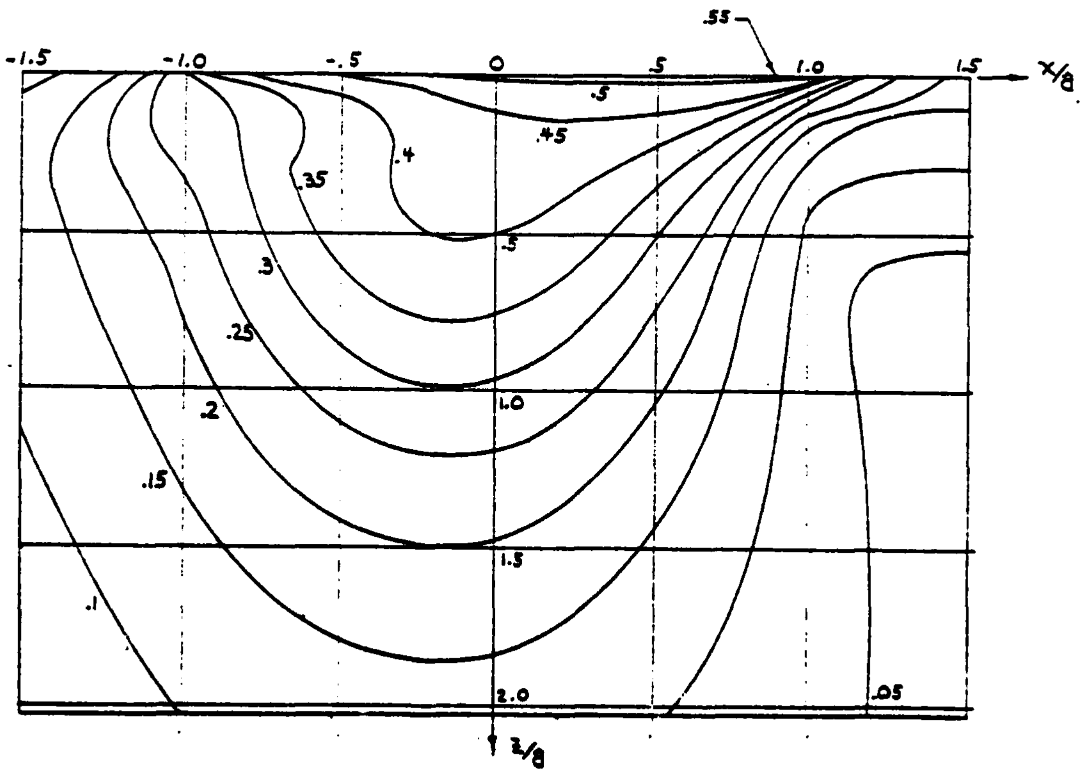

Figure 4.2: Magnesium, f'=.50

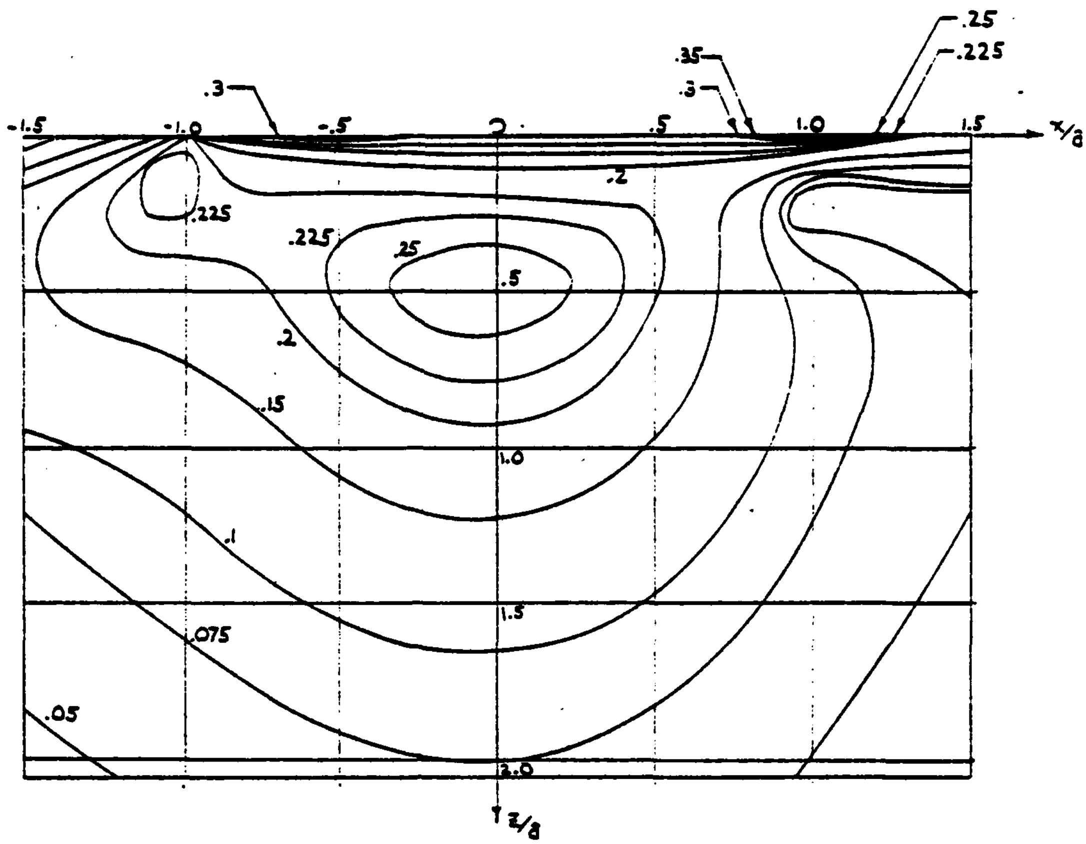

Figure 4.3: Cadmium, f'=.25

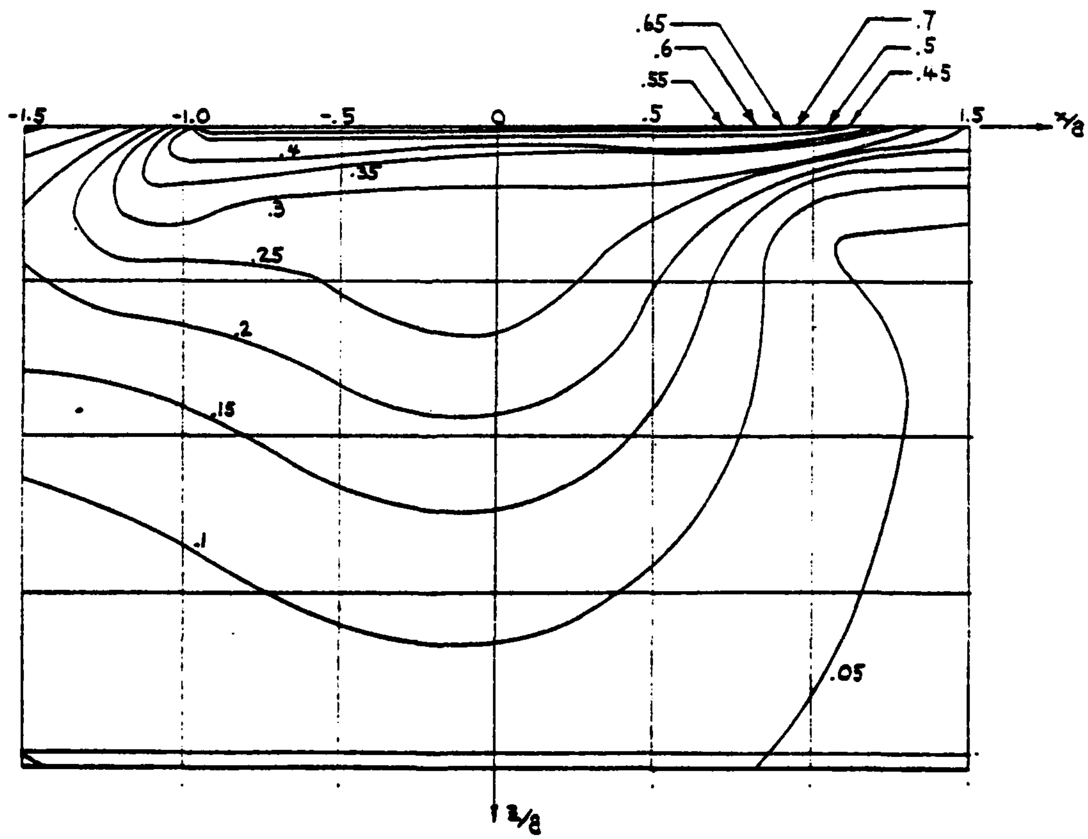

Figure 4.4: Cadmium, f'=.50

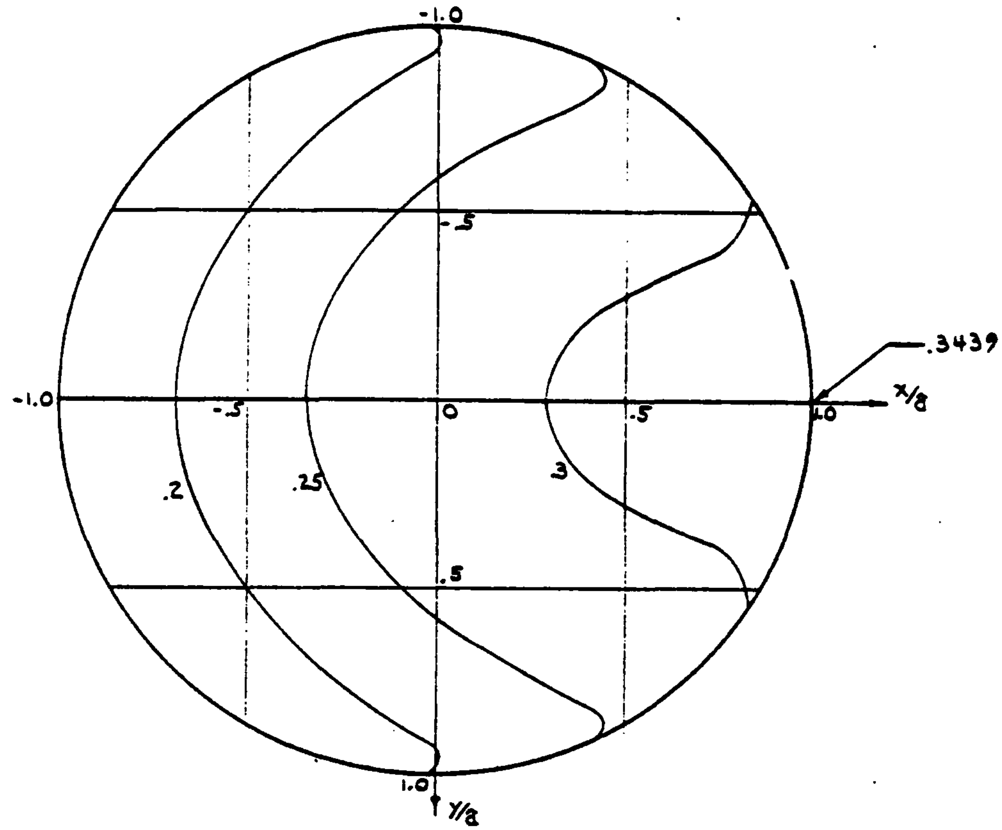

Figure 4.5: Magnesium, f'=.25

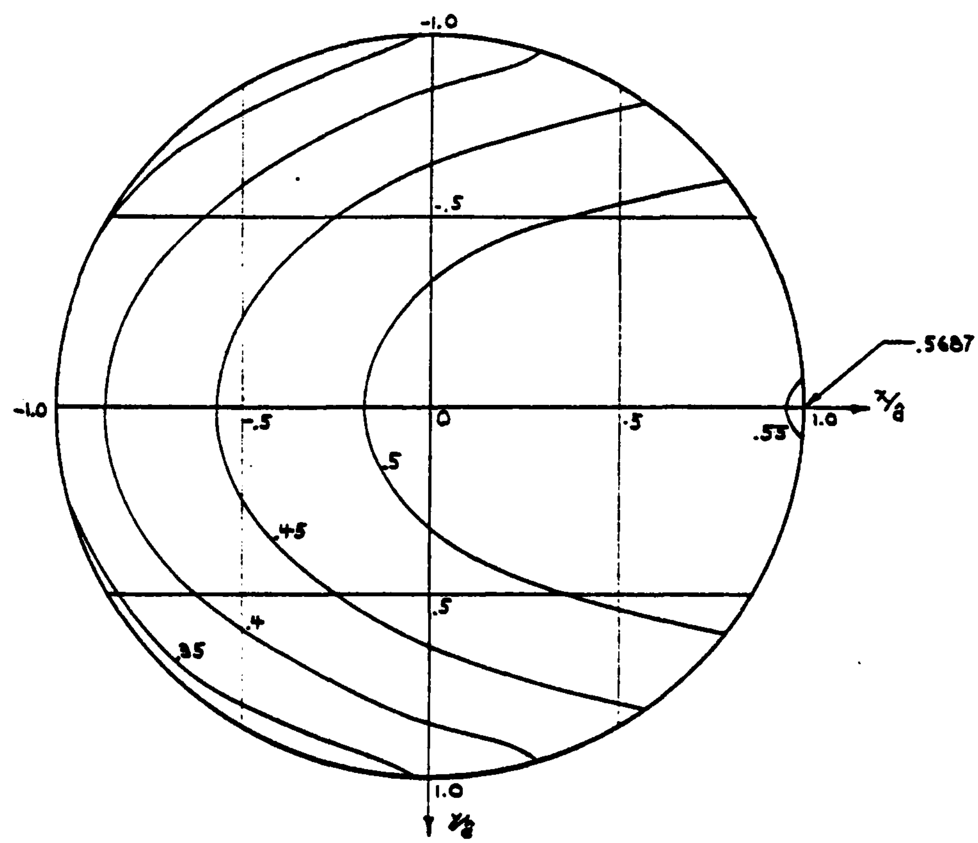

Figure 4.6: Magnesium, f'=.50

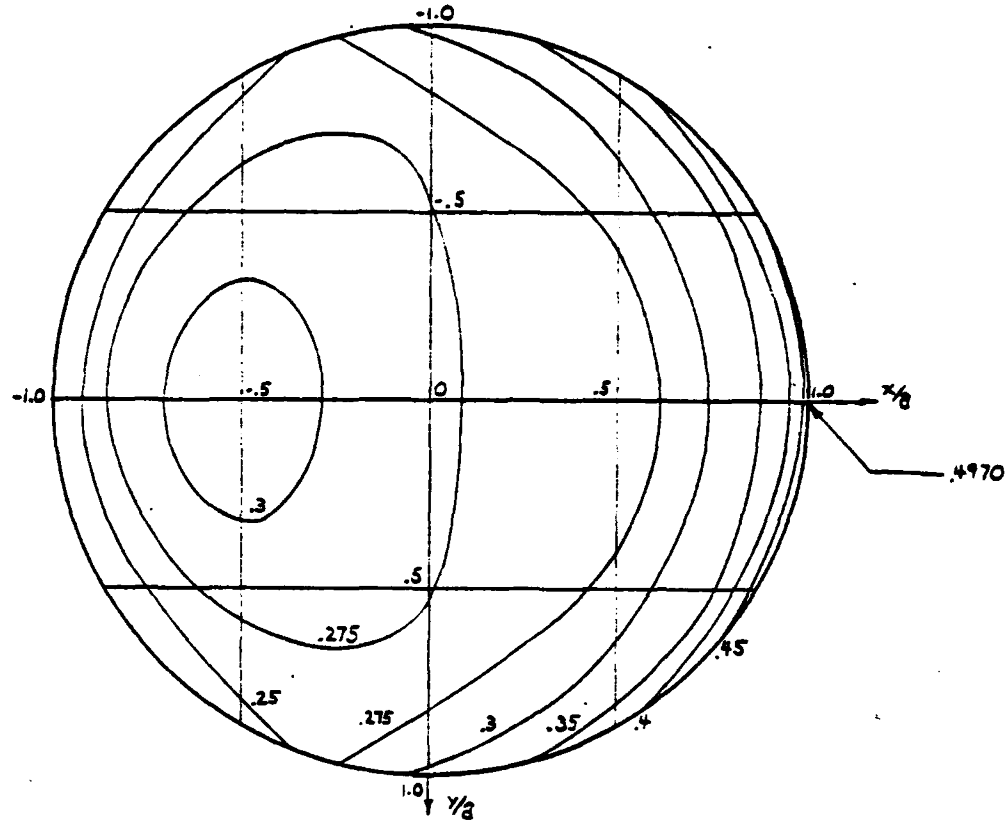

Figure 4.7: Cadmium, f′=.25

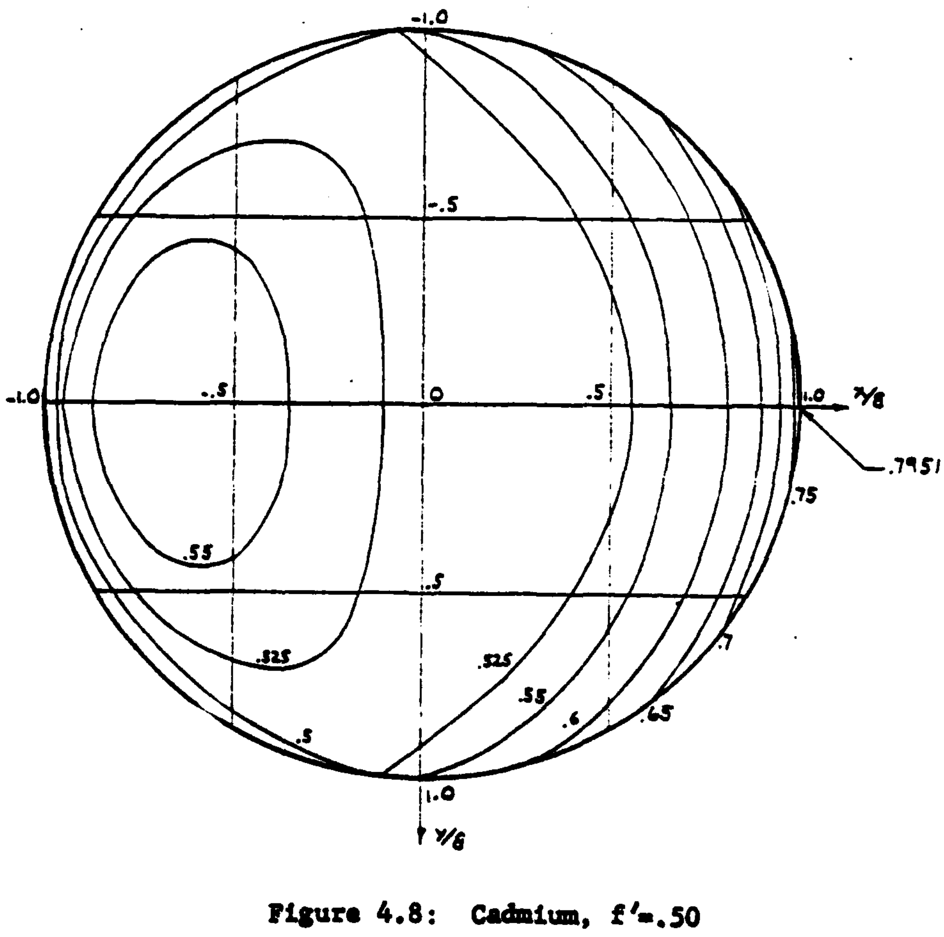

Figure 4.8: Cadmium, f′=.50

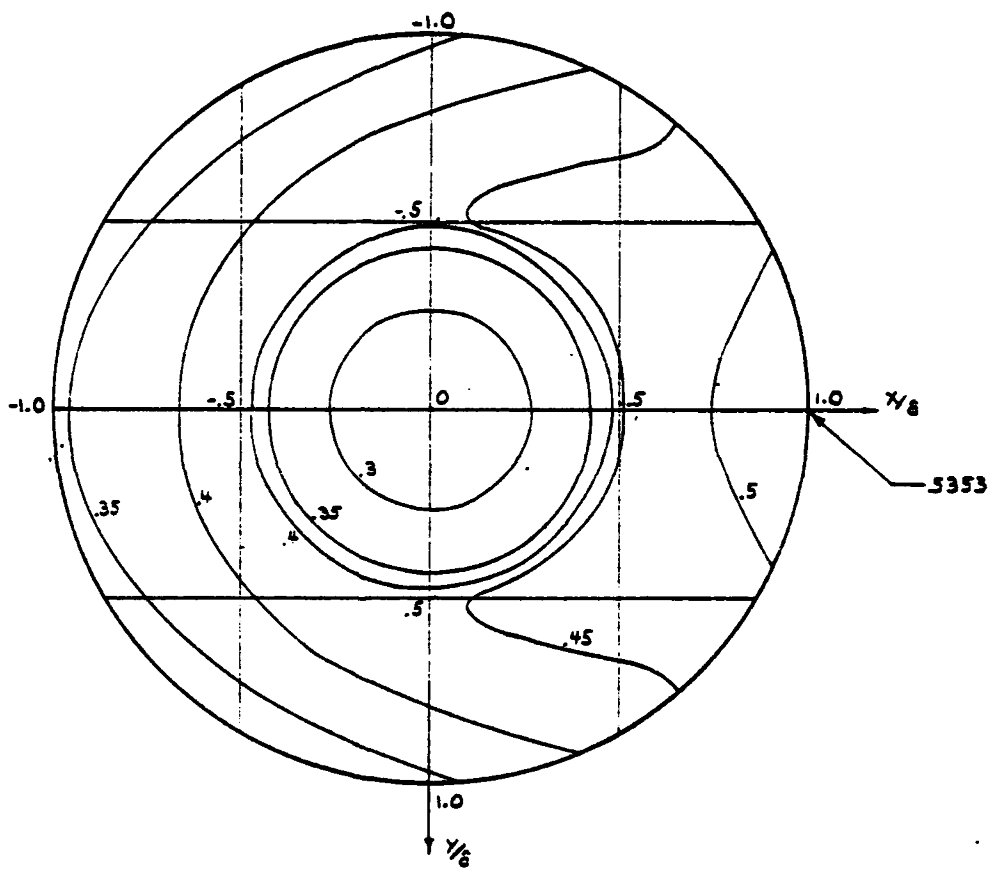

Figure 4.9: Magnesium, â′=.5, ê = .5 â

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Appendix A

$$a' = \frac{a_{13}(a_{11}-a_{12})}{a_{11}a_{33}-a_{13}^2} \qquad b' = \frac{a_{13}(a_{13}+a_{44})-a_{12}a_{33}}{a_{11}a_{33}-a_{13}^2}$$

$$c' = \frac{a_{13}(a_{11}-a_{12})+a_{11}a_{44}}{a_{11}a_{33}-a_{13}^2} \qquad d' = \frac{a_{11}^2-a_{12}^2}{a_{11}a_{33}-a_{13}^2}$$

$$s_1 = \left[\frac{a'+c'+\sqrt{(a'+c')^2-4d'}}{2d'}\right]^{\frac{1}{2}}$$

$$s_2 = \left[\frac{a'+c'-\sqrt{(a'+c')^2-4d'}}{2d'}\right]^{\frac{1}{2}}$$

$$\rho_1 = 1 - a's_1^2$$

$$\rho_2 = 1 - a's_2^2$$

$$q_1 = (b'-a's_2^2)\rho_1$$

$$q_2 = (b'-a's_1^2)\rho_2$$

$$\nu = \frac{(b'-1)\sqrt{d'}}{a'c' - d'}$$

$$\mu = \frac{(b'-1)(a'+\sqrt{d'})}{a'c' - d'}$$

$$\delta_1 = a_{44}\,\frac{s_1 s_2}{s_2 - s_1} + (a_{12} - a_{11})\,\frac{\nu s_1^2 \rho_2}{s_2 - s_1}$$

$$\delta_2 = a_{44}\,\frac{s_1 s_2}{s_2 - s_1} + (a_{12} - a_{11})\,\frac{\nu s_2^2 \rho_1}{s_2 - s_1}$$

$$m_1 = \frac{s_1^2}{(s_2 - s_1)\sqrt{d'}} \qquad m_2 = \frac{s_2^2}{(s_2 - s_1)\sqrt{d'}}$$

$$\ell_1 = \frac{\nu s_1^2 \rho_2}{s_2 - s_1} \qquad \ell_2 = \frac{\nu s_2^2 \rho_1}{s_2 - s_1}$$