Almost Sure Comparisons of Renewal Processes and Poisson Processes, with Application to Reliability Theory

by

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Technical Report No. 63
Department of Statistics
June 1976

Mathematical Sciences
ALMOST SURE COMPARISONS OF
RENEWAL PROCESSES AND POISSON PROCESSES,
WITH APPLICATION TO RELIABILITY THEORY1, 2

BY

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JUNE 1976

1The preparation of this manuscript was supported in part by
the Office of Naval Research under Contract N00014-76-C-0843.
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2Most of the research in this paper was performed prior to
the ONR contract starting date. It was presented at the ORSA/
TIMS Philadelphia meeting on April 2, 1976 under the title
"Almost sure convergence and comparisons of some stochastic
processes occurring in reliability theory."
Abstract

If the interarrival times of a renewal process \( \{ S_i, i=0,1,2,\ldots \} \) have a failure rate function which is bounded away from 0 and \( \infty \), then it is possible to construct (nonhomogeneous) Poisson processes \( \{ T_{i}^{0}, i=0,1,2,\ldots \} \) and \( \{ T_{i}^{1}, i=0,1,2,\ldots \} \) on the same probability space with \( \{ S_i, i=0,1,2,\ldots \} \) such that \( \{ T_{0}^{0}, T_{1}^{0}, T_{2}^{0},\ldots \} \subset \{ S_0, S_1, S_2,\ldots \} \subset \{ T_{0}^{1}, T_{1}^{1}, T_{2}^{1},\ldots \} \) almost surely. This has applications to the reliability theory of maintained systems. An almost sure comparison is also demonstrated for certain alternating renewal processes which arose in Barlow and Proschan's (1976) investigation of maintained systems in which repairs are not instantaneous.

Key words and phrases.
Renewal process, Poisson process, stochastic inequality, almost sure inequality, reliability theory, maintained system.
1. Introduction and Summary

Inequalities play an important role in reliability theory. One example is the following: Let $F$ be the lifetime distribution of a component or system with failure rate function $r(t)$. Suppose $\inf_r r(t) = r_0 > 0$ and $\sup_r r(t) = r_1 < \infty$. If $F_0$ and $F_1$ are cdf's of exponential random variables with means $r_0^{-1}$ and $r_1^{-1}$ respectively, then $F_0(x) \leq F(x) \leq F_1(x)$ for $x \leq 0$. Such inequalities are valuable in situations where the lifetime cdf $F$ is unknown but it is possible to know or establish bounds on the failure rate function.

If a component is instantaneously replaced or repaired when it fails, the sequence of failure times forms a renewal process: Let $X_i, i=1,2,\ldots$ be a sequence of i.i.d. lifetime random variables. Let $S_0 = 0$, $S_1 = X_1$, $S_2 = X_1 + X_2$, $S_3 = X_1 + X_2 + X_3$, ..., then $\{S_i, i=0,1,2,\ldots\}$ is a renewal process. The purpose of this note is to present bounds for the renewal process $S$ similar to the example of the preceding paragraph, namely: Suppose that the distribution of $X$ has failure rate function $r(t)$ such that $\inf_r r(t) = r_0 > 0$ and $\sup_r r(t) = r_1 < \infty$. We show (Corollaries 1 and 2) that it is possible to construct Poisson processes $\{T_i^0, i=0,1,2,\ldots\}$ and $\{T_i^1, i=0,1,2,\ldots\}$ with rates $r_0$ and $r_1$ respectively, such that

$$\{T_i^0, T_i^1, T_i^2, \ldots\} \supset \{S_0, S_1, S_2, \ldots\} \supset \{T_i^0, T_i^1, T_i^2, \ldots\}$$ (1.1)

with probability 1, i.e., the failure times of the respective processes are subsets of one another. (We shall actually construct nonhomogeneous Poisson processes which satisfy (1.1)).
It is possible to use this to compute bounds on expected maintenance costs, for example. This result is probably also of general interest as a contribution to the theory of Poisson processes and renewal theory and thus may be useful for establishing bounds in queueing theory or inventory theory.

It is also possible to obtain a comparison result (Theorem 3) for certain alternating renewal processes which arise when repairs or replacement are not instantaneous. Barlow and Proschan (1976) consider such processes. Theorem 3 and its proof give a slightly different and more general approach to their problem. (We assume NWU repair times; they assume DFR.)

There is a growing literature on comparison results for stochastic processes. There are two general approaches: stochastic inequalities [4,8,11,15,24] and almost sure inequalities [13,14,22]. (The list of references is by no means exhaustive.) Pledger and Proschan (1973), Ross (1974), Keilson (1974), and Barlow and Proschan (1976) have applied stochastic comparison techniques to reliability of maintained systems. The comparisons derived in this paper will be of the "almost sure" variety. These results are special for renewal and Poisson processes and are derived from first principles, not from any theories developed in the above-cited literature, for example [13,22] do not apply. The above theories tend to concentrate on stochastically monotone Markov processes [4]. We take more of a point-process point of view in this paper, focusing on sample paths and intensities rather than transition probabilities. Our approach is constructive and indicates some interesting relationships involving the failure rate functions; it may have pedagogical value.

We shall use the following relationship between stochastic and almost
sure inequalities for the univariate case: Let $X$ and $Y$ be real-valued random variables with cdf's $F$ and $G$, then $X$ is stochastically less than $Y$ (denoted $X \preceq^\text{st} Y$) if $F(z) \geq G(z)$ for all $z$. Let $U$ and $V$ be real-valued random variables defined on the same probability space $(\Omega,F,P)$, then $U$ is almost surely less than $V$ (denoted $U \preceq^\text{as} V$) if $P(\omega \in \Omega : U(\omega) \leq V(\omega)) = 1$. Let $X \overset{D}{=} U$ mean that the random variables $X$ and $U$ have the same distribution.

**Lemma 1.** Let $X$ and $Y$ be real-valued random variables. If $X \preceq^\text{st} Y$, then there exist a probability space $(\Omega,F,P)$ and random variables $U$ and $V$ defined on it such that $U \overset{D}{=} X$, $V \overset{D}{=} Y$, and $U \preceq^\text{as} V$.

**Proof:** Let $\Omega$ be $[0,1]$, $\mathcal{F}$ be the Borel sets and $P$ be Lebesgue measure. If $X$ and $Y$ have cdf's $F$ and $G$, let $U = F^{-1}$ and $V = G^{-1}$.

(This result can be extended to random vectors and random functions [12].)

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2. Poisson Bounds on Renewal Processes

In this section the existence of (dependent) nonhomogeneous Poisson and renewal processes satisfying (1.1) on a common probability space is verified. We will use two representations of point processes on $[0,\infty)$: the counting process $N$ and the partial sum process $S$. $N(t) = \# \text{ points in } (0,t]$. $S_n = \inf \{ t : N(t) \leq n \}$, $n = 1,2,\ldots$. There will be a point at $0 : S_0 = 0$; this is the zeroth point and is not counted by $N$. The necessary background facts about homogeneous Poisson processes, renewal processes, failure rates and other topics which will be used without
reference can be found in most introductory stochastic process texts [1,6,9,19,22]. The necessary facts about nonhomogeneous Poisson processes on \([0,\infty)\) are presented below.

**Definition 1** Let \(r(\cdot)\) be a real-valued function on \([0,\infty)\) which is integrable over bounded intervals, then \(\{N(t), t \geq 0\}\) is the counting process representation of a nonhomogeneous Poisson process with intensity function \(r\) if

1) \(N(0) = 0\)

2) \(N\) has independent increments

3) for \(0 \leq s < t\), \(N(t) - N(s)\) is a Poisson random variable with

\[
E(N(t) - N(s)) = \int_{s}^{t} r(u) \, du.
\]

We define \(\Lambda(t) = \int_{0}^{t} r(u) \, du\) as the cumulative mean function of the Poisson process with intensity \(r(\cdot)\).

**Lemma 2.** Let \(\{N(t), t \geq 0\}\) be a nonhomogeneous Poisson process with intensity function \(r(\cdot)\). Let \(T_n = \inf\{t : N(t) = n\}\), \(n = 0, 1, 2, \ldots\). Given that \(N(t) = k\), then \(T_1, T_2, \ldots, T_k\) are jointly distributed as the order statistics of a sample of size \(k\) from the cdf \(\Lambda(s)/\Lambda(t), 0 \leq s \leq t\). The density of this conditional joint distribution is

\[
f_t(t_1, t_2, \ldots, t_k \mid k) = k! \prod_{i=1}^{k} \frac{r(t_i)}{\Lambda(t)}, 0 \leq t_1 \leq t_2 \leq \ldots \leq t_k \leq t.
\]

**Proof:** Note that the conditional distribution of the interarrival time \(T_{i+1} - T_i\) given \(T_i = t_i\) is
\[
P\{ T_{i+1} - T_i > T_i = t_i \}
\]
\[
= P\{ N(t_i + x) - N(t_i) = 0 \mid N(t_i) = i \}
\]
\[
= \exp(- \int_{t_i}^{t_i + x} r(s)ds)
\]
Thus a version of the density is
\[
f_{T_{i+1} - T_i | T_i}(x | t_i) = r(t_i + x) \exp(- \int_{t_i}^{t_i + x} r(s)ds)
\]
The conditional density \( f_t(t_1, t_2, \ldots, t_k \mid k) \) can be computed now exactly as is done in the proof of the analogous result for homogeneous Poisson processes, see [19] p.17, [1] p.67, or [9] p.126.

**Lemma 3.** Let \( r(\cdot) \) be right continuous, then \( \{N(t), t \geq 0\} \) is a nonhomogeneous Poisson process with intensity function \( r(\cdot) \), if and only if, for \( t \geq 0 \),

\[
\lim_{h \to 0} h^{-1} P\{ N(t+h) - N(t) = 1 \mid N(s), s \leq t \} = r(t) \quad (2.1)
\]
\[
\lim_{h \to 0} h^{-1} P\{ N(t+h) - N(t) \geq 2 \mid N(s), s \leq t \} = 0 \quad (2.2)
\]
almost surely, and \( N(0) = 0 \).

**Proof:** If \( N \) is a nonhomogeneous Poisson process \( N(t+h) - N(t) \) is independent of \( \{ N(s), s \leq t \} \) and is Poisson with mean \( \int_{t}^{t+h} r(u)du \).
\[
P\{ N(t+h) - N(t) = 1 \} = \exp(- \int_{t}^{t+h} r(u)du) \int_{t}^{t+h} r(u)du = hr(t) + o(h).
\]
This verifies (2.1); (2.2) is verified similarly.
Conversely, (2.1) and (2.2) imply $N$ is nonhomogeneous Poisson. Part (iii) of Definition 1 can be verified by solving the Kolmogorov forward differential equations as in the homogeneous case, [9] p.24, [20] p.118. Part (ii) of Definition 1 follows because the limits are independent of the entire past history, $\{ N(s), s \leq t \}$: this implies the Markov property which in turn implies independent increments in this case.

Note that if (2.1) and (2.2) were weakened by conditioning over $\{N(t)\}$ instead of $\{N(s), s \leq t\}$ then (unless the Markov property is assumed) $N$ is not necessarily a Poisson process. Similarly part (ii) of Definition 1 cannot be weakened (to pair-wide independence, for example). Shepp (see [5]) and Szasz (1970) present counterexamples for the homogeneous case; see Renyi (1967) also.

We can now present the first comparison theorem.

**Theorem 1.** Let $\{ S_i, i=0,1,2,\ldots \}$ be a renewal process with inter-arrival cdf $F$ which has failure rate function $r(\cdot)$, i.e. $F(x) = 1 - \exp(-\int_0^x r(s)ds)$, $x \geq 0$. Let $r_1$ be a right-continuous function such that $\sup_{0 \leq s \leq t} r(s) \leq r_1(t) < \infty$ for $t \geq 0$. Then there exists a nonhomogeneous Poisson process $\{ T_i^1, i=0,1,2,\ldots \}$ with cumulative mean function $\Lambda_i(t) = \int_0^t r_i(s)ds$ defined on the same probability space as $\{ S_i, i=0,1,2,\ldots \}$ such that $\{ T_0^1, T_1^1, T_2^1, \ldots \} \triangledown \{ S_0, S_1, S_2, \ldots \}$ almost surely.

**Proof:** As is frequently done, we shall not explicitly define the common probability space on which $S$ and $T_i^1$ are defined; from the constructive definition of $S$ from $T_i^1$ it will be clear how an appropriate probability
could be defined.

Consider \( \{ T_1^i, i=0,1,2,\ldots \} \), a homogeneous Poisson process with cumulative mean function \( \Lambda_1 \). We shall construct a dependent version of \( \{ S_1, i=0,1,2,\ldots \} \), the desired renewal process, by "thinning" \( T_1 \), i.e. certain points \( T_1^i \) will be removed and the remaining points will constitute \( \{ S_1, i=0,1,2,\ldots \} \). Define \( S_0 = T_0^1 = 0 \). If \( T_1^i = t_1 \), remove ("thin") the point \( T_1^i \) with probability \( 1-r(t_1)/r_1(t_1) \), thus \( S_1 = T_1^i \) with probability \( r(t_1)/r_1(t_1) \) conditional on \( T_1^i = t_1 \). If \( T_1 \) is thinned and \( T_1^2 = t_2 \), remove the point \( T_1^2 \) with probability \( 1-r(t_2)/r_1(t_2) \). Thus, if \( T_1^i = t_1, T_1^j = t_2, S_1 = t_2 \) with probability \( (1-r(t_1)/r_1(t_1))(r(t_2)/r_1(t_2)) \). Continue inductively until the first "unthinned" point of \( T_1 \) is reached and define this point to be \( S_1 \). In other words, given that \( T = (0, t_1, t_2, t_3, \ldots ) \) define an integer-valued random variable \( M_1 \):

\[
P( M_1 > m \mid T_1 = (0,t_1,t_2,\ldots )) = \prod_{i=1}^{m} \left( 1 - \frac{r(t_1)}{r_1(t_1)} \right)
\]

Then \( S_1 = T_{M_1}^1 \).

We now consider the distribution of \( S_1 \). Let \( N_1(t) = \max\{ n: T_n^1 \leq t \} \). Then

\[
P( S_1 > t ) = \sum_{k=0}^{\infty} P( S_1 > t, N_1(t) = k )
\]

\[
= \sum_{k=0}^{\infty} \int_{0 \leq t_1 \leq \cdots \leq t_k \leq t} P( S_1 > t \mid t_1,\ldots,t_k ) f(t_1,\ldots,t_k) \times dt_1 \cdots dt_k P( N_1(t) = k )
\]
\[
\begin{align*}
&= \sum_{k=0}^{\infty} \prod_{i=1}^{k} \left(1 - \frac{r(t_i)}{r_1(t_i)}\right) \frac{r(t_i)}{\lambda_1(t)} \frac{r_1(t_i)}{\lambda(t)} \ dt_1 \ldots dt_k \times \\
&\quad \exp(-\lambda_1(t)) \frac{\lambda_1(t)^k}{k!} \\
&= \sum_{k=0}^{\infty} \prod_{i=1}^{k} (r_1(t_i) - r(t_i)) \ dt_1 \ldots dt_k \times \\
&\quad \exp(-\lambda_1(t)) \frac{\lambda_1(t)^k}{k!} \\
&= \sum_{k=0}^{\infty} \prod_{i=1}^{k} (r_1(s) - r(s)) ds \times \\
&\quad \exp(-\lambda_1(t)) \frac{\lambda_1(t)^k}{k!} \\
&= \sum_{k=0}^{\infty} \left(\lambda_1(t) - \lambda(t)\right)^k \frac{\lambda_1(t)^k}{k!} \\
&= \exp(\lambda_1(t) - \lambda(t)) \exp(-\lambda_1(t)) \\
&= \exp(-\lambda(t)) = \exp\left(-\int_{0}^{t} r(s) ds\right).
\end{align*}
\]
The third equality follows from Lemma 2 and (2.3), the fourth by symmetry, the rest are routine. The above analysis implies that the distribution of $S_1$ has failure rate function $r(\cdot)$. Thus $S_1$ has the desired distribution.

Now we must consider the succeeding events in the renewal process $S$. Condition over the partition $\{ S_1 = T_{M_1}^1 = s_1 \}$, $s_1 > 0$: the random index $M_1$ is a stopping time ([3], p.59), therefore the strong Markov property ([3], p.131) implies that the distribution of $\{ T_{M_1}^1 + i - T_{M_1}^1, i=0,1,2,\ldots \}$ is independent of $\{ T_i^1, 0 \leq i \leq M_1 \}$ conditional on $\{ T_{M_1}^1 = s_1 \}$. Given that $\{ T_{M_1}^1 = s_1 \}$, $\{ T_{M_1}^1 + i - T_{M_1}^1, i=0,1,\ldots \}$ is a nonhomogeneous Poisson process with cumulative mean function $A_{1,1}(t) = \int_{s_1}^{s_1+t} r_1(s) ds = \int_0^t r_1(s+s_1) ds$.

Note that $r_1(t+s_1) \geq \sup_{0 \leq s \leq t+s_1} r(s) \geq \sup_{0 \leq s \leq t} r(s)$ for all $t$, thus conditional on $\{ T_{M_1}^1 = s_1 \}$, the process $\{ T_{M_1}^1 + i - T_{M_1}^1, i=0,1,2,\ldots \}$ satisfies the same hypotheses which enabled us to construct $S_1 = T_{M_1}^1$. Repeating the constructive procedure gives $S_2$ conditional on $S_1 = s_1$. Continuing inductively allows construction of the entire renewal process $\{ S_i, i=0,1,2,\ldots \}$. This completes the proof of Theorem 1.

At this junction, some remarks are in order concerning "thinning":

The usual connotation of "thinning" refers to a more random form of deleting points, for example, for $0 < p < 1$, each point is deleted with probability $p$ independently of the process and other deletions. It is well known that a Poisson process which is "thinned" in this sense remains a Poisson process, see Renyi (1970), p.254, P.4.10. Other aspects of thinning are discussed
by Råde (1972) and Jagers and Lindvall (1973).

In Theorem 1 if \( \sup_{0 \leq s < \infty} r(s) < \infty \) then letting \( r_1 = \sup_{0 \leq s < \infty} r(s) \) the hypotheses of theorem are satisfied and we have the following corollary.

**Corollary 1.** Let \( \{ S_i, i=0,1,\ldots \} \) be a renewal process with inter-arrival distribution \( F \) which has failure rate function \( r(\cdot) \). If \( \sup r(t) = r_1 < \infty \) then there exists a (homogeneous) Poisson process \( \{ T_i, i=0,1,\ldots \} \) with intensity \( r_1 \) defined on the same probability space as \( \{ S_i, i=0,1,\ldots \} \) such that \( \{ T_0, T_1, T_2, \ldots \} \subseteq \{ S_0, S_1, S_2, \ldots \} \) almost surely.

**Theorem 2.** Let \( \{ S_i, i=0,1,2,\ldots \} \) be a renewal process with inter-arrival distribution \( F \) which has failure rate function \( r(\cdot) \). Let \( r_0 \) be a right-continuous function such that \( \inf_{s \leq t} r(s) \geq r_0(t) > 0 \) for \( t \geq 0 \). Then there exists a nonhomogeneous Poisson process \( \{ T_i, i=0,1,2,\ldots \} \) with cumulative mean function \( \Lambda_0(t) = \int_0^t r_0(s)ds \) defined on the same probability space as \( \{ S_i, i=0,1,2,\ldots \} \) such that \( \{ T_0, T_1, T_2, \ldots \} \subseteq \{ S_0, S_1, S_2, \ldots \} \) almost surely.

**Proof:** Consider \( \{ S_i, i=0,1,2,\ldots \} \), the renewal process with interarrival cdf \( F \) which has failure rate function \( r \). We shall construct a dependent version of \( \{ T_i, i=0,1,2,\ldots \} \), the desired nonhomogeneous Poisson process, by "thinning" \( F \) as follows: if \( S_i = s_i \) and \( S_i - S_{i-1} = x_i \), delete \( S_i \) with probability \( 1 - r_0(s_i)/r(x_i) \), \( i=1,2,\ldots \). Let \( T_0 = S_0 \); let \( T_1 \) equal the smallest undeleted \( S_i, i \geq 1 \); let \( T_2 \) equal the second smallest, etc. We use Lemma 3 to show that \( \{ T_i, i=0,1,\ldots \} \) is a nonhomogeneous Poisson
process with intensity \( r_0(*) \): Let \( N(t) = \sup_n \{ n : S_n < t \} \), and
\[
N_0(t) = \sup_n \{ n : T_n^0 < t \}
\]
Because \( T^0 \) is a thinning of \( S \), the \( \sigma \)-fields they generate are comparable: \( \sigma(\{ N_0(s), s \leq t \}) \subset \sigma(\{ N(s), s \leq t \}) \). Therefore, in order to verify

\[
\lim_{h \to 0} h^{-1} \Pr \{ N_0(t+h) - N_0(t) = 1 \mid N_0(s), s \leq t \} = r_0(t)
\]

it suffices to verify

\[
\lim_{h \to 0} h^{-1} \Pr \{ N_0(t+h) - N_0(t) = 1 \mid N(s), s \leq t \} = r_0(t) \quad (2.4)
\]

By the strong Markov property for renewal processes, \( \{ N(s), 0 \leq s \leq S_N(t) \} \) is conditionally independent of \( \{ N(s), S_N(t) \leq s \leq t \} \) given \( S_N(t) \), furthermore since \( S_N(t) \) is the last renewal at or before \( t \),
\[
\sigma(\{ N(s), S_N(t) \leq s \leq t \}) = \sigma(\{ S_N(t), N(t) \}) \quad \text{; also, for a renewal process,} \quad N(t+h) - N(t) \text{ is independent of } N(t) \text{ given } S_N(t) \quad \text{.}
\]

Putting all of this together gives

\[
\Pr \{ N_0(t+h) - N_0(t) = 1 \mid N(s), s \leq t \} = \sum_{k=1}^{\infty} \Pr \{ N_0(t+h) - N_0(t) = 1 \mid N(t+h) - N(t) = k, S_N(t) \} \times
\]
\[
\Pr \{ N(t+h) - N(t) = k \mid S_N(t) \}
\]

By renewal theory,

\[
\Pr \{ N(t+h) - N(t) = k \mid S_N(t) = s_* \} \leq \frac{F(t+h-s_*) - F(t-s_*)}{1 - F(t-s_*)} \quad (2.6)
\]
equality holds for \( k = 1 \), strictly inequality for \( k > 2 \). Dividing (2.6) by \( h \) and letting \( h \to 0 \) yields \( r(t-s_\star) \) for \( k = 1 \) and \( 0 \) for \( k > 1 \). Thus in order to verify (2.4) using (2.5), it suffices to show that

\[
\lim_{h \to 0} \frac{\text{P}\{ N_0(t+h) - N_0(t) = 1 \mid N(t) = 1, S_{N(t)} = s_\star \}}{r(t-s_\star)} = \frac{r_0(t)}{r(t-s_\star)} \tag{2.7}
\]

But

\[
\text{P}\{ N_0(t+h) - N_0(t) = 1 \mid N(t+h) - N(t) = 1, S_{N(t)} = s_\star \} = \int_{t}^{t+h} \frac{r_0(s) ds}{r(s-s_\star)} \frac{r(s-s_\star) \exp(- \int_{0}^{s-s_\star} r(u) du)}{\exp(- \int_{0}^{t-s_\star} r(u) du) - \exp(- \int_{0}^{t+h-s_\star} r(u) du)}
\]

\[
= \int_{t}^{t+h} \frac{r_0(s) \exp(- \int_{t-s_\star}^{s-s_\star} r(u) du)}{1 - \exp(- \int_{t-s_\star}^{t+h-s_\star} r(u) du)} \frac{r(s-s_\star) \exp(- \int_{0}^{s-s_\star} r(u) du)}{1 - \exp(- \int_{0}^{t-s_\star} r(u) du)} \frac{1}{r(t-s_\star) + o(h)} \]

\[
\frac{1}{h} \int_{t}^{t+h} \frac{r_0(s)}{r(t-s)} \, ds + o(h)
\]

\[
\frac{r_0(s)}{r(t-s)} + o(h)
\]

This verifies (2.7) and consequently (2.1) of Lemma 3. Eq. (2.2) of the Lemma 3 is verified similarly. Thus \(T^0\) is a nonhomogeneous Poisson process with intensity \(r_0(\cdot)\).

**Corollary 2.** Let \(\{S_i, i=0,1,\ldots\}\) be a renewal process with interarrival cdf \(F\) which has failure rate function \(r(\cdot)\). If \(\inf_{0 \leq t < \infty} r(t) = r_0 > 0\) then there exists a (homogeneous) Poisson process \(\{T^0_i, i=0,1,\ldots\}\) with intensity \(r_0\) defined on the same probability space as \(\{S_i, i=0,1,\ldots\}\) such that \(\{T^0_0, T^0_1, T^0_2, \ldots\} \subset \{S_0, S_1, S_2, \ldots\}\) almost surely.

We shall now present an example from reliability theory which indicates a possible application of the above theorems: bounds on the expected discounted cost of repair/replacement of a component renewal process. Suppose the life time of a component has distribution \(F\) with failure rate function \(r(\cdot)\). Let \(S_1, S_2, \ldots\) be successive failure times. Suppose if a component failures at time \(t\), it costs \(c(t)\) to renew it. No assumptions (except integrability) are placed on \(c\), it can vary daily or seasonally, etc. Let \(d\) be a discount factor. The discounted cost of maintaining the system is

\[
\sum_{i=1}^{\infty} \exp(-dS_i) c(S_i) .
\]
If $T^0$ and $T^1$ are nonhomogeneous Poisson processes with intensity functions $r_0(\cdot)$ and $r_1(\cdot)$ respectively, as described in Theorems 1 and 2 then

$$\sum_{i=1}^{\infty} \exp(-dT^0_i) c(T^0_i) \leq \sum_{i=1}^{\infty} \exp(-dS_i) c(S_i) \leq \sum_{i=1}^{\infty} \exp(-dT^1_i) c(T^1_i)$$

It is often easier to evaluate functionals of Poisson processes than renewal processes. In the above case it is much easier to compute moments:

$$\int_0^\infty e^{-dt} c(t)r_0(t)dt \leq E(\sum_{i=1}^{\infty} \exp(-dS_i)C(S_i)) \leq \int_0^\infty e^{-dt} c(t)r_1(t)dt .$$

In this example it is plausible that $r(\cdot)$ is unknown but can be assumed to be bounded by $r_0$ and $r_1$ thus giving the above bound on expected renewal cost.

3. Comparisons of some alternating renewal processes

In this section we compare some processes which arise when failed components are not instantaneously repaired. Consider the special case of a repairable component with exponential lifetime distribution $F$ and NWU (new worse than used) repair time distribution $G$. The component will alternating spend random times functioning and under repair. Let $X$ be the performance process $[2,21]$

$$X(t) = \begin{cases} 
0, & \text{component functioning at } t \\
1, & \text{component under repair at } t 
\end{cases}$$

If $L_1, L_2, \ldots$ and $R_1, R_2, \ldots$ are the successive lifetimes and repair times, let $T_{2i} = \frac{1}{3i}(L_j + R_j)$ and $T_{2i+1} = T_{2i} + L_{i+1}$, $i = 0,1,2,\ldots$,
Let the auxiliary performance process [2]:

\[
X(t) = \begin{cases} 
0 & \text{if } T_{2i} \leq t < T_{2i+1} \\
1 & \text{if } T_{2i+1} \leq t < T_{2i+2}
\end{cases}
\]

\[i = 0,1,2,\ldots\]

This process describes a system which is functioning at time 0. Let \( Z_u \) describe a system which has been under repair at time 0 for a duration equal to \( u \). Barlow and Proschan (1976) establish a stochastic inequality between \( Z_u \) and \( Z \) under the assumption that repair times are DFR (decreasing failure rate). We shall establish an almost-sure inequality.

Let \( R_0 \) be the initial repair period of a system under repair at time 0. If repair commenced at \( -u \), let \( T_{-1}^u = -u \) and \( T_0^u = -u + R_0 > 0 \).

Define \( T_{1}^u = T_{0}^u + T_{1} \), as defined above. Then

\[
Z_u(t) = \begin{cases} 
0 & \text{if } T_{2i}^u \leq t < T_{2i+1}^u \\
t - T_{2i+1}^u & \text{if } T_{2i+1}^u \leq t < T_{2i+2}^u
\end{cases}
\]

\[i = -1,0,1,\ldots\]

Note that \( Z \) and \( Z_u \) are both Markov processes with the same transition law, only their initial state differs.

**Theorem 3** The processes \( Z \) and \( Z_u \) can be defined on a common probability space such that \( Z(t) \leq Z_u(t) \), \( t \geq 0 \), almost surely.

**Proof:** Let \( R_0^u \) be the remaining repair time in a repair period which has been in progress \( u \) units of time. Let \( L_1 \) be lifetime random
variables and \( R_i \) be repair time random variables, \( i = 1,2, \ldots \). Let \( R_i^t \) be the remaining repair time in a repair period which has been in progress \( t \) units of time. Because \( R \) is NWU, the residual repair time \( R_i^t \) is stochastically greater than \( R_i^t, t > 0, i = 1,2, \ldots \). By Lemma 1, let \( I_i = [0,1], F_i \) be Borel sets and \( P_i \) be Lebesgue measure, define \( R_i, R_i^t, t > 0 \) on \((I_i, F_i, P_i)\) such that \( P( R_i \leq R_i^t ) = 1 \).

Define \( R_0^u \) on \((I_0, F_0, P_0)\) and define \( L_1 \) on similar probability spaces \((J_1, G_1, Q_1)\). Let \( (\Omega, F, P) \) be the product space of all the above spaces; extend the domain of the above define random variables in the obvious way.

\{ \( Z(t), t \geq 0 \) \} is automatically defined on \( (\Omega, F, P) \). We define \{ \( Z_u(t), t \geq 0 \) \} as follows: On \( \{ R_0^u < L_1 \} \), \( Z_u(t) = u + t \) for \( 0 \leq t < R_0^u \), and \( Z_u(t) = Z(t) \) for \( t \geq R_0^u \). Note that the lack of memory property of the exponential distribution guarantees that the first lifetime of the \( Z_u \)-process will have the correct distribution. On \( \{ R_0^u > L_1 \} \), condition over \( L_1 = l_1 \) if \( L_1 = l_1 \) and \( R_0^u \geq l_1 \) then at time \( l_1 \) (when the \( Z \)-process begins its first repair period) the \( Z_u \)-process has been under repair for a duration \( u + l_1 \), thus \( R_1^{u+l_1} \) describes the repair status of \( Z_u \). On the set \( \{ L_1 = l_1 \leq R_0^u, R_1^{u+l_1} \leq R_1 + L_2 \} \) define \( Z_u(t) = u + t \) for \( 0 \leq t \leq l_1 + R_1^{u+l_1} \) and \( Z_u(t) = Z(t) \) for \( t \geq l_1 + R_1^{u+l_1} \). By construction, the \( Z \)-process is repaired before the \( Z_u \)-process; since \( R_1^{u+l_1} \leq R_1 + L_2 \), the \( Z \)-process will be functioning when the repair of the \( Z_u \)-process is completed and the residual lifetime will be exponential. This procedure of defining \( Z_u \) on \((\Omega, F, P)\) can be continued inductively. The next step is to condition over \( \{ L_1 = l_1, R_1 = r_1, L_2 = l_2 \} \) and \( \{ R_0^u \geq l_1 \} \cap \{ R_1^{u+l_1} \geq r_1 = l_2 \} \).
Given these conditions, the $Z_u$-process has been under repair $t_3 + u$ time units at time $t_3 = l_1 + r_1 + l_2$, thus $R_{u+t_3}^u$ is the distribution of residual repair time. On the set \( \{ L_1 = l_1 \geq R_0^u, R_1 = r_1, L_2 = l_2, R_1^{u+t} \geq r_1 + l_2, R_2^{u+t} \leq R_2 + L_3 \} \) define $Z_u(t) = t + u$ for $0 \leq t \leq t_3 + R_{u+t_3}^u$ and $Z_u(t) = Z(t)$ for $t \geq t_3 + R_{u+t_3}^u$. It should be clear from the construction that $Z_u(t) \geq Z(t)$ with probability 1.

Theorem 3 can be used to establish almost sure comparisons between the operating processes of coherent systems with exponential component life times and NWU component repair times similar to the stochastic inequality established by Barlow and Proschan (1976) and then show that the time until first system failure is NBU.
References


Almost sure comparisons of renewal processes and Poisson processes with application to reliability theory.

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renewal process, Poisson process, stochastic inequality, almost sure inequality, reliability theory, maintained system