We consider plane-wave motion at normal incidence in a horizontally layered system. The system is assumed lossless, and only the compressional waves are treated. A procedure is introduced for determining the reflection coefficients of the layered system when the observed seismic data may contain random noise. No deconvolution of the measured seismic data is required by the procedure when the input is a narrow wavelet.
ABSTRACT

We consider plane-wave motion at normal incidence in a horizontally layered system. The system is assumed lossless, and only the compressional waves are treated. A procedure is introduced for determining the reflection coefficients of the layered system when the observed seismic data may contain random noise. No deconvolution of the measured seismic data is required by the procedure when the input is a narrow wavelet.

1. INTRODUCTION

In recent years much attention has been given to the problem of determining reflection coefficients for a layered medium from the observed seismic data [1-4]. In line with the customary assumptions and restrictions, we also limit our attention to a horizontally stratified nonabsorptive earth with vertically traveling plane compressional waves. Such a system is completely described by a set of reflection coefficients and travel times within layers.

A fundamental procedure described in detail in the above references for deriving values of the reflection coefficients can be summarized by the following assumptions and steps.

Standard Assumptions:

(A1) The input wavelet is assumed known.

(A2) The data is assumed noise free.

(A3) The layered system is assumed to have uniform travel times between layers where a number of the layers are hypothetical, i.e., they may not correspond to an actual interface of the substructure and are associated with zero reflection and transmission coefficients.

Standard Steps:

(S1) The observed seismic data is deconvolved using the input waveform to produce the system response to a unit spike input.

(S2) The number of layers is chosen high enough to result in travel times short compared with the inverse of the bandwidth of the observed seismic data.

(S3) The deconvolved data is sampled with sampling interval equal to the chosen one-way travel time between layers.

(S4) The system structure is used to arrive at a set of normal equations (linear simultaneous equations) in terms of reflection coefficients and the discretized and deconvolved observed data.

(S5) The normal equations of the preceding step have the Toeplitz structure which makes it possible to utilize the very efficient Levinson algorithm to recursively solve for the reflection coefficients.

In this paper the method of solution to the inverse problem stated above is fundamentally modified to cope with the existence of the noise in the measurement data, often without need for any deconvolution. More specifically, although again a uniform layered system is assumed, the choice of number of layers can now be made independent of the sampling rate requirement of the data (step (S2) above) often resulting in the need for far fewer layers. No deconvolution is necessary (step (S1)) for wavelets of duration of the order of twice the layer travel times. The exact deconvolution of step (S1) is either not possible in practice or, at the least, will further aggravate the harmful effects of the noise in the observation [5]. Furthermore, the deconvolution is a time consuming operation. Finally, the procedure is very simple to derive and does not need the concepts of z-transforms, minimum phase, forward and backward polynomials, spectral factorization, etc.

The results reduce to the existing solution of the inverse problem in the absence of noise and with a spike input signal (wavelet) [1].

2. STATEMENT OF THE PROBLEM

We are considering a uniform K layered system and normal incident compressional waves. Figure 1 represents such a system where \( d_j(t) \) is the down-going wave at the bottom of the \( j \)-th layer and \( u_j(t) \) is the up-going wave at the top of the layer. The reflection, downward transmission and upward transmission coefficients associated with the interface at the bottom of \( j \)-th layer are denoted \( r_j, t_j, \) and \( t_j' \) respectively where \( t_j, t_j', r_j \) are given in terms of reflection coefficients and the one way travel time between layers is denoted by \( \tau \).

The input to the system, \( d_0(t) \), is assumed known (the wavelet) and the output may be either \( u_0(t) \) (in the marine environment) or \( y_j(t) \). The measured seismic data, \( y(t) \), consists of the output and an additive noise component \( n(t) \). The source of this noise may be the instrument measurement noise, the uncertainty in the knowledge of the input wavelet or response to unwanted inputs (ambient noise). It is desired to process \( y(t) \), \( t \geq 0 \) and derive values for the reflection coefficients \( r_j, j = 1, ..., K \) or \( r_j \) may be assumed known in cases such as the marine environment.

3. STATE EQUATIONS

Using the notation of Fig. 1, for a general \( j \)-th layer...
we have \[ u_j(t+c) = t_j u_{j-1}(t) + r_j d_j(t) \]

(1)

\[ d_{j+1}(t+c) = -r_{j+1} u_{j+1}(t) + t_j d_j(t) \]

(2)

These equations are valid for \( j = 1, \ldots, K-1 \). The

They should be augmented at the surface with

\[ u_0(t) = t_0 u_{-1}(t) + r_0 d_0(t) \]

(3)

and at the basement with

\[ u_K(t+c) = t_K u_{K-1}(t) + r_K d_K(t) \]

(4)

and

\[ d_{K+1}(t+c) = -t_{K+1} u_K(t) + t_K d_K(t) \]

(5)

Equations (3, 4) and (5, 6) can be derived from (1) and (2) [letting \( j = 0, 1, \ldots, K \)] by noting that \( u_j(t) \) is taken at the bottom of layer 0 and \( d_{j+1}(t) \) represents the down-going wave leaving the last interface and is not reflected by any other interface; hence \( u_{K+1}(t) = 0 \). These equations, called causal functional, are not difference equations since \( t \) is the continuous time variable.

Using the state equations given above, it can be shown \cite{Appendix B, 9} that the function \( d_{j+1} \) satisfies the equation

\[ d_{j+1}(t+c) = d_{j+1}(t) + r_j u_j(t) + t_j d_j(t) \]

\[ + r_j u_j(t) + t_j d_j(t) \]

(6)

Note that the coefficient of the highest term of the left hand side is unity and that of the lowest terms is \( r_j t_j \). The precise form of the other coefficients is not important. Only the structural form of (7) will be utilized in the sequel. In this equation, the unknowns are the reflections coefficients which are embedded in the coefficients \( a_j, \ldots, a_{j+1}, r_j t_j \) and \( a_j \). The input \( u_0(t) \) is assumed known. Equation (7) is the starting point for our inverse procedure; however, it is in terms of a signal which, in general, is not measurable. In the following two sections we relate \( d_{j+1}(t) \) to measured seismic data. We do this so that we will be able to extract the reflection coefficients from measured data.

4. A GENERALIZED ENERGY TRANSFER KUNETZ RELATION

Consider \( t \) to be a non-negative continuous or discrete variable with dimension of time. Equations (1) and (2) where \( j = 0, 1, \ldots, K \) are multiplied

by \( u_j(t+c) \) and \( d_j(t+c) \) respectively resulting in

\[ u_j(t+c) u_{j+1}(t+c) + u_j(t+c) d_j(t+c) \]

\[ + r_j u_j(t+c) u_{j+1}(t+c) + r_j d_j(t+c) u_{j+1}(t+c) \]

(8)

\[ d_{j+1}(t+c) u_{j+1}(t+c) + d_{j+1}(t+c) d_j(t+c) \]

\[ - r_{j+1} u_{j+1}(t+c) u_{j+1}(t+c) - t_{j+1} d_{j+1}(t+c) d_{j+1}(t+c) \]

(9)

Multiplying (8) by \( t_j / t_{j+1} \) and adding the resulting expression to (9) yields

\[ d_{j+1}(t+c) d_j(t+c) = (t_j / t_{j+1}) u_j(t+c) u_{j+1}(t+c) \]

(10)

Let us define the following correlation-type functions

\[ D_j(c) = \int_{-\infty}^{+\infty} d_j(t) d_j(t+c) dt \]

(11)

Integrating both sides of Eq. (10) from \( -\infty \) to \( +\infty \), we obtain

\[ D_j(c) = \int_{-\infty}^{+\infty} [D_j(t) - u_j^2(t)] dt \]

(12)

\[ D_j(c) = \int_{-\infty}^{+\infty} d_j(t) d_j(t+c) dt \]

Integrating both sides of Eq. (10) from \(-\infty\) to \(+\infty\), we find that

\[ D_j(c) = \int_{-\infty}^{+\infty} [D_j(t) - u_j^2(t)] dt \]

(13)

where \( j = 0, 1, 2, \ldots, K \). This is a generalization of the well-known \cite{1} energy transfer (Kunetz) relation. Note that in our derivation, input \( d_j(t) \) is not assumed to be an impulse and the seismic data is not discretized.

Iterating (13), starting with \( j = 1 \) and ending with \( j = K \), we obtain

\[ D_j(c) = \frac{1}{2} [D_j(c) - u_j^2(c)] \]

(14)

where \( j \) can take on the values of \( 0, 1, 2, \ldots, K \). In the marine case this relationship is used with \( j = 1 \). In the non-marine case it is used with \( j = 0 \).

5. APPLICATION TO MARINE ENVIRONMENT

In this section we will direct our attention to the marine case and shall express \( D_j(c) - u_j^2(c) \) in terms of measured signals. To do this, we set \( r_0 = 1 \) and we see from (14) that we must express \( D_j(c) - u_j^2(c) \) in terms of the measured signals. The first layer can be depicted as in Fig. 2. Observe that (4) becomes

\[ d_1(t+c) = u_1(t+c) + t_1 d_1(t+c) \]

(15)

From (11, 12, 13), we can evaluate the difference in terms \( D_j(c) - u_j^2(c) \),

\[ D_j(c) - u_j^2(c) = \int_{-\infty}^{+\infty} [2d_j(t) - u_j(t)] dt \]

(16)

or

\[ D_j(c) - u_j^2(c) = \int_{-\infty}^{+\infty} [2d_j(t) - u_j(t)] dt \]

(17)

hence, \( P(c) \) can be evaluated from a knowledge of \( d_j(t) \) and \( u_j(t) \) for any desired \( c \). Observe, also, that (14) with \( j = 1 \) can be written in terms of \( P(c) \), using (16), as

\[ D_1(c) = \frac{1}{2} \int_{-\infty}^{+\infty} P(c) \]

(18)

We should point out at this stage that the quantity \( u_j(t) \) needed in (17) is only available through the observation

\[ y(t) = u_j(t) + n(t) \]

(19)

where \( n(t) \) is the additive noise. Consequently, \( P(c) \) is not physically available; however, we can define \( \tilde{P}(c) \) by replacing \( y(t) \) for \( u_j(t) \) in (17),

\[ \tilde{P}(c) = \int_{-\infty}^{+\infty} 4d_j(t)d_j(t+c) dt - \int_{-\infty}^{+\infty} 2d_j(t+y(t)) dt \]

(20)

Because of the range of integration in (11), we can also express \( D_j(c) \) as \( D_j(c) = \int_{-\infty}^{+\infty} d_j(t) d_j(t+c) dt \). We use this form of (11) in our development of \( D_j(c) - u_j^2(c) \).
which can also be written as
\[ P(c) = P(c) + N(c), \quad (21) \]
where
\[ N(c) = \sum_{t=1}^{\infty} d_\alpha(t) n(t+c) dt - \sum_{t=1}^{\infty} 2n(t) d_\alpha(t+c) dt. \quad (22) \]
The statistics of noise term \( N(c) \) can be determined in terms of those of \( n(t) \). Using \( P(c) \) in (18) yields
\[ \hat{d}_\alpha(c) = \frac{1}{P(c)} \cdot (23) \]
where \( P(c) \) is a known quantity. Equation (23) is a fundamental relationship which will be used in the derivation of the inverse procedure.

6. DERIVATION OF THE NORMAL EQUATIONS

Equation (7) is the main relation which will be used to derive the reflection coefficients. Dividing both sides by \( \sum t_1 \), and identifying the resulting coefficients by \( \hat{d}_0, \ldots, \hat{d}_{K-2} \), we get
\[ 1 = \sum_{t=1}^{\infty} 2 \hat{d}_\alpha(t) n(t-K) dt + \sum_{t=1}^{\infty} 2 \hat{d}_\beta(t) n(t-K) dt. \quad (24) \]
We compute the coefficients of this equation by means of the following "least-squares" criterion:
\[ \min_{\hat{d}_0, \ldots, \hat{d}_{K-2}} \left\{ \sum_{t=1}^{\infty} 2 \hat{d}_\alpha(t) n(t-K) dt + \sum_{t=1}^{\infty} 2 \hat{d}_\beta(t) n(t-K) dt \right\}. \quad (25) \]
The result of this minimization is equivalent to multiplying both sides of (7) by \( d_\alpha(t[K-K-1]) \) and integrating from \( -\infty \) to \( \infty \) for \( i = 0, 1, \ldots, K-2 \). By either approach, we obtain \( K+1 \) simultaneous equations, which, using (11), become
\[ \begin{bmatrix} D_{\alpha,0} & D_{\alpha,1} & \cdots & D_{\alpha,2K-1} \\ D_{\alpha,1} & D_{\alpha,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ D_{\alpha,2K-1} & \vdots & \cdots & D_{\alpha,0} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{K-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (26) \]
where we have substituted \( t_1 = 1 + t_0 = 2 \) to represent the marine environment and
\[ a_{K-1} = \sum_{t=1}^{\infty} d_\alpha(t) d_{\alpha,0}(t[K-K-1]) dt, \quad i = 0, 1, 2, \ldots, K. \quad (27) \]
Substituting for \( D_{\alpha,i} \) from (23), we find that
\[ (26) \] reduces to
\[ \begin{bmatrix} \hat{P}(0) & \hat{P}(2) & \cdots & \hat{P}(2K) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{K-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (28) \]
Note that the \( (K+1) \times (K+1) \) matrix on the left is the Toeplitz structure. The terms \( N(0), N(2), \ldots \) which appear in \( \hat{P}(0), \hat{P}(2), \ldots \), are random variables with known statistics; they will be zero if the noise is not deterministic; however, we will show in the following that the input wavelet is narrow enough (not necessarily a spike), (28) has a unique solution for the reflection coefficients. The absence of this useful property in the non-marine case renders the procedure of this paper inapplicable in that case.

7. SPECIAL CASE OF NARROW WAVELET

Let us now consider the case where \( d_\alpha(t) \) does not extend beyond \( 2\tau \), i.e.,
\[ d_\alpha(t) = 0, \quad t < 0, \quad t > 2\tau. \quad (29) \]
Since the time of arrival at the \( K+1 \) interface is \( K\tau \) and the time of arrival of the first reflections is \( K\tau + \tau \),
\[ d_\alpha(t) = 0, \quad t < K\tau \quad (30a) \]
\[ \hat{d}_\alpha(t) = 2\tau^2 \int_0^t d_\alpha(t-K\tau) dt, \quad K\tau < t < (K+2)\tau \quad (30b) \]
\[ \ldots \]
\[ \text{more complicated terms } t > (K+2)\tau. \quad (30c) \]
From (27, 30a, 30b, and 29) we see that
\[ a_i = 2\tau^2 \int_0^t d_\alpha(t-K\tau) dt, \quad i = 0, 1, \ldots, K. \quad (31) \]
Note that (28) will have precisely \( K+1 \) unknowns. From (27, 30a, 30b) and (29) we see that
\[ a_i = 2\tau^2 \int_0^t d_\alpha(t-K\tau) dt, \quad i = 0, 1, \ldots, K. \quad (32) \]
Finally, Normal Equation (28) can be written in a compact matrix form as
\[ \begin{bmatrix} \hat{P}(0) & \hat{P}(2) & \cdots & \hat{P}(2K) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{K-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \quad (33) \]
where \( \hat{P}(t) \) is a \((K+1) \times (K+1)\) Toeplitz matrix with the first column consisting of \( \hat{P}(0), \hat{P}(2), \ldots, \hat{P}(2K) \); \( a_0 \) is a \((K+1) \times 1\) column vector with first and last elements 1 and \( r_0 \), respectively; and, \( \hat{d}_\alpha = (\hat{d}_\alpha(0), 0, \ldots, 0) \)
\[ \hat{d}_\alpha = 2\tau^2 \int_0^t d_\alpha(t) dt. \quad (34) \]
The Normal Equation (33) can be solved for \( a_0 \). This only produces one of the \( K \) reflection coefficients, namely \( r_0 \). We will show, in the following, that in the case of the marine environment, nested within (33) are a set of normal equations, the solutions of which produce each one of the reflection coefficients. The absence of this useful property in the non-marine case renders the procedure of this paper inapplicable in that case.

Let us now hypothesize a \( j \)-layer system (i.e., \( P_0, P_2, \ldots, P_{2K+1} \)), consisting of the top \( j \)-layers of the above \( K \)-layer system \( (K \geq j) \). Clearly, from (33), we have
\[ \hat{P}(t)_j = \hat{P}(t). \quad (35) \]
If this condition is not satisfied, we can always deconvolve the data to achieve this. Since the requirement here is not to deconvolve down to an impulse function (only \( 2\tau \) has to be satisfied), this results in a more practical solution.

For a narrow wavelet and \( \hat{d}_\alpha = 2\tau^2 \), the calculation of \( \hat{P}(t) \) simplifies since (20) reduces to
\[ \hat{P}(t) = -2\tau^2 \int_0^t d_\alpha(t) dt. \quad (36) \]
where \( a_j \) will again have 1 and \( r_j \) as first and last elements. We shall now show that, in the case of the marine environment, \( P_l \) is a \((j+1) \times (j+1)\) Toeplitz matrix, composed of the left corner of \( P_l \), i.e., its first row is given by \([P(0), P(2), \ldots, P(2i-2)]\).

For the moment let us ignore the additive noise term in (20). Let us denote by \( u(t)\) the response of the j-layer system (i.e., the term \( u(t) \) in Fig. 2 is replaced by \( u(i) \)). In [17], due to the fact that \( d(t) = 0 \) for \( t > 2i \), the last value of \( u(t) \) contributing to \( P_l \) is \( u(2j+1) \). In determination of \( P_l \), with elements \( P_l(a, c) \), \( c \geq 0, \ldots, 2j, \) for a j-layered system, therefore the last value of \( u(t) \) contributing to \( P_l \) is \( u(2j+1) \). On the other hand \( u_0(t) \) is the response of the K-layer system, and

\[
u_0(t) = u(t) \quad (35)
\]

since the first return from the interfaces below the \( j^{th}\) will not appear earlier than \( t = 2j+1 \). Hence, the elements of \( P_l \), which are functions of \( u(t) \) and \( P_l \), which are functions of \( u(t) \), are identical when (35) is satisfied. In other words, the numerical values of \( P(0), \ldots, P(2i-1) \) will be identical to those of the K-layer system for all \( j \leq K \). Furthermore, the additive noise term in (21) is independent of the number of layers, as is evident from (21).

The set of normal equations given by (34), for \( j = 1, \ldots, K \), can now be solved for the vectors \( a_j \) and \( r_j \), \( j = 1, \ldots, K \). The matrix \( P_l \) is Toeplitz and consequently, the Levinson algorithm [1] can be used to solve for the vectors \( a_j, r_j \) recursively.

Since the \( r_j \)'s are reflection coefficients, for the solution to this problem to be sensible, each \( r_j \) must be less than unity in magnitude. It is shown in [9] that any solution of (34) with \( \beta_j > 0 \) for all \( j \) yields a set of \( r_j \)'s which satisfy this condition. Moreover, if \( \beta_j > 0 \), it is guaranteed that \( |r_j| < 1 \) has nothing to do with a specific method of solution of the normal equations (i.e., the Levinson procedure). This result is different from the comparable result in [1-3], where one is left with the impression that a specific method of solution leads to \( |r_j| < 1 \).

8. EXPERIMENTAL RESULTS

A seven layer system was chosen with the following reflection coefficients: \( r_1, r_1 = 0.01, r_2 = 0.15, r_3 = -0.3, r_4 = 0.25, r_5 = 0.12, r_6 = 0.05, r_7 = 0.2 \). We used a one-way layer travel time and data sampling rate of 20 msec and 2 msec, respectively.

The input wavelet was chosen as depicted in Figure 3. Note that this is a non-minimum phase function. It was specifically chosen so as to indicate that the method introduced in this paper is not limited to minimum phase wavelets. Figure 4 is the synthetic seismogram response of the system. Figure 5 is obtained by adding white Gaussian noise with variance 1 to the sampled seismogram. Simulated results were obtained for variances of 0.1 and 10. These responses were then utilized to produce estimates of reflection coefficient values. The results of a Monte-Carlo simulation for 100 different samples of noise appear in Tables 1 and 2. Table 1 presents the mean value of error variance. As seen, the results are exact for zero noise variance (as expected) and are quite good for variances of 0.1 and 1. For noise variance of 10, although the average is not poor, the error variance indicates that the estimates are not very reliable.

A comparison between the procedure of this paper and the standard procedures described in [1-4] is warranted. Let us consider the noise term \( n(t) \) to be white. Clearly, (22) indicates that, for the random variables \( N(t) \) have finite variances. For this case \( n(t) \) (white), had we performed the necessary deconvolution and sampling required by the classical approach to the inverse problem, the resulting \( N(t) \) random variables would have infinite variance, clearly rendering the approach meaningless. Of course, an approximate deconvolution will eliminate this problem but at a great sacrifice in the information available within the seismic data. It should also be noted that, for the narrow wavelet, no deconvolution is required by the procedure outlined in this paper.

9. CONCLUSIONS

We have developed a procedure for extracting reflection coefficients from noisy data which we feel is a substantial generalization of similar procedures which have been reported in the literature. Associated with these earlier procedures are Standard Assumptions and Steps (see Introduction, p. 1) which include requirements that the data be noise free and that the observed seismic data be deconvolved. The procedure of our paper avoids these restrictive requirements. Furthermore, our procedure totally avoids the concepts of z-transforms, minimum phase, spectral factorization, etc., which appear in the literature on this subject. Finally, since our derivation is so straightforward, it suggests a number of extensions, including the following, which are presently under study: (1) nonstandard locations of source and sensors (e.g., both in the first layer); (2) minimum mean-square estimation in the non-marine environment; and (3) optimal prefiltering of noisy data.

ACKNOWLEDGEMENTS

The work reported on in this paper was performed at the Univ. of Southern Calif., Los Angeles, CA, under National Science Foundation Grants NSF ENG 74-02297 (A01 and ENG 75-03423), Air Force Office of Scientific Research Grant AFOSR-75-2797, Chevron Oil Field Research Co., Contract-76 and Teledyne Exploration Co., Contract TEC-76. The simulations were performed by Mr. Alan Yu, a graduate student.

REFERENCES


If \( n(t) \) is not white, then the variance of \( N(t) \) may not be infinite, but will be very large.


---

![Figure 1. K Layered System](image1)

![Figure 2. First Layer in the Marine Case](image2)

![Figure 3. Source Wavelet](image3)

![Figure 4. Synthetic Seismogram](image4)

![Figure 5. Noisy Sampled Seismogram ($\sigma_n = 1.0$)](image5)

---

**Table 1**

<table>
<thead>
<tr>
<th>$\sigma^2$</th>
<th>0</th>
<th>0.1</th>
<th>1.0</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
<td>1.0</td>
</tr>
<tr>
<td>$r_1$</td>
<td>0.1</td>
<td>0.1001863</td>
<td>0.1002852</td>
<td>0.1003953</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0.15</td>
<td>0.1507439</td>
<td>0.1508406</td>
<td>0.1509444</td>
</tr>
<tr>
<td>$r_3$</td>
<td>0.25</td>
<td>0.2515463</td>
<td>0.2516402</td>
<td>0.2517364</td>
</tr>
<tr>
<td>$r_4$</td>
<td>0.35</td>
<td>0.3538597</td>
<td>0.3539544</td>
<td>0.3540492</td>
</tr>
<tr>
<td>$r_5$</td>
<td>0.45</td>
<td>0.4570198</td>
<td>0.4571146</td>
<td>0.4572094</td>
</tr>
<tr>
<td>$r_6$</td>
<td>0.55</td>
<td>0.5580678</td>
<td>0.5581626</td>
<td>0.5582574</td>
</tr>
<tr>
<td>$r_7$</td>
<td>0.65</td>
<td>0.6649337</td>
<td>0.6650285</td>
<td>0.6651232</td>
</tr>
<tr>
<td>$r_8$</td>
<td>0.75</td>
<td>0.7748307</td>
<td>0.7749255</td>
<td>0.7750203</td>
</tr>
</tbody>
</table>

---

**Table 2**

<table>
<thead>
<tr>
<th>$\sigma^2$</th>
<th>0</th>
<th>0.1</th>
<th>1.0</th>
<th>10.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_0$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$r_1$</td>
<td>0</td>
<td>0.0010858</td>
<td>0.0011893</td>
<td>0.0012828</td>
</tr>
<tr>
<td>$r_2$</td>
<td>0</td>
<td>0.0021768</td>
<td>0.0022755</td>
<td>0.0023726</td>
</tr>
<tr>
<td>$r_3$</td>
<td>0</td>
<td>0.0032678</td>
<td>0.0033662</td>
<td>0.0034652</td>
</tr>
<tr>
<td>$r_4$</td>
<td>0</td>
<td>0.0043548</td>
<td>0.0044523</td>
<td>0.0045504</td>
</tr>
<tr>
<td>$r_5$</td>
<td>0</td>
<td>0.0054318</td>
<td>0.0055293</td>
<td>0.0056275</td>
</tr>
<tr>
<td>$r_6$</td>
<td>0</td>
<td>0.0064446</td>
<td>0.0065421</td>
<td>0.0066403</td>
</tr>
<tr>
<td>$r_7$</td>
<td>0</td>
<td>0.0071530</td>
<td>0.0072494</td>
<td>0.0073354</td>
</tr>
<tr>
<td>$r_8$</td>
<td>0</td>
<td>0.0080920</td>
<td>0.0081784</td>
<td>0.0082623</td>
</tr>
</tbody>
</table>