ON DECOMPOSITIONS OF A MULTI-GRAH INTO SPANNING SUBGRAPHS,

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1. Let $G$ be a multi-graph, i.e., a finite graph with no loops. $V(G)$ and $E(G)$ denote the vertex-set and edge-set of $G$, respectively. For $x \in V(G)$, $d(x, G)$ denotes the degree (or valency) of $x$ in $G$ and $m(x, G)$ denotes the multiplicity of edges at $x$ in $G$, i.e. the minimum number $m$ such that $x$ is joined to any other vertex in $G$ by at most $m$ edges.

A graph $H$ is called a spanning subgraph of $G$ if $V(H) = V(G)$ and $E(H) \subseteq E(G)$. Let $k$ be any positive integer. Let

$$
\sigma : G = H_1 \cup H_2 \cup \ldots \cup H_k
$$

be a decomposition of $G$ into $k$ spanning subgraphs so that (1) $H_1, H_2, \ldots, H_k$ are spanning subgraphs of $G$, (2) $H_1, H_2, \ldots, H_k$ are pairwise edge-disjoint, and (3) $\bigcup_{1 \leq i \leq k} E(H_i) = E(G)$. For each $x \in V(G)$, let $\nu(x, \sigma)$ denote the number of subgraphs $H_i$ in $\sigma$ such that $d(x, H_i) \geq 1$. Evidently, $\nu(x, \sigma) \leq \min(k, d(x, G))$ for all $x \in V(G)$.

2. Given a multi-graph $G$ and any positive integer $k$, we consider the problem of determining a decomposition $\sigma$ of $G$ into $k$ spanning subgraphs such that $\nu(x, \sigma)$ is as large as possible for each vertex $x \in V(G)$. In particular, we have proved the following two theorems.
Theorem 2.1: If \( G \) is a bipartite graph, then, for every positive integer \( k \), there exists a decomposition \( \sigma \) of \( G \) into \( k \) spanning subgraphs such that

\[
\forall(x, \sigma) = \min(k, d(x, G)) \quad \text{for all} \quad x \in V(G).
\]

Theorem 2.2: If \( G \) is a multi-graph, then for every positive integer \( k \), there exists a decomposition \( \sigma \) of \( G \) into \( k \) spanning subgraphs such that

\[
\forall(x, \sigma) \geq \begin{cases} 
\min(k - m(x, G), d(x, G)) & \text{if } d(x, G) \leq k \\
\min(k, d(x, G) - m(x, G)) & \text{if } d(x, G) > k
\end{cases} \quad \text{for all} \quad x \in V(G).
\]

Moreover, if \( W \subset V(G) \) is such that

\[
W \cap \{x \in V(G): k - m(x, G) < d(x, G) < k + m(x, G)\}
\]

is independent, then \( \sigma \) can be so chosen that in addition to (2.2), we have

\[
\forall(x, \sigma) = \min(k, d(x, G)) \quad \text{for all} \quad x \in W.
\]

3. The above theorems generalize some well-known theorems in graph theory.

Let \( G \) be a multi-graph; let \( H \) be a spanning subgraph of \( G \). \( H \) is said to be a matching of \( G \) if for every vertex \( x \), \( d(x, H) \leq 1 \); \( H \) is said to be a cover of \( G \) if for every vertex \( x \), \( d(x, H) \geq 1 \).
The chromatic index of $G$, denoted by $\chi_1(G)$, is defined to be the minimum number $k$ such that there exists a decomposition of $G$ into $k$ spanning subgraphs each of which is a matching of $G$. The cover index of $G$, denoted by $\kappa_1(G)$ is the maximum number $k$ such that there exists a decomposition of $G$ into $k$ spanning subgraphs each of which is a cover of $G$.

Theorems 3.1 and 3.2 below are obtained from Theorem 2.1 by taking $k = \max \limits_{x \in V(G)} d(x, G)$ and $k = \min \limits_{x \in V(G)} d(x, G)$, respectively.

**Theorem 3.1 [1]:** If $G$ is a bipartite graph, then,

$$\chi_1(G) = \max \limits_{x \in V(G)} d(x, G).$$

**Theorem 3.2 [2]:** If $G$ is a bipartite graph, then,

$$\kappa_1(G) = \min \limits_{x \in V(G)} d(x, G).$$

Similarly, Theorems 3.3 and 3.4 are seen to be special cases of Theorem 2.2.

**Theorem 3.3 [3, 4]:** If $G$ is a multi-graph, then,

$$\chi_1(G) \leq \max \limits_{x \in V(G)} (d(x, G) + m(x, G)).$$

**Theorem 3.4 [5]:** If $G$ is a multi-graph, then,

$$\kappa_1(G) \geq \min \limits_{x \in V(G)} (d(x, G) - m(x, G)).$$
Remark: We have also generalized Theorem 2.1 to a theorem for balanced hypergraphs which contains as special cases some theorems due to C. Berge [6].
References


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