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ABSTRACT

We state upper and lower bound formulas for the torsional stiffness of shafts of varying circular cross section, in accordance with the classical Michell formulation of this problem, through use of the principles of minimum potential and complementary energy. The general results are used to obtain explicit first-approximation bounds which, for the limiting case of the cylindrical shaft, reproduce the known elementary exact results. It is conjectured that the first-approximation lower bound is significantly closer to the exact result than the first-approximation upper bound.

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We consider, in terms of cylindrical coordinates $r$ and $z$, a homogeneous isotropic linear elastic solid bounded by plane surfaces $z = 0$ and $z = L$, and by a surface of revolution $r = r_o(z)$ for $0 \leq z \leq L$. We assume that the surface portion $z = 0$ is held fixed, and that the boundary conditions for the surface portion $z = L$ are mixed conditions, of prescribed circumferential displacement $v = \Theta r$, and of vanishing normal stress $\sigma_z$ and shear stress $\tau_{rz}$. We further assume that the surface portion $r = r_o(z)$ is traction free.

The solution of this problem can be obtained, as first shown by J. H. Michell, by assuming the vanishing of all radial and axial displacements and of all stresses except $\tau_{r\theta}$ and $\tau_{z\theta}$. With this there remain the stress strain relations $\tau_{r\theta} = G\gamma_{r\theta}$ and $\tau_{z\theta} = G\gamma_{z\theta}$, with strain displacement relations

$$\gamma_{z\theta} = v, \quad \gamma_{r\theta} = r(r^{-1}v),$$

where $v$ is independent of $\theta$, and with stress stress-function relations

$$\tau_{z\theta} = r^{-2}\Psi, \quad \tau_{r\theta} = -r^{-2}\Psi_z,$$

where $\Psi$ is independent of $\theta$.

While the boundary conditions for $z = 0$ and $z = L$ are now directly given in terms of $v$, the corresponding conditions of no tractions for $r = r_o(z)$ are readily shown to be equivalent to the one condition $\Psi = \text{const}$.

In what follows we are interested, in particular, in the values of the stiffness coefficient $K$ in the torque-twist relation $T = K\Theta$ where

$$T = 2\pi \int_0^r r_o \tau_{z\theta} r^2 dr.$$
Specifically, our object is to obtain upper and lower bounds for $K$, through use of both the principles of minimum potential and minimum complementary energy, analogous to what has recently been done for the problem of the end loaded cantilever beam, treated within the framework of the theory of plane stress [1].

The appropriate basic inequalities for the special case under consideration are readily shown to be of the form

$$\tilde{T} \Theta - 2\pi \int_0^L \int_0^r \tilde{B} r \, dz \, dr \leq \frac{1}{2} T \Theta \leq 2\pi \int_0^L \int_0^r \tilde{A} r \, dz \, dr. \quad (4)$$

In this we have

$$\tilde{A} = \frac{1}{2} G \left( \tilde{\gamma}_{z\theta}^2 + \tilde{\gamma}_{r\theta}^2 \right), \quad \tilde{B} = \frac{1}{2} G \left( \tilde{\tau}_{z\theta}^2 + \tilde{\tau}_{r\theta}^2 \right), \quad (5)$$

where $\tilde{\gamma}_{z\theta}$ and $\tilde{\gamma}_{r\theta}$ are given in terms of a function $\tilde{\nu}$ in accordance with (1), and $\tilde{\tau}_{z\theta}$ and $\tilde{\tau}_{r\theta}$ are given in terms of a function $\tilde{\Psi}$ in accordance with (2). The function $\tilde{\nu}$ must be differentiable and satisfy the boundary conditions $\tilde{\nu}(r,0) = 0$, $\tilde{\nu}(r,L) = \Theta r$. The function $\tilde{\Psi}$ must also be differentiable and satisfy the boundary condition $\tilde{\Psi}[r_0(z),z] = \text{const.}$ for $0 \leq z \leq L$. Furthermore, $\tilde{\nu}$ and $\tilde{\Psi}$ must be such that the double integrals in (4) do in fact exist. We note that the equal signs in (4) apply in the event that $\tilde{\nu}$ and $\tilde{\Psi}$, respectively, are the actual solution functions $\nu$ and $\Psi$ of the Michell boundary value problem.

**First - Approximation Upper Bound.** We assume

$$\tilde{\nu} = r \nu(z) \quad (6)$$

giving $\tilde{\gamma}_{r\theta} = 0$ and $\tilde{\gamma}_{z\theta} = r \nu'(z)$ and therewith

$$\tilde{\Pi}_d = 2\pi \int_0^L \int_0^r \tilde{A} r \, dz \, dr = \frac{1}{2} \pi G \int_0^L \left[ \nu'(z) \right]^2 r^2 \, dz. \quad (7)$$

We determine $\nu(z)$ by setting $\delta \tilde{\Pi}_d = 0$, subject to the constraint conditions $\nu(0) = 0$, $\nu(L) = \Theta$, and obtain
\[ V(z) = \Theta \frac{\int_0^z r_o^{-4} \, dz}{\int_0^L r_o^{-4} \, dz}, \]  

and therewith

\[ \bar{\Pi}_d = \frac{\pi G \Theta^2}{4 \int_0^L r_o^{-4} \, dz}, \]  

and then, from (4),

\[ K_{Ul} = \frac{\pi G}{2 \int_0^L r_o^{-4} \, dz} \equiv K_0. \]  

We note that \( K_0 \) is the approximate value of \( K \) which follows by an elementary combination of the solutions for a large number of short shafts of appropriate differing uniform circular cross section.

**First-Approximation Lower Bound.** We assume

\[ \bar{\Psi} = c(r/r_o)^4, \]  

which satisfies \( \bar{\Psi}(r, z) = c \), is exact for the uniform diameter case, and gives

\[ \bar{\tau}_z = 4cr_o^{-5}r, \quad \bar{\tau}_\theta = 4cr_o^{-5}r' r^2. \]  

Therewith

\[ \bar{\Pi}_s = \bar{\tau} \Theta - 2\pi \left[ \int_0^L \int_0^r \tilde{B} \, rdrdz \right] = 2\pi c \Theta - \frac{4\pi c^2}{G} \int_0^L \left[ 1 + \frac{2}{3} (r')^2 \right] \frac{dz}{r_o^4}. \]  

We determine the smallest value of \( \bar{\Pi}_s \) by setting \( \partial \bar{\Pi}_s / \partial c = 0 \). This gives

\[ \bar{\Pi}_s = \frac{\pi G \Theta^2}{4 \int_0^L \left[ 1 + \frac{2}{3} (r')^2 \right] r_o^{-4} \, dz}, \quad \bar{\Pi}_s = \frac{\pi G \Theta^2}{4 \int_0^L \left[ 1 + \frac{2}{3} (r')^2 \right] r_o^{-4} \, dz}. \]
and then, from (4),

$$K_{L1} = \frac{\pi G}{2 \int_0^L \left[ 1 + \frac{2}{3}(\frac{r'}{r_0})^2 \right] \frac{r_-^4}{r_0} \, dz} \quad (15)$$

We note that $K_{L1} = K_{U1} = K_o$, as it must be, for the case of the uniform circular cylinder for which $r'_0 = 0$, throughout. For other cases we expect, on the basis of experience with other problems, that $K_{L1}$ will be a better approximation than $K_{U1}$ to the exact value of $K$.

**Results for conical shafts.** Setting $r'_0 = a + z \tan \alpha$, where $a$ and $\alpha$ are constants we find, from (10) and (15),

$$\frac{K_{L1}}{K_{U1}} = \frac{1}{1 + \frac{2}{3} \tan^2 \alpha} \quad (16)$$

showing a significant difference between $K_{L1}$ and $K_{U1}$ for sufficiently large values of $\alpha$. For example, when $\alpha = 30^\circ$ then $K_{L1}/K_{U1} = 9/11 \approx .82$, and when $\alpha = 45^\circ$ then $K_{L1}/K_{U1} = .6$. We note that the known closed-form solution for conical shafts, [2], is not an exact solution of the problem as stated here inasmuch as this closed-form solution implies rigid rotations of spherical end surfaces, rather than of plane end surfaces.

**Higher-Approximation Bounds.** It appears to be easier to derive improved lower bounds of relatively attractive appearance than to do the same for the problem of upper bounds.

In order to obtain a sequence of lower bounds $K_{L1}$ for $N = 1, 2, \ldots$ we may set

$$\Phi = (r/r_0)^4 \left[ c_1 + c_2 (r/r_0)^2 + \ldots + c_N (r/r_0)^{2N-2} \right] \quad (17)$$

and then determine values of the coefficients $c_n$ from the $N$ simultaneous linear equations $\frac{\partial n}{\partial c_n} = 0$, $n = 1, 2, \ldots, N$.

In order to obtain improved upper bounds $K_{U1}$ we may set, in generalization of (6)
\[ \tilde{\nu} = \Theta [ r V_1(z) + r^3 V_2(z) + \ldots + r^{2N-1} V_N(z) ] \]  

(18)

In this the functions \( V_n \) have to satisfy the constraint boundary conditions
\[ V_n(0) = 0, \quad V_1(L) = 1, \quad V_2(L) = 0, \ldots, \quad V_N(L) = 0, \]
with the variational equation \( \delta \tilde{\Pi}_d = 0 \) then leading to a system of simultaneous linear second-order differential equations for the functions \( V_n \).

To illustrate the nature of the problem we consider the case
\[ \tilde{\nu} = \Theta (r V_1 + r^3 V_2) \] where we will assume, additionally, that \( V_1 \) coincides with the solution function \( V \) in our first-approximation calculation. We now obtain, in place of equation (7)
\[ \tilde{\Pi}_d = \frac{\pi}{4} GE^2 \int_0^L \left[ r^4 (V_1')^2 + \frac{8}{3} r^6 V_2^3 + \frac{1}{2} r^8 (V_2')^2 + \frac{4}{3} r^6 V_1 V_2' \right] dz \]  

(19)

and with this, as differential equation for \( V_2 \),
\[ (r^8 V_2')' - \frac{16}{3} r^6 V_2 = - \frac{4}{3} (r^6 V_1')' = - \frac{8 r r'}{3 \int_0^L r^{-4} dz} \]  

(20)

The associated minimum value of \( \tilde{\Pi}_d \) follows from (19), with the help of (20) and (8), in the form
\[ \tilde{\Pi}_d = \frac{GE^2}{4 \int_0^L r^{-4} dz} \left[ 1 - \frac{4}{3} \int_0^L r r' V_2 dz \right] \]  

(21)

and then from (4), and with \( K_0 \) as in (10),
\[ K_{U2} = K_0 \left[ 1 - \frac{4}{3} \int_0^L r r' V_2 dz \right] \]  

(22)

Equation (22) may be written, with the help of (20) and upon making use of the boundary conditions for \( V_2 \), in the alternate form
\[ K_{U2} = K_0 \left\{ 1 - \frac{8}{3} \int_0^L \left[ r^6 V_2^2 + \frac{3}{16} r^6 (V_2')^2 \right] dz \right\} \]  

(22')
which makes it evident that \( K_{U_2} \) is in fact a better upper bound for \( K \) than the first approximation bound \( K_{U_1} \).

As regards the determination of the function \( V_2 \) we note the possibility of an explicit solution, as a combination of powers of \( r_o^+ \), for the case \( r_o^+ = a + z \tan \alpha \). The resulting ratio \( K_{U_2}/K_{U_1} \) is a less simple expression than the ratio \( K_{L_1}/K_{U_1} \) in (16), and we refrain from stating it here, in the hope that some alternate, simpler, upper bound improvement might be obtained later, in a different way.

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