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STABILITY IN NEUTRAL EQUATIONS

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STABILITY IN NEUTRAL EQUATIONS

by

Jack K. Hale and Pedro Martinez-Amores

Abstract

↙ Coupled systems of differential-difference and ordinary difference equations occur in various applications including the theory of transmission lines [1] and gas dynamics [2]. Stability of linear systems has been discussed by Brayton [1] using Laplace transform and the problem of absolute stability by Rasvan [12] using the frequency domain method of Popov.

In this paper, the same problems are discussed by the following method. By differentiating the difference equation, one obtains a system of neutral differential-difference equations. The desired solutions of the original problem are obtained by restricting the initial data to lie on certain manifolds in the space of all initial data. In this way, this class of problems can be treated in a natural manner using the known methods of neutral equations. Generalizations to arbitrary functional differential equations is also immediate when this approach is employed.

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1. Notation and Background

Let $R = (-\infty, \infty)$ and let R^n be an n -dimensional linear vector space with norm $|\cdot|$. For $r \geq 0$, let $C = C([-r, 0], R^n)$ be the space of continuous functions mapping $[-r, 0]$ into R^n with the topology of uniform convergence. The norm in C will also be designated by $|\phi| = \sup_{-r \leq \theta \leq 0} |\phi(\theta)|$, $\phi \in C$. Suppose D, L are bounded linear operators from C to R^n ,

$$(1.1) \quad \begin{aligned} D(\phi) &= H\phi(0) - \int_{-\gamma}^0 [d\mu(\theta)]\phi(\theta) \\ L(\phi) &= \int_{-\gamma}^0 [d\eta(\theta)]\phi(\theta) \end{aligned}$$

where H is an $n \times n$ matrix, $\det H \neq 0$, μ, η are $n \times n$ matrix functions of bounded variation on $[-r, 0]$ with μ nonatomic at zero. This latter hypothesis is equivalent to the existence of a continuous, nondecreasing function $\gamma: [0, r] \rightarrow R$ such that $\gamma(0) = 0$ and

$$\left| \int_{-\varepsilon}^0 [d\mu(\theta)]\phi(\theta) \right| \leq \gamma(\varepsilon) |\phi|$$

for $\varepsilon \in [0, r]$, $\phi \in C$.

If x is a function from $[\sigma-r, \infty)$ to R^n , let x_t , $t \in [0, \infty)$, be the function from $[-r, 0]$ to R^n defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-r, 0]$. An autonomous linear homogeneous neutral functional differential equation (NFDE) is defined to be

$$(1.2) \quad \frac{d}{dt} D(x_t) = L(x_t).$$

A solution $x = x(\phi)$ through $\phi \in C$ at $t = 0$ is a continuous function taking $[-r, A)$, $A > 0$, into R^n such that $x_0 = \phi$, $D(x_t)$ is continuously differentiable on $[0, A)$ and equation (1.2) is satisfied on this interval. It follows from Hale and Mayer [9] that a solution through ϕ exists on $[-r, \infty)$, is unique and depends continuously in ϕ .

If $T(t): C \rightarrow C$, $t \geq 0$, is defined by $T(t)\phi = x_t(\phi)$, then $T(t)$, $t \geq 0$ is a strongly continuous semigroup with infinitesimal generator $A: \mathcal{D}(A) \rightarrow C$, $A\phi(\theta) = \dot{\phi}(\theta)$, $-r \leq \theta \leq 0$, and

$$\mathcal{D}(A) = \{\phi \in C: \dot{\phi} \in C, D\dot{\phi} = L\phi\}.$$

The spectrum $\sigma(A)$ of A consists of all λ which satisfy the characteristic equation

$$(1.3) \quad \det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda D(e^{\lambda \cdot} I) - L(e^{\lambda \cdot} I) \\ = \lambda H - \lambda \int_{-r}^0 e^{\lambda \theta} d\mu(\theta) - \int_{-r}^0 e^{\lambda \theta} d\eta(\theta).$$

The fundamental matrix solution $X(t)$ of (1.2) is defined to be the $n \times n$ matrix solution of the equation

$$(1.4) \quad D(X_t) = I + \int_0^t L(X_s) ds, \quad t \geq 0$$

$$X_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0 \\ H^{-1}, & \theta = 0 \end{cases}$$

The following results of Henry [11] will be fundamental to our investigation.

Lemma 1.1. If $\operatorname{Re} \lambda \leq \delta$ for all λ satisfying (1.3), then, for any $\epsilon > 0$, there is a $K = K(\epsilon)$ such that

$$(1.5) \quad |T(t)|, |X(t)|, |\dot{X}(t)| \leq Ke^{(\delta+\epsilon)t}, \text{ a.e. for } t \geq 0.$$

Definition 1.1. The operator D is said to be stable if there is a $\nu > 0$ such that all roots of the equation

$$(1.6) \quad \det D(e^{\lambda \cdot} I) = 0$$

satisfy $\operatorname{Re} \lambda \leq -\nu$.

From the results of Cruz and Hale [4] and Henry [11], an operator D is stable if and only if the zero solution of the functional equation

$$(1.7) \quad D(Y_t) = 0, \quad t \geq 0$$

is uniformly asymptotically stable; that is, there are constants $K, \alpha > 0$ such that

$$(1.8) \quad |Y_t(\phi)| \leq Ke^{-\alpha t} |\phi|, \quad t \geq 0, \phi \in C, D\phi = 0.$$

If $D\phi = H\phi(0) - J\phi(-r)$, then D is stable if the roots of the polynomial equation

$$\det [H - \rho J] = 0$$

satisfy $|\rho| < 1$.

An important property of equation (1.2) when D is stable is the following (see [5]): If D is stable, there is a constant $a_D < 0$ such that for any $a > a_D$, there are only a finite number of roots λ of (1.3) with $\operatorname{Re} \lambda > a$.

If $F, G: R \rightarrow R^n$ are continuous, a nonhomogeneous linear NFDE is defined as

$$(1.9) \quad \frac{d}{dt} [D(x_t) - G(t)] = L(x_t) + F(t).$$

A solution through ϕ at $t = \sigma$ is defined as before and is known to exist on $[\sigma - r, \infty)$.

The variation of constants formula for (1.9) (see [8]) states that the solution of (1.9) through (σ, ϕ) is given by

$$(1.10) \quad x(t) = T(t-\sigma)\phi(0) + \int_{\sigma}^t X(t-s)F(s)ds \\ - \int_{\sigma}^{t^+} [d_s X(t-s)][G(s) - G(\sigma)]$$

for $t \geq \sigma$, where X is the fundamental matrix solution.

Another convenient equivalent form for equation (1.10), is the following:

$$(1.11) \quad x(t) - X(0)G(t) = T(t-\sigma)\phi(0) - X(t-\sigma)G(\sigma) \\ + \int_{\sigma}^t X(t-s)F(s)ds - \int_{\sigma}^t [d_s X(t-s)]G(s), \quad t \geq \sigma.$$

Let us make a few other observations on the variation of constants formula which suggest changes of variables which will be useful in later sections. Let PC be the space of functions taking $[-r,0]$ into R^n which are uniformly continuous on $[-r,0)$ and may be discontinuous at zero. With the matrix X_0 as defined before, it is clear that

$$PC = C + \langle X_0 \rangle$$

where $\langle X_0 \rangle = \text{span} \{X_0\}$; that is, any $\psi \in PC$ is given as

$$\psi = \phi + X_0 b$$

where $\phi \in C$, $b \in R^n$. We make PC a normed vector space by defining the norm $|\psi| = \sup_{-r \leq \theta \leq 0} |\psi(\theta)|$.

Let us define $x_t(\psi) = T(t)\psi$ where $\psi \in PC$ and $x(\psi)$ is the solution of (1.2) through ψ . The operator $T(t): PC \rightarrow$ (functions on $[-r,0]$) is linear, but $T(t)$ does not take $PC \rightarrow PC$. The operator $T(t)$ is an extension of the original semigroup $T(t)$ on C . If we use this notation, then the variation of constants formulas (1.10), (1.11) can be written as

$$(1.12) \quad x_t = T(t-\sigma)\phi + \int_{\sigma}^t T(t-s)X_0F(s)ds \\ + \int_0^{t^+} [d_s T(t-s)X_0] [G(s) - G(\sigma)]$$

$$(1.13) \quad x_t - X_0G(t) = T(t-\sigma) [\phi - X_0G(\sigma)] + \int_{\sigma}^t T(t-s)X_0F(s)ds \\ - \int_0^t [d_s T(t-s)X_0]G(s)$$

for $t \geq \sigma$, $\phi \in C$. As usual in the theory of functional differential equations, these integrals are in R^n ; that is, each integral is evaluated at each $\theta \in [-r,0]$ as an integral in R^n .

Formula (1.13) certainly suggests the change of variables

$$(1.14) \quad x_t - X_0G(t) = z_t, \quad \phi - X_0G(\sigma) = \psi$$

from $C \rightarrow PC$. If this is done, equation (1.13) becomes

$$(1.15) \quad z_t = T(t-\sigma)\psi + \int_{\sigma}^t T(t-s)X_0F(s)ds - \int_{\sigma}^t [d_s T(t-s)X_0]G(s)$$

a formula much simpler than either (1.12) or (1.13). This remark will play an important role in the subsequent discussion.

2. Stability in Nonlinear Equations

In this section, we give some elementary results on the stability of nonlinear equations in order to show the previous transformation from C to PC can be of assistance. To keep the notation at a minimum, the most general results are not given.

Suppose $F: C \rightarrow R^n$, $G: C \rightarrow R^n$ are given continuous functions and $G(\phi)$ depends only upon values of $\phi(\theta)$ for $\theta < 0$; that is, for any $a \in R^n$ and any sequence $\phi_n \in C$, $\phi_n(0) = a$, $n = 1, 2, \dots$, which converges to ϕ uniformly on compact subsets of $[-r, 0)$, the limit of $G(\phi_n)$ exists as $n \rightarrow \infty$ and $\lim_{n \rightarrow \infty} G(\phi_n) = G(\phi)$. The relation

$$(2.1) \quad \frac{d}{dt} [D(x_t) - G(x_t)] = L(x_t) + F(x_t)$$

with D, L as in Section 1 defines a neutral functional differential equation. Existence of solutions for initial data in C follows from [6], [9].

The variation of constants formula (1.13) for this equation (2.1) is

$$(2.2) \quad x_t - X_0 G(x_t) = T(t) [\phi - X_0 G(\phi)] \\ + \int_0^t T(t-s) X_0 F(x_s) ds - \int_0^t [d_s T(t-s) X_0] G(x_s)$$

for $t \geq 0$, $\phi \in C$.

Consider the map

$$h: C \rightarrow C + \langle X_0 \rangle$$

given by

$$h(\phi)(\theta) = \begin{cases} \phi(\theta) & \theta < 0 \\ \phi(0) - H^{-1}G(\phi), & \theta = 0 \end{cases}$$

Since $G(\phi)$ does not depend on $\phi(0)$, the mapping h has a continuous inverse, that is, h is a homeomorphism, [8]

If $z_t = h(x_t)$, $\psi \in h(\phi)$, equation (2.2) becomes

$$(2.3) \quad z_t = T(t)\psi + \int_0^t T(t-s) X_0 F(h^{-1}(z_s)) ds \\ - \int_0^t [d_s T(t-s) X_0] G(h^{-1}(z_s)).$$

We are now in a position to prove the following theorem.

Theorem 2.1. Suppose $F(0) = 0$, $G(0) = 0$ and the first derivatives $DF(\phi)$, $DG(\phi)$ are continuous and vanish at $\phi = 0$.

If the linear equation (1.2) is uniformly asymptotically stable, then the system (2.1) is uniformly exponentially asymptotically stable.

Proof: From the hypothesis on G , the mapping $h: C \rightarrow C + \langle X_0 \rangle$ is a homeomorphism in a fixed neighborhood of $\phi = 0 \in C$, $\psi = 0 \in C + \langle X_0 \rangle$. Furthermore, there are constants $k_1, k_2 > 0$ such that in this neighborhood $\psi = h(\phi)$ implies $|\psi| \leq k_1 |\phi|$, $|\phi| \leq k_2 |\psi|$. Applying Lemma 1.1 and the hypotheses on F, G , we have there a $\delta > 0$ (as small as desired) such that z in (2.3) satisfies

$$|z_t| \leq Ke^{-\alpha t} |\psi| + \int_0^t Ke^{-\alpha(t-s)} \delta k_2 |z_s| ds$$

as long as $|z_s| < \epsilon(\delta)$. Applying Gronwall's inequality to $|z_t|e^{\alpha t}$, we obtain

$$\frac{1}{k_2} |x_t| \leq |z_t| \leq Ke^{-(\alpha - Kk_2\delta)t} |\psi| \leq Kk_1 e^{-(\alpha - Kk_2\delta)t} |\phi|$$

as long as $|z_t| < \epsilon(\delta)$. Since this clearly can be assured for all $t \geq 0$ if $|\phi|$ is sufficiently small, we obtain the result stated in the theorem.

As one sees from the above proof, the transformation from $C \rightarrow C + \langle X_0 \rangle$ reduces the discussion to an argument very similar to the one for ordinary differential equations. One could easily generalize the above results to obtain the more general stability

properties given by Hale and Izé [10].

Another interesting remark about the above transformation concerns the manner in which difference equations are included in neutral equations. Suppose $F = 0$, $L = 0$ in (2.1). Then the equation is equivalent to the functional equation (no derivatives)

$$(2.4) \quad D(x_t) - G(x_t) = D(\phi) - G(\phi), \quad t \geq 0$$

with $x_0 = \phi$. The usual difference equations are homogeneous,

$$(2.5) \quad D(x_t) - G(x_t) = 0, \quad t \geq 0, \quad x_0 = \phi.$$

That is, the initial function ϕ satisfies

$$(2.6) \quad D(\phi) - G(\phi) = 0.$$

It is a well known fact that if the zero solution of the linear homogeneous equation

$$(2.7) \quad D(x_t) = 0$$

is uniformly asymptotically stable (i.e. D is stable) then the nonhomogeneous equation (2.5) satisfies the same properties if G satisfies the conditions of Theorem 2.1. On the other hand, Theorem 2.1 does not imply the result since the homogeneous linear equation

$$\frac{d}{dt} D(x_t) = 0$$

is not uniformly asymptotically stable as a consequence of the fact $D(\phi) \neq 0$ for all $\phi \in C$. The question is the following: How can we use the theory of neutral equations to obtain the above stability theorem for the functional equation (2.5)?

Let us now show how the transformation $h: C + C + \langle X_0 \rangle$ solves this problem. Let

$$(2.8) \quad PC_0 = \{\psi \in C + \langle X_0 \rangle : D(\psi) = 0\}.$$

Using Laplace transform and the same type of arguments as in Henry [11], one can prove the following

Lemma 2.1. If D is stable, then there are positive constants K, α such that if $x(\psi)$ is the solution of

$$D(x_t) = 0, \quad t \geq 0, \quad x_0 = \psi \in C + \langle X_0 \rangle$$

then

$$|x(\psi)(t)| \leq Ke^{-\alpha t} |\psi| \quad t \geq 0.$$

Now consider the equation

$$\frac{d}{dt} [D(x_t) - G(x_t)] = 0.$$

The variation of constants formula implies

$$x_t - X_0 G(x_t) = T(t) [\phi - X_0 G(\phi)] - \int_0^t [d_s T(t-s) X_0] G(x_s).$$

If we let $z_t = h(x_t)$, $\psi = h(\phi)$, then, a direct evaluation yields

$$D(\psi) = D[\phi - X_0 G(\phi)] = D(\phi) - G(\phi).$$

If we assume ϕ satisfies (2.6), then $D(\psi) = 0$ and $|T(t)\psi| \leq K|\psi|e^{-\alpha t}$ by Lemma 2.1. The kernel in the integral above also has an exponential bound of the same type since $T(t)X_0$ satisfies $D(T(t)X_0) = I$ and we are only interested in the variation of $T(t)X_0$. Consequently, the stability results for equation (2.5) is easily obtained exactly as in the proof of Theorem 2.1 after we have made the above elementary observations about PC_0 .

These observations about functional equations are the motivation for the discussion of the mixed differential and difference equations of the next section.

3. A Special Equation

In this section, we consider the system

$$(3.1) \quad \begin{aligned} \text{a) } \dot{x}(t) &= Ax(t) + By(t-r) \\ \text{b) } y(t) - E'x(t) - Jy(t-r) &= 0 \end{aligned}$$

where x, y are k, m vectors, respectively, all matrices are constants, E' is the transpose of E . For any $a \in R^k$, $\phi \in C$, one can define a solution of (3.1) with initial value $x(0) = a$, $y_0 = \phi$. If we define $C = C([-r, 0], R^m)$,

$$(3.2) \quad D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} : R^k \times C \rightarrow R^{k \times m}$$

$$D_1(a, \phi) = a$$

$$D_2(a, \phi) = \phi(0) - E'a - J\phi(-r)$$

and

$$(3.3) \quad L = \begin{pmatrix} L_1 \\ 0 \end{pmatrix} : R^k \times C \rightarrow R^{k \times m}$$

$$L_1(a, \phi) = Aa + B\phi(-r)$$

then equation (3.1) is a special case of the NFDE

$$(3.4) \quad \frac{d}{dt} D(x(t), y_t) = L(x(t), y_t)$$

and one obtains the equation (3.1) by requiring that

$$(3.5) \quad D_2(a, \phi) = 0.$$

Observe that $D(a, \phi) = H\bar{\phi}(0) - M\bar{\phi}(-r)$ where

$$\bar{\phi} = \begin{pmatrix} a \\ \phi \end{pmatrix}, \quad H = \begin{bmatrix} I & 0 \\ -E' & I \end{bmatrix}, \quad M = \begin{bmatrix} 0 & 0 \\ 0 & J \end{bmatrix}.$$

Hence, if we assume the eigenvalues of the matrix J have moduli less than 1, then D is stable.

The characteristic equation of (3.1) is

$$\det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \begin{bmatrix} \lambda I - A & -B e^{-\lambda r} \\ -E' & I - J e^{-\lambda r} \end{bmatrix}$$

Equation (3.4) generates a semigroup $T(t)$ on $R^k \times C$. If we define

$$(R^k \times C)_0 = \{(a, \phi) \in R^k \times C: D_2(a, \phi) = 0\}$$

then $(R^k \times C)_0$ can be considered as a Banach space. Furthermore, for any $(a, \phi) \in (R^k \times C)_0$, the solution of (3.4) through (a, ϕ) will be in $(R^k \times C)_0$ since it corresponds to the solution of (3.1) through (a, ϕ) . Consequently,

$$T_0(t) \stackrel{\text{def}}{=} T(t) \Big|_{(R^k \times C)_0} : (R^k \times C)_0 \rightarrow (R^k \times C)_0$$

and is a strongly continuous semigroup. The infinitesimal generator \mathcal{A}_0 of $T_0(t)$ is $\mathcal{A}_0 = \mathcal{A} \Big|_{(R^k \times C)_0}$ where \mathcal{A} is the infinitesimal generator of $T(t)$. One can easily show that

$$(3.6) \quad \sigma(\mathcal{A}_0) = \{\lambda \in \mathbb{C}: \det \Delta(\lambda) = 0\}$$

The fundamental matrix solution $X(t)$ of (3.4) is defined by

$$X_0(\theta) = \begin{bmatrix} I & 0 \\ E' & I \end{bmatrix} = H^{-1}, \quad \theta = 0, \quad X_0(\theta) = 0, \quad -r \leq \theta < 0.$$

If

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix}$$

where X_{11} is a $k \times k$ matrix, etc, then X must be a solution of (3.4) with the initial data specified above. Therefore, the matrices X_{ij} must automatically satisfy

$$(3.7) \quad D_2(X_{11}(t), X_{21}, t) = 0, \quad t \geq 0$$

$$(3.8) \quad D_2(X_{12}(t), X_{22}, t) = I, \quad t \geq 0.$$

Notice that (3.7) implies X_{11}, X_{21} are solutions of (3.1). The functions X_{12}, X_{22} do not satisfy the equation (3.1b), but a nonhomogeneous version of it. However, it is important to notice that if these functions were differentiable, the derivatives

would satisfy (3.1b). This is an important remark since it essentially implies that the variation of $X(t)$ satisfies (3.1).

Now we consider the nonhomogeneous system

$$(3.9) \quad \begin{aligned} \dot{x}(t) &= Ax(t) + by(t-r) + f(t) \\ y(t) - E'x(t) - Jy(t-r) - g(t) &= 0 \end{aligned}$$

where f, g are continuous functions from $[0, \infty)$ to R^k, R^m , respectively. If

$$G = \begin{pmatrix} 0 \\ g \end{pmatrix} \in R^{k \times m}, \quad F = \begin{pmatrix} f \\ 0 \end{pmatrix} \in R^{k \times m}$$

then the equation (3.9) is a special case of the NFDE

$$(3.10) \quad \frac{d}{dt} [D(x(t), y_t) - G(t)] = L(x(t), y_t) + F(t)$$

with D, L defined as in (3.2), (3.3) respectively. One obtains the equation (3.9) from (3.10) by requiring that

$$(3.11) \quad D_2(a, \phi) = g(0)$$

If we let $w_t = \text{col}(x(t), y_t)$, $\psi = \text{col}(a, \phi)$, then the general solution of (3.10) is given by the variation of constants formula

$$(3.12) \quad w_t - X_0 G(t) = T(t)\psi - X_t G(0) \\ + \int_0^t X_{t-s} F(s) ds - \int_0^t [d_s X(t-s)] G(s).$$

As in Section 2, we extend the definition of $T(t)$ to $R^k \times C + \langle X_0 \rangle \stackrel{\text{def}}{=} Y$. Then (3.12) can be written as

$$(3.13) \quad w_t - X_0 G(t) = T(t) [\psi - X_0 G(0)] + \int_0^t T(t-s) X_0 F(s) ds \\ - \int_0^t [d_s T(t-s) X_0] G(s).$$

An element of the space Y can be represented as $(a, \bar{\phi})$, $\bar{\phi} = (\phi, b)$ where, $a \in R^k$, $b \in R^m$, $\phi \in C$. Let

$$Y_0 = \{(a, \bar{\phi}) \in Y: D_2(a, \bar{\phi}) = 0\}.$$

The analogue of Lemma 2.1 for this situation is obtained by using Laplace transform and applying the arguments in Henry [11] and is stated precisely as

Lemma 3.1. If the eigenvalues of J have moduli less than one and all roots of (3.6) satisfy $\text{Re } \lambda \leq -\delta < 0$, then there are positive constants K, α such that

$$|x_{11}(t)|, |x_{21}(t)|, |\dot{x}_{ij}(t)| \leq Ke^{-\alpha t}, \quad \text{a.e. } t \geq 0$$

$i, j = 1, 2$, and, for any $(a, \bar{\phi}) \in Y_0$, the solution $x(a, \bar{\phi})$ of the equation (3.1) satisfies

$$|x(a, \bar{\phi})(t)| \leq Ke^{-\alpha t} |(a, \bar{\phi})|, \quad t \geq 0.$$

As a first application of the previous results, let us consider the stability of the solution $(x(t), y_t) = 0$ of the equation

$$(3.14) \quad \frac{d}{dt} [D(x(t), y_t) - G(x(t), y_t)] = L(x(t), y_t) + F(x(t), y_t)$$

$$G(a, \phi) = \begin{pmatrix} 0 \\ g(a, \phi) \end{pmatrix}, \quad F(a, \phi) = \begin{pmatrix} f(a, \phi) \\ 0 \end{pmatrix}$$

where f, g are continuous functions with

$$(3.15) \quad \begin{aligned} f(0, 0) &= 0, & y(0, 0) &= 0 \\ |f(\psi) - f(\psi_1)| &\leq \mu(\sigma) |\psi - \psi_1| \\ |g(\psi) - g(\psi_1)| &\leq \mu(\sigma) |\psi - \psi_1| \end{aligned}$$

for $|\psi|, |\psi_1| \leq \sigma$, where $\mu(\sigma)$ is a continuous nondecreasing function such that $\mu(\sigma) \rightarrow 0$ as $\sigma \rightarrow 0$.

If $D_2(a, \phi) = g(a, \phi)$ then the corresponding solution of

(3.14) satisfies

$$(3.16) \quad \begin{aligned} \dot{x}(t) &= Ax(t) + By(t-r) + f(x(t), y_t) \\ y(t) &= E'x(t) + Jy(t-r) + f(x(t), y_t). \end{aligned}$$

Theorem 3.1. If D is stable and there is a $\delta > 0$ such that all solutions of (3.6) satisfy $\operatorname{Re} \lambda \leq -\delta$ then the zero solution of (3.16) is uniformly asymptotically stable.

Proof: The variation of constants formula (3.13) is

$$\begin{aligned} w_t - X_0 G(w_t) &= T(t) [\psi - X_0 G(\psi)] + \int_0^t T(t-s) X_0 F(w_s) ds \\ &\quad - \int_0^t [d_s T(t-s) X_0] G(w_s) \end{aligned}$$

where, $w_t = \operatorname{col}(x(t), y_t)$, $\psi = \operatorname{col}(a, \phi)$. If we let

$$z_t = w_t - X_0 G(w_t), \quad \xi = \psi - X_0 G(\psi)$$

then we have defined the transformation $h: \mathbb{R}^{k \times m} \times \mathbb{C} \rightarrow Y$,

$h(\psi) = \xi$, which is a homeomorphism. Hence, there are constants

$k_1, k_2 > 0$ such that $|\xi| \leq k_1 |\psi|$, $|\psi| \leq k_2 |\xi|$ for $|\xi|, |\psi|$

sufficiently small. Also,

$$D(\xi) = D(\psi) - G(\psi) = 0$$

since equation (3.16) is satisfied. Since $F = \text{col}(f, 0)$, only the $\text{col}(X_{11}, X_{21})$ of X is used to evaluate $T(t-s)X_0 F(w_s)$. Therefore, we may apply Lemma 3.1 and the same arguments as in the proof of Theorem 2.1 to complete the proof of the theorem.

It is obvious that one could generalize the results in this section in many ways. The perturbations g, f could depend upon t as long as all estimates are uniform in t . Also, and more importantly, the linear equation can be much more general. In fact, we could have considered an equation of the form

$$\frac{d}{dt} D(x(t), y_t) = L(x(t), y_t)$$

where $L = \begin{pmatrix} L_1 \\ 0 \end{pmatrix}: \mathbb{R}^k \times C \rightarrow \mathbb{R}^{k \times m}$ is an arbitrary continuous linear functional and

$$D(a, \phi) = H \begin{pmatrix} a \\ \phi(0) \end{pmatrix} - d_1(\phi)$$

where H is a $(k \times m) \times (k \times m)$ nonsingular matrix and

$$d_1(\phi) = \int_{-r}^0 [d\mu(\theta)] \phi(\theta)$$

with μ an $m \times m$ matrix of bounded variation with $\mu(0) = \mu(0^-)$. This means the results apply to the more general equation

$$\dot{x}(t) = L_1(x(t), y_t) + f(x(t), y_t)$$

$$D_2(x(t), y_t) = g(x(t), y_t)$$

where

$$D(a, \phi) = \text{col}(a, D_2(a, \phi)).$$

These same remarks apply equally as well to the results in latter sections.

4. Absolute Stability

Let D, L be as in section 3; let $h > 0$ be given and let $f: R \rightarrow R$ be a given continuous function satisfying

$$(4.1) \quad h_1 \sigma^2 \leq \sigma f(\sigma) \leq h_2 \sigma^2, \quad 0 < h_1 \leq h_2 < h.$$

Let c be a k -dimensional row vector, $b_1 \in R^k$, $b_2 \in R^m$ be constant vectors and consider the system

$$(4.2) \quad \dot{x}(t) = L_1(x(t), y_t) + b_1 f(\sigma(t))$$

$$D_2(x(t), y_t) = b_2 f(\sigma(t))$$

$$\sigma = cx.$$

Our objective is to apply the method of Popov to determine sufficient conditions for the absolute stability of system (4.2). The variation of constants formula implies the solution of (4.2)

through $\psi = (a, \phi) \in R^{k \times m} \times C$ satisfies the equation

$$(4.3) \quad w_t - X_0 G(\sigma(t)) = T(t) [\psi - X_0 G(\sigma(0))] + \int_0^t T(t-s) X_0 F(\sigma(s)) ds \\ - \int_0^t [d_s T(t-s) X_0] G(\sigma(s))$$

where $w_t = \text{col}(x(t), y_t)$, $G = \text{col}(0, b_2 f)$, $F = \text{col}(b_1 f, 0)$ and $\sigma = cx$.

As in the previous sections, let us simplify the formula

(4.3) by putting

$$(4.4) \quad z_t = w_t - X_0 G(cx(t)).$$

If we let $z_t = (u(t), v_t)$ then (4.4) is equivalent to

$$(4.5) \quad u(t) = x(t) \\ v_t = y_t - X_{22,0} b_2 f(cx(t))$$

or

$$(4.6) \quad u(t) = x(t) \\ v_t(0) = y_t(0), \quad 0 < t \\ v_t(0) = y_t(0) - b_2 f(cx(t))$$

Equation (4.3) for z_t becomes

$$(4.7) \quad z_t = T(t)z_0 + \int_0^t T(t-s)X_0 F(\sigma(s))ds - \int_0^t [d_s T(t-s)X_0]G(\sigma(s))$$

since $\sigma(t) = cx(t) = cu(t)$.

For the following discussion, it is convenient to have this equation (4.7) explicitly in $R^{k \times m}$. It is easily seen that (4.7) is equivalent to the equations

$$(4.8) \quad \begin{aligned} \text{a) } u(t) &= u^0(t) + \int_0^t X_{11}(t-s)b_1 f(\sigma(s))ds - \\ &\quad - \int_0^t [d_s X_{12}(t-s)]b_2 f(\sigma(s)) \\ \text{b) } v(t) &= v^0(t) + \int_0^t X_{21}(t-s)b_1 f(\sigma(s))ds \\ &\quad - \int_0^t [d_s X_{22}(t-s)]b_2 f(\sigma(s)). \end{aligned}$$

By Lemma 3.1, we know that if D is stable and there is $\delta > 0$ such that $\text{Re } \lambda \leq -\delta$ for all $\lambda \in \sigma(D)$ then $T(t)z_0$, that is, $u^0(t), v^0(t)$, and $X_{11}(t), X_{21}(t), \dot{X}_{ij}(t)$, $i, j = 1, 2$ approach zero exponentially as $t \rightarrow \infty$.

Consider (4.8a). As $\sigma(t) = cx(t) = cu(t)$, we have

$$\begin{aligned} \sigma(t) &= cu^0(t) + \int_0^t cX_{11}(t-s)b_1 f(\sigma(s))ds \\ &\quad + \int_0^t c\dot{X}_{12}(t-s)b_2 f(\sigma(s))ds. \end{aligned}$$

Let $cu^0(t) = \gamma(t)$, $cX_{11}(t)b_1 = k_{11}(t)$, $c\dot{X}_{12}(t)b_2 = k_{12}(t)$.

Then

$$(4.9) \quad \sigma(t) = \gamma(t) + \int_0^t k_{11}(t-s)f(\sigma(s))ds + \int_0^t k_{12}(t-s)f(\sigma(s))ds$$

Notice that $\gamma(t)$, $\frac{d\gamma(t)}{dt}$, $k_{11}(t)$, $k_{12}(t)$ tend to zero exponentially as $t \rightarrow \infty$.

Now, define

$$f_T(t) = \begin{cases} f(\sigma(t)), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

$$\sigma_T(t) = \begin{cases} \sigma(t), & 0 \leq t \leq T \\ 0, & t > T \end{cases}$$

$$\gamma_T(t) = \begin{cases} \gamma(t), & 0 \leq t \leq T \\ \int_0^T k_{11}(t-s)f(\sigma(s))ds + \int_0^T k_{12}(t-s)f(\sigma(s))ds, & t > T \end{cases}$$

Thus

$$(4.10) \quad \sigma_T(t) - \gamma_T(t) = \int_0^t k_{11}(t-s)f_T(\sigma)ds + \int_0^t k_{12}(t-s)f_T(\sigma)ds.$$

Let

$$\begin{aligned}
(4.11) \quad \chi(T) &= \int_0^T \left\{ \sigma(t) - \gamma(t) - \frac{1}{h} f(\sigma(t)) \right. \\
&\quad \left. + q \left(\frac{d\sigma(t)}{dt} - \frac{d\gamma(t)}{dt} \right) \right\} f(\sigma(t)) dt \\
&= \int_0^\infty \left\{ \sigma_T - \gamma_T - \frac{1}{h} f_T + q \left(\frac{d\sigma_T}{dt} - \frac{d\gamma_T}{dt} \right) \right\} f_T dt \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \operatorname{Re} \{ [\tilde{\sigma}_T - \tilde{\gamma}_T - \frac{1}{h} \tilde{f}_T + q i \omega (\tilde{\sigma}_T - \tilde{\gamma}_T)] \tilde{f}_T^* d\omega
\end{aligned}$$

where $\tilde{}$ denotes the Fourier transform and f^* is the transpose of f . From (4.10), by the convolution theorem, we have

$$\tilde{\sigma}_T - \tilde{\gamma}_T = \tilde{k}_{11} \cdot \tilde{f}_T + \tilde{k}_{12} \tilde{f}_T = \tilde{k}_{11} \tilde{f}_T + i\omega \tilde{k}_{12} \tilde{f}_T.$$

Hence, (4.11) becomes

$$\chi(T) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \{ \operatorname{Re} [(1+i\omega q) (\tilde{k}_{11} + i\omega \tilde{k}_{12})] - \frac{1}{h} \} |\tilde{f}_T|^2 d\omega.$$

If the condition $\operatorname{Re} [(1+i\omega q) (\tilde{k}_{11} + i\omega \tilde{k}_{12})] - \frac{1}{h} \leq 0$ is fulfilled, then $\chi(T) \leq 0$. This means that

$$\begin{aligned}
&\int_0^T \left\{ \sigma(t) - \frac{1}{h} f(\sigma(t)) + q \frac{d\sigma(t)}{dt} \right\} f(\sigma(t)) dt \\
&\leq \int_0^T \left\{ \gamma(t) + q \frac{d\gamma(t)}{dt} \right\} f(\sigma(t)) dt.
\end{aligned}$$

From here the proof further continues as in Halanay's book, Chapter 4, Section 4.6, and we obtain $\lim_{t \rightarrow \infty} \sigma(t) = 0$ and

$\lim_{t \rightarrow \infty} f(\sigma(t)) = 0$. This implies, by (4.8a), that

$\lim_{t \rightarrow \infty} u(t) = \lim_{t \rightarrow \infty} x(t) = 0$. From (4.8b) it follows that

$\lim_{t \rightarrow \infty} v(t) = 0$, since $\lim_{t \rightarrow \infty} f(\sigma(t)) = 0$. Hence $\lim_{t \rightarrow \infty} y(t) = 0$.

Thus, the following theorem is proved:

Theorem 4.1. If D is stable, all roots of (3.6) satisfy $\text{Re } \lambda \leq -\delta < 0$ and there is a $q > 0$ such that

$$\text{Re}[(1+i\omega q)(\tilde{k}_{11} + i\omega\tilde{k}_{12})] - \frac{1}{h} \leq 0$$

where $\tilde{k}_{11} = cX_{11}b_1$, $\tilde{k}_{12} = c\tilde{X}_{12}b_2$, then the system (4.2) is absolutely stable for every function f satisfying (4.1).

Rasvan [12] has studied the same problem. He gives the condition on q in terms of the transfer function $\eta(i\omega)$.

If we compute $\tilde{k}_{11} + i\omega\tilde{k}_{12}$ we see that

$$\eta(i\omega) = \tilde{k}_{11} + i\omega\tilde{k}_{12}.$$

One could easily study the first critical case using the same arguments as in Halanay [7] and Datko [3].

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