

U.S. DEPARTMENT OF COMMERCE
National Technical Information Service

AD-A035 080

A PARAMETER ESTIMATION TECHNIQUE FOR THE
GENERALIZED RAYLEIGH-RICIAN DISTRIBUTION
AND LAHA'S BESSEL DISTRIBUTION

ARMY MISSILE RESEARCH, DEVELOPMENT AND
ENGINEERING LABORATORY
REDSTONE ARSENAL, ALABAMA

8 DECEMBER 1976

ADA 035080



TECHNICAL REPORT RD-77-11

**A PARAMETER ESTIMATION TECHNIQUE FOR THE
GENERALIZED RAYLEIGH-RICIAN DISTRIBUTION
AND LAHA'S BESSEL DISTRIBUTION**

Jerry W. Vickers
Aeroballistics Directorate
US Army Missile Research, Development and Engineering Laboratory
US Army Missile Command
Redstone Arsenal, Alabama 35809

8 December 1976

Approved for public release; distribution unlimited.



U.S. ARMY MISSILE COMMAND

Redstone Arsenal, Alabama 35809

REPRODUCED BY
NATIONAL TECHNICAL
INFORMATION SERVICE
U. S. DEPARTMENT OF COMMERCE
SPRINGFIELD, VA. 22161

DDC
REPRODUCED
FEB 7 1977
REGULATED
B

DISPOSITION INSTRUCTIONS

DESTROY THIS REPORT WHEN IT IS NO LONGER NEEDED. DO NOT RETURN IT TO THE ORIGINATOR.

DISCLAIMER

THE FINDINGS IN THIS REPORT ARE NOT TO BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION UNLESS SO DESIGNATED BY OTHER AUTHORIZED DOCUMENTS.

TRADE NAMES

USE OF TRADE NAMES OR MANUFACTURERS IN THIS REPORT DOES NOT CONSTITUTE AN OFFICIAL INDORSEMENT OR APPROVAL OF THE USE OF SUCH COMMERCIAL HARDWARE OR SOFTWARE.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER RD-77-11	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) A PARAMETER ESTIMATION TECHNIQUE FOR THE GENERALIZED RAYLEIGH-RICIAN DISTRIBUTION AND LAHA'S BESSEL DISTRIBUTION		5. TYPE OF REPORT & PERIOD COVERED Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Jerry W. Vickers		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Commander US Army Missile Command Attn: DRSMI-RD Redstone Arsenal, Alabama 35809		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS (DA) 1W362303A214 AMCMS Code 632303,2140511
11. CONTROLLING OFFICE NAME AND ADDRESS Commander US Army Missile Command Attn: DRSMI-RPR Redstone Arsenal, Alabama 35809		12. REPORT DATE 8 December 1976
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		13. NUMBER OF PAGES 26
		15. SECURITY CLASS. (of this report) Unclassified
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited.		
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)		
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Maximum likelihood estimators Method of moments Method of quantiles Minimum variance		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) The purpose of this report is to provide a means by which the parameters of the Generalized Rayleigh Distribution can be estimated from experimental data. This report is an outgrowth of research in the area of laser radar technology; however, the results should be of general interest and find application in other disciplines where the Generalized Rayleigh Distribution is encountered.		

CONTENTS

	Page
I. INTRODUCTION	3
II. BACKGROUND	4
III. METHODOLOGY	5
IV. PROPERTIES OF THE ESTIMATORS	9
V. PERFORMANCE OF THE ESTIMATORS.	10
VI CONCLUSIONS	13
Appendix A. DERIVATION OF THE MOMENTS FOR THE BESSEL DISTRIBUTION.	15
Appendix B. SIMULATION RESULTS.	21

ACCESSION for		
NTIS	White Section	<input checked="" type="checkbox"/>
GDC	Buff Section	<input type="checkbox"/>
UNANNOUNCED		<input type="checkbox"/>
JUSTIFICATION		
BY		
DISTRIBUTION, AVAILABILITY CODES		
Dist.	AVAIL. and/or	SPECIAL
A		

ACKNOWLEDGMENT

The author wishes to express appreciation for the contributions made to this effort by Dr. Raymond Meyers, Virginia Polytechnic Institute, and by Latricha Greene, Aeroballistics Directorate.

I. INTRODUCTION

The purpose of this report is to provide a means by which the parameters of the Generalized Rayleigh Distribution can be estimated from experimental data.

The Generalized Rayleigh Distribution can be expressed as

$$p(R) = \frac{A}{\psi_0} \left(\frac{R}{A}\right)^{1/2n} \text{EXP} \left\{ -\frac{(R^2 + A^2)}{2\psi_0} \right\} I_{1/2(n-2)} \left(\frac{RA}{\psi_0}\right) \quad (1)$$

where

$$R \geq 0,$$

$$R = \sqrt{X_1^2 + X_2^2 + \dots + X_n^2},$$

$\begin{Bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{Bmatrix}$ is a random normal vector distributed in n dimensions,

$\begin{Bmatrix} A_1 \\ A_2 \\ \vdots \\ A_n \end{Bmatrix}$ is the mean vector,

$$A = \sqrt{A_1^2 + A_2^2 + \dots + A_n^2},$$

M_n is a positive definite covariance matrix = $I_n \psi_0$ where ψ_0 is a positive constant and I_n is the identity matrix,

I_ν is the modified Bessel function of the first kind of order ν .

The derivation of the probability density function in Equation (1) is given by Miller¹.

II. BACKGROUND

Special cases of the Generalized Rayleigh Distribution appear in many physical environments in electrical engineering, radar engineering, and some areas of optics.

This report is an outgrowth of research in the area of laser radar technology. However, the results should be of general interest and find application in other disciplines where the Generalized Rayleigh Distribution is encountered.

If n , the number of dimensions, in Equation (1) is set equal to 2 then the probability density function can be expressed as

$$p(R) = \left(\frac{R}{\psi_0}\right) \text{EXP} \left\{ -\frac{(R^2 + A^2)}{2\psi_0} \right\} I_0 \left(\frac{RA}{\psi_0}\right) \quad (2)$$

This distribution is sometimes called the Rician Distribution². Radar engineers will recognize it as the output of an IF amplifier when the input is Gaussian noise and a sinusoidal signal of constant amplitude.

If A is allowed to approach zero in Equation (2) then it can be shown that $I_0 \left(\frac{RA}{\psi_0}\right)$ is approximately 1 and the probability density function can be expressed as

$$p(R) = \left(\frac{R}{\psi_0}\right) \text{EXP} \left\{ \frac{-R^2}{2\psi_0} \right\} \quad (3)$$

This is the well-known Rayleigh Distribution. To radar engineers it is the familiar output of an IF amplifier when the input is Gaussian noise.

In many practical situations it becomes necessary to have the capability to estimate the two parameters (A, ψ_0) of Equation (2) from experimental data. This was the problem that had to be addressed during the research in Laser Radar Technology.

¹Miller, K. S., Multivariate Distributions, Robert E. Krieger Publishing Company, Inc., New York, 1974.

²Ibid.

The remainder of this report presents the development of the methodology to estimate the two parameters and an analysis of the performance of the estimation technique in a simulated environment.

III. METHODOLOGY

Some of the more common statistical techniques employed to estimate the parameters of a known distribution are:

- a) Maximum likelihood estimators.
- b) Method of moments.
- c) Method of quantiles.
- d) Minimum variance.

Each of these methods was explored and analytical and practical problems were encountered in all four methods.

Applying the technique of maximum likelihood requires the determination of the log-likelihood function of a sample drawn from the distribution. Given a sample of size n drawn from the population whose distribution function is $p(R; A, \psi_0)$ the likelihood function is

$$\begin{aligned}
 L(R_1, \dots, R_n; A, \psi_0) &= p_n(R_1, \dots, R_n; A, \psi_0) \\
 &= p(R_1; A, \psi_0) p(R_2; A, \psi_0) \dots p(R_n; A, \psi_0) \\
 &= \prod_{i=1}^n p(R_i; A, \psi_0) \quad . \quad (4)
 \end{aligned}$$

The log of the likelihood function is usually easier to work with and is

$$\ln L = \ln \left\{ \prod_{i=1}^n p(R_i; A, \psi_0) \right\} \quad . \quad (5)$$

The estimates of ψ_0 and A are determined by solving the following simultaneous equations

$$\begin{aligned}
 \frac{\partial L}{\partial A} &= 0 \\
 \frac{\partial L}{\partial \psi_0} &= 0 \quad . \quad (6)
 \end{aligned}$$

The problem with this approach is the difficulty in taking the partial derivative of the log of a Bessel function.

The moments of the Generalized Rayleigh Distribution proved difficult to work with. The expressions for the first two population moments are (for $n = 2$)

$$\begin{aligned}\mu_1' &= (2\psi_0)^{1/2} \text{EXP} \left\{ \frac{-A^2}{2\psi_0} \right\} \Gamma\left(\frac{3}{2}\right) {}_1F_1\left(\frac{3}{2}, 1, \frac{A^2}{2\psi_0}\right) \\ \mu_2' &= (2\psi_0) \text{EXP} \left\{ \frac{-A^2}{2\psi_0} \right\} \Gamma(2) {}_1F_1\left(2, 1, \frac{A^2}{2\psi_0}\right)\end{aligned}\quad (7)$$

where $\Gamma(x)$ is the gamma function and ${}_1F_1$ is the confluent hypergeometric function.

The first moment can also be expressed in terms of Bessel functions. That is,

$$\mu_1' = \sqrt{\frac{\pi\psi_0}{2}} \text{EXP} \left\{ \frac{-A^2}{2\psi_0} \right\} \left[\left(1 + \frac{A^2}{2\psi_0}\right) I_0\left(\frac{A^2}{4\psi_0}\right) + \left(\frac{A^2}{2\psi_0}\right) I_1\left(\frac{A^2}{4\psi_0}\right) \right] \quad (8)$$

These population moments are equated to the sample moments and the resulting system of equations is solved for the parameter estimates. These equations could be solved only through some approximation procedure.

Quantile estimators require that the probability density function be integrated to yield the cumulative density function. The integral of Equation (3) can not be easily evaluated and a numerical or series approximation is needed.

Linear estimators have been developed³ but this technique (at least the one revealed by this author) requires that ψ_0 be a known constant.

Due to the computational complexity of the assumptions involved the previous methods were eliminated from consideration as viable candidates.

Normally, in a situation such as this where the distribution is of a form that does not yield simple estimators, an alternative approach is to attempt to apply some simple transformation which results in a simplification of the estimation process.

³Miller, loc cit.

A transformation of the distribution in Equation (2) yields a distribution form for which simple parameter estimates can be found and which characterizes a device known to the radar engineer. The transformation is

$$X = R^2 \quad . \quad (9)$$

The result is the same as passing the output of the IF amplifier through a square law detection device.

Applying the transformation to the probability density function in Equation (2) yields the following density function:

$$p(X) = \left(\frac{1}{2\psi_0} \right) \text{EXP} \left\{ - \frac{(X + A^2)}{2\psi_0} \right\} I_0 \left(\frac{\sqrt{X}A}{\psi_0} \right) \quad . \quad (10)$$

Letting $p = 1$, $\alpha = \frac{1}{2\psi_0}$, and $\beta = \frac{A}{\psi_0}$ results in

$$p(X) = \alpha^{p-1} \text{EXP} \left\{ - \frac{\beta^2}{4\alpha} \right\} \text{EXP} \left\{ -\alpha X \right\} I_{p-1} \left(\beta \sqrt{X} \right) \quad (11)$$

which is the Bessel Distribution as defined by Laha⁴.

This form of the density function allows one to develop moment estimators for the parameters (A^2 and ψ_0) of the Generalized Rayleigh Distribution.

The moment estimators can be developed from the characteristic function given by Laha⁵.

$$\phi_x(t) = \left(1 - \frac{it}{\alpha} \right)^{-p} \text{EXP} \left\{ \frac{it \beta^2}{4\alpha^2 (1 - it/\alpha)} \right\} \quad (12)$$

The moments of the distribution are found by taking the derivative of $\phi_x(t)$ with respect to t . The first four moments are

⁴Laha, R. O., "On Some Properties of the Bessel Function Distributions," Bull-Calcutta Mathematical Society, Vol. 16, p. 59-71.

⁵Ibid.

$$\begin{aligned}
\mu_1'(X) &= \frac{d \phi_x(t)/dt}{i} = \frac{\beta^2}{4\alpha^2} + \frac{p}{\alpha} \\
\mu_2'(X) &= \frac{d^2 \phi_x(t)/dt^2}{i^2} = \frac{\beta^4}{16\alpha^4} + \frac{\beta^2(p+1)}{2\alpha^3} + \frac{p(p+1)}{\alpha^2} \\
\mu_3'(X) &= \frac{d^3 \phi_x(t)/dt^3}{i^3} = \frac{\beta^6}{64\alpha^6} + \frac{\beta^4(3p+6)}{16\alpha^5} + \frac{\beta^2(3p^2+9p+6)}{4\alpha^4} \\
&\quad + \frac{(p^3+3p^2+2p)}{\alpha^3} \\
\mu_4'(X) &= \frac{d^4 \phi_x(t)/dt^4}{i^4} = \frac{\beta^8}{256\alpha^8} + \frac{\beta^6(p+3)}{16\alpha^7} + \frac{\beta^4(3p^2+15p+18)}{8\alpha^6} \\
&\quad + \frac{\beta^2(p^3+6p^2+11p+6)}{\alpha^5} + \frac{(p^4+6p^3+11p^2+6p)}{\alpha^4} .
\end{aligned} \tag{13}$$

The complete derivation of these moments is shown in Appendix A. Letting $p = 1$, $\alpha = \frac{1}{2\psi_0}$, and $\beta = \frac{A}{\psi_0}$ yields:

$$\begin{aligned}
\mu_1'(X) &= A^2 + 2\psi_0 \\
\mu_2'(X) &= A^4 + 8\psi_0 A^2 + 8\psi_0^2 \\
\mu_3'(X) &= A^6 + 18\psi_0 A^4 + 72\psi_0^2 A^2 + 48\psi_0^3 \\
\mu_4'(X) &= A^8 + 32\psi_0 A^6 + 288\psi_0^2 A^4 + 768\psi_0^3 A^2 + 384\psi_0^4 .
\end{aligned} \tag{14}$$

The first two central moments are:

$$\begin{aligned}
\mu_1(X) &= \mu_1'(X) = A^2 + 2\psi_0 \\
\mu_2(X) &= \mu_2'(X) - [\mu_1'(X)]^2 = 4\psi_0 A^2 + 4\psi_0^2
\end{aligned} \tag{15}$$

Estimators for the two parameters can be found by equating the equations for the population moments in Equation (15) to the sample moments and solving the two equations for ψ_0 and A^2 . The sample estimates are

$$\hat{A}^2 = \pm \sqrt{M_1(X)^2 - M_2(X)} \tag{16}$$

$$\hat{\psi}_0 = \frac{M_1(X) - \hat{A}^2}{2}$$

where $M_1(X)$ and $M_2(X)$ are the two sample moments, i.e., $M_1(X)$ is the sample mean and $M_2(X)$ is the sample variance. A^2 cannot be a negative quantity; therefore, the sign of the radical in \hat{A}^2 is positive.

IV. PROPERTIES OF THE ESTIMATORS

In most statistical analyses it is desirable to know something about the properties of the different estimators so that some logical choice can be made between competing estimators. Although only one set of estimators has been developed it is still desirable to investigate some of the more common optimality properties of the estimators.

An analytic investigation of the properties of \hat{A}^2 and $\hat{\psi}_0$ was not performed because the approaches necessary to such an investigation proved to be analytically intractable. However, an empirical analysis of some of the properties was performed by simulating sample data from the Generalized Rayleigh Distribution.

The first property investigated was that of consistency. An estimator is said to be consistent if:

- a) The expected value of the estimator approaches the true value of the parameter as the sample size increases.
- b) The dispersion (variance) of the estimator decreases as the sample size increases.

The following simulation runs were made varying the parameters and the sample size. Seven samples for each set of conditions were run by varying the seed number for the random number generator.

A^2 : 2.0, 5.0, 10.0, 17.0, 26.0

ψ_0 : 1.0

n : 25, 50, 75, 100, 200, 300, 400, 500, 750, 1000.

The results of the analysis are shown in Appendix B. An examination of the results indicates that the estimators appear to be consistent. Figures 1 and 2 show a plot of the expected value of \hat{A}^2 and standard error of \hat{A}^2 , respectively, for one case.

A second property that is desirable for an estimator to possess is unbiasedness. For an estimator to be unbiased its expected (average) value must equal the true parameter value. Consistency implies asymptotic ($n \rightarrow \infty$) unbiasedness but it does not imply unbiasedness for any specific sample size. One cannot conclude from the simulation data in Appendix B that the estimators are unbiased for any sample size; however, it can be concluded that the bias is relatively small for sample sizes greater than 75. A more accurate estimate of the true bias could be obtained by generating more than 7 samples with which to calculate the expected value.

V. PERFORMANCE OF THE ESTIMATORS

To determine how well the parameter estimators \hat{A}^2 and $\hat{\psi}_0$ perform in a practical situation a number of random samples from Generalized Rayleigh Distributions were generated with various parameter values. From these samples the estimators were calculated and a chi-square goodness-of-fit test was performed to determine if the sample data were really drawn from a Generalized Rayleigh Distribution. The rationale behind this approach is that since the distribution is known to be the Generalized Rayleigh then the parameter estimators should lead to an acceptance of the hypothesis of the chi-square test if they adequately estimate the true parameters.

The only problem with this approach is that the distribution parameters used in the test must be either known or estimated by the maximum likelihood method. Since the method of moments was used to estimate the parameters, the chi-square test is not strictly applicable. This problem arises quite often in practice where the usual procedure is to use the chi-square test and to hope that any error introduced is small.

Twenty-one different cases were examined using a 1000-sample data points. The results of the chi-square test are shown in Table 1.

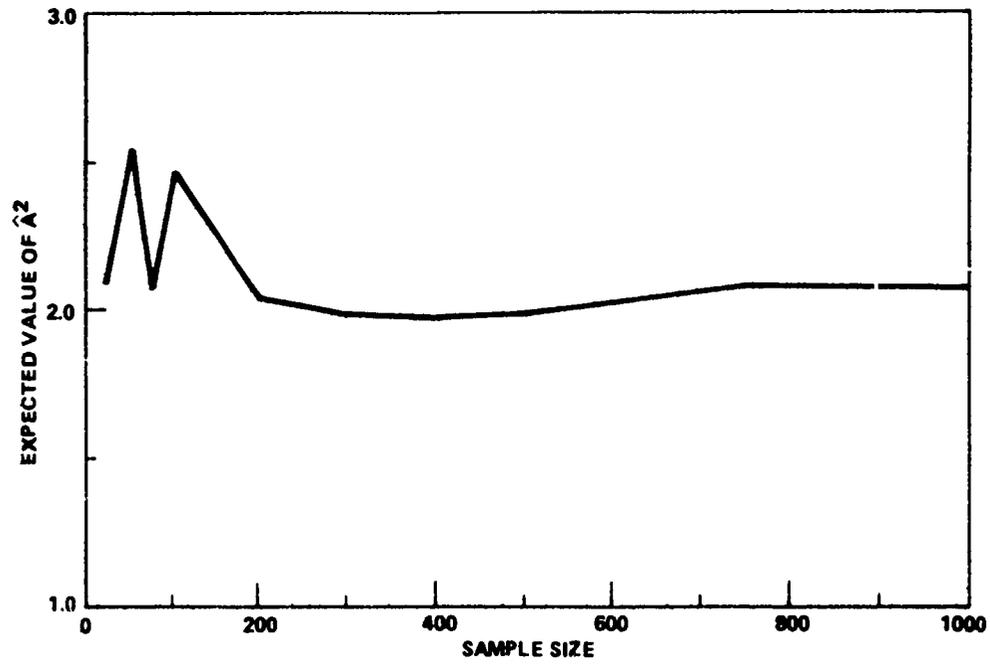


Figure 1. Expected value of \hat{A}^2 versus sample size.
Average of 7 samples.

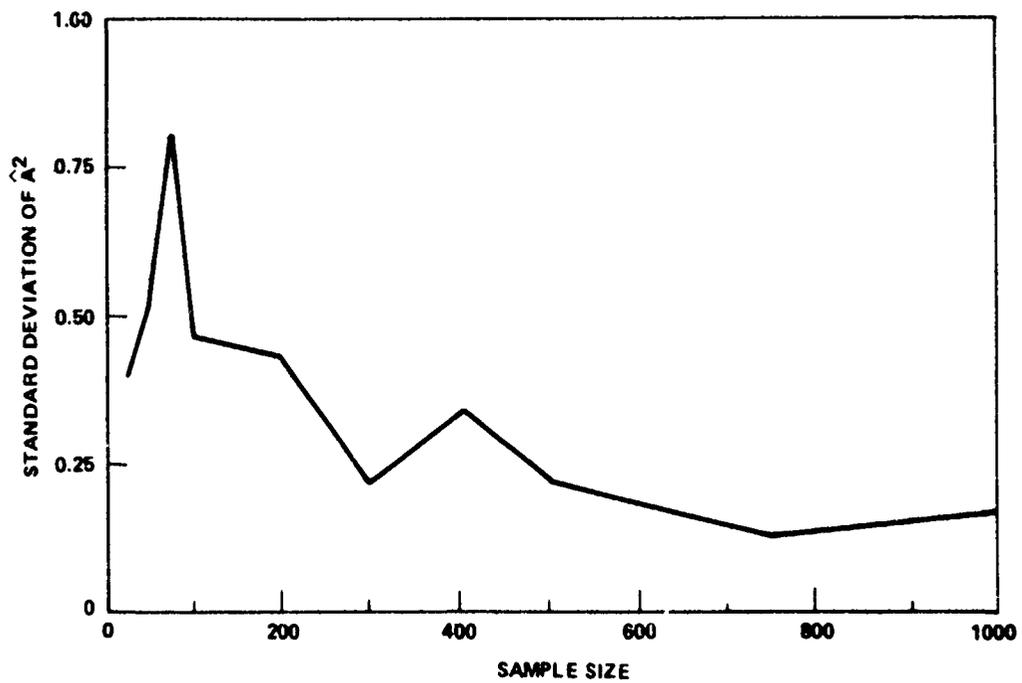


Figure 2. Standard deviation of \hat{A}^2 versus sample size
taken from 7 samples.

TABLE 1. RESULTS OF CHI-SQUARE GOODNESS-OF-FIT TESTS.
 SAMPLE SIZE = 1000, CELLS = 25.

True Parameter Values		Estimated Parameter Values		Critical α Level
A^2	ψ_0	\hat{A}^2	$\hat{\psi}_0$	$\hat{\alpha}$
0.0	0.5	0.28	0.35	0.76
0.0	1.0	0.55	0.69	0.76
0.0	2.0	1.12	1.39	0.76
1.0	0.5	1.12	0.43	0.62
1.0	1.0	1.33	0.81	0.02
1.0	2.0	1.85	1.52	0.66
2.0	0.5	2.10	0.46	0.53
2.0	1.0	2.24	0.90	0.37
2.0	2.0	2.61	1.70	0.37
5.0	0.5	5.10	0.46	0.42
5.0	1.0	5.20	0.91	0.78
5.0	2.0	5.47	1.77	0.52
10.0	0.5	10.10	0.47	0.57
10.0	1.0	10.18	0.92	0.39
10.0	2.0	10.39	1.81	0.48
17.0	0.5	17.09	0.47	0.25
17.0	1.0	17.17	0.93	0.06
17.0	2.0	17.35	1.83	0.02
26.0	0.5	26.09	0.47	0.50
26.0	1.0	26.16	0.93	0.23
26.0	2.0	26.32	1.84	0.55

To apply the chi-square tests that were used requires that the sample data be divided into equiprobable frequency cells. The number of cells in this analysis was 25.

The larger the critical α level in Table 1, the better the fit; i.e., the hypothesis that the data are from a Generalized Rayleigh Distribution with parameters \hat{A}^2 and $\hat{\psi}_0$ is more likely to be accepted.

A common α level is 0.05 or 0.10. As can be seen there are three cases where the α level was below 0.10. For these cases the hypothesis must be rejected.

A better fit for the data in these three cases could possibly be obtained by a slight shift in some of the cell limits since the cells containing abnormally high and low frequency counts were adjacent to one another. The difference in some of the data points in the adjacent cells was in the second and third decimal places.

VI. CONCLUSIONS

It is believed that the parameter estimators developed in this report provide a simple, viable means by which the parameters of the Generalized Rayleigh Distribution can be estimated. Furthermore, from the empirical study it would appear that the estimators offer consistent and relatively unbiased estimates of the true parameters. The practical value of the estimators is realized with the ease with which the necessary calculations can be made.

Appendix A.

DERIVATION OF THE MOMENTS FOR THE BESSEL DISTRIBUTION

The probability density function of the Bessel variate is given by Laha⁶ as:

$$p(X) = \alpha^{p-1} \text{EXP} \left\{ \frac{-\beta^2}{4\alpha} \right\} \text{EXP} \left\{ -\alpha X \right\} I_{p-1} (\beta\sqrt{X}) \quad . \quad (\text{A-1})$$

The associated characteristic function is given as:

$$\phi_x(t) = \left(1 - \frac{it}{\alpha} \right)^{-p} \text{EXP} \left\{ \frac{it\beta^2}{4\alpha^2 (1 - it/\alpha)} \right\} \quad . \quad (\text{A-2})$$

The jth factorial moment of the distribution can be found by

$$\mu'_j(X) = \left[\frac{d^j \phi_x(t) / dt^j}{i^j} \right]_{t=0} \quad (\text{A-3})$$

$$\mu'_j(X) = \left[\frac{d^j \left[\left(1 - it/\alpha \right)^{-p} \text{EXP} \left\{ \frac{it\beta^2}{4\alpha^2 (1 - it/\alpha)} \right\} \right] / dt^j}{i^j} \right]_{t=0} \quad . \quad (\text{A-4})$$

Letting $w = 1 - \frac{it}{\alpha}$

and

$$y = \frac{it\beta^2}{4\alpha^2 (1 - it/\alpha)}$$

then for $j = 1$

$$\begin{aligned} \frac{d \phi_x(t)}{dt} &= \frac{d w^{-p} e^y}{dt} \\ &= w^{-p} \frac{de^y}{dt} + e^y \frac{dw^{-p}}{dt} \\ &= w^{-p} e^y \frac{dy}{dt} + e^y (-pw^{-p-1}) \frac{dw}{dt} \end{aligned}$$

⁶Laha, loc. cit.

$$\begin{aligned}
&= w^{-p} e^y \left(\frac{dy}{dt} - pw^{-1} \frac{dw}{dt} \right) \\
&= \phi_x(t) \left(\frac{dy}{dt} - pw^{-1} \frac{dw}{dt} \right) \\
&= \phi_x(t) \left(\frac{i\beta^2}{4\alpha^2(1-it/\alpha)^2} - \frac{p(-i/\alpha)}{(1-it/\alpha)} \right)
\end{aligned}$$

The first moment is

$$\mu_1'(X) = \left[\frac{\phi_x(t) \left(\frac{i\beta^2}{4\alpha^2(1-it/\alpha)^2} + \frac{pi/\alpha}{(1-it/\alpha)} \right)}{1} \right]_{t=0} \quad (A-5)$$

$$\mu_1'(X) = \frac{\beta^2}{4\alpha^2} + \frac{p}{\alpha}$$

For $j = 2$ one finds

$$\begin{aligned}
\frac{d^2 \phi_x(t)}{dt^2} &= \frac{d(d\phi_x(t)/dt)}{dt} \\
&= \frac{d\phi_x(t) \left(\frac{i\beta^2}{4\alpha^2(1-it/\alpha)^2} + \frac{ip/\alpha}{(1-it/\alpha)} \right)}{dt}
\end{aligned}$$

Letting $z = z_1 + z_2$

$$= \frac{i\beta^2}{4\alpha^2(1-it/\alpha)^2} + \frac{ip/\alpha}{(1-it/\alpha)}$$

then

$$\frac{d^2 \phi_x(t)}{dt} = \frac{d\phi_x(t)z}{dt}$$

$$\begin{aligned}
&= \phi_x(t) \frac{dz}{dt} + z \frac{d\phi_x(t)}{dt} \\
&= \phi_x(t) \frac{dz}{dt} + z^2 \phi_x(t) \\
&= \phi_x(t) \left(z^2 + \frac{dz}{dt} \right) \\
&= \phi_x(t) \left\{ \frac{i\beta^4}{16\alpha^4(1-it/\alpha)^4} + \frac{i^2\beta^2(p+1)}{2\alpha^3(1-it/\alpha)^3} + \frac{i^2p(p+1)}{\alpha^2(1-it/\alpha)^2} \right\} .
\end{aligned}$$

The second moment is

$$\begin{aligned}
\mu_2'(X) &= \left[\frac{\phi_x(t) \left\{ \frac{i\beta^4}{16\alpha^4(1-it/\alpha)^4} + \frac{i^2\beta^2(p+1)}{2\alpha^3(1-it/\alpha)^3} + \frac{i^2p(p+1)}{\alpha^2(1-it/\alpha)^2} \right\}}{i^2} \right]_{t=0} \\
\mu_2'(X) &= \left\{ \frac{\beta^4}{16\alpha^4} + \frac{\beta^2(p+1)}{2\alpha^3} + \frac{p(p+1)}{\alpha^2} \right\} . \quad (A-6)
\end{aligned}$$

For $j = 3$ one finds

$$\begin{aligned}
\frac{d^3\phi_x(t)}{dt} &= \frac{d(d^2\phi_x(t)/dt^2)}{dt} \\
&= \frac{d\phi_x(t) \left\{ \frac{i\beta^4}{16\alpha^4(1-it/\alpha)^4} + \frac{i^2\beta^2(p+1)}{2\alpha^3(1-it/\alpha)^3} + \frac{i^2p(p+1)}{\alpha^2(1-it/\alpha)^2} \right\}}{dt} .
\end{aligned}$$

Letting $a = a_1 + a_2 + a_3$

$$= \frac{i\beta^4}{16\alpha^4(1-it/\alpha)^4} + \frac{i^2\beta^2(p+1)}{2\alpha^3(1-it/\alpha)^3} + \frac{i^2p(p+1)}{\alpha^2(1-it/\alpha)^2}$$

then

$$\begin{aligned}\frac{d^3 \phi_x(t)}{dt^3} &= \frac{d^3 \phi_x(t)a}{dt^3} \\ &= \phi_x(t) \frac{da}{dt} + \frac{ad\phi_x(t)}{dt}.\end{aligned}$$

Now $d\phi_x(t)/dt$ has been shown to be

$$\frac{d\phi_x(t)}{dt} = \phi_x(t) \left\{ \frac{i\beta^2}{4\alpha^2(1-it/\alpha)^2} + \frac{ip/\alpha}{(1-it/\alpha)} \right\}$$

which we previously defined to be

$$\begin{aligned}&= \phi_x(t) z \\ &= \phi_x(t) (z_1 + z_2).\end{aligned}$$

Therefore

$$\begin{aligned}\frac{d^3 \phi_x(t)}{dt^3} &= \phi_x(t) \frac{da}{dt} + a\phi_x(t) (z_1 + z_2) \\ &= \phi_x(t) \left\{ \frac{da}{dt} + a(z_1 + z_2) \right\} \\ &= \phi_x(t) \left\{ \frac{da_1}{dt} + \frac{da_2}{dt} + \frac{da_3}{dt} + a_1z_1 + a_1z_2 + a_2z_1 + a_2z_2 \right. \\ &\quad \left. + a_3z_1 + a_3z_2 \right\} \\ &= \phi_x(t) \left\{ \frac{i^3\beta^6}{64\alpha^6(1-it/\alpha)^6} + \frac{i^3\beta^4(3p+6)}{16\alpha^5(1-it/\alpha)^5} \right. \\ &\quad \left. + \frac{i^3\beta^2(3p^2+9p+6)}{4\alpha^4(1-it/\alpha)^4} + \frac{i^3(p^3+3p^2+2p)}{\alpha^3(1-it/\alpha)^3} \right\}\end{aligned}$$

$$\mu'_3(X) = \left[\frac{d^3 \phi_x(t)/dt^3}{i^3} \right]_{t=0}$$

$$= \left\{ \frac{\beta^6}{64\alpha^6} + \frac{\beta^4(3p+6)}{16\alpha^5} + \frac{\beta^2(3p^2+9p+6)}{4\alpha^4} + \frac{(p^3+3p^2+2p)}{\alpha^3} \right\}.$$

For $j = 4$ one finds

$$\frac{d^4 \phi_x(t)}{dt} = \frac{d(d^3 \phi_x(t)/dt^3)}{dt}$$

$$= d \left[\frac{\phi_x(t) \left\{ \frac{i^3 \beta^6}{64\alpha^6 (1-it/\alpha)^6} + \frac{i^3 \beta^4 (3p+6)}{16\alpha^5 (1-it/\alpha)^5} + \frac{i^3 \beta^2 (3p^2+9p+6)}{4\alpha^4 (1-it/\alpha)^4} + \frac{i^3 (p^3+3p^2+2p)}{\alpha^3 (1-it/\alpha)^3} \right\}}{dt} \right].$$

Letting $b = b_1 + b_2 + b_3 + b_4$

$$= \frac{i^3 \beta^6}{64\alpha^6 (1-it/\alpha)^6} + \frac{i^3 \beta^4 (3p+6)}{16\alpha^5 (1-it/\alpha)^5} + \frac{i^3 \beta^2 (3p^2+9p+6)}{4\alpha^4 (1-it/\alpha)^4}$$

$$+ \frac{i^3 (p^3+3p^2+2p)}{\alpha^3 (1-it/\alpha)^3}$$

then

$$\frac{d^4 \phi_x(t)}{dt^4} = \frac{d \phi_x(t) b}{dt}$$

$$= \phi_x(t) \frac{db}{dt} + \frac{bd \phi_x(t)}{dt}$$

$$\begin{aligned}
&= \phi_{\mathbf{x}}(t) \frac{db}{dt} + b \phi_{\mathbf{x}}(t)(z_1 + z_2) \\
&= \phi_{\mathbf{x}}(t) \left\{ \frac{db}{dt} + b(z_1 + z_2) \right\} \\
&= \phi_{\mathbf{x}}(t) \left\{ \frac{db_1}{dt} + \frac{db_2}{dt} + \frac{db_3}{dt} + \frac{db_4}{dt} + b_1 z_1 + b_1 z_2 + b_2 z_1 + b_2 z_2 + b_3 z_1 \right. \\
&\quad \left. + b_3 z_2 + b_4 z_1 + b_4 z_2 \right\} \\
&= \phi_{\mathbf{x}}(t) \left\{ \frac{i^4 \beta^8}{256 \alpha^8 (1 - it/\alpha)^8} + \frac{i^4 \beta^6 (p+3)}{16 \alpha^7 (1 - it/\alpha)^7} \right. \\
&\quad \left. + \frac{i^4 \beta^4 (3p^2 + 15p + 18)}{8 \alpha^6 (1 - it/\alpha)^6} + \frac{i^4 \beta^2 (p^3 + 6p^2 + 11p + 6)}{\alpha^5 (1 - it/\alpha)^5} \right. \\
&\quad \left. + \frac{i^4 (p^4 + 6p^3 + 11p^2 + 6p)}{\alpha^4 (1 - it/\alpha)^4} \right\} \\
\mu_4^1(X) &= \left[\frac{d^4 \phi_{\mathbf{x}}(t)/dt^4}{i^4} \right]_{t=0} \\
&= \frac{\beta^8}{256 \alpha^8} + \frac{\beta^6 (p+3)}{16 \alpha^7} + \frac{\beta^4 (3p^2 + 15p + 18)}{8 \alpha^6} \\
&\quad + \frac{\beta^2 (p^3 + 6p^2 + 11p + 6)}{\alpha^5} + \frac{(p^4 + 6p^3 + 11p^2 + 6p)}{\alpha^4}
\end{aligned}$$

**Appendix B.
SIMULATION RESULTS**

**TABLE B-1. SIMULATION RESULTS - EXPECTED VALUE AND STANDARD
DEVIATION OF \hat{A}^2 AND $\hat{\psi}_0$.**

True Parameter Values		Sample Size	Parameter Estimates			
			\hat{A}^2		$\hat{\psi}_0$	
A^2	ψ_0		Mean	Standard Deviation	Mean	Standard Deviation
2.0	1.0	25	2.10	0.40	1.17	0.63
		50	2.55	0.52	0.94	0.20
		75	2.06	0.80	1.02	0.37
		100	2.48	0.46	0.90	0.11
		200	2.04	0.43	0.91	0.13
		300	1.99	0.22	0.95	0.13
		400	1.98	0.34	0.97	0.20
		500	1.99	0.22	0.99	0.16
		750	2.08	0.13	0.97	0.02
		1000	2.08	0.17	0.95	0.10
5.0	1.0	25	5.39	0.82	1.19	0.43
		50	5.72	0.38	0.97	0.20
		75	5.33	0.67	0.98	0.23
		100	5.48	0.56	0.95	0.12
		200	4.97	0.51	0.93	0.13
		300	5.05	0.26	0.94	0.10
		400	4.99	0.33	0.98	0.17
		500	5.00	0.27	0.98	0.15
		750	5.08	0.10	0.98	0.12
		1000	5.06	0.15	0.97	0.08
10.0	1.0	25	10.70	1.17	1.18	0.38
		50	10.92	0.37	0.98	0.18
		75	10.50	0.79	0.99	0.18
		100	10.53	0.77	0.97	0.13

TABLE B-1. (Continued).

True Parameter Values		Sample Size	Parameter Estimates			
			\hat{A}^2		ψ_0	
A^2	ψ_0		Mean	Standard Deviation	Mean	Standard Deviation
10.0	1.0	200	9.93	0.60	0.93	0.12
		300	10.08	0.35	0.93	0.09
		400	10.0	0.35	0.97	0.14
		500	10.0	0.32	0.98	0.13
		750	10.01	0.08	0.99	0.10
		1000	10.05	0.17	0.97	0.07
17.0	1.0	25	18.02	1.5	1.17	0.37
		50	18.12	0.45	0.98	0.17
		75	17.67	0.96	1.0	0.17
		100	17.61	1.01	0.98	0.14
		200	16.89	0.70	0.93	0.12
		300	17.11	0.47	0.93	0.07
		400	17.02	0.40	0.97	0.13
		500	17.00	0.36	0.98	0.13
		750	17.1	0.09	0.99	0.09
1000	17.04	0.20	0.98	0.07		
26.0	1.0	25	26.82	1.87	1.16	0.40
		50	27.33	0.59	0.99	0.17
		75	26.85	1.15	1.0	0.16
		100	26.69	1.25	0.99	0.15
		200	25.86	0.82	0.94	0.12
		300	26.13	0.61	0.93	0.07
		400	26.03	0.46	0.97	0.12
		500	26.01	0.43	0.98	0.12
		750	26.12	0.11	0.99	0.09
		1000	26.05	0.24	0.98	0.07

DISTRIBUTION

	No. of Copies
Defense Documentation Center Cameron Station Alexandria, Virginia 22314	6
Commander US Army Materiel Development and Readiness Command Attn: DRCRD	1
DRCDL	1
5001 Eisenhower Avenue Alexandria, Virginia	
DRSMI-FR, Mr. Strickland	1
-LP, Mr. Voigt	1
-R, Dr. McDaniel	1
Dr. Kobler	1
-RD, Dr. Hartman	1
-RR, Dr. Wilkerson	1
Dr. Stettler	1
-RDF, Mr. Vickers	12
-RBD	3
-RPR (Record Set)	1
(Reference Copy)	1