RADC-TR-77-109
Interim Technical Report
March 1977

H-FIELD, E-FIELD, AND COMBINED FIELD SOLUTIONS FOR BODIES OF REVOLUTION

Department of Electrical and Computer Engineering
Syracuse University

Approved for public release;
distribution unlimited.

ROME AIR DEVELOPMENT CENTER
AIR FORCE SYSTEMS COMMAND
GRIFFISS AIR FORCE BASE, NEW YORK 13441

Professor Roger F. Harrington and Dr. J. R. Mautz are the co-responsible investigators for this contract. Peter R. Franchi (EEER) is the RADC Project Engineer.

This report has been reviewed by the RADC Information Office (OI) and is releasable to the National Technical Information Service (NTIS). At NTIS it will be releasable to the General public, including foreign nations.

This technical report has been reviewed and approved for publication.

APPROVED:

Peter R. Franchi

PETER R. FRANCHI
Project Engineer

APPROVED:

Allan C. Schell

ALLAN C. SCHELL
Acting Chief
Electromagnetic Sciences Division

FOR THE COMMANDER: John P. Huss

Plans Office

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: RADC-TR-77-109	2. GOVT ACCESSION NO.
3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle): H-FIELD, E-FIELD, AND COMBINED FIELD SOLUTIONS FOR BODIES OF REVOLUTION	5. TYPE OF REPORT & PERIOD COVERED: Interim Technical Report #1
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Joseph R. Mautz, Roger F. Harrington	8. CONTRACT OR GRANT NUMBER(s): F19628-76-C-0300
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Syracuse University, Department of Electrical and Computer Engineering, Syracuse, New York 13210	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 61102F/23040221
11. CONTROLLING OFFICE NAME AND ADDRESS: Deputy of Electronic Technology (RADC), Hanscom AFB, Massachusetts 01731, Contract Monitor: Peter R. Franchi/ETER	12. REPORT DATE: Mar[illegible]977
	13. NUMBER OF PAGES: 48
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS. (of this report): UNCLASSIFIED
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

A - Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

This work was partially supported by the Air Force Post Doctoral Program under Contract No. F30602-75-C-0121.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

- Bodies of Revolution
- Combined field equation
- Conducting bodies
- E-field equation
- Electromagnetic Scattering
- H-field equation
- Method of Moments
- Radar cross section

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

H-field, E-field, and combined field solutions are developed for the electric surface current and far scattered fields of a perfectly conducting body of revolution excited by an incident plane wave. These solutions are obtained by applying the method of moments to the H-field, E-field, and combined field integral equations, respectively. The H-field integral equation is obtained by requiring the tangential magnetic field to be zero just inside the surface S of the body of revolution. The E-field integral equation is obtained by requiring

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

S/N 0102-014-6601

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) the tangential electric field to be zero on S. The combined field integral equation is a linear combination of the H-field and E-field integral equations.

Computations show that both the H-field and the E-field solutions deteriorate near internal resonances of the conducting surface S, but that the combined field solution does not. The computer program subroutines used to perform these computations will appear in a forthcoming report.

EVALUATION

This report is the first Technical Report. It is a theoretical analysis of a method for obtaining solutions for conducting bodies of revolution with an unspecified cross section. The approach involves applying the method of moments to the E-field, H-field and the combined field integral equations, respectively. The final solution is a linear combination of the H-field and E-field integral equations. This technique is an improvement over previous single field techniques, because at certain critical frequencies the single field techniques appear to become unstable. This does not happen with the combined field approach.

A second report giving the computer program for this technique is being prepared. These reports will enable the practical computation of the backscatter cross section of many complex bodies of revolution.

Peter R Franchi

PETER R. FRANCHI
Project Engineer

CONTENTS

PRECEDING PAGE BLANK-NOT FILMED

I. INTRODUCTION

Formulas for the computation of the electric surface current and far scattered field of a perfectly conducting body of revolution are derived for arbitrary plane wave excitation. Computer program subroutines which implement these formulas in the resonance region will appear in a forthcoming report. Computations show that both the H-field solution and the E-field solution deteriorate near internal resonances of the conducting surface, but that the combined field solution does not.

The field solutions are obtained by applying the method of moments to the H-field integral equation, the E-field integral equation, and the combined field integral equation for a perfectly conducting body of revolution. Although the computer program subroutines are written explicitly for a perfectly conducting body of revolution, they are directly applicable to the more difficult problem of plane wave scattering by dielectric bodies considered by Wu [1].

Our H-field solution is similar to that of Uslenghi [2], generalized to oblique incidence and with expansion and testing functions equal to four impulse approximations to triangle functions divided by the cylindrical coordinate radius. We use Gaussian quadrature instead of Simpson's rule for integration. Our treatment of coincident impulses is simpler than Uslenghi's. The impulses are combined in groups of four, not to make the H-field solution more efficient, but to make it compatible with the E-field solution. Actually, for low order solutions where much more effort is required to obtain the matrix elements than to solve the system of linear equations, it is wasteful to combine the impulses if all one wants is the H-field solution alone.

Our E-field solution is in some respects a simplification and in other respects a refinement of an earlier E-field solution [3].

---

[1] T. K. Wu, "Electromagnetic Scattering from Arbitrarily-Shaped Lossy Dielectric Bodies," Ph.D. Thesis, University of Mississippi, May 1976.

[2] P.L.E. Uslenghi, "Computation of Surface Currents on Bodies of Revolution," Alta Frequenza, vol. 39, No. 8, 1970, pp. 1-12.

[3] J. R. Mautz and R. F. Harrington, "Radiation and Scattering from Bodies of Revolution," Appl. Sci. Res., vol. 20, June 1969, pp. 405-435.

The interaction between impulse portions of the expansion and testing functions is calculated in the same way as in the H-field solution. Computationally, this new E-field solution is roughly three times faster than that of [3] with comparable accuracy.

Our combined field formulation is that proposed by Oshiro et al. [4,5]. It is obtained by the method of moments applied to a weighted average of the H and E field integral equations for a perfectly conducting body of revolution. The matrix operator for the combined field solution is a linear combination of the matrix operators for the H and E field solutions. The excitation vector for the combined field solution is the same linear combination of the excitation vectors for the H and E field solutions.

II. STATEMENT OF THE PROBLEM

We seek the electric surface current and the far scattered field of the perfectly conducting body of revolution of Fig. 1 excited by an incident plane wave. In Fig. 1, ρ,ϕ,z are cylindrical coordinates, and t,ϕ form an orthogonal curvilinear coordinate system on the surface S of the body of revolution. Also, $\underset{\sim}{u}_t$ and $\underset{\sim}{u}_\phi$ are orthogonal unit vectors in the t and ϕ directions, respectively. The coordinate origin is on the axis of the body of revolution but not necessarily at the lower pole as in Fig. 1. Figure 2 defines the propagation vector $\underset{\sim}{k}_t$ of the incident plane wave, the transmitter coordinate θ_t, the coordinates θ_r, ϕ_r (receiver coordinates) at which the far scattered field is observed, and the propagation vector $\underset{\sim}{k}_r$ of a hypothetical measurement plane wave which travels from the receiver location (θ_r, ϕ_r) toward the origin. Note that the ϕ coordinate of the transmitter is zero such that $\underset{\sim}{k}_t$ is in the xz plane. In Fig. 2, $\underset{\sim}{u}_\theta^t$, $\underset{\sim}{u}_\phi^t$, $\underset{\sim}{u}_\theta^r$, and $\underset{\sim}{u}_\phi^r$ are unit vectors in the θ_t, y, θ_r, and ϕ_r directions respectively.

We consider separately a θ polarized incident plane wave defined

[4] F.K. Oshiro, K.M. Mitzner, and S.S. Locus et al., "Calculation of Radar Cross Section," Air Force Avionics Laboratory Tech. Rept. AFAL-TR-70-21, Part II, April 1970.

[5] A.J. Poggio and E.K. Miller, "Integral Equation Solutions of Three-Dimensional Scattering Problems," Chap. 4 of Computer Techniques for Electromagnetics, edited by R. Mittra, Pergamon Press, 1973.

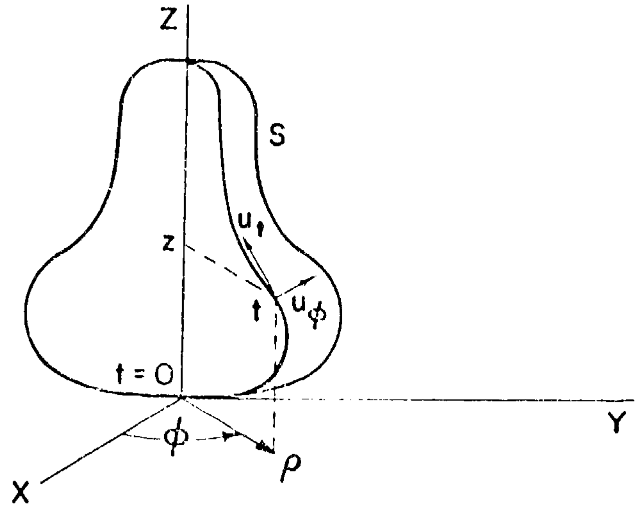

Fig. 1. Body of revolution and coordinate system.

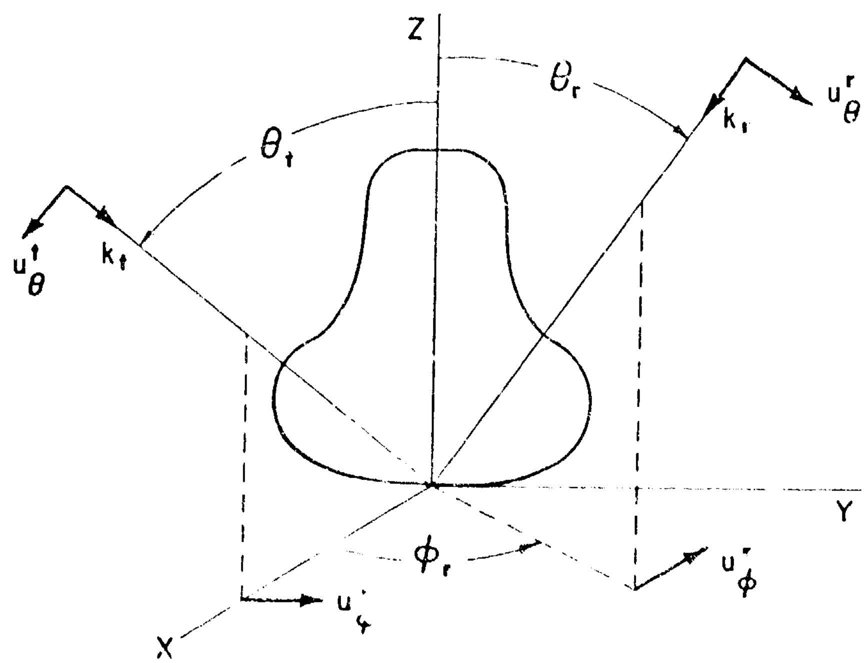

Fig. 2. Plane wave scattering by a conducting body of revolution.

by

$$\underset{\sim}{E}^i = \underset{\sim}{u}_\theta^t\, k\eta\, e^{-j\underset{\sim}{k}_t \cdot \underset{\sim}{r}}$$
$$\underset{\sim}{H}^i = (\underset{\sim}{k}_t \times \underset{\sim}{u}_\theta^t) e^{-j\underset{\sim}{k}_t \cdot \underset{\sim}{r}} \tag{1}$$

and a ϕ polarized incident plane wave defined by

$$\underset{\sim}{E}^i = \underset{\sim}{u}_\phi^t\, k\eta\, e^{-j\underset{\sim}{k}_t \cdot \underset{\sim}{r}}$$
$$\underset{\sim}{H}^i = (\underset{\sim}{k}_t \times \underset{\sim}{u}_\phi^t) e^{-j\underset{\sim}{k}_t \cdot \underset{\sim}{r}} \tag{2}$$

where $\underset{\sim}{E}^i$ and $\underset{\sim}{H}^i$ denote incident electric and magnetic fields respectively, $\underset{\sim}{r}$ is the radius vector from the origin, k is the propagation constant, and η is the intrinsic impedance. Either plane wave gives rise to t and ϕ directed electric surface currents on S and θ_r and ϕ_r directed far scattered fields.

III. H-FIELD SOLUTION

The H-field solution is obtained by applying the method of moments to the H-field integral equation. The H-field integral equation is derived by setting the component tangential to S of the total magnetic field equal to zero just inside S.

The boundary condition that the total tangential magnetic field is zero just inside S is written as

$$-\underset{\sim}{n} \times \underset{\sim}{H}^s = \underset{\sim}{n} \times \underset{\sim}{H}^i \quad \text{just inside } S \tag{3}$$

where $\underset{\sim}{n}$ is the unit outward normal vector to S, $\underset{\sim}{H}^s$ is the magnetic field due to the electric surface current on S and $\underset{\sim}{H}^i$ is the incident magnetic field given by either (1) or (2).

To obtain an expression for $\underset{\sim}{n} \times \underset{\sim}{H}^s$, we note first that from page 98 of [6]

[6] R. F. Harrington, Time-Harmonic Electromagnetic Fields, McGraw-Hill Book Co., 1961.

$$
\underset{\sim}{n} \times \underset{\sim}{H}^{S} = \lim_{\delta \to 0} \underset{\sim}{n} \times \underset{\sim}{\nabla} \times \iint_{S} \underset{\sim}{J}(\underset{\sim}{r}') \frac{e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|}}{4\pi |\underset{\sim}{r}-\underset{\sim}{r}'|} ds' \tag{4}
$$

where δ is the distance between the field point $\underset{\sim}{r}$ and the inside face of the surface S, $\underset{\sim}{r}'$ is a running source point on S, and $\underset{\sim}{J}(\underset{\sim}{r}')$ is the electric surface current on S. Next, we view $\underset{\sim}{J}(\underset{\sim}{r}')$ in (4) as the current which resides on ΔS plus the current on S minus ΔS where ΔS is that portion of S inside a sphere of radius ε centered at the point on S nearest $\underset{\sim}{r}$. Let ε be so small that ΔS is essentially flat and that the electric surface current on it is constant. If δ is appreciably less than ε, then, from Ampére's law, the contribution to $\underset{\sim}{n} \times \underset{\sim}{H}^{S}$ from the current on ΔS is $-\underset{\sim}{J}/2$ where $\underset{\sim}{J}$ is the value of the current on ΔS. Moreover, this contribution to $\underset{\sim}{n} \times \underset{\sim}{H}^{S}$ comes exclusively from a small portion of ΔS in the immediate vicinity of $\underset{\sim}{r}$. The current on that portion of ΔS for which the distance to the field point $\underset{\sim}{r}$ is appreciably greater than δ does not contribute to $\underset{\sim}{n} \times \underset{\sim}{H}^{S}$. We now let $\delta \to 0$ in which case $\underset{\sim}{r}$ becomes a point of S, the contribution $-\underset{\sim}{J}/2$ comes from the value of $\underset{\sim}{J}$ at the single point $\underset{\sim}{r}$, and the current on any portion of ΔS which does not contain $\underset{\sim}{r}$ contributes nothing to $\underset{\sim}{n} \times \underset{\sim}{H}^{S}$. Hence,

$$
\underset{\sim}{n} \times \underset{\sim}{H}^{S} = -\frac{\underset{\sim}{J}(\underset{\sim}{r})}{2} + \iint_{S} \underset{\sim}{n} \times \underset{\sim}{\nabla} \times [\underset{\sim}{J}(\underset{\sim}{r}') \frac{e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|}}{4\pi |\underset{\sim}{r}-\underset{\sim}{r}'|}] ds' \tag{5}
$$

where $\underset{\sim}{r}$ is exactly on S and where the improper integral in (5) is convergent.

In view of (5) and the vector identity

$$
\underset{\sim}{\nabla} \times [\underset{\sim}{J}(\underset{\sim}{r}') \frac{e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|}}{|\underset{\sim}{r}-\underset{\sim}{r}'|}] = -(\frac{1 + jk|\underset{\sim}{r}-\underset{\sim}{r}'|}{|\underset{\sim}{r}-\underset{\sim}{r}'|^{3}}) e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|} (\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}') \tag{6}
$$

(3) becomes

$$\frac{\underset{\sim}{J}(\underset{\sim}{r})}{2} + \frac{1}{4\pi} \iint_S \left(\frac{1+jk|\underset{\sim}{r}-\underset{\sim}{r}'|}{|\underset{\sim}{r}-\underset{\sim}{r}'|^3}\right) e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|} \underset{\sim}{n} \times [(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}')]ds' = \underset{\sim}{n} \times \underset{\sim}{H}^i(\underset{\sim}{r}) \tag{7}$$

for $\underset{\sim}{r}$ on S. Henceforth, we assume that the outward normal vector $\underset{\sim}{n}$ is given by $\underset{\sim}{n} = \underset{\sim}{u}_\phi \times \underset{\sim}{u}_t$. If $\underset{\sim}{u}_t$ is chosen such that $\underset{\sim}{n} = -\underset{\sim}{u}_\phi \times \underset{\sim}{u}_t$, then our evaluations of the terms proportional to $\underset{\sim}{n}$ in (7) will have the wrong signs throughout the remainder of this report.

If

$$\underset{\sim}{J}(\underset{\sim}{r}') = \underset{\sim}{u}_t' J^t(t',\phi') + \underset{\sim}{u}_\phi' J^\phi(t',\phi') \tag{8}$$

where $\underset{\sim}{u}_t'$ and $\underset{\sim}{u}_\phi'$ are unit vectors in the t' and ϕ' directions respectively, then, as shown in Appendix A, (7) becomes

$$\begin{aligned}
&\underset{\sim}{u}_t \Bigg\{ \frac{J^t(t,\phi)}{2} + \frac{k^3}{4\pi} \int \rho' dt' \int_0^{2\pi} d\phi' G J^t(t',\phi'+\phi) [((\rho'-\rho)\cos v' - (z'-z)\sin v') \\
&\cos \phi' - 2\rho \cos v' \sin^2 (\tfrac{\phi'}{2})] + \frac{k^3}{4\pi} \int \rho' dt' \int_0^{2\pi} d\phi' G J^\phi(t',\phi'+\phi)(z'-z) \sin \phi' \Bigg\} \\
&+ \underset{\sim}{u}_\phi \Bigg\{ \frac{J^\phi(t,\phi)}{2} + \frac{k^3}{4\pi} \int \rho' dt' \int_0^{2\pi} d\phi' G J^t(t',\phi'+\phi)(\rho' \sin v \cos v' - \rho \sin v' \cos v \\
&- (z'-z) \sin v \sin v') \sin \phi' + \frac{k^3}{4\pi} \int \rho' dt' \int_0^{2\pi} d\phi' G J^\phi(t',\phi'+\phi) \\
&\quad [((\rho'-\rho)\cos v - (z'-z)\sin v)\cos \phi' + 2\rho' \cos v \sin^2 (\tfrac{\phi'}{2})] \Bigg\} = \underset{\sim}{n} \times \underset{\sim}{H}^i
\end{aligned} \tag{9}$$

where

$$G = \frac{1 + jkR}{k^3 R^3} e^{-jkR} \tag{10}$$

$$R = \sqrt{(\rho-\rho')^2 + (z-z')^2 + 4\rho\rho' \sin^2 (\frac{\phi'}{2})} \tag{11}$$

In (9), both $\underset{\sim}{n}$ and $\underset{\sim}{H}^i$ are to be evaluated at (t,ϕ) on S and v is the angle between $\underset{\sim}{u}_t$ and the z axis. v is positive if $\underset{\sim}{u}_t$ points away from the z axis and v is negative if $\underset{\sim}{u}_t$ points toward the z axis. The variables ρ', z', and v' are respectively ρ, z, and v evaluated at t'. If the electric surface current is bounded and if S curves smoothly, then all the iterated integrals in (9) converge because the integrands are at least as well behaved as R^{-1}.

According to the method of moments, we let

$$\underset{\sim}{J}(\underset{\sim}{r}) = \sum_{n,j} (I^t_{nj} \underset{\sim}{J}^t_{nj}(t,\phi) + I^\phi_{nj} \underset{\sim}{J}^\phi_{nj}(t,\phi)) \tag{12}$$

where $\underset{\sim}{J}^t_{nj}(t,\phi)$ and $\underset{\sim}{J}^\phi_{nj}(t,\phi)$ are expansion functions defined by

$$\underset{\sim}{J}^t_{nj} = \underset{\sim}{u}_t f_j(t)\ e^{jn\phi} \tag{13}$$

$$\underset{\sim}{J}^\phi_{nj} = \underset{\sim}{u}_\phi f_j(t)\ e^{jn\phi} \tag{14}$$

Expansion functions whose t and ϕ components are proportional to $e^{jn\phi}$ are especially suitable because they make the t and ϕ components of the left-hand side of (9) proportional to $e^{jn\phi}$. The coefficients I^t_{nj} and I^ϕ_{nj} are determined by solving the matrix equation which results when (12) is substituted via (8) into (9) and the inner product of (9) with testing functions $\underset{\sim}{W}^t_{mi}(t,\phi)$ and $\underset{\sim}{W}^\phi_{mi}(t,\phi)$ defined by

$$\underset{\sim}{W}^t_{mi} = \underset{\sim}{u}_t f_i(t)\ e^{-jm\phi} \tag{15}$$

$$\underset{\sim}{W}^\phi_{mi} = \underset{\sim}{u}_\phi f_i(t)\ e^{-jm\phi} \tag{16}$$

is taken. For an inner product equal to the dot product integrated over S, the matrix equation decomposes into

$$\begin{bmatrix} Y_n^{tt} & Y_n^{t\phi} \\ Y_n^{\phi t} & Y_n^{\phi\phi} \end{bmatrix} \begin{bmatrix} \vec{I}_n^t \\ \vec{I}_n^\phi \end{bmatrix} = \begin{bmatrix} \vec{\hat{I}}_n^t \\ \vec{\hat{I}}_n^\phi \end{bmatrix} \quad , \; n=0, \pm 1, \pm 2 \ldots \tag{17}$$

where $\vec{I}_n^t$ and $\vec{I}_n^\phi$ are column vectors whose jth elements are I_{nj}^t and I_{nj}^ϕ respectively. Also, $\vec{\hat{I}}_n^t$ and $\vec{\hat{I}}_n^\phi$ are column vectors whose ith elements are given by

$$\hat{I}_{ni}^t = \int dt\, \rho\, f_i(t) \int_0^{2\pi} d\phi (\underset{\sim}{u}_t \times \underset{\sim}{n}) \cdot \underset{\sim}{H}^i\, e^{-jn\phi} \tag{18}$$

and

$$\hat{I}_{ni}^\phi = \int dt\, \rho\, f_i(t) \int_0^{2\pi} d\phi (\underset{\sim}{u}_\phi \times \underset{\sim}{n}) \cdot \underset{\sim}{H}^i\, e^{-jn\phi} \tag{19}$$

respectively. Finally, Y_n^{tt}, $Y_n^{\phi t}$, $Y_n^{t\phi}$, and $Y_n^{\phi\phi}$ are square matrices whose ijth elements are given by

$$(Y_n^{tt})_{ij} = \pi \int dt\, \rho\, f_i(t) f_j(t) + k^3 \int dt\, \rho\, f_i(t) \int dt' \rho' f_j(t') [((\rho' - \rho)\cos v'$$

$$- (z'-z) \sin v')\, G_2 - G_1\, \rho \cos v'] \tag{20}$$

$$(Y_n^{\phi t})_{ij} = jk^3 \int dt\, \rho\, f_i(t) \int dt' \rho' f_j(t')\, (\rho' \sin v \cos v' - \rho \sin v' \cos v$$

$$- (z'-z) \sin v \sin v') G_3 \tag{21}$$

$$(Y_n^{t\phi})_{ij} = jk^3 \int dt\, \rho\, f_i(t) \int dt' \rho' f_j(t') (z'-z) G_3 \tag{22}$$

$$(Y_n^{\phi\phi})_{ij} = \pi \int dt\, \rho\, f_i(t) f_j(t) + k^3 \int dt\, \rho\, f_i(t) \int dt' \rho' f_j(t') [((\rho'-\rho)\cos v$$

$$- (z'-z) \sin v) G_2 + G_1 \rho' \cos v] \tag{23}$$

where

$$G_1 = 2 \int_0^{\pi} d\phi' G \sin^2(\phi'/2) \cos (n\phi') \tag{24}$$

$$G_2 = \int_0^{\pi} d\phi' G \cos \phi' \cos (n\phi') \tag{25}$$

$$G_3 = \int_0^{\pi} d\phi' G \sin \phi' \sin (n\phi') \tag{26}$$

We define $\rho f_i(t)$ to be a four impulse approximation to a triangle function in the following manner. Letting $t = (\rho, z)$ denote that ρ and z are cylindrical coordinates of the point t, we define an odd number greater than or equal to 5 of consecutive points $t = (\rho, z) = \bar{t}_i = (\bar{\rho}_i, \bar{z}_i)$, i=1,2,...P on the generating curve of the body of revolution such that $(\bar{\rho}_1, \bar{z}_1)$ and $(\bar{\rho}_P, \bar{z}_P)$ are the poles. If the body of revolution has no poles because the generating curve closes upon itself as with a torus, then three points must be overlapped such that

$$(\bar{\rho}_{P-3+i}, \bar{z}_{P-3+i}) = (\bar{\rho}_i, \bar{z}_i), \quad i = 1,2,3 .$$

Preferably, the points $\bar{t}_i$ should be such that $\underset{\sim}{n} = \underset{\sim}{u}_\phi \times \underset{\sim}{u}_t$ where $\underset{\sim}{u}_t$ points from $\bar{t}_i$ to $\bar{t}_{i+1}$. If the points $\bar{t}_i$ are chosen such that $\underset{\sim}{n} = - \underset{\sim}{u}_\phi \times \underset{\sim}{u}_t$, then all expressions which can be traced back to the terms proportional to $\underset{\sim}{n}$ in (7) will have the wrong signs.

We now approximate the generating curve by drawing straight lines between the points $(\bar{\rho}_i, \bar{z}_i)$, i=1,2,...P and define points

$$t = t_i = (\rho_i, z_i) = \left(\frac{\bar{\rho}_i + \bar{\rho}_{i+1}}{2}, \frac{\bar{z}_i + \bar{z}_{i+1}}{2}\right) \tag{27}$$

on this approximate generating curve. The length d_i of the interval centered about t_i is given by

$$d_i = \sqrt{(\bar{\rho}_{i+1} - \bar{\rho}_i)^2 + (\bar{z}_{i+1} - \bar{z}_i)^2} \tag{28}$$

In terms of coefficients T_{p+4i-4} defined by

$$\begin{aligned}
T_{4i-3} &= \frac{k d_{2i-1}^2}{2(d_{2i-1} + d_{2i})} \\
T_{4i-2} &= \frac{k(d_{2i-1} + \frac{1}{2} d_{2i}) d_{2i}}{d_{2i-1} + d_{2i}} \\
T_{4i-1} &= \frac{k(d_{2i+2} + \frac{1}{2} d_{2i+1}) d_{2i+1}}{d_{2i+1} + d_{2i+2}} \\
T_{4i} &= \frac{k d_{2i+2}^2}{2(d_{2i+1} + d_{2i+2})} ,
\end{aligned} \tag{29}$$

we construct

$$\rho f_i(t) = \frac{1}{k} \sum_{p=1}^{4} T_{p+4i-4}\, \delta(t - t_{p+2i-2}) \tag{30}$$

where $\delta(t)$ is the unit impulse function. The right-hand side of (30) is the desired four impulse approximation to a triangle function (see Fig. 3).

Substitution of (30) into (20) - (23) yields

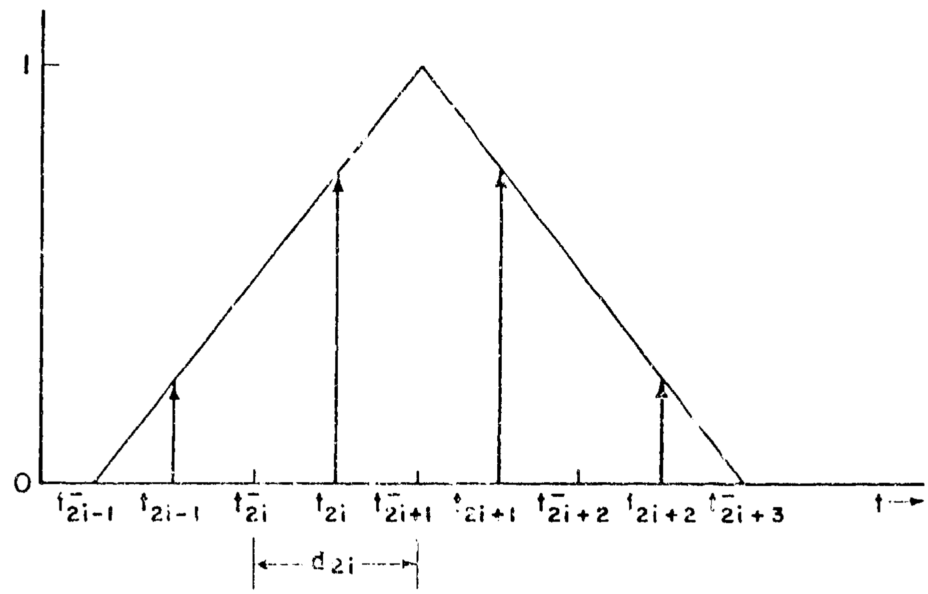

Fig. 3. Triangle function (solid) and four impulse approximation (arrows).

$$\begin{bmatrix} (Y_n^{tt})_{ij} \\ (Y_n^{\phi t})_{ij} \\ (Y_n^{t\phi})_{ij} \\ (Y_n^{\phi\phi})_{ij} \end{bmatrix} = \sum_{p=1}^{4} T_{p+4i-4} \sum_{q=1}^{4} T_{q+4j-4} \begin{bmatrix} (Y1)_s \\ (Y2)_s \\ (Y3)_s \\ (Y4)_s \end{bmatrix} \tag{31}$$

where s denotes the double subscript p+2i-2, q+2j-2 and

$$(Y1)_{ij} = \begin{cases} k((\rho_j-\rho_i)\cos v_j - (z_j-z_i)\sin v_j)G_2 - k\rho_i G_1 \cos v_j, & i \neq j \\ \dfrac{\pi}{k^2 d_i \rho_i} - k\rho_i\, G_1 \cos v_i, & i=j \end{cases} \tag{32}$$

$$(Y2)_{ij} = jk(\rho_j \sin v_i \cos v_j - \rho_i \sin v_j \cos v_i - (z_j-z_i)\sin v_i \sin v_j)G_3 \tag{33}$$

$$(Y3)_{ij} = jk(z_j - z_i)G_3 \tag{34}$$

$$(Y4)_{ij} = \begin{cases} k((\rho_j-\rho_i)\cos v_i - (z_j-z_i)\sin v_i)G_2 + k\rho_j G_1 \cos v_i, & i \neq j \\ \dfrac{\pi}{k^2 d_i \rho_i} + k\rho_i G_1 \cos v_i, & i=j \end{cases} \tag{35}$$

Here, v_i is the angle between the approximate generating curve at (ρ_i, z_i) and the z axis. The term $\frac{\pi}{k^2 d_i \rho_i}$ in (32) and (35) was obtained by replacing one of the coincident impulse functions in the first integral on the right-hand sides of both (20) and (23) by an equivalent pulse over the interval of length d_i.

In (32)-(35), G_1, G_2, and G_3 are given by (24) to (26) in which G is given by (10) with R of (11) evaluated at $(\rho,z,\rho',z') = (\rho_i,z_i,\rho_j,z_j)$. When $i = j$ in (32)-(35), none of the integrals G_1,G_2, and G_3 converge because

$$R = 2\rho_i \sin(\phi'/2)$$

This lack of convergence is ascribed to use of the impulse representation (30) rather than to any deficiency in the H-field integral equation (9). To obtain convergence for $i = j$, we replace the above R by an equivalent distance R_e given by

$$R_e = \sqrt{(d_i/4)^2 + 4\rho_i^2 \sin^2(\phi'/2)} \tag{36}$$

Expression (36) may be obtained by displacing the field point a distance $d_i/4$ perpendicular to the plane of the source loop. Now, $d_i/4$ is the equivalent radius [7] of a flat strip whose width is d_i. Expression (36) can also be obtained by averaging R^2 for field points displaced a distance $d_i/4$ in either direction along the approximate generating curve from the source loop.

An N_ϕ point Gaussian quadrature formula is used to calculate the integrals G_1,G_2, and G_3 defined by (24)-(26) in which G is given by (10) and R by either (11) or (36). According to this quadrature formula,

$$\int_0^\pi f(\phi')d\phi' = \frac{\pi}{2} \sum_{k=1}^{N_\phi} A_k f(\frac{\pi}{2}(x_k + 1)) \tag{37}$$

where $f(\phi')$ is the function being integrated and x_k and A_k are constants tabulated by Krylov [8]. In (37), the multiplier $\frac{\pi}{2}$ and argument $\frac{\pi}{2}(x_k+1)$

[7] E. A. Wolff, Antenna Analysis, John Wiley and Sons, Inc., New York, 1966, p. 61.

[8] V. I. Krylov, Approximate Calculation of Integrals, translated by A. H. Stroud, Macmillan Co., New York, 1962, Appendix A.

instead of just x_k are due to the transformation of Krylov's interval from -1 to 1 into the interval from 0 to π.

The $\frac{\pi}{k^2 d_i \rho_i}$ terms in (32) and (35) destroy some symmetry properties of (31). If the $\frac{\pi}{k^2 d_i \rho_i}$ terms were absent, then

$$
\begin{aligned}
(Y1)_{ij} &= -\ (Y4)_{ji} \\
(Y2)_{ij} &= -\ (Y2)_{ji} \\
(Y3)_{ij} &= -\ (Y3)_{ji}
\end{aligned}
\tag{38}
$$

and replacement of (i,j,p,q) by (j,i,q,p) in (31) would show that

$$
\begin{aligned}
(Y_n^{tt})_{ij} &= -\ (Y_n^{\phi\phi})_{ji} \\
(Y_n^{\phi t})_{ij} &= -\ (Y_n^{\phi t})_{ji} \\
(Y_n^{t\phi})_{ij} &= -\ (Y_n^{t\phi})_{ji}
\end{aligned}
\tag{39}
$$

An efficient method of computing (31) which takes (38) and (39) into account is described in a subsequent report.

IV. E-FIELD SOLUTION

The E-field solution is obtained by applying the method of moments to the E-field integral equation. The E-field integral equation is derived by setting the component tangential to S of the total electric field equal to zero on S.

The boundary condition that the total tangential electric field is zero on S is written as

$$
-\frac{1}{\eta}\,\underset{\sim}{E}^{s}_{tan} = \frac{1}{\eta}\,\underset{\sim}{E}^{i}_{tan} \qquad \text{on } S \tag{40}
$$

where $\underset{\sim}{E}^s$ is the electric field due to the electric surface current on S, $\underset{\sim}{E}^i$ is the incident electric field given by either (1) or (2) and η is the intrinsic impedance. The subscript tan denotes tangential components on S. The $1/\eta$ terms are included in (40) to give it the dimensions of current.

The field $\underset{\sim}{E}^s$ can be expressed in terms of a vector potential $\underset{\sim}{A}$ and a scalar potential Φ as

$$\underset{\sim}{E}^s = -j\omega\underset{\sim}{A}(\underset{\sim}{J}) - \underset{\sim}{\nabla}\Phi(\underset{\sim}{J}) \tag{41}$$

where

$$\underset{\sim}{A}(\underset{\sim}{J}) = \mu \iint_S \underset{\sim}{J}(\underset{\sim}{r}') \frac{e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|}}{4\pi|\underset{\sim}{r}-\underset{\sim}{r}'|} ds' \tag{42}$$

$$\Phi(\underset{\sim}{J}) = \frac{1}{\epsilon} \iint_S \sigma \frac{e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|}}{4\pi|\underset{\sim}{r}-\underset{\sim}{r}'|} ds' \tag{43}$$

Here, $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ are vectors to the field and source points respectively, $\underset{\sim}{J}(\underset{\sim}{r}')$ is the electric surface current on S, k is the propagation constant, μ is the permeability, ϵ is the permittivity, and σ is the surface charge given by

$$\sigma = -\frac{1}{j\omega} \lim_{\Delta S \to 0} \left(\frac{\int_C \underset{\sim}{J}(\underset{\sim}{r}) \cdot \underset{\sim}{u}_n \, dc}{\Delta S} \right) = -\frac{1}{j\omega} \underset{\sim}{\nabla}_s \cdot \underset{\sim}{J}(\underset{\sim}{r}) \tag{44}$$

where $\underset{\sim}{u}_n$ is the unit vector tangential to S and normal to the curve C which bounds the small portion ΔS of S. $\underset{\sim}{u}_n$ points away from ΔS. The operator $\underset{\sim}{\nabla}_s \cdot$ is the surface divergence on S.

Following the method of moments, we write

$$\underset{\sim}{J} = \sum_{n,j} \left(I^t_{nj} \underset{\sim}{J}^t_{nj} + I^\phi_{nj} \underset{\sim}{J}^\phi_{nj} \right) \tag{45}$$

where I_{nj}^{t} and I_{nj}^{ϕ} are coefficients to be determined and $\underset{\sim}{J}_{nj}^{t}$ and $\underset{\sim}{J}_{nj}^{\phi}$ are expansion functions defined by

$$\underset{\sim}{J}_{nj}^{t} = \underset{\sim}{u}_t f_j(t)e^{jn\phi}$$

$$\underset{\sim}{J}_{nj}^{\phi} = \underset{\sim}{u}_\phi f_j(t)e^{jn\phi} \tag{46}$$

Next, we take the integral over S of the dot product of (40) with each one of a collection of testing functions $\underset{\sim}{W}_{mi}^{t}$, $\underset{\sim}{W}_{mi}^{\phi}$ defined by

$$\underset{\sim}{W}_{mi}^{t} = \underset{\sim}{u}_t f_i(t)e^{-jm\phi}$$

$$\underset{\sim}{W}_{mi}^{\phi} = \underset{\sim}{u}_\phi f_i(t)e^{-jm\phi} \tag{47}$$

to obtain the matrix equation

$$\sum_n ([Z_{mn}^{tt}]\ \vec{I}_n^{t} + [Z_{mn}^{t\phi}]\ \vec{I}_n^{\phi}) = \vec{V}_m^{t}$$

$$\sum_n ([Z_{mn}^{\phi t}]\ \vec{I}_n^{t} + [Z_{mn}^{\phi\phi}]\ \vec{I}_n^{\phi}) = \vec{V}_m^{\phi} \tag{48}$$

where the Z's are square matrices whose ij-th elements are defined by

$$(Z_{mn}^{pq})_{ij} = \frac{1}{\eta}\iint_S \underset{\sim}{W}_{mi}^{p} \cdot (j\omega \underset{\sim}{A}(\underset{\sim}{J}_{nj}^{q}) + \underset{\sim}{\nabla}\Phi(\underset{\sim}{J}_{nj}^{q}))ds \tag{49}$$

where p may be either t or ϕ and q may be either t or ϕ and $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance. Also, $\vec{V}_m^{t}$ and $\vec{V}_m^{\phi}$ are column vectors whose i-th elements are given by

$$V_{mi}^{p} = \frac{1}{\eta}\iint_S \underset{\sim}{W}_{mi}^{p} \cdot \underset{\sim}{E}^{i} ds \tag{50}$$

where p may be either t or ϕ. Lastly, $\vec{I}_n^t$ and $\vec{I}_n^\phi$ are column vectors of the coefficients I_{nj}^t and I_{nj}^ϕ appearing in (45).

The following manipulations serve to transfer the differential operator on Φ in (49) to $\underset{\sim}{W}_{mi}^p$. Since S is closed,

$$\iint_S \underset{\sim}{\nabla}_s \cdot (\Phi\, \underset{\sim}{W})\, ds = 0 \tag{51}$$

where $\underset{\sim}{W}$ denotes $\underset{\sim}{W}_{mi}^p$. Now, the representation

$$\underset{\sim}{\nabla}_s \cdot \underset{\sim}{W} = \frac{1}{\rho}\frac{\partial}{\partial t}(\rho\, \underset{\sim}{W}\cdot \underset{\sim}{u}_t) + \frac{1}{\rho}\frac{\partial}{\partial \phi}(\underset{\sim}{W}\cdot \underset{\sim}{u}_\phi) \tag{52}$$

of the surface divergence and the definition

$$\underset{\sim}{\nabla}_s \Phi = \underset{\sim}{u}_t \frac{\partial \Phi}{\partial t} + \underset{\sim}{u}_\phi \frac{\partial \Phi}{\rho \partial \phi} \tag{53}$$

of the surface gradient imply that

$$\underset{\sim}{\nabla}_s \cdot (\Phi\, \underset{\sim}{W}) = \Phi \underset{\sim}{\nabla}_s \cdot \underset{\sim}{W} + \underset{\sim}{W} \cdot \underset{\sim}{\nabla}_s \Phi \tag{54}$$

In (54), the surface gradient of Φ can be replaced by the ordinary three-dimensional gradient of Φ because (53) is the component of the three-dimensional gradient tangential to S and $\underset{\sim}{W}$ is tangential to S. Substitution of (54) into (51) and then (51) into (49) yields

$$(Z_{mn}^{pq})_{ij} = \frac{j\omega}{\eta}\iint_S \{\underset{\sim}{W}_{mi}^p \cdot \underset{\sim}{A}(\underset{\sim}{J}_{nj}^q) + \sigma_{mi}^p\ \Phi(\underset{\sim}{J}_{nj}^q)\}\, ds \tag{55}$$

where

$$\sigma_{mi}^p = -\frac{1}{j\omega}\underset{\sim}{\nabla}_s \cdot \underset{\sim}{W}_{mi}^p \tag{56}$$

Since, as shown in Appendix B, (55) is zero for $m \neq n$, (48) reduces to

$$\begin{bmatrix} Z_n^{tt} & Z_n^{t\phi} \\ Z_n^{\phi t} & Z_n^{\phi\phi} \end{bmatrix} \begin{bmatrix} \vec{I}_n^{t} \\ \vec{I}_n^{\phi} \end{bmatrix} = \begin{bmatrix} \vec{V}_n^{t} \\ \vec{V}_n^{\phi} \end{bmatrix}, \quad n=0, \pm 1, \pm 2, \ldots \tag{57}$$

where Z_n^{pq} is Z_{nn}^{pq} of (55). It is also shown in Appendix B that the elements of Z_n^{pq} are given by

$$(Z_n^{tt})_{ij} = j \int dt \int dt' \{k^2 \rho f_i(t) \rho' f_j(t')(G_5 \sin v \sin v' + G_4 \cos v \cos v')$$
$$- \frac{\partial}{\partial t} (\rho f_i(t)) \frac{\partial}{\partial t'} (\rho' f_j(t')) G_4\} \tag{58}$$

$$(Z_n^{\phi t})_{ij} = - \int dt \rho f_i(t) \int dt' (k^2 \rho' f_j(t') \; G_6 \sin v' + \frac{n}{\rho} \frac{\partial}{\partial t'} (\rho' f_j(t')) G_4) \tag{59}$$

$$(Z_n^{t\phi})_{ij} = \int dt \int dt' \rho' f_j(t') (k^2 \rho f_i(t) \; G_6 \; \sin v + \frac{n}{\rho'} \frac{\partial}{\partial t} (\rho f_i(t)) \; G_4) \tag{60}$$

$$(Z_n^{\phi\phi})_{ij} = j \int dt \rho f_i(t) \int dt' \rho' f_j(t') (k^2 G_5 - \frac{n^2}{\rho\rho'} G_4) \tag{61}$$

where v is the angle between the tangent to the generating curve and the z axis and where

$$G_4 = \int_0^{\pi} d\phi' \frac{e^{-jkR}}{kR} \cos (n\phi') \tag{62}$$

$$G_5 = \int_0^{\pi} d\phi' \frac{e^{-jkR}}{kR} \cos \phi' \cos (n\phi') \tag{63}$$

$$G_6 = \int_0^{\pi} d\phi' \frac{e^{-jkR}}{kR} \sin \phi' \sin (n\phi') \tag{64}$$

$$R = \sqrt{(\rho-\rho')^2 + (z-z')^2 + 4\rho\rho' \sin^2(\frac{\phi'}{2})} \tag{65}$$

Here, ρ, z, and v depend on t while ρ', z', and v' depend on t'.

To evaluate (58)-(61), we choose for $\rho f_i(t)$ the four impulse approximation (30) to a triangle function which reads

$$\rho f_i(t) = \frac{1}{k} \sum_{p=1}^{4} T_{p+4i-4}\, \delta(t-t_{p+2i-2}) \tag{66}$$

where $\delta(t)$ is the unit impulse function and T_i and t_i are defined by (29) and (27) respectively. For $\frac{d}{dt}(\rho f_i(t))$, we choose the four impulse approximation

$$\frac{d}{dt}(\rho f_i(t)) = \sum_{p=1}^{4} T'_{p+4i-4}\, \delta(t-t_{p+2i-2}) \tag{67}$$

to the derivative of the triangle function as shown in Fig. 3. Figure 4 illustrates (67). The coefficients T'_{p+4i-4} appearing in (67) are given by

$$\begin{aligned} T'_{4i-3} &= \frac{d_{2i-1}}{d_{2i-1}+d_{2i}} \\ T'_{4i-2} &= \frac{d_{2i}}{d_{2i-1}+d_{2i}} \\ T'_{4i-1} &= \frac{-d_{2i+1}}{d_{2i+1}+d_{2i+2}} \\ T'_{4i} &= \frac{-d_{2i+2}}{d_{2i+1}+d_{2i+2}} \end{aligned} \tag{68}$$

where d_i is defined by (28).

Substituting (66) and (67) into (58)-(61), we obtain

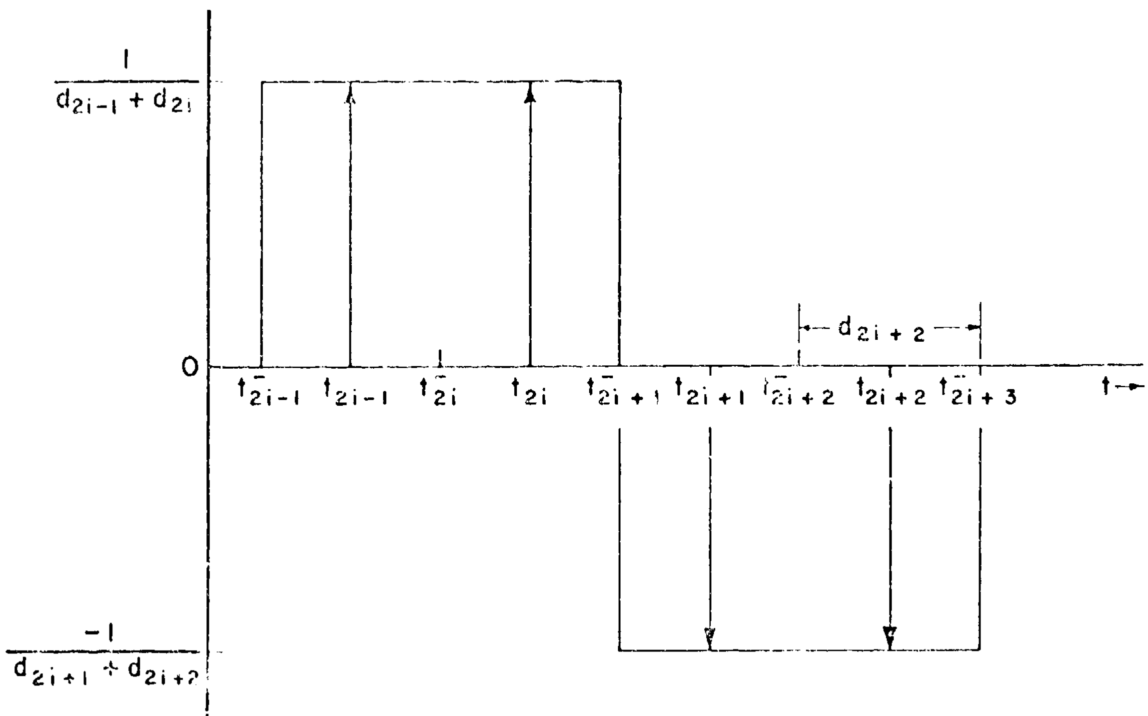

Fig. 4. Derivative of triangle function (solid) and four impulse approximation (arrows).

$$(Z_n^{tt})_{ij} = j \sum_{p=1}^{4} \sum_{q=1}^{4} \{T_{p'} T_{q'} (G_5 \sin v_{i'} \sin v_{j'} + G_4 \cos v_{i'} \cos v_{j'}) - T'_{p'} T'_{q'} G_4\} \tag{69}$$

$$(Z_n^{\phi t})_{ij} = - \sum_{p=1}^{4} \sum_{q=1}^{4} \{T_{p'} T_{q'} G_6 \sin v_{j'} + \frac{n}{k\rho_{i'}} T_{p'} T'_{q'} G_4\} \tag{70}$$

$$(Z_n^{t\phi})_{ij} = \sum_{p=1}^{4} \sum_{q=1}^{4} \{T_{p'} T_{q'} G_6 \sin v_{i'} - \frac{n}{k\rho_{j'}} T'_{p'} T_{q'} G_4\} \tag{71}$$

$$(Z_n^{\phi\phi})_{ij} = j \sum_{p=1}^{4} \sum_{q=1}^{4} T_{p'} T_{q'} (G_5 - \frac{n^2}{k^2 \rho_{i'} \rho_{j'}} G_4) \tag{72}$$

where

$$\begin{aligned} p' &= p + 4i - 4 \\ q' &= q + 4j - 4 \\ i' &= p + 2i - 2 \\ j' &= q + 2j - 2 \end{aligned} \tag{73}$$

The subscript i' denotes evaluation at $t_{i'}$. The subscript j' denotes evaluation at $t_{j'}$. In (69) - (72), G_4, G_5, and G_6 are given by (62) - (64) in which R of (65) is evaluated at $t = t_{i'}$, $t' = t_{j'}$ which, in terms of cylindrical coordinates, is at $\rho, z, \rho', z' = \rho_{i'}, z_{i'}, \rho_{j'}, z_{j'}$. If $i' = j'$, we replace R by the equivalent distance R_e given by (36) which, with i replaced by i' reads

$$R_e = \sqrt{(d_{i'}/4)^2 + 4\rho_{i'}^2 \sin^2(\phi'/2)} \tag{74}$$

The N_ϕ point Gaussian quadrature formula (37) which reads

$$\int_0^{\pi} f(\phi')d\phi' = \frac{\pi}{2} \sum_{k=1}^{N_\phi} A_k f(\frac{\pi}{2} (x_k + 1)) \tag{75}$$

is used to calculate the integrals G_4, G_5, and G_6 defined by (62) - (64).

Since replacement of (i,j,p,q) by (j,i,q,p) in (73) implies replacement of (i',j',p',q') by (j',i',q',p') in (69) - (72), and since G_4, G_5, and G_6 are symmetric in i' and j', it is evident that

$$\begin{aligned}(Z_n^{tt})_{ij} &= (Z_n^{tt})_{ji} \\ (Z_n^{\phi t})_{ij} &= -(Z_n^{t\phi})_{ji} \\ (Z_n^{\phi\phi})_{ij} &= (Z_n^{\phi\phi})_{ji}\end{aligned} \tag{76}$$

An efficient method of computing (69) - (72) which takes advantage of (76) is described in a subsequent report.

V. COMBINED FIELD SOLUTION

In this section, we assume that the incident field $(\underset{\sim}{E}^i, \underset{\sim}{H}^i)$ due to sources outside the perfectly conducting body whose closed surface is S induces a unique electric surface current $\underset{\sim}{J}$ on S and that this $\underset{\sim}{J}$ satisfies (3) and (40) which read

$$-\underset{\sim}{n} \times \underset{\sim}{H}^s(\underset{\sim}{J}) = \underset{\sim}{n} \times \underset{\sim}{H}^i \qquad \text{just inside S} \tag{77}$$

$$-\frac{1}{\eta}\underset{\sim}{E}^s_{tan}(\underset{\sim}{J}) = \frac{1}{\eta}\underset{\sim}{E}^i_{tan} \qquad \text{on S} \tag{78}$$

where $\underset{\sim}{n}$ is the unit outside normal vector to S, $(\underset{\sim}{E}^s, \underset{\sim}{H}^s)$ is the field due to $\underset{\sim}{J}$, and the subscript tan denotes tangential components on S. The question arises whether (77) alone is sufficient to determine $\underset{\sim}{J}$, whether (78) alone is sufficient, or whether information must be drawn simultaneously from both (77) and (78).

If both $\underset{\sim}{J}$ and $\underset{\sim}{J} + \hat{\underset{\sim}{J}}$ satisfy (77), then

$$\underset{\sim}{H}^s_{tan}(\hat{\underset{\sim}{J}}) = 0 \qquad \text{just inside S} \tag{79}$$

Maxwell's equations will be satisfied if

$$\nabla \times \nabla \times H^s(\hat{J}) = k^2 H^s(\hat{J}) \quad \text{inside } S \tag{80}$$

The solution J to (77) is not unique for values of k at which (79) and (80) admit a nontrivial solution $\hat{J}$. This $\hat{J}$ will be called a magnetic cavity mode. Similarly the solution J to (78) is not unique for values of k at which

$$E^s_{tan}(\hat{J}) = 0 \quad \text{on } S \tag{81}$$

$$\nabla \times \nabla \times E^s(\hat{J}) = k^2 E^s(\hat{J}) \tag{82}$$

admit a nontrivial solution $\hat{J}$ called an electric cavity mode. Comparison of (79) and (80) with (81) and (82) shows that the magnetic and electric cavity modes occur at the same values of k and that the magnetic field of the magnetic mode is proportional to the electric field of the electric mode inside S. The electric mode field vanishes outside S because its tangential electric field is zero just outside S, but the magnetic mode field does not vanish outside S because its tangential magnetic field is not zero just outside S.

The assumed existence of a unique solution J to the physical problem implies that the incident field is orthogonal to a nontrivial solution, if it exists, of the adjoint field problem. The adjoint magnetic field operator has been determined by Marin [9].

We will show that the solution J to the combined field formulation [4,5]

$$-n \times H^s(J) - \frac{\alpha}{\eta} E^s_{tan}(J) = n \times H^i + \frac{\alpha}{\eta} E^i_{tan} \quad \text{just inside } S \tag{83}$$

is unique and satisfies both (77) and (78) whenever α is a positive real number.

---

[9] L. Marin, "Natural-Mode Representation of Transient Scattered Fields," IEEE Trans. Antennas Propagat., vol. AP-21, pp. 809-818, Nov. 1973.

The solution to (83) is unique if

$$-\underset{\sim}{n} \times \underset{\sim}{H}^{s}(\underset{\sim}{J}) - \frac{\alpha}{\eta} \underset{\sim}{E}^{s}_{tan}(\underset{\sim}{J}) = 0 \tag{84}$$

implies that $\underset{\sim}{J} = 0$. Scalar multiplication of (84) by its complex conjugate and integration over S lead to

$$\iint_S \left(|\underset{\sim}{H}^{s}_{tan}(\underset{\sim}{J})|^2 + \frac{\alpha^2}{\eta^2} |\underset{\sim}{E}^{s}_{tan}(\underset{\sim}{J})|^2\right) ds + \frac{2\alpha}{\eta} \left[\text{Real} \iint_S (\underset{\sim}{E}^{s}(\underset{\sim}{J}) \times \underset{\sim}{H}^{s*}(\underset{\sim}{J})) \cdot (-\underset{\sim}{n})\, ds\right] = 0 \tag{85}$$

The bracketed quantity in (85) is the real power flowing inside S and hence is either zero if the media is loss-free or greater than zero if the media is lossy. Thus, (85) implies that

$$\underset{\sim}{H}^{s}_{tan}(\underset{\sim}{J}) = 0 \qquad \text{just inside S} \tag{86}$$

$$\underset{\sim}{E}^{s}_{tan}(\underset{\sim}{J}) = 0 \qquad \text{on S} \tag{87}$$

Since (87) implies that $\underset{\sim}{H}^{s}_{tan}(\underset{\sim}{J})$ is also zero just outside S, we obtain the desired result that $\underset{\sim}{J} = 0$. Hence the solution to (83) is unique. The statement on page 224 of [5] that (83) has an infinite number of solutions at eigenfrequencies is not correct.

If (83) is true, then (84) - (87) are valid with $\underset{\sim}{H}^{s}(\underset{\sim}{J})$ replaced by $\underset{\sim}{H}^{s}(\underset{\sim}{J}) + \underset{\sim}{H}^{i}$ and $\underset{\sim}{E}^{s}(\underset{\sim}{J})$ replaced by $\underset{\sim}{E}^{s}(\underset{\sim}{J}) + \underset{\sim}{E}^{i}$. Therefore, (83) implies both (77) and (78).

Since (83) is the linear combination of (3) and (40) with relative weight α, the method of moments formulation obtained from (83) is the same linear combination of (17) and (57). Hence,

$$\left[\begin{bmatrix} Y^{tt}_n & Y^{t\phi}_n \\ Y^{\phi t}_n & Y^{\phi\phi}_n \end{bmatrix} + \alpha \begin{bmatrix} Z^{tt}_n & Z^{t\phi}_n \\ Z^{\phi t}_n & Z^{\phi\phi}_n \end{bmatrix}\right] \begin{bmatrix} \vec{I}^{t}_n \\ \vec{I}^{\phi}_n \end{bmatrix} = \begin{bmatrix} \vec{I}^{*t}_n \\ \vec{I}^{*\phi}_n \end{bmatrix} + \alpha \begin{bmatrix} \vec{V}^{t}_n \\ \vec{V}^{\phi}_n \end{bmatrix}, \quad n=0,\pm1,\pm2\ldots \tag{88}$$

where all matrices and column vectors have the same meaning as in Sections III and IV except that $\vec{I}_n^t$ and $\vec{I}_n^\phi$ now, when substituted into (45), give the combined field solution for $\underset{\sim}{J}$.

VI. FAR FIELD MEASUREMENT AND PLANE WAVE EXCITATION

In this section, measurement vectors are used to obtain the far field of the surface current $\underset{\sim}{J}$. The plane wave excitation vectors needed for the H-field, E-field, and combined field solutions are then expressed in terms of these measurement vectors.

By reciprocity,

$$\underset{\sim}{E}^s(\underset{\sim}{J}) \cdot I\underset{\sim}{\ell}_r = \iint_S \underset{\sim}{J}(\underset{\sim}{r}) \cdot \underset{\sim}{E}(I\underset{\sim}{\ell}_r)ds \tag{89}$$

where $\underset{\sim}{E}^s(\underset{\sim}{J})$ is the far electric field due to $\underset{\sim}{J}$, $I\underset{\sim}{\ell}_r$ is a receiving electric dipole at the far field measurement point, and $\underset{\sim}{E}(I\underset{\sim}{\ell}_r)$ is the electric field due to $I\underset{\sim}{\ell}_r$ evaluated at $\underset{\sim}{r}$ on S. If $\underset{\sim}{\ell}_r$ is tangent to the radiation sphere,

$$\underset{\sim}{E}(I\underset{\sim}{\ell}_r) = \frac{-jk\eta e^{-jkr_r}}{4\pi r_r} I\underset{\sim}{\ell}_r\, e^{-j\underset{\sim}{k}_r \cdot \underset{\sim}{r}} \tag{90}$$

where r_r is the distance between the measurement point and the origin in the vicinity of S and $\underset{\sim}{k}_r$ is the propagation vector of the plane wave coming from $I\underset{\sim}{\ell}_r$. Substi ting (45), (46), and (90) into (89) and letting $\underset{\sim}{\ell}_r$ be either $\underset{\sim}{u}_\theta^r$ or $\underset{\sim}{u}_\phi^r$, we obtain

$$\begin{bmatrix} E_\theta^s(\underset{\sim}{J}) \\ E_\phi^s(\underset{\sim}{J}) \end{bmatrix} = \frac{-j\eta e^{-jkr_r}}{4\pi r_r} \sum_n \begin{bmatrix} \tilde{R}_n^{t\theta} & \tilde{R}_n^{\phi\theta} \\ \tilde{R}_n^{t\phi} & \tilde{R}_n^{\phi\phi} \end{bmatrix} \begin{bmatrix} \vec{I}_n^t \\ \vec{I}_n^\phi \end{bmatrix} e^{jn\phi_r} \tag{91}$$

where $\tilde{R}_n^{pq}$ is the transpose of a column vector $\vec{R}_n^{pq}$ whose i-th element is given by

$$R_{ni}^{pq} = k\int dt\rho f_i(t)\int_0^{2\pi} d\phi\ (\underset{\sim}{u}_p \cdot \underset{\sim}{u}_q^r)e^{j(-\underset{\sim}{k}_r \cdot \underset{\sim}{r} + n(\phi-\phi_r))} \tag{92}$$

where ϕ_r is the azimuth of the far field measurement point. In (91), $E_\theta^s(\underset{\sim}{J})$ and $E_\phi^s(\underset{\sim}{J})$ are respectively the $\underset{\sim}{u}_\theta^r$ and $\underset{\sim}{u}_\phi^r$ components of $\underset{\sim}{E}^s(\underset{\sim}{J})$.

With a view toward evaluation of (92), we note from Figs. 2, A-1, and A-2 that

$$\begin{aligned}
\underset{\sim}{u}_t \cdot \underset{\sim}{u}_\theta^r &= -\sin\theta_r\cos v + \cos\theta_r\sin v\cos(\phi-\phi_r) \\
\underset{\sim}{u}_\phi \cdot \underset{\sim}{u}_\theta^r &= -\cos\theta_r\sin(\phi-\phi_r) \\
\underset{\sim}{u}_t \cdot \underset{\sim}{u}_\phi^r &= \sin v\sin(\phi-\phi_r) \\
\underset{\sim}{u}_\phi \cdot \underset{\sim}{u}_\phi^r &= \cos(\phi-\phi_r) \\
-\underset{\sim}{k}_r \cdot \underset{\sim}{r} &= kz\cos\theta_r + k\rho\sin\theta_r\cos(\phi-\phi_r)
\end{aligned} \tag{93}$$

Substituting (93) and (30) into (92) and taking advantage of the integral formula

$$J_n(k\rho\sin\theta_r) = \frac{j^{-n}}{2\pi}\int_0^{2\pi} e^{j(k\rho\sin\theta_r\cos\phi + n\phi)}\, d\phi \tag{94}$$

deduced from (9.1.21) of [10] for Bessel functions, we obtain

---

[10] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," U. S. Government Printing Office, Washington, D.C. (Natl. Bur. Std. U. S. Appl. Math. Ser. 55), 1964, p. 360.

$$R^{t\theta}_{ni} = \pi j^n \sum_{p=1}^{4} T_{p+4i-4}(-2J_n \sin\theta_r \cos v + j(J_{n+1} - J_{n-1})\cos\theta_r \sin v)e^{jkz\cos\theta_r}$$

$$R^{\phi\theta}_{ni} = -\pi j^n \sum_{p=1}^{4} T_{p+4i-4}(J_{n+1} + J_{n-1})\cos\theta_r \, e^{jkz\cos\theta_r}$$

$$R^{t\phi}_{ni} = \pi j^n \sum_{p=1}^{4} T_{p+4i-4}(J_{n+1} + J_{n-1})\sin v \, e^{jkz\cos\theta_r} \tag{95}$$

$$R^{\phi\phi}_{ni} = \pi j^{n+1} \sum_{p=1}^{4} T_{p+4i-4}(J_{n+1} - J_{n-1}) \, e^{jkz\cos\theta_r}$$

where

$$J_n = J_n(k\rho \sin\theta_r) \tag{96}$$

In (95), ρ, z, and v are to be evaluated at $t = t_{p+2i-2}$.

Substitute (1) and (2) into (18) and (19) to obtain

$$\hat{T}^{pq}_{ni} = \int dt\rho f_i(t) \int_0^{2\pi} d\phi(\underset{\sim}{u}_p \times \underset{\sim}{n}) \cdot (\underset{\sim}{k}_t \times \underset{\sim}{u}^t_q) \, e^{j(-\underset{\sim}{k}_t \cdot \underset{\sim}{r} - n\phi)} \tag{97}$$

and then substitute (1) and (2) into (50) where $\underset{\sim}{W}^p_{mi}$ is given by (47) to obtain

$$V^{pq}_{ni} = k \int dt\rho f_i(t) \int_0^{2\pi} d\phi(\underset{\sim}{u}_p \cdot \underset{\sim}{u}^t_q)e^{j(-\underset{\sim}{k}_t \cdot \underset{\sim}{r} - n\phi)} \tag{98}$$

where p is either t or ϕ. The additional superscript q on the left-hand sides of (97) and (98) is either θ or ϕ according as the incident electric field is θ polarized as in (1) or ϕ polarized as in (2). Comparison of (97) with (92) shows that

$$\begin{bmatrix} \vec{I}_n^{t\theta} & \vec{I}_n^{t\phi} \\ \vec{I}_n^{\phi\theta} & \vec{I}_n^{\phi\phi} \end{bmatrix} = \begin{bmatrix} -\vec{R}_{-n}^{\phi\phi} & \vec{R}_{-n}^{\phi\theta} \\ \vec{R}_{-n}^{t\phi} & -\vec{R}_{-n}^{t\theta} \end{bmatrix} \tag{99}$$

where the R's on the right-hand side of (99) are to be evaluated at $\theta_r = \theta_t$. Comparison of (98) with (92) leads to

$$\vec{V}_n^{pq} = \vec{R}_{-n}^{pq} \tag{100}$$

where p is either t or ϕ, q is either θ or ϕ, and $\vec{R}_{-n}^{pq}$ is to be evaluated at $\theta_r = \theta_t$.

The expressions (45) and (91) for the electric surface current and far field can be simplified by combining the +n and -n terms. Substituting (46) into (45), we obtain

$$\underset{\sim}{J}^q = \sum_{n=-\infty}^{\infty} e^{jn\phi} \{(\tilde{f}\, \vec{I}_n^{tq})\underset{\sim}{u}_t + (\tilde{f}\, \vec{I}_n^{\phi q})\underset{\sim}{u}_\phi\} \tag{101}$$

where $\tilde{f}$ is the transpose of the column vector $\vec{f}$ of the $f_j(t)$, and $\vec{I}_n^{tq}$ and $\vec{I}_n^{\phi q}$ are column vectors of the coefficients I_{nj}^t and I_{nj}^ϕ respectively. The additional superscript q is either θ or ϕ according as the incident electric field is θ polarized as in (1) or ϕ polarized as in (2). The column vectors $\vec{I}_n^{tq}$ and $\vec{I}_n^{\phi q}$ appearing in (101) are obtained by solving either the H-field matrix equation (17), the E-field matrix equation (57), or the combined field matrix equation (88) with the additional superscript q on the column vectors therein to denote the polarization of the incident electric field.

Inspection of (20) - (26) and (58) - (64) reveals that

$$\begin{bmatrix} Y_{-n}^{tt} & Y_{-n}^{t\phi} \\ Y_{-n}^{\phi t} & Y_{-n}^{\phi\phi} \end{bmatrix} = \begin{bmatrix} Y_n^{tt} & -Y_n^{t\phi} \\ -Y_n^{\phi t} & Y_n^{\phi\phi} \end{bmatrix}, \quad n=0,1,2,\ldots \tag{102}$$

and

$$\begin{bmatrix} Z^{tt}_{-n} & Z^{t\phi}_{-n} \\ Z^{\phi t}_{-n} & Z^{\phi\phi}_{-n} \end{bmatrix} = \begin{bmatrix} Z^{tt}_{n} & -Z^{t\phi}_{n} \\ -Z^{\phi t}_{n} & Z^{\phi\phi}_{n} \end{bmatrix}, \quad n=0,1,2,\ldots \tag{103}$$

From (95), it is apparent that

$$\begin{bmatrix} \vec{R}^{t\theta}_{-n} & \vec{R}^{t\phi}_{-n} \\ \vec{R}^{\phi\theta}_{-n} & \vec{R}^{\phi\phi}_{-n} \end{bmatrix} = \begin{bmatrix} \vec{R}^{t\theta}_{n} & -\vec{R}^{t\phi}_{n} \\ -\vec{R}^{\phi\theta}_{n} & \vec{R}^{\phi\phi}_{n} \end{bmatrix}, \quad n=0,1,2,\ldots \tag{104}$$

Substitution of (104) into (99) and (100) gives

$$\begin{bmatrix} \vec{I}^{t\theta}_{-n} & \vec{I}^{t\phi}_{-n} \\ \vec{I}^{\phi\theta}_{-n} & \vec{I}^{\phi\phi}_{-n} \end{bmatrix} = \begin{bmatrix} \vec{I}^{t\theta}_{n} & -\vec{I}^{t\phi}_{n} \\ -\vec{I}^{\phi\theta}_{n} & \vec{I}^{\phi\phi}_{n} \end{bmatrix}, \quad n=0,1,2,\ldots \tag{105}$$

and

$$\begin{bmatrix} \vec{V}^{t\theta}_{-n} & \vec{V}^{t\phi}_{-n} \\ \vec{V}^{\phi\theta}_{-n} & \vec{V}^{\phi\phi}_{-n} \end{bmatrix} = \begin{bmatrix} \vec{V}^{t\theta}_{n} & -\vec{V}^{t\phi}_{n} \\ -\vec{V}^{\phi\theta}_{n} & \vec{V}^{\phi\phi}_{n} \end{bmatrix}, \quad n=0,1,2,\ldots \tag{106}$$

Since the properties (102) and (103) survive matrix inversion, it is evident from (17), (57), (88), (105), and (106) that

$$\begin{bmatrix} \vec{I}^{t\theta}_{-n} & \vec{I}^{t\phi}_{-n} \\ \vec{I}^{\phi\theta}_{-n} & \vec{I}^{\phi\phi}_{-n} \end{bmatrix} = \begin{bmatrix} \vec{I}^{t\theta}_{n} & -\vec{I}^{t\phi}_{n} \\ -\vec{I}^{\phi\theta}_{n} & \vec{I}^{\phi\phi}_{n} \end{bmatrix}, \quad n=0,1,2,\ldots \tag{107}$$

In view of (107), (101) becomes

$$J^{\theta} = (\tilde{f}\,\vec{I}_o^{t\theta})u_t + \sum_{n=1}^{\infty} \{2(\tilde{f}\,\vec{I}_n^{t\theta})u_t \cos(n\phi) + 2j(\tilde{f}\,\vec{I}_n^{\phi\theta})u_{\phi} \sin(n\phi)\}$$

$$J^{\phi} = (\tilde{f}\,\vec{I}_o^{\phi\phi})u_{\phi} + \sum_{n=1}^{\infty} \{2j(\tilde{f}\,\vec{I}_n^{t\phi})u_t \sin(n\phi) + 2(\tilde{f}\,\vec{I}_n^{\phi\phi})u_{\phi} \cos(n\phi)\} \tag{108}$$

where $\bar{f}$ is a row vector of the $f_j(t)$. If $\rho f_j(t)$ is the triangle function itself rather than the four impulse approximation (30) to the triangle function, then

$$J^{\theta}\Big|_{t=t_{2i+1}^-} = \frac{I_{oi}^{t\theta} u_t + \sum_{n=1}^{\infty} \{2I_{ni}^{t\theta}u_t \cos(n\phi) + 2j\, I_{ni}^{\phi\theta}u_{\phi} \sin(n\phi)\}}{\rho_{2i+1}^-}$$

$$J^{\phi}\Big|_{t=t_{2i+1}^-} = \frac{I_{oi}^{\phi\phi}u_{\phi} + \sum_{n=1}^{\infty} \{2jI_{ni}^{t\phi}u_t \sin(n\phi) + 2I_{ni}^{\phi\phi}u_{\phi} \cos(n\phi)\}}{\rho_{2i+1}^-} \tag{109}$$

Equations (104) and (107) reduce the specializations of (91) for θ and ϕ polarized incident electric fields to

$$E_{\theta\theta}^s(J) = \frac{-j\eta e^{-jkr_r}}{2\pi r_r} \{\frac{1}{2}\tilde{R}_o^{t\theta}\vec{I}_o^{t\theta} + \sum_{n=1}^{\infty} (\tilde{R}_n^{t\theta}\vec{I}_n^{t\theta} + \tilde{R}_n^{\phi\theta}\vec{I}_n^{\phi\theta})\cos(n\phi_r)\}$$

$$E_{\phi\theta}^s(J) = \frac{\eta e^{-jkr_r}}{2\pi r_r} \{\sum_{n=1}^{\infty} (\tilde{R}_n^{t\phi}\vec{I}_n^{t\theta} + \tilde{R}_n^{\phi\phi}\vec{I}_n^{\phi\theta})\sin(n\phi_r)\}$$

$$E_{\theta\phi}^s(J) = \frac{\eta e^{-jkr_r}}{2\pi r_r} \{\sum_{n=1}^{\infty} (\tilde{R}_n^{t\theta}\vec{I}_n^{t\phi} + \tilde{R}_n^{\phi\theta}\vec{I}_n^{\phi\phi})\sin(n\phi_r)\} \tag{110}$$

$$E_{\phi\phi}^s(J) = \frac{-j\eta e^{-jkr_r}}{2\pi r_r} \{\frac{1}{2}\tilde{R}_o^{\phi\phi}\vec{I}_o^{\phi\phi} + \sum_{n=1}^{\infty} (\tilde{R}_n^{t\phi}\vec{I}_n^{t\phi} + \tilde{R}_n^{\phi\phi}\vec{I}_n^{\phi\phi})\cos(n\phi_r)\}$$

In (110), the first subscript on E^s denotes the component under consideration whereas the second subscript denotes the polarization of the incident electric field.

The scattering cross section σ is that area for which the incident wave contains enough power to produce by omnidirectional radiation the scattered power density at the far field point. Specializing σ to the four different polarizations, we obtain

$$\frac{\sigma^{pq}}{\lambda^2} = \frac{1}{4\pi^3} \left| \frac{2\pi r_r E^s_{pq}(\underset{\sim}{J})}{\eta} \right|^2 \tag{111}$$

where p is either θ or ϕ and q is either θ or ϕ. In (111), p is the receiver polarization, q is the transmitter polarization, $\lambda = 2\pi/k$ is the wavelength, and $E^s_{pq}(\underset{\sim}{J})$ is given by (110).

VII. EXAMPLES

Computer program subroutines have been written to calculate the square matrices and measurement vectors needed for the H-field, E-Field and combined field solutions. These subroutines will be described and listed in a subsequent report. Some computational results obtained with these subroutines are given in this section.

We compare computed approximations to the electric current induced on the surface of a conducting sphere by an axial incident plane wave with the known exact solution. Figure 5 is a plot of Λ versus ka where k is the propagation constant, a is the radius of the sphere, and Λ is defined by

$$\Lambda = \sqrt{\frac{\iint_S |\underset{\sim}{J} - \underset{\sim}{J}^e|^2 ds}{|\underset{\sim}{H}^i|^2 \iint_S ds}} \tag{112}$$

Here, S is the surface of the sphere and $\underset{\sim}{H}^i$ is the incident magnetic field. Also, $\underset{\sim}{J}^e$ is the exact electric current given by (6-103) of [6]

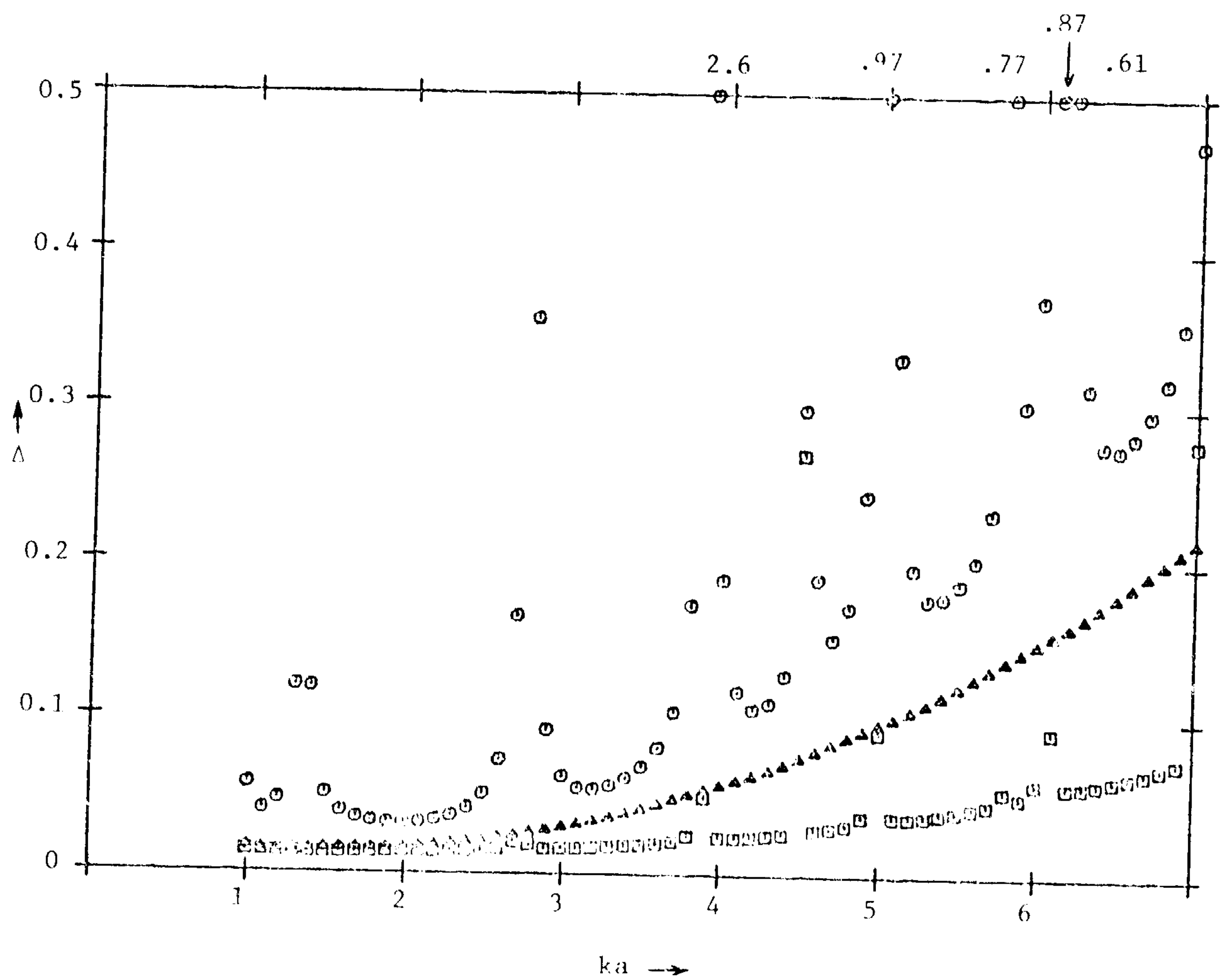

Fig. 5. Electric current error Λ versus ka for a conducting sphere of radius a excited by an incident plane wave. Squares denote H-field solution, circles denote E-field solution, and triangles denote combined field solution. Values of Λ > 0.5 are plotted at the top.

and $\underset{\sim}{J}$ is the computed approximation to $\underset{\sim}{J}^c$. The squares, circles, and triangles in Fig. 5 tell which field solution (E-field, H-field, or combined field with $\alpha = 1$ in (88)) $\underset{\sim}{J}$ is obtained from. In (112), the expression

$$|\underset{\sim}{J} - \underset{\sim}{J}^c|^2 = (\underset{\sim}{J} - \underset{\sim}{J}^c) \cdot (\underset{\sim}{J} - \underset{\sim}{J}^c)^* \tag{113}$$

where * denotes complex conjugate is the time average of the square of the length of the time dependent vector whose root-mean-square phasor is $(\underset{\sim}{J}-\underset{\sim}{J}^c)$.

The number P (maximum value of (i+1) in (27)) of data points on the generating curve is 31 for all the examples of this section. These data points, equally spaced from the lower pole to the upper pole of the sphere, give 14 expansion functions for the t directed electric current and 14 expansion functions for the ϕ directed current. Both surface integrals in (112) are evaluated by integrating analytically in ϕ and by sampling in t at the 30 points defined by (27). The number N_ϕ of points used in the Gaussian quadrature integration (37) is 20.

In Fig. 5, the H-field solution is generally the best and the E-field solution is the worst as far as $\underset{\sim}{J}$ is concerned. For both E-field and H-field solutions, the error in $\underset{\sim}{J}$ is large at resonances of the spherical cavity which are tabulated on page 270 of [6]. The combined field solution is not affected by these resonances. Note that the error in $\underset{\sim}{J}$ for the E-field solution has a peak around ka = 1.35 which is far from any resonance. This peak disappeared when the number P of data points was reduced from 31 to 21.

Figure 6 compares the normalized radar cross section $\sigma/\pi a^2$ in the backscattering direction obtained from the H-field solution, the E-field solution, and the combined field solution with the exact $\sigma/\pi a^2$. The exact $\sigma/\pi a^2$ is calculated from (6-105) of [6] in which A_e is our α. In Fig. 6, the squares denote the H-field solution, the circles denote the E-field solution, the triangles denote the combined field solution, and the solid line denotes the exact solution. The E-field, H-field and

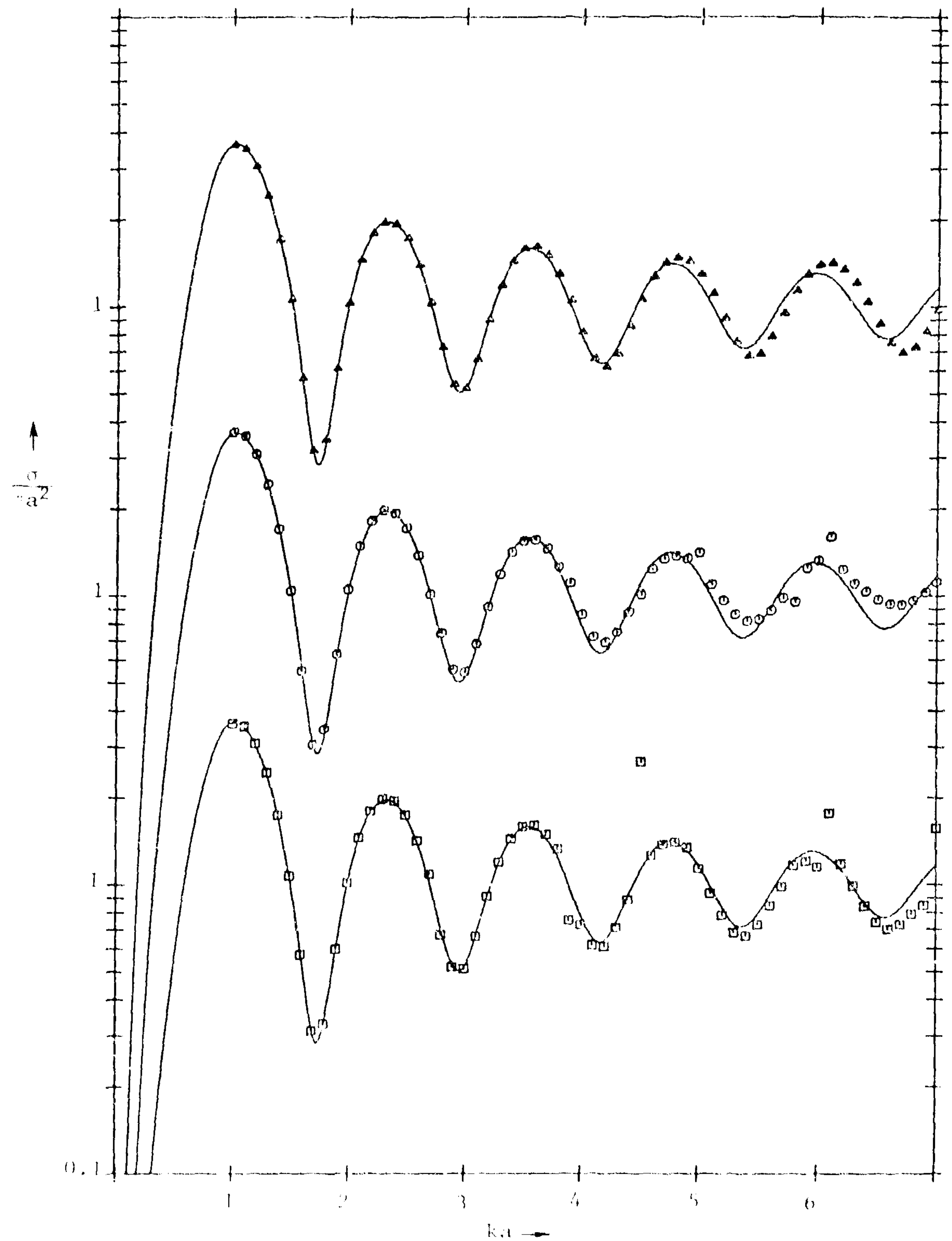

Fig. 6. Normalized radar cross section $\sigma/\pi a^2$ versus ka for a conducting sphere of radius a. Solid line denotes exact solution, squares denote H-field solution, circles denote E-field solution, and triangles denote combined field solution.

combined field solutions for $\sigma/\pi a^2$ are beginning to deteriorate for the larger values of ka in Fig. 6. At ka = 6, there are only 5 expansion functions per wavelength.

Figures 7 and 8 show the curves of Figs. 5 and 6 respectively in more detail in the vicinity of the first resonance which occurs at ka = 2.744. The disturbance in the H-field solution occurs quite close to the resonant frequency, but the disturbance in the E-field solution occurs at a slightly higher frequency. Although the error in J for the E-field solution is tremendous, the error in $\sigma/\pi a^2$ for the E-field solution is quite small except at two or three points. The following explanation is offered. According to Section V, the E-field solution does not determine how much of the electric cavity mode current is contained in J. Hence, we suspect that our numerical E-field solution does not contain the right amount of the electric cavity mode. If this suspicion is true, then the radar cross section can still be quite accurate because the electric cavity mode does not radiate any external field.

Figures 7 and 8 show that the combined field solution is much better than either the H-field solution or the E-field solution in the vicinity of the first resonance.

VIII DISCUSSION

An H-field solution, an E-field solution and a combined field solution for plane wave scattering from a perfectly conducting body of revolution have been developed and expounded. The H-field solution is a modification and generalization to oblique incidence of Uslenghi's [2] impulse solution. Instead of impulse expansion functions and Simpson's rule for integration with respect to the azimuth ϕ, we use four impulse approximations to triangle functions and Gaussian quadrature integration. The E-field solution is that of [3] with impulse Green's functions obtained from Gaussian quadrature integration in the azimuth ϕ instead of pulse Green's functions obtained from equal interval, equal weight sampling. The

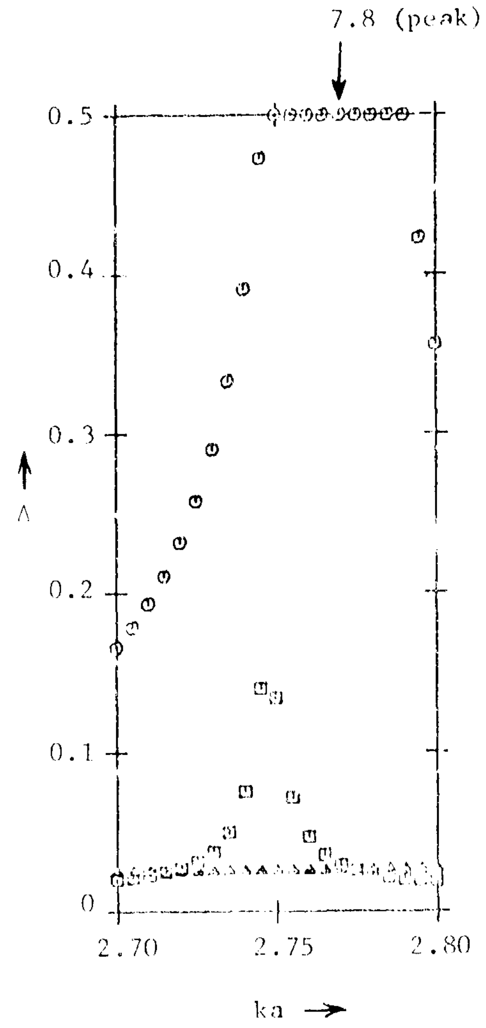

Fig. 7. Electric current error Λ versus ka for a conducting sphere of radius a excited by an incident plane wave. Squares denote H-field solution, circles denote E-field solution, and triangles denote combined field solution. The first resonance of the spherical cavity is at ka = 2.744. Values of Λ > 0.5 are plotted at the top.

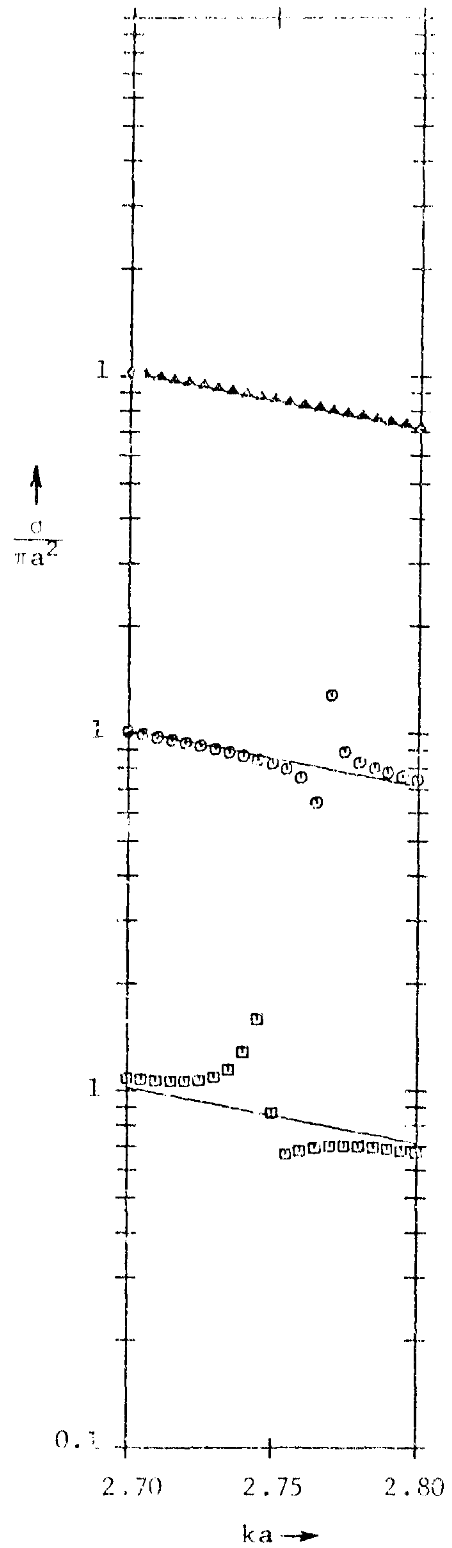

Fig. 8. Normalized radar cross section $\sigma/\pi a^2$ versus ka for a conducting sphere of radius a. Solid line denotes exact solution, squares denote H-field solution, circles denote E-field solution, and triangles denote combined field solution. The first resonance of the spherical cavity is at ka = 2.744.

combined field formulation is a linear combination of the equations for the H-field and E-field solutions. Computer program subroutines which calculate the square matrices for the H-field and E-field solutions and the plane wave measurement vectors will appear in a forthcoming report. The subroutine which calculates the square matrix for the E-field solution executes appreciably faster and requires considerably less storage to obtain the same kind of accuracy as a previous solution [11].

The H-field and E-field solutions deteriorate in the vicinity of cavity resonances because their homogeneous equations admit nontrivial solutions at these resonances. In Section V, it is shown that the homogeneous equation associated with the combined field formulation has no nontrivial solution if the relative weight α of the E-field equation is real and, if the inside media is lossy, positive. For this reason the combined field solution is much better than either the H-field or E-field solutions in the vicinity of cavity resonances. Figures 7 and 8 bear this out.

For the examples of Section VII, the relative weight α of the E-field equation in the combined field formulation is unity. This puts the H-field and E-field equations on a more or less equal footing because the magnitude of the excitation due to the H-field equation is then that due to the E-field equation rotated 90° in space. According to Fig. 6, the H-field solution for $\sigma/\pi a^2$ is generally a bit more accurate than the E-field solution away from the cavity resonances. This suggests that one weights the E-field equation less than the H-field equation in the combined field formulation.

Oshiro et al. [4,5] conclude from their plots of mean error versus α for $0 \leq \alpha \leq 1$ that an α value on the order of 0.2 is best. However, there is no logical reason for ruling out negative values of α when the electric surface current radiates into a loss-free inside media because then the left-hand side of (85) does not depend on the sign of α. We see little significance in the facts that the magnitude of the combined field excitation on the right-hand side of (83) is generally larger on the illuminated portion of the surface of the body of revolution than in the shadow zone for $\alpha > 0$ and that the opposite is true for $\alpha < 0$.

---

[11] R.F. Harrington and J.R. Mautz, "Radiation and Scattering from Bodies of Revolution," Report AFCRL-69-0305, Contract No. F-19628-67-C-0233 between Syracuse University and Air Force Cambridge Research Laboratories, July 1969.

APPENDIX A

DERIVATION OF THE H-FIELD INTEGRAL EQUATION

The purpose of Appendix A is to obtain (9) from (7) and (8). In view of (8),

$$(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}') = (\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{u}_t' J^t(t',\phi') + (\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{u}_\phi' J^\phi(t',\phi') \qquad \text{(A-1)}$$

The cross products on the right-hand side of (A-1) are evaluated by expressing all vectors in terms of unit vectors $\underset{\sim}{u}_\rho$, $\underset{\sim}{u}_\phi$, and $\underset{\sim}{u}_z$ in the ρ, ϕ, and z directions respectively.

$$\underset{\sim}{r} = \underset{\sim}{u}_\rho \rho + \underset{\sim}{u}_z z \qquad \text{(A-2)}$$

$$\underset{\sim}{r}' = \underset{\sim}{u}_\rho \rho' \cos(\phi'-\phi) + \underset{\sim}{u}_\phi \rho' \sin(\phi'-\phi) + \underset{\sim}{u}_z z' \qquad \text{(A-3)}$$

$$\underset{\sim}{u}_t' = \underset{\sim}{u}_\rho \sin v' \cos(\phi'-\phi) + \underset{\sim}{u}_\phi \sin v' \sin(\phi'-\phi) + \underset{\sim}{u}_z \cos v' \qquad \text{(A-4)}$$

$$\underset{\sim}{u}_\phi' = -\underset{\sim}{u}_\rho \sin(\phi'-\phi) + \underset{\sim}{u}_\phi \cos(\phi'-\phi) \qquad \text{(A-5)}$$

Equation (A-3) has been obtained by first writing

$$\underset{\sim}{r}' = \underset{\sim}{u}_\rho' \rho' + \underset{\sim}{u}_z z' \qquad \text{(A-6)}$$

and then using Fig. A-1 to express $\underset{\sim}{u}_\rho'$ in terms of $\underset{\sim}{u}_\rho$ and $\underset{\sim}{u}_\phi$. To verify (A-4), use Fig. A-2 to express $\underset{\sim}{u}_t'$ in terms of $\underset{\sim}{u}_\rho'$ and $\underset{\sim}{u}_z$ and then use Fig. A-1 to express $\underset{\sim}{u}_\rho'$ in terms of $\underset{\sim}{u}_\rho$ and $\underset{\sim}{u}_\phi$.

Substitution of (A-2) - (A-5) into (A-1) yields

$$\begin{aligned}(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}') = \{&\underset{\sim}{u}_\rho (-\rho'\cos v' + (z'-z)\sin v')\sin(\phi'-\phi) \\ &+ \underset{\sim}{u}_\phi(-(\rho-\rho'\cos(\phi'-\phi))\cos v' - (z'-z)\sin v' \cos(\phi'-\phi)) \\ &+ \underset{\sim}{u}_z \rho \sin v' \sin(\phi'-\phi)\} J^t(t',\phi') + \{\underset{\sim}{u}_\rho (z'-z)\cos(\phi'-\phi) \\ &+ \underset{\sim}{u}_\phi (z'-z)\sin(\phi'-\phi) + \underset{\sim}{u}_z(-\rho' + \rho\cos(\phi'-\phi))\} J^\phi(t',\phi')\end{aligned} \qquad \text{(A-7)}$$

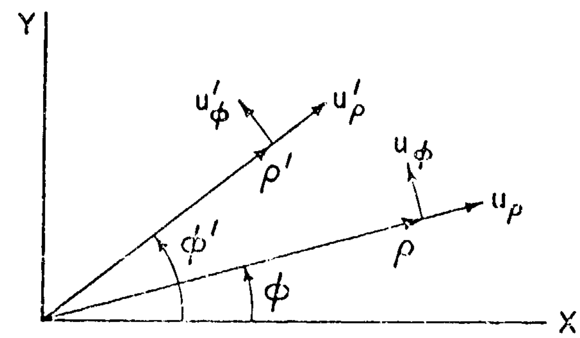

Fig. A-1. Unit vectors $\underline{u}_\rho$, $\underline{u}_\phi$, $\underline{u}'_\rho$, and $\underline{u}'_\phi$ in xy plane.

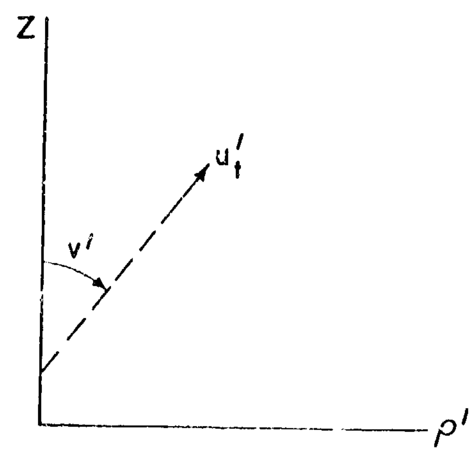

Fig. A-2. Unit vector $\underline{u}'_t$ in $\rho'z$ plane.

To find the $\underset{\sim}{u}_t$ and $\underset{\sim}{u}_\phi$ components of (7), we need

$$\underset{\sim}{u}_t \cdot \underset{\sim}{n} \times [(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}')] = \underset{\sim}{u}_\phi \cdot [(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}')] \tag{A-8}$$

$$\underset{\sim}{u}_\phi \cdot \underset{\sim}{n} \times [(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}')] = -\underset{\sim}{u}_t \cdot [(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}')] \tag{A-9}$$

With the help of

$$\underset{\sim}{u}_t = \underset{\sim}{u}_\rho \sin v + \underset{\sim}{u}_z \cos v \tag{A-10}$$

and (A-7), (A-8) and (A-9) become

$$\begin{aligned} \underset{\sim}{u}_t \cdot \underset{\sim}{n} \times [(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}')] &= \{((\rho'-\rho)\cos v' - (z'-z)\sin v')\cos(\phi'-\phi) \\ &- 2\rho \cos v' \sin^2(\tfrac{\phi'-\phi}{2})\}\, J^t(t',\phi') + (z'-z)\sin(\phi'-\phi) J^\phi(t',\phi') \end{aligned} \tag{A-11}$$

$$\begin{aligned} \underset{\sim}{u}_\phi \cdot \underset{\sim}{n} \times [(\underset{\sim}{r}-\underset{\sim}{r}') \times \underset{\sim}{J}(\underset{\sim}{r}')] &= (\rho' \sin v \cos v' - \rho \sin v' \cos v \\ &- (z'-z)\sin v \sin v')\sin(\phi'-\phi)\, J^t(t',\phi') + \{((\rho'-\rho)\cos v \\ &- (z'-z)\sin v)\cos(\phi'-\phi) + 2\rho' \cos v \sin^2(\tfrac{\phi'-\phi}{2})\}\, J^\phi(t',\phi') \end{aligned} \tag{A-12}$$

The distance $|\underset{\sim}{r}-\underset{\sim}{r}'|$ appearing in (7) is the square root of the sum of the squares of $(z-z')$ and the projection of $(\underset{\sim}{r}-\underset{\sim}{r}')$ in the xy plane. Hence,

$$|\underset{\sim}{r}-\underset{\sim}{r}'| = \sqrt{(z-z')^2 + \rho'^2 + \rho^2 - 2\rho\rho' \cos(\phi'-\phi)} = \sqrt{(\rho-\rho')+(z-z')^2 + 4\rho\rho' \sin^2(\tfrac{\phi'-\phi}{2})} \tag{A-13}$$

An integral with respect to ϕ' results when the surface integral in (7) is iterated. Because this integral with respect to ϕ' is an integral of a 2π periodic function of ϕ' over the period 2π, ϕ' may be replaced by $\phi' + \phi$ without changing the value of the integral. Substitution of (A-11) - (A-13) into (7) leads to the desired H-field integral equation (9).

APPENDIX B

DERIVATION OF THE E-FIELD MATRIX EQUATION

The testing functions $\underset{\sim}{W}^{p}_{mi}$ appearing in (55) are defined by (47). From (56), (47), and (52), we obtain

$$\sigma^{t}_{mi} = \frac{-1}{j\omega\rho}\frac{\partial}{\partial t}(\rho f_i(t))e^{-jm\phi}$$

$$\sigma^{\phi}_{mi} = \frac{m}{\omega\rho} f_i(t)e^{-jm\phi} \tag{B-1}$$

The vector and scalar potentials $\underset{\sim}{A}$ and Φ appearing in (55) are given by (42) and (43) with $\underset{\sim}{J}^{q}_{nj}$ defined by (46). In agreement with (44), (46), and (52), and in analogy with (B-1), the charge density σ appearing in (43) is specialized to either σ^{t}_{nj} or σ^{ϕ}_{nj} given by

$$\sigma^{t}_{nj} = \frac{-1}{j\omega\rho'}\frac{\partial}{\partial t'}(\rho' f_j(t'))e^{jn\phi'}$$

$$\sigma^{\phi}_{nj} = \frac{-n}{\omega\rho'} f_j(t')e^{jn\phi'} \tag{B-2}$$

In view of the above considerations, (55) can be rewritten as

$$\begin{bmatrix} (Z^{tt}_{mn})_{ij} \\ (Z^{\phi t}_{mn})_{ij} \\ (Z^{t\phi}_{mn})_{ij} \\ (Z^{\phi\phi}_{mn})_{ij} \end{bmatrix} = \int dt \int dt' \int_0^{2\pi} d\phi \int_0^{2\pi} d\phi' \frac{e^{-jk|\underset{\sim}{r}-\underset{\sim}{r}'|}e^{j(n\phi'-m\phi)}}{4\pi|\underset{\sim}{r}-\underset{\sim}{r}'|} \times$$

(cont. on next page)

$$
\begin{bmatrix}
jk\rho f_i(t)\rho' f_j(t')\underset{\sim}{u}_t \cdot \underset{\sim}{u}'_t - \frac{j}{k}\frac{\partial}{\partial t}(\rho f_i(t))\frac{\partial}{\partial t'}(\rho' f_j(t')) \\
jk\rho f_i(t)\rho' f_j(t')\underset{\sim}{u}_\phi \cdot \underset{\sim}{u}'_t - \frac{m}{k\rho}\rho f_i(t)\frac{\partial}{\partial t'}(\rho' f_j(t')) \\
jk\rho f_i(t)\rho' f_j(t')\underset{\sim}{u}_t \cdot \underset{\sim}{u}'_\phi + \frac{n}{k\rho'}\frac{\partial}{\partial t}(\rho f_i(t))\rho' f_j(t') \\
j\rho f_i(t)\rho' f_j(t')\ (k\underset{\sim}{u}_\phi \cdot \underset{\sim}{u}'_\phi - \frac{mn}{k\rho\rho'})
\end{bmatrix}
\qquad (B\text{-}3)
$$

To facilitate evaluation of the dot products appearing in (B-3), we write

$$
\begin{aligned}
\underset{\sim}{u}_t &= \underset{\sim}{u}_\rho \sin v + \underset{\sim}{u}_z \cos v \\
\underset{\sim}{u}'_t &= \underset{\sim}{u}'_\rho \sin v' + \underset{\sim}{u}_z \cos v'
\end{aligned}
\qquad (B\text{-}4)
$$

where $\underset{\sim}{u}_\rho$, $\underset{\sim}{u}'_\rho$, and v' are defined in Figs. A-1 and A-2. With the help of (B-4), we obtain

$$
\begin{aligned}
\underset{\sim}{u}_t \cdot \underset{\sim}{u}'_t &= \sin v \sin v' \cos(\phi'-\phi) + \cos v \cos v' \\
\underset{\sim}{u}_\phi \cdot \underset{\sim}{u}'_t &= \sin v' \sin(\phi'-\phi) \\
\underset{\sim}{u}_t \cdot \underset{\sim}{u}'_\phi &= -\sin v \sin(\phi'-\phi) \\
\underset{\sim}{u}_\phi \cdot \underset{\sim}{u}'_\phi &= \cos(\phi'-\phi)
\end{aligned}
\qquad (B\text{-}5)
$$

The distance $|\underset{\sim}{r}-\underset{\sim}{r}'|$ is given by (A-13) which reads

$$
|\underset{\sim}{r}-\underset{\sim}{r}'| = \sqrt{(\rho-\rho')^2 + (z-z')^2 + 4\rho\rho'\sin^2\left(\frac{\phi'-\phi}{2}\right)} \qquad (B\text{-}6)
$$

In view of (B-5) and (B-6), the integrands of (B-3) are periodic functions of ϕ' with period 2π. Hence, ϕ' can be replaced by $(\phi'+\phi)$ without changing the values of any of the integrals. When this is done, the ϕ dependence of the integrands becomes $e^{j(n-m)\phi}$ which when integrated gives 2π for $m=n$ and zero for $m \neq n$. Taking the liberty of replacing the double subscript mn by the single subscript n, we obtain (58) - (61). The forms (62) - (64) of the ϕ' integrals follow from the even or odd symmetry of the terms in (B-5) about $(\phi'-\phi) = 0$ and the even symmetry of (B-6) about $(\phi'-\phi) = 0$.

REFERENCES

[1] T. K. Wu, "Electromagnetic Scattering from Arbitrarily-Shaped Lossy Dielectric Bodies," Ph.D. thesis, University of Mississippi, May 1976.

[2] P.L.E. Uslenghi, "Computation of Surface Currents on Bodies of Revolution," Alta Frequenza, vol. 39, No. 8, 1970, pp. 1-12.

[3] J. R. Mautz and R. F. Harrington, "Radiation and Scattering from Bodies of Revolution," Appl. Sci. Res., vol. 20, June 1969, pp. 405-435.

[4] F. K. Oshiro, K. M. Mitzner, and S. S. Locus et al., "Calculation of Radar Cross Section," Air Force Avionics Laboratory Tech. Rept. AFAL-TR-70-21, Part II, April 1970.

[5] A. J. Poggio and E. K. Miller, "Integral Equation Solutions of Three-Dimensional Scattering Problems," Chap. 4 of Computer Techniques for Electromagnetics, edited by R. Mittra, Pergamon Press, 1973.

[6] R. F. Harrington, Time-Harmonic Electromagnetic Fields, McGraw-Hill Book Co., 1961.

[7] E. A. Wolff, Antenna Analysis, John Wiley and Sons, Inc., New York, 1966, p. 61.

[8] V. I. Krylov, Approximate Calculation of Integrals, translated by A. H. Stroud, Macmillan Co., New York, 1962, Appendix A.

[9] L. Marin, "Natural-Mode Representation of Transient Scattered Fields," IEEE Trans. Antennas Propagat., vol. AP-21, pp. 809-818, Nov. 1973.

[10] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," U.S. Govt. Printing Office, Washington, D.C. (Natl. Bur. Std. U.S. Appl. Math. Ser. 55), 1964, p. 360.

[11] R. F. Harrington and J. R. Mautz, "Radiation and Scattering from Bodies of Revolution," Report AFCRL-69-0305, Contract No. F-19628-67-C-0233 between Syracuse University and Air Force Cambridge Research Laboratories, July 1969.

MISSION
of
Rome Air Development Center

RADC plans and conducts research, exploratory and advanced development programs in command, control, and communications (C^3) activities, and in the C^3 areas of information sciences and intelligence. The principal technical mission areas are communications, electromagnetic guidance and control, surveillance of ground and aerospace objects, intelligence data collection and handling, information system technology, ionospheric propagation, solid state sciences, microwave physics and electronic reliability, maintainability and compatibility.

Printed by
United States Air Force
Hanscom AFB, Mass. 01731