A RAYLEIGH-RITZ DETERMINATION OF UPPER AND LOWER BOUNDS FOR THE DEFLECTION OF ORTHOTROPIC CANTILEVER BEAMS

by S. Nair

April 1977

Prepared for
OFFICE OF NAVAL RESEARCH
Washington, D. C.
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A Rayleigh-Ritz Determination of Upper and Lower Bounds
for the Deflection of Orthotropic Cantilever Beams

by

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ABSTRACT

A numerical scheme based on polynomial approximation is
presented for the calculation of upper and lower bounds for the flexibility
coefficient of end-loaded orthotropic cantilever beams in a state of plane
stress, through use of the principles of minimum potential energy and
of minimum complementary energy. The degree-of-freedom of the two-
dimensional polynomials employed along with the elastic constants and
the depth-to-span ratio of the beam are external parameters of the
problem.
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INTRODUCTION

A formulation of the problem of the end-loaded cantilever beam, which makes it possible to obtain upper and lower bounds for the flexibility coefficient through the use of the principles of minimum potential energy and of minimum complementary energy, has been reported earlier [1].

A later work [2] contains improved bounds obtained through the use of arbitrary functions of the axial coordinate in the assumed admissible expressions for displacements and stresses in the calculation of the energy functionals. Although these improved bounds provide reliable corrections to the elementary theory flexibility coefficient when the parameter \( \mu = \frac{G}{\sqrt{E\alpha}} \) is not too large, the differences between the bounds becomes significant for large values of \( \mu \).

In what follows we use a Rayleigh-Ritz procedure in conjunction with the Lagrange multiplier method to improve the bounds obtained in [2]. We approximate the displacement components in the potential energy calculations and the stress components in the complementary energy calculations by polynomials in the coordinate variables. The coefficients in these polynomials are obtained by solving the simultaneous algebraic equations which result upon extremizing the energy functionals subject to the constraint conditions on these polynomials.
Numerical results obtained employing sixth-degree polynomials in two variables show that for isotropic beams when $\nu = 1/3$, and the depth-to-length ratio is as high as 0.6 the maximum error in using the average value of the bounds instead of the actual value of the flexibility coefficient is less than 0.3%. Using polynomials of the same degree for orthotropic beams this error comes out to be less than 1% when $\nu = 1/2$ and $\mu = 100$ for all depth-to-span ratios.

FORMULATION

We briefly restate the problem of the end-loaded orthotropic beam as formulated in [1]. The inequality for the flexibility coefficient $C$ reads

$$\frac{V^2}{2\tilde{I}_d} \leq C \leq \frac{V^2}{2\tilde{I}_s},$$

where $V$ is the prescribed uniform end deflection and

$$\tilde{I}_d = \frac{1}{2} \int_{-c}^{c} \int_{0}^{a} \left\{ \frac{E\tilde{u}^2 + 2\nu E \tilde{u} \tilde{v} + E \tilde{v}^2}{1 - \nu^2} + E \gamma + \frac{G}{1 - \nu^2} \tilde{u} \tilde{\gamma} \right\} dx dy,$$  \hspace{1cm} (2)

$$\tilde{I}_s = V \int_{-c}^{c} \tilde{\gamma}(0, y) dy - \frac{1}{2} \int_{-c}^{c} \int_{0}^{a} \left\{ \frac{\tilde{x}^2}{E} - 2\nu \frac{\tilde{x} \tilde{\gamma} + \tilde{\gamma} \tilde{\gamma}}{E \gamma} + \frac{\tilde{\gamma}^2}{G} \right\} dx dy,$$  \hspace{1cm} (3)

with $2c$ and $a$ representing, respectively, the depth and the length of the beam, and with $E_m = \sqrt{EE_y}$.

In equation (2), $\tilde{u}$ and $\tilde{v}$ are assumed axial and lateral displacement components which are differentiable functions satisfying the displacement boundary conditions

$$\tilde{u}(a, y) = \tilde{v}(a, y) = 0 , \hspace{1cm} \tilde{v}(0, y) = -V ,$$  \hspace{1cm} (4)
and in equation (3) \( \tilde{\sigma} \) and \( \tilde{\tau} \) are stress components which satisfy the equilibrium equations

\[
\tilde{\sigma}_{x,x} + \tilde{\tau}_{x,y} = 0 , \quad \tilde{\sigma}_{y,y} + \tilde{\tau}_{x,x} = 0 ,
\]

and the stress boundary conditions

\[
\tilde{\tau}(x, \pm c) = \tilde{\sigma}_y(x, \pm c) = \tilde{\sigma}_x(0, y) = 0 .
\]

In what follows, we obtain upper and lower bounds for \( C \) by maximizing \( \tilde{I}_s \) and by minimizing \( \tilde{I}_d \) over a suitably restricted class of polynomials.

**UPPER BOUND CALCULATION**

We introduce the dimensionless quantities

\[
\xi = \frac{x}{a} , \quad \eta = \frac{y}{c} ; \quad \rho = \frac{c}{a} ,
\]

\[
\left( \tilde{\sigma}_x, \tilde{\tau}, \tilde{\sigma}_y \right) = \frac{a}{V E \rho} \left( \tilde{\sigma}_x, \frac{\tilde{\tau}}{\rho}, \frac{\tilde{\sigma}_y}{\rho^2} \right) ,
\]

\[
\lambda = \sqrt{\frac{E}{G}} \rho , \quad \mu = \frac{G}{E_m} ,
\]

and have therewith, from (3), that \( \tilde{I}_s = V^2 E \rho^3 \tilde{I}_s \), where

\[
\tilde{I}_s = 2 \int_0^1 \int_0^{1/2} \left\{ \tilde{\sigma}_x^2 - 2 \nu \lambda^2 \tilde{\sigma}_x \tilde{\sigma}_y - \mu \lambda^2 \tilde{\sigma}_y^2 + \lambda^2 \tilde{\tau}^2 \right\} d\xi d\eta .
\]

We satisfy the equilibrium equations (5) identically by writing

\[
\tilde{\sigma}_x = F, \xi \eta , \quad \tilde{\tau} = -F, \xi \eta , \quad \tilde{\sigma}_y = F, \xi \xi ,
\]
and assume \( F \) in the form

\[
F = \sum_{m=1}^{M} \sum_{n=1}^{N} A_{mn} \xi^m \eta^{2n-1}, \tag{12}
\]

where the \( A_{mn} \) are arbitrary constants, except for the relations

\[
\sum_{n=1}^{N} (2n - 1) A_{mn} = 0, \quad m = 1, 2, \ldots, M, \tag{13}
\]

\[
\sum_{n=1}^{N} A_{mn} = 0, \quad m = 2, 3, \ldots, M, \tag{14}
\]

which result upon satisfying the stress boundary conditions \( \tau = \sigma_y = 0 \) at \( \eta = \pm 1 \), with the condition \( \tilde{\sigma}_x = 0 \) for \( \xi = 0 \) being satisfied automatically.

Extremization of \( I_s \) with respect to the \( A_{mn} \), subject to the constraint relations (13) and (14), is accomplished as follows.

We define a quantity \( I_s^* \) by

\[
I_s^* = I_s + 2 \sum_{m=2}^{M} B_m \left( \sum_{n=1}^{N} A_{mn} \right) + 2 \sum_{m=1}^{N} C_m \left( \sum_{n=1}^{N} (2n - 1) A_{mn} \right), \tag{15}
\]

with \( B_m \) and \( C_m \) being Lagrange multipliers.

Setting \( \delta I_s^* = 0 \), we obtain

\[
\sum_{k=1}^{M} \sum_{\ell=1}^{N} \left\{ \frac{\delta I_s}{\delta A_{k\ell}} + 2B_k + 2C_k (2\ell - 1) \right\} \delta A_{k\ell} = 0, \tag{16}
\]

where \( B_1 = 0 \), in view of the form of (15). From this we have further,
\[
\frac{1}{2} \frac{\delta \tilde{I}_{s}}{\delta A_{k \ell}} + B_{k} + (2t - 1) C_{k} = 0, \quad k = 1, \ldots, M
\]
\[
B_{k} = 0, \quad t = 1, \ldots, N.
\]

Equations (17), together with (13) and (14), represent a system of equations of just the right number to determine the unknown constants \(A_{mn}, B_{m}, C_{m}\).

We next obtain an expression for \(\tilde{I}_{s \text{ max}}\) in terms of the coefficients \(A_{k \ell}\). To do this we make use of the fact that we may set \(\delta A_{k \ell} = A_{k \ell}\) in equation (16) and write, using equations (13) and (14),

\[
\sum_{k=1}^{M} \sum_{\ell=1}^{N} \frac{\delta \tilde{I}_{s}}{\delta A_{k \ell}} A_{k \ell} = 0.
\]

(18)

Considering the linear and quadratic terms present in equation (10), we may deduce from (10) in conjunction with (18) that

\[
\tilde{I}_{s \text{ max}} = \int_{0}^{1} \tau (0, \eta) \, d\eta.
\]

(19)

With \(\tau\) from (11) and (12), this becomes

\[
\tilde{I}_{s \text{ max}} = - \sum_{n=1}^{N} A_{1n}.
\]

(20)

We introduce the associated \(\tilde{I}_{s \text{ max}}\) on the right of equation (1), and therewith obtain the upper bound formula

\[
\frac{C_{U}}{C_{o}} = \frac{-1}{\sum_{n=1}^{N} A_{1n}},
\]

(21)

where \(C_{o}\) is the elementary-theory flexibility coefficient \(1/2 E \pi^{3}\).
It remains to write the system (17) explicitly. To do this we introduce $\sigma_x$, $\tau$, and $\sigma_y$ from equations (11) and (12) into equation (10) for $\bar{s}$ and carry out the differentiations with respect to $A_{k\ell}$ and the integrations with respect to $\xi$ and $\eta$. In this way equation (17) becomes

$$
\sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \frac{(2n-1)(2n-2)(2\ell-1)(2\ell-2)}{(m+k+1)(2n+2\ell-5)} \right\} A_{mn}
$$

$$
- \nu \mu \lambda^2 \frac{(2n-1)(2n-2)k(k-1) + (2\ell-1)(2\ell-2)m(m-1)}{(m+k-1)(2n+2\ell-3)}
$$

$$
+ \mu^2 \lambda^4 \frac{mk(m-1)(k-1)}{(m+k-3)(2n+2\ell-1)} + \lambda^2 \frac{mk(2n-1)(2\ell-1)}{(m+k-1)(2n+2\ell-3)} A_{mn}
$$

$$
- B_k (2\ell-1) C_k + \delta \xi_k = 0, \quad k = 1, 2, \ldots, M
$$

$$
- B_k (2\ell-1) C_k + \delta \eta_k = 0, \quad \ell = 1, 2, \ldots, N
$$

again with $B_1 = 0$.

Setting $k = 1$, $\ell = 1$ in equation (22), we may sum the series which occurs in (21), as

$$
\sum_{n=1}^{N} A_{1n} = \frac{C_1 - 1}{\lambda^2}, \quad (23)
$$

and this enables us to write equation (21) in the final form

$$
\frac{C_{U}}{C_0} = \frac{\lambda^2}{1 - C_1}
$$

$$
(24)
$$

It remains to solve (22), (13), and (14) in order to obtain the value of the Lagrange multipliers $C_1$.

We note that for $M = 1$, $N = 2$, the above procedure gives
$C_U/C_0 = 1 + 6\lambda^2/5$ which coincides, as it must, with a previously obtained upper bound [1].

**LOWER BOUND CALCULATION**

Introducing the nondimensional displacements

$$\bar{v} = \frac{v}{V}, \quad \bar{u} = \frac{u}{V_P},$$

we write $\overline{I_d} = V^2 E\rho^3 \overline{I_d}$ where

$$\overline{I_d} = \int_0^1 \int_0^1 \left\{ \frac{1}{1 - \nu^2} \left( \bar{u}_\xi \bar{u}_\xi + \frac{2\nu}{\mu\lambda^2} \bar{u}_\xi \bar{\nu} + \frac{1}{\mu^2\lambda^4} \bar{\nu}_\xi \bar{\nu}_\xi \right) + \frac{1}{\lambda^2} \left( \bar{u}_\eta \bar{\eta} + \bar{\nu}_\eta \bar{\nu}_\eta \right)^2 \right\} d\xi d\eta.$$  

(25)

Expressions for displacements are now assumed in the form

$$\bar{v} = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} (1 - \xi)^m \eta^{2n-2}, \quad \bar{u} = \sum_{m=1}^{M'} \sum_{n=1}^{N'} b_{mn} (1 - \xi)^m \eta^{2n-1},$$

(26)

where, since $\bar{v}(0, \eta) = -1$, we have the constraint condition

$$\sum_{m=1}^{M} a_{mn} + \delta_{ln} = 0, \quad n = 1, 2, \ldots, N.$$  

(27)

We obtain $a_{mn}$ and $b_{mn}$ by minimizing $\overline{I_d}$ with respect to these coefficients, subject to (28), that is by extremizing

$$\overline{I_d}^* = \overline{I_d} + 2 \sum_{n=1}^{N} c_n \left( \sum_{m=1}^{M} a_{mn} + \delta_{ln} \right),$$

(29)

with the $c_n$ being Lagrange multipliers.
Setting $\delta I_d^* = 0$ gives

$$\sum_{k=1}^{M} \sum_{l=1}^{N} \left\{ \frac{\delta I_d}{\delta a_{k,l}} + 2c_l \right\} \delta a_{k,l} + \sum_{k=1}^{M'} \sum_{l=1}^{N'} \frac{\delta I_d}{\delta b_{k,l}} \delta b_{k,l} = 0 . \quad (30)$$

Equating the coefficient of each $\delta a_{k,l}$ and $\delta b_{k,l}$ in (30) equal to zero, we obtain the equations

$$\frac{1}{2} \frac{\delta I_d}{\delta a_{k,l}} + c_l = 0 , \quad k = 1, 2, \ldots, M , \quad l = 1, 2, \ldots, N \quad (31)$$

$$\frac{1}{2} \frac{\delta I_d}{\delta b_{k,l}} = 0 , \quad k = 1, 2, \ldots, M' , \quad l = 1, 2, \ldots, N' \quad (32)$$

which, together with (28), allow the determination of the unknown constants $a_{mn}$, $b_{mn}$ and $c_n$.

We next obtain an expression for $I_d^\text{min}$ first in terms of the coefficients $a_{mn}$ and ultimately in terms of $c_1$ alone. To do this, we make use of the fact that we may set in equation (30) $\delta a_{k,l} = a_{k,l}$ and $\delta b_{k,l} = b_{k,l}$, and write

$$\sum_{k=1}^{M} \sum_{l=1}^{N} \frac{\delta I_d}{\delta a_{k,l}} a_{k,l} + \sum_{k=1}^{M'} \sum_{l=1}^{N'} \frac{\delta I_d}{\delta b_{k,l}} b_{k,l} + 2 \sum_{k=1}^{M} \sum_{l=1}^{N} c_l a_{k,l} = 0 . \quad (33)$$

Since $I_d$ is a quadratic form, the first two terms in (33) add up to $2I_d^\text{min}$ and we have then

$$I_d^\text{min} = - \sum_{k=1}^{M} \sum_{l=1}^{N} c_l a_{k,l} . \quad (34)$$
Equation (34) gives with the aid of the constraint equations (28),

$$\bar{I}_d \min = c_1,$$  \hspace{1cm} (35)

which, in connection with equation (1), gives the lower bound formula

$$\frac{C_L}{C_0} = \frac{1}{c_1}.$$  \hspace{1cm} (36)

It remains to write equations (31) and (32) in explicit form. We accomplish this by introducing \(\bar{u}\) and \(\bar{v}\) from (27) into the equation (26), and by carrying out the differentiations with respect to \(a_{k \ell}\) and \(b_{k \ell}\) as well as the integrations with respect to \(\xi\) and \(\eta\). This gives

$$\sum_{m=1}^{M'} \sum_{n=1}^{N'} \left\{ \frac{\lambda^a}{1 - \nu^b} \frac{mk}{(m+k-1)(2n+2\ell-1)} + \frac{(2n-1)(2\ell-1)}{(m+k+1)(2n+2\ell-3)} \right\} b_{mn}$$

$$- \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \frac{\nu}{\mu(1-\nu^b)} \left( \frac{k(2n-2) + m(2\ell-1)}{(m+k)(2n+2\ell-3)} \right) \right\} a_{mn} = 0,$$  \hspace{1cm} (37)

for \(k = 1, 2, \ldots, M'\) and \(\ell = 1, 2, \ldots, N'\), and

$$\sum_{m=1}^{M'} \sum_{n=1}^{N'} \left\{ \frac{\nu}{\mu(1-\nu^b)} \frac{mk}{m(2\ell-2) + k(2n-1)} \right\} \frac{b_{mn}}{(m+k)(2n+2\ell-3)}$$

$$- \sum_{m=1}^{M} \sum_{n=1}^{N} \left\{ \frac{1}{\mu^a \lambda^2(1-\nu^b)} \frac{(2n-2)(2\ell-2)}{(m+k+1)(2n+2\ell-5)} \right\} \frac{mk}{(m+k-1)(2n+2\ell-3)} \right\} a_{mn} = \lambda^2 c_\ell,$$  \hspace{1cm} (38)

for \(k = 1, 2, \ldots, M\) and \(\ell = 1, 2, \ldots, N\).
Equation (37), (38) and (28) must now be solved in order to obtain the value of $c_1$ in (36). Numerical results reported in the next section are obtained by setting $M' = M - 1$, $N' = N$, so as to have an equal number of a-terms and b-terms in the expressions for the shear-strains which, numerical evidence indicates, is the most effective way to carry out this work. We note that the choice $M = 3$, $N = 1$ corresponds to the "semi-elementary" polynomial used in the calculation of the lower bound, \( C_L/C_0 = 1 - \nu^2 + \lambda^2 \) [1].

**DISCUSSION**

In Table 1, values of upper and lower bounds are presented for isotropic beams when $\nu = 1/3$, and $c/a = 0.3$ and $0.5$, as functions of the integers $M$ and $N$ representing the number of degrees-of-freedom of the associated admissible polynomials. The Hilbert-type matrices involved in the numerical work, require double precision arithmetic in the solution procedures for large values of $M$ and $N$. The results show that in the extreme case of a beam which is as deep as it is long ($c/a = 0.5$), the maximum percentage error in employing the mean value of the present bounds with $M = N = 6$, instead of the actual flexibility coefficient is only $0.22\%$.

Numerical values of the upper and lower bounds obtained for orthotropic beams, with the large value $\mu = G/E_m = 100$ when $\nu = 1/2$, are plotted against $\lambda = \sqrt{E/G}$ $c/a$ in Figure 1. To obtain a
maximum error of less than one percent it is found necessary to use
$M = N = 6$ in both upper and lower bound calculations. These results
are represented by the curves $U_{66}$ and $L_{66}$. For the sake of comparison,
we also include the bound curves $U_2$ and $L_4$ obtained in [2]. The
bounds $U_{12}$ and $L_{31}$ are equivalent to the earlier results obtained
in [1]. The bound curves $U_{66}$ and $L_{66}$ show that there is a decrease
in flexibility for small $\lambda$. As shown previously in [2], this decrease is
due to the end constraint $v = 0$ at $x = a$. The shear deformation effects
dominate when $\lambda$ is large and the flexibility progressively increases with
increasing values of $\lambda$.

REFERENCES

1. Reissner, E., "Upper and Lower Bounds for Deflections of
   Laminated Cantilever Beams Including the Effect of

2. Nair, S., and E. Reissner, "Improved Upper and Lower Bounds
   for Deflections of Orthotropic Cantilever Beams,"
Table 1. Convergence of the bounds for isotropic beams when $\nu = 1/3$.

<table>
<thead>
<tr>
<th>M, N</th>
<th>$c/a = 0.3$</th>
<th>$c/a = 0.5$</th>
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</tr>
<tr>
<td>6, 6</td>
<td>1.2642</td>
<td>1.2715</td>
</tr>
</tbody>
</table>

$^\dagger$ $\%$ Error = \( \frac{C_U - C_L}{2C_L} \times 100 \).
Fig. 1. Dependence of the bounds on $\lambda = \sqrt{E/G} \ c/a$
for orthotropic beams when $\nu = 1/2$ and
$\mu = G/m = 100$. 
A numerical scheme based on polynomial approximation is presented for the calculation of upper and lower bounds for the flexibility coefficient of end-loaded orthotropic cantilever beams in a state of plane stress, through use of the principles of minimum potential energy and of minimum complementary energy. The degree-of-freedom of the two-dimensional polynomials employed along with the elastic constants and the depth-to-span ratio of the beam are external parameters of the problem.