A CLASS OF CONTROLLABLE NONLINEAR SYSTEMS

by

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Abstract

Sufficient conditions for global and local controllability of the nonlinear system

\[
\frac{d}{dt} x(t) = A(t,x(t),u(t))x(t) + B(t)u(t) + f(t,x(t),u(t))
\]

and its perturbed system are given. These conditions extend some previous results through the removal of certain boundedness conditions involving the functions \((A,f)\) and their partial derivatives.

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I. INTRODUCTION

The purpose of this correspondence is to extend the results of [1] by considering a more general class of nonlinear control systems. The results obtained in this work provide sufficient conditions for global and local controllability of perturbed nonlinear systems.

Consider the nonlinear time-varying system

$$\frac{d}{dt} x(t) = A(t,x(t),u(t))x(t) + B(t)u(t) + f(t,x(t),u(t))$$

where $x(t)$ is a $n \times 1$ state vector; $u(t)$ is a $m \times 1$ input vector; $A, B, f$ are $n \times n$, $n \times m$, $n \times 1$ matrix-valued functions, respectively.

Denote $C_{nm}[t_0,t_1]$ as the Banach space of continuous $R^n \times R^m$ valued functions on $[t_0,t_1]$ with the uniform norm $|[x(t),u(t)]| = |x(t)| + |u(t)|$, where $|w(t)| = \max_{i \in [t_0,t_1]} |w_i(t)|$ and $|w_i(t)|$ is the absolute value of $w_i(t)$, the element of $w(t)$.

Define the norm of a continuous $n \times m$ matrix-valued function $S(t)$ by

$$||S(t)|| = \max_{i \in [t_0,t_1]} \max_{j=1}^m |S_{ij}(t)|,$$

where $S_{ij}$ are elements of $S$.

Given $(x_0, x_1)$ as the initial and final state, respectively of (1). The problem is to find a continuous input function $u(t)$, defined in $[t_0,t_1]$, which steers system (1) from $x_0$ at $t_0$ to $x_1$ at $t_1$. The usual definitions of globally and locally completely and totally controllable are assumed [2].

For each fixed element $[z,v] \in C_{nm}[t_0,t_1]$, consider the following system

$$\frac{d}{dt} x(t) = A(t,z(t),v(t))x(t) + B(t)u(t) + f(t,z(t),v(t))$$

The solution of the system (2) with $x(t_0) = x_0$ is given by
\[ x(t) = \phi(t, t_0; z, v)x_0 + \int_{t_0}^{t} \phi(t, s; z, v)B(s)u(s)ds + \int_{t_0}^{t} \phi(t, s; z, v)f(s, z, v)ds \]  
(3)

where \( \phi(t, t_0; z, v) \) is the state transition matrix of the homogeneous system

\[
\frac{d}{dt} x(t) = A(t, z(t), v(t))x(t), \quad \phi(t_0, t_0; z, v) = I \text{ the identity matrix.}
\]

Define the controllability matrix by

\[
G(t_0, t; z, v) = \int_{t_0}^{t} \phi(t, s; z, v)B(s)B'(s)\phi'(t, s; z, v)ds
\]
(4)

the prime denotes the matrix transpose operation. Obviously, \( G(t_0, t; z, v) \) is symmetric and non-negative definite.

II. GLOBAL CONTROLLABILITY RESULT

**Theorem 1:** The system (1) is globally (a) completely controllable at \( t_0 \), or (b) totally controllable if the following conditions are satisfied.

(i) \( B(t) \) has a continuous first derivative with respect to \( t \),

(ii) \( A(t, x, u), A_x(t, x, u), A_u(t, x, u), f(t, x, u), f_x(t, x, u), \) and \( f_u(t, x, u) \) are continuous and bounded in \([t_0, t_1] \times \mathbb{R}^n \times \mathbb{R}^m \),

(iii) there exists a positive constant \( q \) such that

\[
\inf_{[z, v] \in C_{nm}^{n \times m} \times [t_0, t_1]} \det G(t_0, t_1; z, v) > q
\]
(5)

(a) for some \( t_1 > t_0 \), or (b) for all \( t_0 \) and all \( t_1 > t_0 \).

**Proof:** The proof of the theorem is based on the Schauder's fixed point theorem of the following version, "Every continuous map which maps a compact convex subset of a Banach space into itself has a fixed point".
For each fixed element \([z,v] \in C_{nm} [t_0,t_1]\), consider the control function \(u(t)\) for \(t \in [t_0,t_1]\)

\[
u(t) = B'(t) \phi'(t_1,t;v) G^{-1}(t_0,t_1;z,v)[x_1 - \phi(t_1,t_0;z,v)x_0
\]

\[
= \int_{t_0}^{t_1} \phi(t_1,s;z,v) f(s,z,v) ds
\]

where \(\phi(t,t_0;z,v)\) is defined as that in Eq. (3). With this expression, Eq. (3) can be rewritten as

\[
x(t) = \phi(t,t_0;z,v)x_0
\]

\[
+ \int_{t_0}^{t} \phi(t,s;z,v) B(s) B'(s) \phi'(t_1,s;z,v) G^{-1}(t_0,t_1;z,v) ds 
\]

\[
- \int_{t_0}^{t_1} \phi(t_1,s;z,v) f(s,z,v) ds + \int_{t_0}^{t} \phi(t,s;z,v) f(s,z,v) ds
\]

It should be noted that by hypothesis (iii) \(G^{-1}(t_0,t_1;z,v)\) is well-defined in above expression. It is easily seen that \(x(t)\) in Eq. (7) satisfies both boundary conditions at \(t = t_0\) and \(t = t_1\).

Now the right sides of Eqs. (6)-(7) can be viewed as a pair of operators, \(P_2([z,v])(t)\) and \(P_1([z,v])(t)\), respectively. Define the nonlinear mapping

\[
P([z,v])(t) = [P_1([z,v])(t), P_2([z,v])(t)].
\]

Obviously, \(P_1([z,v])(t)\) and \(P_2([z,v])(t)\) are continuous in \(t\) by the uniform continuity of \(\phi(t,t_0;z,v)\) in \(t\). Therefore, \(P\) maps \(C_{nm} [t_0,t_1]\) into \(C_{nm} [t_0,t_1]\). It can also be easily verified that, by hypothesis (ii) and the definition of \(\phi(t,t_0;z,v)\), \(P\) is continuous in \([z,v]\).
Considering the subset of $C_{nm}[t_0,t_1]$

$$S = \left\{ (z,v) \in C_{nm}[t_0,t_1] : ||[z,v]|| \leq K , \right\}$$

$$||[z(t),v(t)] - [\tilde{z}(t),\tilde{v}(t)]|| \leq K|t-\tilde{t}| , \forall t, \tilde{t} \in [t_0,t_1]$$

where $K$ is certain positive constant depending upon the bounds of $A,B,f$ and their partial derivatives, it can be easily shown that the image set $P(S) \subseteq S$. Besides, $S$ is closed and convex by this construction. Furthermore, each sequence $\{s_i\} \subseteq S$ constitutes a uniformly bounded equicontinuous family. Hence by the Arzela-Ascoli theorem [3], $S$ is relative compact and therefore compact.

Then, the Schauder's theorem [3] can be applied to conclude that $P$ has a fixed point $[z^*,v^*]$, i.e.,

$$P([z^*,v^*])(t) = [P_1([z^*,v^*]), P_2([z^*,v^*])] = [z^*,v^*]$$

Substitute this fixed point into Eqs. (6)-(7), a direct differentiation of Eq. (7) with respect to $t$ shows that $z^*(t)$ is a solution to the system (1) for the control function $u(t)$ given by $v^*(t)$.

If condition (iii)(a) holds, $v^*(t)$ drives the system (1) from $x_0$ to $x_1$ on some interval $[t_0,t_1]$ for all $x_0,x_1 \in \mathbb{R}^n$, system (1) is globally completely controllable at $t_0$. If condition (iii)(b) holds, we have global total controllability.

**Corollary 1:** Given the system (1) with conditions (i)-(iii), then the perturbed system

$$\frac{d}{dt} x(t) = [A(t,x,u) + \varepsilon \tilde{A}(t,x,u)]x(t) + [B(t) + \varepsilon \tilde{B}(t)]u(t) + f(t,x,u), \quad (8)$$

with $\tilde{A}, \tilde{B}$ satisfying the same type of conditions imposed on $A,B$, is globally controllable provided $\varepsilon$ is sufficiently small.
Proof: One needs only to show that the determinant of the modified controllability matrix $G^*$ has a positive infimum. By observing that the determinant of $G^*(t_1,t_0;z,v,\varepsilon)$ can be expanded about $\varepsilon = 0$ [5] into the sum of $\det G(t_1,t_0;z,v)$, $\epsilon H(t_1,t_0;z,v)$ and $o(\varepsilon^2)$ where $H(t_1,t_0;z,v)$ is a bounded scalar, we know that by taking $\varepsilon$ small enough there exists some $q^* > 0$ such that

$$\det G^*(t_1,t_0;z,v,\varepsilon) \geq q^* \quad \text{for all} \quad [z,v] \in C_{nm}[t_0,t_1].$$

Therefore, Theorem 1 concludes the result of controllability.

Remarks: (i) If $(A,f)$ do not explicitly depend on $u(t)$, the theorem is applicable to the class of systems with $B = B(t,x(t))$. These results then reduce to the work reported in [4].

(ii) Uniform boundedness of partial derivatives can be slightly weakened by considering a Lipschitz condition on corresponding variables.

III. LOCAL CONTROLLABILITY RESULT

Consider the following subset of $[t_0,t_1) \times \mathbb{R}^n \times \mathbb{R}^m$,

$$D = \{(t,x,u) : t_0 \leq t \leq t_1, \quad |x| + |u| \leq d, \quad d \text{ is some positive constant}\}.$$

Theorem 2: The system (1) is locally (a) completely controllable at $t_0$, or (b) totally controllable if the following conditions are satisfied.

(i) $B(t)$ has a continuous first derivative with respect to $t$,

(ii) $A(t,x,u)$, $A_x(t,x,u)$, $A_u(t,x,u)$, $f(t,x,u)$, $f_x(t,x,u)$ and $f_u(t,x,u)$ are continuous in $D$. 
(iii) There exists a positive constant $q$ such that

$$\inf_{[z,v] \in E} \det G(t_1,t_0;z,v) \geq q$$

where

$$E = \{(z,v) \in C_{nm}[t_0,t_1] : \|[z,v]\| \leq d\}$$

and

$$M < \min \left\{ \frac{d}{k_1}, \frac{d}{k_2} \right\}$$

where

$M$ is the bound of $f$ in $D$, $k_1$ and $k_2$ are certain positive constants depending upon the bounds of $A,B$ and their partial derivatives in $D$.

(a) for some $t_1 > t_0$, or (b) for all $t_0$ and all $t_1 > t_0$.

**Proof:** The proof is similar to that of Theorem 1. We leave it to the reader.

**Corollary 2:** Given the system (1) with conditions (i)-(iii) in Theorem 2, the perturbed system

$$\frac{d}{dt} x(t) = [A(t,x,u) + \varepsilon A(t,x,u)]x(t) + [B(t) + \varepsilon B(t)]u(t) + f(t,x,u) \quad (9)$$

with $\tilde{A},\tilde{B}$ satisfying the same type of conditions imposed upon $A,B$ is locally controllable provided $\varepsilon$ is sufficiently small.

**Proof:** The proof is similar to that of Corollary 1 and follows immediately from Theorem 2.
Example: Consider the system

\[ \begin{align*}
    \dot{x}_1 &= x_2 + [\sin^2(x_1 + u)]x_2 \\
    \dot{x}_2 &= x_1 + u + [\sin^2(x_1 + u)]x_1 
\end{align*} \]

In matrix form, we have

\[ A(t,x,u) = \begin{bmatrix}
    0 & 1 + \sin^2(x_1 + u) \\
    1 + \sin^2(x_1 + u) & 0
\end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \]

It is easily seen that \( B, dB/dt, A, \partial A/\partial x, \partial A/\partial u \) exist and are bounded in \( \mathbb{R} \) and \( \mathbb{R}^2 \times \mathbb{R} \), respectively. Furthermore, for each fixed \([z,v] \in C_{21}[t_0,t_1]\), the controllability matrix \( G \) is

\[ G(t_0,t_1;z,v) = \frac{1}{4} \begin{bmatrix}
    \int_{t_0}^{t_1} (a-b)^2 dy & \int_{t_0}^{t_1} (a^2-b^2) dy \\
    \int_{t_0}^{t_1} (a^2-b^2) dy & \int_{t_0}^{t_1} (a+b)^2 dy
\end{bmatrix} \]

where

\[ a = \exp \int_{t_0}^{t_1} [1 + \sin^2(z_1 + v)] dy, \quad b = \exp \int_{t_0}^{t_1} [-1 + \sin^2(z_1 + v)] dy. \]

It can be easily shown that \( \det G(t_0,t_1;z,v) > q(t_0,t_1) \) for all \([z,v] \) where

\[ q(t_0,t_1) = \frac{1}{16} \left[ e^{2(t_1-t_0)} - 1 - 2(t_1-t_0) - 2(t_1-t_0)^2 - \frac{4}{3} (t_1-t_0)^3 \right] > 0 \]

if \( t_1 > t_0 \). Hence, by Theorem 1 the system is globally totally controllable.

However, the result of [1] cannot be applied here because in this case the non-linearity \( f(t,x,u) = [x_2 \sin^2(s_1 + u), x_1 \sin(x_1 + u)]' \) is not bounded in \( \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \).
IV. CONCLUSION

Sufficient conditions for global and local controllability of a class of nonlinear systems have been given. A natural question arises as to whether or not these results hold by the same fixed point arguments if the system matrices \((A,f)\) are parameterized as \(A(t,p(t),q(t))\) and \(f(t,p(t),q(t))\) where \([p,q] \in C_{nm}[t_0,t_1]\) so that for each given \([p,q]\) the parametrized linear time-varying system is controllable. The answer is negative because boundedness and continuity of the partial derivatives of relevant matrices are required in addition to the nonsingularity of the controllability matrix.

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References


Sufficient conditions for global and local controllability of the nonlinear system

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\]

as its perturbed system are given. These conditions extend some previous results through the removal of certain boundedness conditions involving the functions \((A,f)\) and their partial derivatives.