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**Analysis of the Displacement Field for  
Localized Disturbances in a Stratified  
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# Analysis of the Displacement Field for Localized Disturbances in a Stratified Fluid With Shear

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## ANALYSIS OF THE DISPLACEMENT FIELD FOR LOCALIZED DISTURBANCES IN A STRATIFIED FLUID WITH SHEAR

### INTRODUCTION

Many theoretical and experimental studies have been devoted to various aspects of the two-dimensional collapse of a mixed region in a stratified fluid. Bell and Dugan [1] and Merritt [2] describe work in this area up to about 1974. Interest in the phenomenon is due in large part to its relevance as a model for collapse of the turbulent wake of a self-propelled body moving through a stratified ocean.

The complexity of the phenomenon, which involves localized turbulent mixing in a stratified fluid and the generation of internal waves, has led many investigators [3-10] to study an initial-value problem involving a two-dimensional disturbance. Implicit in this simplification is the assumption that no mixing occurs following the initial time. Schooley and Stewart [11], in the first demonstration of the wake collapse phenomenon behind a moving self-propelled body, introduced the use of such initial-value problems in order to gain some understanding of the salient features of the collapse process.

We present here an analysis of an initial-value problem involving a localized disturbance in a stratified fluid which has a background shear. Although shear flows are generally present in the ocean, only the recent work of Hartman [6] incorporates shear in a mathematical model of collapse. Our objective is to demonstrate that certain features of the displacement field of fluid particles can be calculated directly from the initial data of the problem. The calculation is made with the Lagrangian viewpoint and is valid for finite-amplitude disturbances in an incompressible, viscous stratified fluid which is supposed nondiffusive.

In addition we consider the inverse question whereby inferences about the nature of the initial disturbance are drawn from data describing the shape of the initial perturbed region. The results obtained may prove useful to experimentalists as a means for estimating the degree of homogeneity of mixed regions.

The present results may be viewed as a generalization of earlier work [12] in which the collapse occurred in fluid which was at rest outside the localized disturbance. From a theoretical point of view both works give examples of the fact that the Lagrangian view can be used to obtain exact information in some rather complicated situations.

An elementary application of the results is made to simplify an argument of Stockhausen, Clark, and Kennedy [13] concerning zero net convergence above the region of initial disturbance.

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Manuscript submitted August 16, 1976.

## THE INITIAL-VALUE PROBLEM

## Case of Laminar Motion

The mathematical model we use is that of an incompressible, viscous, stratified fluid which is regarded as nondiffusive. The assumption that the fluid is nondiffusive means that those quantities affecting the density, such as temperature and salinity, are not altered by molecular diffusion. Consequently the density of each fluid particle remains unchanged during its motion. The model is widely used in the study of stratified fluids [14].

The Eulerian form of the governing equations is

$$\nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$\frac{d\rho}{dt} = 0, \quad (2)$$

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \rho \mathbf{g} + \nabla \cdot \overleftrightarrow{\mathbf{T}} \quad (3)$$

for all times  $t \geq 0$ . The fluid velocity is designated by  $\mathbf{u}$ , the density by  $\rho$ , the pressure by  $p$ , the constant gravitational field by  $\mathbf{g}$ , and the viscous stress tensor by  $\overleftrightarrow{\mathbf{T}}$ . The time derivative in Eqs. (2) and (3) is the material derivative.

For simplicity we assume an infinite fluid, with no free surface and no bottom boundary. Although the analysis to be carried out can be applied to many situations involving a free surface or bottom boundary, the discussion of all possible cases would be unnecessarily long and might conceal the simplicity of the actual considerations involved in the analysis.

We use two coordinate frames  $(x, y, z)$  and  $(X, Y, Z)$  that are coincident to describe the motion of the fluid. A time-dependent background shear motion described by the velocity field  $(U(y, t), 0, W(y, t))$  is permitted. Superposed on this background motion we suppose there to be an initial localized disturbance described by a local velocity disturbance and a local deformation of the isopycnals, both independent of  $z$ . Figure 1 is a sketch of the situation. We make the basic assumption that the background motion is stable with respect to the initial disturbance.

Our interest is in the motion of fluid particles  $\mathbf{x} = \mathbf{x}(X, t)$ , defined by

$$\mathbf{x} = \mathbf{X} + \mathbf{d}(X, t), \quad (4)$$

and our particular interest is in the displacement of fluid particles  $\mathbf{d}(X, t)$ . Since particles in any  $z = \text{constant}$  plane experience the same forces, the displacement field  $\mathbf{d}$  is independent of  $Z$ .

We assume that at  $t = 0$  the localized disturbance can be expressed as

$$\rho(X, Y, 0) = \rho_e(Y) + \delta\rho(X, Y) \quad (5)$$

and the initial velocity field as

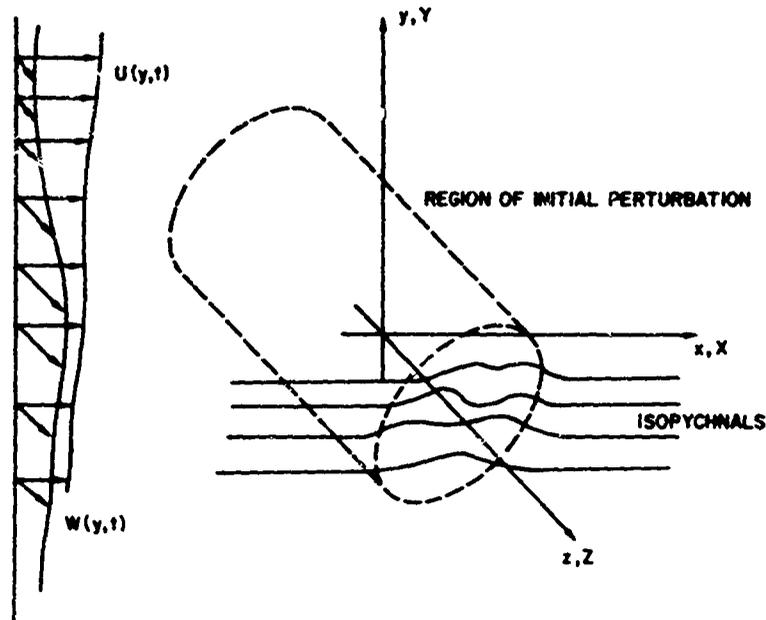


Fig. 1—Infinite fluid with time-dependent background shear flow and an initial localized disturbance.  $U(y, t)$  and  $W(y, t)$  are components of the shear flow.

$$u(X, Y, 0) = U(Y, 0) + \delta u(X, Y). \quad (6)$$

The finite-amplitude initial disturbance described by  $\delta\rho$  and  $\delta u$  is assumed to vanish as  $|X| \rightarrow \infty$ . The density  $\rho_e(Y)$  represents the stratification prevailing at large horizontal distances from the localized disturbance.

From the Eulerian equations we find that the background motion must satisfy

$$\rho_e \frac{\partial U}{\partial t} = \mu \frac{\partial^2 U}{\partial y^2} \quad (7)$$

and

$$\rho_e \frac{\partial W}{\partial t} = \mu \frac{\partial^2 W}{\partial y^2}, \quad (8)$$

where  $\mu$  is the mechanical viscosity of the fluid. Equations (7) and (8) state that the background motion is modified by diffusion. Since we shall deal with an infinite fluid, we regard the background motion as a specified solution of Eqs. (7) and (8).

This completes our description of the formal initial-value problem for laminar flows. Before proceeding with its study, we first make some remarks about a similar initial-value problem which arises in the case of localized regions of turbulent mixing produced by the motion of a self-propelled body through a stratified fluid.

**Remarks on the Disturbance Created by the Motion of a Self-Propelled Body**

We give a brief discussion of a mathematical model for the generation of internal waves by the turbulent wake of a self-propelled body. The model originated with the work of Schooley and Stewart [11] and has received much attention from theorists and experimentalists [1-13]. Our aim is to demonstrate that the governing equations for the mean disturbance (the ensemble average disturbance) bear a close resemblance to those for the laminar flow already discussed and that useful results follow from this fact.

When a self-propelled body moves rapidly through a stratified ocean environment, its turbulent wake mixes the fluid and initially increases in cross section through entrainment of fluid outside the wake. Eventually the intensity of the turbulence diminishes to such an extent that buoyancy forces the mixed fluid back toward an appropriate equilibrium level and the vertical dimension of the wake reduces in size. This phase of the wake behavior is referred to as the gravitational-collapse phase and is accompanied by the propagation of internal waves away from the wake region. A primary assumption of the present mathematical model is that significant mixing of the fluid no longer occurs once the gravitational collapse has begun.

Since the rate at which the moving body generates mixed fluid is usually large in comparison to the rate at which gravitational collapse occurs, the assumption is made in the model that the mean or ensemble average characteristics of the collapse process can be regarded as independent of distance along the track of the body.

We express the disturbance as

$$u = U(x, y, t) + u', \quad (9)$$

$$\rho = \rho_{av}(x, y, t) + \rho', \quad (10)$$

$$p = P(x, y, t) + p', \quad (11)$$

where  $\rho_{av}$ ,  $U$ ,  $P$  are ensemble average quantities. We use  $\langle \cdot \rangle$  to indicate an ensemble average; consequently the fluctuations satisfy

$$\langle u' \rangle = 0,$$

$$\langle \rho' \rangle = 0,$$

$$\langle p' \rangle = 0.$$

Time  $t$  is measured from the initiation of the collapse process.

The ensemble average quantities contain the mean background shear flow, the mean density anomaly due to prior turbulent mixing of the fluid, and the mean internal wave disturbance resulting from collapse of the density anomaly. The basic equations governing the evolution of the ensemble average disturbance are

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$$\nabla \cdot \mathbf{U} = 0, \quad (12)$$

$$\frac{d\rho_{av}}{dt} = 0, \quad (13)$$

$$\left\langle \rho \frac{d\mathbf{u}}{dt} \right\rangle = -\nabla P + \rho_{av} \mathbf{g} + \nabla \cdot \overline{\overline{\mathbf{T}}}. \quad (14)$$

The absence from the right side of Eq. (13) of terms involving the fluctuating quantities reflects our assumption of  $\ominus$  mixing during the collapse phase.

We can interpret Eqs. (12) through (14) as describing the motion of particles which move about in accord with the ensemble average motion  $\mathbf{U}(x, y, t)$ . Particle paths for the ensemble average motion are calculated from

$$\frac{dx}{dt} = \mathbf{U}.$$

Equations (12) and (13) correspond to Eqs. (1) and (2) for the laminar motion discussed earlier.

It shall become clear that our results for net displacements in the laminar case can also be deduced for the ensemble average motion just described. The reason for this is that the results do not depend on the explicit form of the momentum equation, only on its qualitative features. The features required are that localized disturbances eventually decay, thus allowing the disturbed fluid to return to an equilibrium state specified by the background shear and density field.

## THE DIRECT PROBLEM

### Preliminaries

We now return to the initial-value problem for the laminar flow outlined earlier. In the direct problem the initial configuration is assumed known. The aim here is to demonstrate how certain features of the motion can be deduced without having to solve for the complex flow field. The Lagrangian view is used.

On physical grounds it is clear that as time progresses, the energy contained in the localized disturbance will be radiated into the surrounding fluid and ultimately dissipated by viscous forces. Thus eventually the motion will again be horizontal in accord with Eqs. (7) and (8), with  $\rho_e(y)$  describing the stratification throughout the fluid. At sufficiently large times the velocity of particles can be written in the form

$$\frac{dx}{dt} = U(y^*, t), \quad (15)$$

$$\frac{dy}{dt} = 0, \quad (16)$$

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$$\frac{dz}{dt} = W(y^*, t), \quad (17)$$

where

$$y^* = y^*(X, Y) \quad (18)$$

specifies the final vertical level of the fluid particle initially at  $X, Y$ .

The forms of Eqs. (15) through (17) suggest that we write the particle motion in the forms

$$x = X + \xi(X, Y, t) + \int_0^t U(y^*, t) dt, \quad (19)$$

$$y = Y + \eta(X, Y, t), \quad (20)$$

$$z = Z + \zeta(X, Y, t) + \int_0^t W(y^*, t) dt. \quad (21)$$

If there were no initial disturbance to the fluid, the particle motion would be specified by Eqs. (19) through (21) with the  $\xi$ ,  $\eta$ , and  $\zeta$  terms absent. Consequently,  $\xi$ ,  $\eta$ , and  $\zeta$  specify the full contribution to the displacement field arising as a consequence of the gravitational collapse of the initial localized disturbance

From Eqs. (15) through (17) we conclude that at large times (large in comparison to the effective collapse time, which is controlled by the Brunt-Väisälä period) we must have

$$\lim_{t \rightarrow \infty} \xi(X, Y, t) = \xi^*(X, Y), \quad (22)$$

$$\lim_{t \rightarrow \infty} \eta(X, Y, t) = \eta^*(X, Y), \quad (23)$$

$$\lim_{t \rightarrow \infty} \zeta(X, Y, t) = \zeta^*(X, Y). \quad (24)$$

We shall demonstrate in the following that it is possible to compute  $\xi^*$  and  $\eta^*$  directly from the initial data of the problem.

We are making some formal assumptions here. We assume that the initial-value problem possesses a well-behaved solution for  $0 \leq t < \infty$ . (A rigorous proof does not exist.) This requires, in part, that the background flow must be stable, and, from stability theory, this places requirements on the Richardson number, as discussed by Turner [15, pp. 100 ff].

We must also have the background flow be stable with respect to the localized disturbance of finite amplitude which we prescribe through the initial conditions. The collapse

may, in some cases, produce internal waves which interact with the background shear in such a fashion as to lead to amplification of the waves (critical layer behavior) and to the possibility of wave breakdown [15, p. 124]. Such behavior seems to imply that our initial-value problem does not always possess a well-behaved solution; however general criteria do not exist. In a recent paper Hartman [16] treats a linearized initial-value problem for a two-dimensional, unbounded, exponentially stratified, plane Couette flow. No difficulties are encountered because of critical layer behavior.

### Net Vertical Displacement

From Eq. (2) the density of a particle at time  $t$  must be the same as the density of the particle initially. This yields

$$\rho[x(t), y(t), z(t), t] = \rho_e(Y) + \delta\rho(X, Y). \quad (25)$$

But as  $t \rightarrow \infty$ ,  $y(t) \rightarrow y^*(X, Y)$ , and, since the ultimate density field is again described by  $\rho_e$ , Eq. (25) becomes

$$\rho_e(y^*) = \rho_e(Y + \eta^*) = \rho_e(Y) + \delta\rho(X, Y). \quad (26)$$

Equation (26) is the determining condition for the net vertical displacement  $\eta^*(X, Y)$  in terms of the initial data.

Although this implicit condition for  $\eta^*$  can be solved explicitly for many stratifications of interest, we record here the solution

$$\eta^*(X, Y) = \left( \frac{d\rho_e}{dY} \right)^{-1} \delta\rho(X, Y), \quad (27)$$

which is valid for those cases in which  $\rho_e(Y)$  is linear over the vertical extent of the initial disturbance.

### Net Horizontal Displacement

For the determination of  $\xi^*(X, Y)$  we use the Lagrangian condition

$$\frac{\partial(x, y, z)}{\partial(X, Y, Z)} = 1, \quad (28)$$

which is a consequence of incompressibility and conservation of mass. The notation indicates the Jacobian of the transformation expressed by Eqs. (19) through (21). Since the  $x$  and  $y$  coordinates of a particle do not depend on  $Z$ , Eq. (28) can be simplified to

$$\begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} & 0 \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} & 0 \\ \frac{\partial z}{\partial X} & \frac{\partial z}{\partial Y} & 1 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial X} & \frac{\partial x}{\partial Y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial Y} \end{vmatrix} = 1. \quad (29)$$

For the class of problems under consideration Eq. (29) places no restriction on  $\zeta$  and our methods will yield no information about this component of the displacement field.

Substitution of the expressions in Eqs. (19) through (21) into Eq. (29) yields, when written out,

$$\left(1 + \frac{\partial \xi}{\partial X} + \frac{\partial}{\partial X} \int_0^t U(y^*, t) dt\right) \left(1 + \frac{\partial \eta}{\partial Y}\right) - \left(\frac{\partial \xi}{\partial Y} + \frac{\partial}{\partial Y} \int_0^t U(y^*, t) dt\right) \frac{\partial \eta}{\partial X} = 1.$$

It is preferable to write this result in the form

$$\left(1 + \frac{\partial \xi}{\partial X}\right) \left(1 + \frac{\partial \eta}{\partial Y}\right) - \frac{\partial \xi}{\partial Y} \frac{\partial \eta}{\partial X} + \left[\frac{\partial}{\partial X} \int_0^t U(y^*, t) dt\right] \frac{\partial y}{\partial Y} - \left[\frac{\partial}{\partial Y} \int_0^t U(y^*, t) dt\right] \frac{\partial y}{\partial X} = 1,$$

and carrying out the differentiation of the integrals provides

$$\left(1 + \frac{\partial \xi}{\partial X}\right) \left(1 + \frac{\partial \eta}{\partial Y}\right) - \frac{\partial \xi}{\partial Y} \frac{\partial \eta}{\partial X} + \left[\int_0^t \frac{\partial U(y^*, t)}{\partial y^*} dt\right] \left(\frac{\partial y^*}{\partial X} \frac{\partial y}{\partial Y} - \frac{\partial y^*}{\partial Y} \frac{\partial y}{\partial X}\right) = 1. \quad (30)$$

Because

$$\lim_{t \rightarrow \infty} \left(\frac{\partial y^*}{\partial X} \frac{\partial y}{\partial Y} - \frac{\partial y^*}{\partial Y} \frac{\partial y}{\partial X}\right) = \frac{\partial y^*}{\partial X} \frac{\partial y^*}{\partial Y} - \frac{\partial y^*}{\partial Y} \frac{\partial y^*}{\partial X} = 0,$$

the limiting form of Eq. (30) as  $t \rightarrow \infty$  becomes

$$\left(1 + \frac{\partial \eta^*}{\partial Y}\right) \frac{\partial \xi^*}{\partial X} - \frac{\partial \eta^*}{\partial X} \frac{\partial \xi^*}{\partial Y} = -\frac{\partial \eta^*}{\partial Y}. \quad (31)$$

We have already seen how  $\eta^*$  can be determined from the initial data. Equation (31) then represents a linear first-order partial-differential equation for the determination of  $\xi^*(X, Y)$ .

### Comparison to the Case of No Shear

The equations for the determination of  $\xi^*(X, Y)$  and  $\eta^*(X, Y)$  are precisely the same equations as found earlier [12] in the case of no background motion. In the present case  $\xi^*$  has a different interpretation, since the shear motion itself produces a contribution to the displacement field. From Ref. 12 we have at once that  $\xi^*$  can be solved for by using the method of characteristics, with the characteristic equations given by

$$\frac{d\xi^*}{d\sigma} = -\frac{\partial\eta^*}{\partial Y}, \quad (32)$$

$$\frac{dX}{d\sigma} = 1 + \frac{\partial\eta^*}{\partial Y}, \quad (33)$$

$$\frac{dY}{d\sigma} = -\frac{\partial\eta^*}{\partial X}. \quad (34)$$

The parameter  $\sigma$  identifies points on a characteristic. We further have that the characteristics in the  $XY$  plane are those curves specifying the initial forms of the isopycnals.

In the earlier case of no shear [12] we confined ourselves to symmetric disturbances. We then had  $\xi^*(C, Y) = 0$  as initial data for the solution of Eqs. (32) through (34). In the presence of shear however we no longer have such a condition even for symmetric disturbances. Consequently the initial data available allow the determination of only the relative quantity  $\xi^*(X, Y) - \xi_0^*$ , where  $\xi_0^*$  is the unknown value of  $\xi^*(X, Y)$  for some point on the same isopychnal that the point  $(X, Y)$  is on.

### A Simple Application to the Net Convergence of Fluid Particles

Stockhausen, Clark, and Kennedy [13] describe a laboratory experiment in which a self-propelled body moved through a stratified fluid. They give an argument [13, pp. 72 and 78] that a net convergence between two fluid particles can occur only when the particles are at a depth level where mixing has taken place.

We give here a rigorous proof that the statements of Stockhausen, Clark, and Kennedy apply to the initial-value problem we have been examining. Consider two fluid particles  $P$  and  $Q$  initially on an undisturbed isopychnal as shown in Fig. 2. We have  $\delta\rho(X, Y) = 0$  for all points on the isopychnal, and Eq. (26) provides the result that  $\eta^*(X, Y) = 0$  for all points on the isopychnal. Using this result in Eqs. (32) through (34), we get  $d\xi^*/dX = 0$ , and this can be integrated to obtain  $\xi^*(P) = \xi^*(Q)$ .

Since  $y^*(X, Y) = Y$  for all points on the isopychnal, Eqs. (19) through (21) provide

$$x(P, t) = X_P + \xi(P, t) + \int_0^t U(Y_P, t) dt$$

and, since  $Y_Q = Y_P$ ,

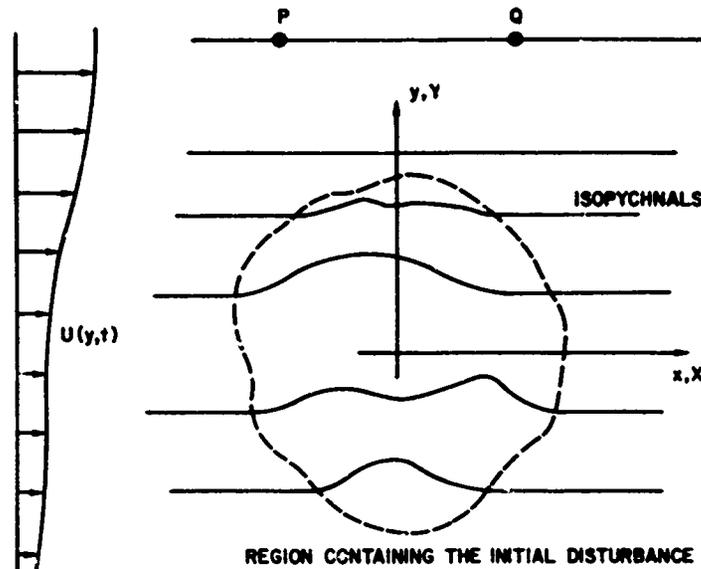


Fig. 2—State of disturbance at  $t = 0$  in the  $xy$  plane, with the dashed lines indicating the region containing the initial disturbance.  $P$  and  $Q$  identify two fluid particles lying initially on an undisturbed isopychnal. The component of the background shear flow normal to the  $xy$  plane is not drawn.

$$x(Q, t) = X_Q + \xi(Q, t) + \int_0^t U(Y_p, t) dt .$$

Subtracting the two expressions, we get

$$x(P, t) - x(Q, t) = X_P - X_Q + \xi(P, t) - \xi(Q, t) .$$

The limit of this as  $t \rightarrow \infty$  provides the ultimate separation of the particles  $P$  and  $Q$  in the  $x$  direction as

$$\begin{aligned} x^*(P) - x^*(Q) &= X_P - X_Q + \xi^*(P) - \xi^*(Q) \\ &= X_P - X_Q , \end{aligned}$$

which is precisely equal to their initial separation.

We conclude that no net convergence occurs between particles such as  $P$  and  $Q$  located on an isopychnal outside the region of initial disturbance. A net convergence can occur only if  $P$  and  $Q$  are at the level of the initial disturbance region. This is perhaps an unexpected result. One's initial expectation is that as the initial disturbance resolves itself into a system of internal waves which propagate through the body of the fluid, the end result would be particles whose locations have been shifted about relative to one another. The severe constraint conferred on the fluid by stratification prevents such relative shifting.

## REMARKS ON THE INVERSE PROBLEM

A matter of great practical importance is the nature of the disturbance in the fluid at the end of a brief period of localized, turbulent mixing. The complexity of the phenomena makes such information difficult to obtain by direct measurement.

We consider, within the context of our initial-value problem, an indirect method for obtaining information about the initial deformation of the isopycnals. In effect we consider an inverse problem in which it is assumed that the background stratification is known as well as the shapes of the disturbed region at  $t = 0$  and at some time  $t$  large in comparison to the time scale of collapse. An earlier report [12] dealt with this question in the case of a zero shear environment. Although the earlier results can be generalized to the situation of collapse in a shear environment, we prefer to give instead a new result which seems to be more generally useful.

Consider the initial configuration of the fluid as shown in Fig. 3a. By hypothesis we know the boundary  $C$  within which the initial disturbance is confined. A typical isopycnal of density  $\rho_e$  which passes through  $C$  has a form as shown in Fig. 3a. Outside  $C$  the isopycnal is horizontal at its equilibrium level, while inside  $C$  it is deformed. Let  $A(\rho_e, t)$  denote the area enclosed between the material particles initially composing the isopycnal  $\rho_e$  and the particles which initially form the boundary  $C$  lying above the isopycnal (Fig. 3). As a consequence of the incompressibility of the fluid this area remains constant throughout the motion, so that

$$A(\rho_e, t) = A(\rho_e, 0)$$

for all time.

We now prove that the area  $A(\rho_e, t)$  is always bounded by  $C$  and the isopycnal  $\rho_e$ . From the nondiffusive nature of the fluid we know that the fluid particles retain their density throughout the motion. Consequently isopycnals are material curves, as is the curve  $C$ . We conclude that  $A(\rho_e, t)$  is always bounded by the particles comprising  $C$  and the isopycnal  $\rho_e$ .

By hypothesis we know the initial and "final" configurations of the material contour  $C$ . A final configuration of the fluid is one in which the isopycnals must be horizontal and at the same level as given by the known initial data. Therefore we can evaluate  $A(\rho_e, \infty)$  for any isopycnal passing through  $C$  and, by the above argument, we then have  $A(\rho_e, 0)$  for each isopycnal passing through the initial contour  $C$ .

From Fig. 3a it is clear that knowledge of  $A(\rho_e, 0)$  can be used to calculate the area enclosed between the position of the deformed isopycnal  $\rho_e$  and its final equilibrium level. This information immediately provides the average level of the isopycnal  $\rho_e$  within the initial region.

We thus have a means for using the initial and final shapes of the disturbed fluid to estimate average characteristics of the initial disturbance. This method may prove to be of use in estimating such quantities as the degree of homogeneity of localized disturbances.

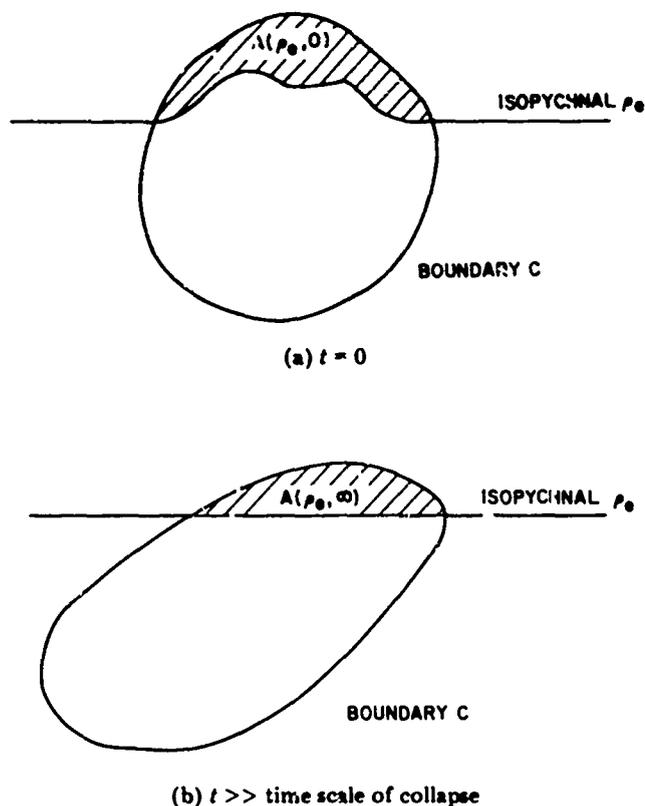


Fig. 3—Initial and final configurations of the region containing the initial disturbance.  $C$  denotes the boundary of the region in the  $xy$  plane.  $A(\rho_e, t)$  is the area at time  $t$  enclosed between the isopycnal of density  $\rho_e$  and the boundary  $C$ .

## CONCLUSION

We considered an initial-value problem involving a two-dimensional localized disturbance in an incompressible, viscous, stratified fluid. A time-dependent background shear flow was incorporated in the analysis. Although the problem was considered within the framework of laminar flow theory, we pointed out that similar results can be obtained for the ensemble-average motion resulting from the collapse of a localized region of turbulent mixed fluid.

The results are perhaps best thought of as an extension of earlier results [1,2] obtained for the case of no background motion. Again the Lagrangian view has been used to advantage.

It was demonstrated for the general class of initial-value problems considered that certain components of the net displacement field could be calculated directly from the initial data. This calculation is effectively exact and holds for finite-amplitude disturbances. A simple application was made to the net convergence of fluid particles. A potential application of the results is in the partial verification of computer algorithms devised for the solution of initial-value problems of the type considered.

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We also examined the inverse problem, which is of particular interest in the experimental investigation of localized mixed regions in stratified fluids. The intent of the inverse problem is to use limited knowledge of the net displacement field, such as might be obtained by simple measurements, to infer characteristics of the initial disturbance. We demonstrated that knowledge of the initial and final material configurations of the shape of the region containing the initial disturbance provides a means for estimating mean characteristics of the initial isopycnal deformation. Thus our study of the mathematical problem suggests that a certain correlation must exist between the shape of the mixed region and the initial disturbance.

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