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II. When are exponential smoothing forecast procedures optimal?

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When are exponential smoothing forecast procedures optimal?

2.1 Introduction and summary:

This chapter takes a critical look at exponential smoothing methods as described in section 1.6, particularly those of Brown which received broad attention in the literature and which have been widely used in practice. It will be shown that:

i) Brown's forecasting procedures are optimal in terms of achieving minimum mean squared error forecasts only if the underlying stochastic process is a specific member of the class of ARIMA(p,d,q) processes. Hence, it will be shown what assumptions are made when using these procedures.

ii) The implication of point (i) is that the stochastic processes which occur in the real world are from a specific subclass of ARIMA models. We, therefore, discuss the question of whether there are any reasons why these particular models should occur more frequently than others. There seems to be no satisfactory explanation.

iii) It is shown that even if the stochastic process which would lead to Brown's models occurred, the methods which he uses for making the forecasts are clumsy, and much simpler procedures can be obtained.

A related study has recently been reported independently by McKenzie (1973). The approach he takes is, however, different from the one adopted in this thesis.

2.2 Updating formulae for the coefficients in the forecast function:

Updating formula for Brown's forecasting procedures:

We saw in section 1.6 that Brown derives forecasts by specifying fitting functions (forecast functions) from the class of functions which satisfies

$$f(t + 1) = Lf(t). \quad (2.2.1)$$

The transition matrix

$$L = \begin{bmatrix} L_1 & \cdots & L_m \\ \vdots & \ddots & \vdots \\ L_m & \cdots & L_1 \end{bmatrix}$$

is assumed to be nonsingular, and $$\hat{f}(0) = [f_1(0), \ldots, f_m(0)]$$ is specified. The coefficients

$$b'(t) = [b_1(t), \ldots, b_m(t)]$$

of the forecast function

$$\hat{z}_t(k) = \hat{P}(t,k) = \sum_{i=1}^m b_i(t) \hat{f}_i(t) = b'(t) \hat{f}(t) \quad (2.2.2)$$
are fitted by discounted least squares, minimizing

$$\sum_{j=0}^{t-1} p_j (t - j)]^2.$$  \hspace{2cm} \text{(2.2.3)}$$

In the steady state case ($t \to \infty$), the solution is given by

$$\hat{b}(t) = F^{-1} r(t) \quad \text{where:} \quad F = \sum_{j=0}^{\infty} p_j (t - j) f(t - j)$$

and $$r(t) = \sum_{j=0}^{t} p_j (t - j) f(t - j).$$  \hspace{2cm} \text{(2.2.5)}$$

Brown (1962) develops an updating formula for the coefficients $b(t)$ from one time origin to the other:

$$\hat{b}(t) = L^* b(t-1) + h$$  \hspace{2cm} \text{(2.2.4)}$$

where $h$ is the one step ahead forecast error, and $h = F^{-1} f(0).$

We note that $h$ is a function of the smoothing coefficient $\beta$ only.

Updating formula for ARIMA (p,d,q) models:

The eventual forecast function of an ARIMA (p,d,q) process is the solution of $\phi_p (B) \hat{z}_t (t) = 0$ for $t > q.$

It is given by

$$\hat{z}_t (t) = b_1 (t) f_1 (t) + \ldots + b_{p+d} (t) f_{p+d} (t) \quad \text{for} \quad t > q-p-d.$$

Box and Jenkins (1970) show that the updating formula for the coefficients $b^*(t) = [b_1^*(t), \ldots, b_{p+d}^*(t)]$ is given by:

$$b^*(t) = L^* b^*(t-1) + g [z_t - z_{t-1}]$$

where:

$$L^* = F^{-1} \Psi$$

with

$$F^* = \begin{bmatrix} f_1^*(t) & \ldots & f_{p+d}^*(t) \\ f_1^* (t+1) & \ldots & f_{p+d}^* (t+1) \\ \vdots & \vdots & \vdots \\ f_1^* (t+p+d-1) & \ldots & f_{p+d}^* (t+p+d-1) \end{bmatrix}$$

and $g = F^{-1} \Psi$ with $\Psi = [\Psi_1, \Psi_{1+1}, \ldots, \Psi_{p+d-1}]$

for any $t > q - p - d.$

$\Psi_i$ are the coefficients of $\psi(B) = \sum_{i=0}^{\infty} \psi_i \Phi^* = \sum_{i=0}^{\infty} \phi_i (B)$

Whenever $p + d = m$ and $f_i^*(t) = f_i (t)$ for $i = 1, 2, \ldots, m$

it follows that

$$L^* = F^{-1} \Psi = L^*.$$

The matrix $L$ appears, therefore, in the updating formula for the Brown model (2.2.4) as well as in the updating formula for the ARIMA model (2.2.5). This is because the transition matrix merely allows for changes in the coefficients arising from the change to a new origin, and has to occur in any reasonable updating formula. The essential difference between
the updating formulae (2.2.4) and (2.2.6), however, lies in the vectors \( h \) and \( g \). Whereas \( h \) is a function of the smoothing parameter \( \beta \) only, \( g \) is a function of all the parameters \((\phi, \theta)\) in the ARIMA model.

2.3 Equivalence theorems for Brown's forecast procedures and forecasts from ARIMA models:

We prove the following theorem.

**Theorem 2.1:**

**Model A:** (Brown model)

Consider the fitting functions
\[
f'(t) = [u_1^{-1}, \ldots, u_n^{-1}] \quad \text{with} \quad |u_1| > 1; \quad u_i \neq u_j \quad \text{for} \quad i \neq j;
\]
and \( |u_i u_j| < 1/\beta \) for \( 1 \leq i, j \leq n. \)

The coefficients of the forecast function
\[
\hat{p}(t, k) = b_1(t)u_1^{-k} + b_2(t)u_2^{-k} + \ldots + b_n(t)u_n^{-k} \tag{2.3.1}
\]

are estimated by discounted least squares with smoothing constant \( \beta; \quad 0 < \beta < 1. \)

**Model B:** ARIMA model

\[
\sum_{i=1}^{n} \frac{1}{u_i^2} (1 - u_i \beta) z_t = \sum_{i=1}^{n} (1 - u_i \beta \theta) a_t. \tag{2.3.2}
\]

Then: Brown's forecasting procedure using the fitting function as specified in model A will provide optimal forecasts in terms of minimizing the mean squared forecast error if the underlying stochastic process follows the ARIMA model given in (2.3.2).

**Comment:** This theorem will be proved by showing that model A and model B are equivalent in terms of having
i) the same form of the forecast function
ii) the same updating formula for the coefficients of the forecast function.

In order to show theorem 2.1 we will use the following lemma about the inverse of a Vandermonde matrix.

**Lemma 2.1:** Consider the matrix

\[
A = \begin{bmatrix}
1 & 1 & \ldots & 1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1^{n-1} & \alpha_2^{n-1} & \ldots & \alpha_n^{n-1}
\end{bmatrix}
\]

where \( \alpha_i \neq \alpha_j \) (for \( i \neq j \)).

Then it is shown that the inverse of \( A \) is given by

\[
A^{-1} = \begin{bmatrix}
\alpha_1 & \alpha_1 & \ldots & \alpha_1 \\
\alpha_1 & \alpha_2 & \ldots & \alpha_2 \\
\vdots & \vdots & \ddots & \vdots \\
\alpha_1 & \alpha_2 & \ldots & \alpha_n
\end{bmatrix}
\]

where \( a_{ij} \) are the coefficients in the expansion.
\[ p_i(x) = \sum_{k=1}^{n} \frac{x^{a_k}}{a_k} = a_{i1}x^0 + a_{i2}x^1 + \ldots + a_{in}x^{n-1} \quad (1 \leq i \leq n) \]

The proof of Lemma 2.1 is given in the appendix to this chapter.

Proof of Theorem 2.1:

a) The eventual forecast function for Model B is the solution of

\[ \sum_{i=1}^{n} (1 - \frac{1}{u_i} B) \hat{z}_t(t) = 0 \quad \text{for } t > n \]

and it is given by

\[ \hat{z}_t(t) = b_1^*(t)u_1^t + b_2^*(t)u_2^t + \ldots + b_n^*(t)u_n^t \quad \text{for } t > 0. \]

\[ \text{(2.3.4)} \]

The eventual forecast function (2.3.4) coincides with the forecast function of Model A given in (2.3.1).

b) The updating algorithm for the coefficients of the forecast function \( b'(t) \hat{f}(t) \) for Model A is given by

\[ b'(t) = L' b'(t-1) + h[z_{t-1}^p(t-1,1)] . \]

Dobbie (1963) showed that for the case of exponential fitting

functions \( f^*(t) = [u_1^t, \ldots, u_n^t] \) where \( u_i \neq u_j \) for \( i \neq j \)

and \( |u_i u_j| < 1/e \) \( (1 \leq i, j \leq n) \)

\[ h' = [h_1, \ldots, h_n] \]

is given by:

\[ h_i = (1 - \beta u_i^2) \sum_{k=1}^{n} \frac{1 - \beta u_i u_k}{1 - u_i u_k} . \]

\[ \text{(2.3.6)} \]

The updating algorithm for the coefficients of the eventual forecast function of the ARIMA model in (2.3.2) is given by:

\[ b^*(t) = L' b^*(t-1) + g[z_{t-1}^p(t-1,1)] . \]

\[ \text{(2.3.7)} \]

Choosing \( t = 1 \) in (2.2.6), it is seen that

\[ \psi_k = F_1^{-1} \phi_1 = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^n & a_2^n & \cdots & a_n^n \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{bmatrix} \]

\[ \text{(2.3.8)} \]

where \( a_i = \frac{1}{u_i} \), \( 1 \leq i \leq n \)

and \( \psi_k (1 \leq k \leq n) \) are the \( \psi \)-weights in.
\[ \phi(B) = 1 + \sum_{k=1}^{n} \psi_k B^k = \frac{\prod_{k=1}^{n} (1 - B u_k)}{\prod_{k=1}^{n} (1 - \frac{B}{a_k})} \]

(2.3.9)

In order to prove the theorem we have to show that:

\[ g = h \quad \text{or equivalently} \]

\[
\begin{bmatrix}
\frac{1}{a_1} & 0 & \ldots & 0 \\
0 & \frac{1}{a_2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & \frac{1}{a_n}
\end{bmatrix}
\begin{bmatrix}
1 & 1 & \ldots & 1 \\
1 & a_2 & \ldots & a_n \\
\vdots & \vdots & \ddots & \vdots \\
1 & a_{n-1} & a_{n-2} & \ldots & a_n
\end{bmatrix}
\psi_1 = h
\]

(2.3.10)

We, therefore, have to show that the coefficient of \( x^0 \) in

\[ V_1(x) = \frac{\prod_{k=1}^{n} (1 - \frac{a_k}{x})}{x^{n}(1-a_k x)} \]

where \( d^* = [d_1, \ldots, d_n] \) and \( d_1 = a_{i1} \psi_1 + a_{i2} \psi_2 + \ldots + a_{i n} \psi_n \)

and \( a_{ij} \) are the elements of the inverse of matrix \( A \).
equals \[ a_1 \prod_{k \neq i} (a_1 - a_k) h_i \] (2.3.12)

Using (2.3.6) we write

\[
a_1 \prod_{k \neq i} (a_1 - a_k) h_i = a_1 (1 - \frac{1}{a_1}) \prod_{k \neq i} (a_1 - a_k) \prod_{k \neq i} \frac{1 - \frac{\beta}{a_1 a_k}}{1 - \frac{\beta}{a_k}} = a_1^n (1 - \frac{\beta}{a_1 a_k}) \prod_{k \neq i} \frac{1 - \frac{\beta}{a_1 a_k}}{1 - \frac{\beta}{a_k}} \prod_{k \neq i} (1 - \frac{\beta}{a_1 a_k}) .
\]

It, therefore, remains to show that the coefficient of \( x^0 \) in \( V_i(x) \) is equal to

\[ a_1^n \prod_{k \neq i} (1 - \frac{\beta}{a_1 a_k}) . \] (2.3.13)

Now

\[
V_i(x) = \prod_{k=1}^{n} \frac{1 - \frac{\beta}{a_k}}{1 - \frac{\beta}{a_k} (1 - \frac{\beta}{a_k})} = [1 + \alpha_1 x + \alpha_1^2 x^2 + \ldots] \times \left( \frac{\beta}{a_1} \right)^{n-1} c_1 \left( \frac{\beta}{a_1} \right)^{n-2} \ldots c_{n-1} \left( \frac{\beta}{a_1} \right) c_n
\]

where the \( c_j \)'s are the coefficients in the expansion of

\[
\prod_{k=1}^{n} \frac{1 - \frac{\beta}{a_k}}{1 - \frac{\beta}{a_k} (1 - \frac{\beta}{a_k})} = \left( \frac{1}{x} \right)^{\frac{\beta}{a_1 a_k}} \left[ e_1^{(i)} \left( \frac{1}{x} \right) \right] e_2^{(i)} \left( \frac{1}{x} \right) \ldots e_n^{(i)} \left( \frac{1}{x} \right) e_n^{(i)}
\]

where the coefficients \( e_j^{(i)} \) (1 \( \leq j \leq n-1 \)) are given by

\[
\begin{align*}
e_1^{(i)} &= -\frac{1}{a_i} \sum_{k \neq i} \frac{\beta}{a_k} \\
e_2^{(i)} &= \frac{1}{a_i} \sum_{k \neq i} \frac{\beta^2}{a_k a_l} \\
e_3^{(i)} &= \frac{1}{a_i} \sum_{k \neq i} \frac{\beta^3}{a_k a_l a_m} \\
&\vdots \\
e_n^{(i)} &= (-1)^{n-1} \frac{1}{a_i} \beta^{n-1} \prod_{k \neq i} \frac{1}{a_k}.
\end{align*}
\] (2.3.15)
It is readily shown that the relation between the coefficients in (2.3.14) and (2.3.15) is given by

\[
\begin{align*}
\mathbf{c}_1 &= a_1 c_1^{(1)} - \frac{\beta}{a_1} \\
\mathbf{c}_2 &= a_1^2 c_2^{(1)} - \beta c_1^{(1)} \\
\mathbf{c}_3 &= a_1^3 c_3^{(1)} - \beta a_1 c_2^{(1)} \\
&\vdots \\
\mathbf{c}_{n-1} &= a_1^{n-1} c_{n-1}^{(1)} - \beta a_1^{n-2} c_{n-2}^{(1)} \\
\mathbf{c}_n &= -\beta a_1^{n-2} c_{n-1}^{(1)} 
\end{align*}
\tag{2.3.16}
\]

Evaluating \( \Pi \left( 1 - \frac{\theta}{a_1 a_k} \right) \) at \( x = 1 \) gives

\[
\Pi \left( 1 - \frac{\theta}{a_1 a_k} \right) = 1 + \mathbf{c}_1^{(1)} + \cdots + \mathbf{c}_{n-1}^{(1)}
\]

and, therefore,

\[
a_1^n \left( 1 - \frac{\theta}{a_1} \right) \Pi \left( 1 - \frac{\theta}{a_1 a_k} \right) = a_1^n \left( 1 - \frac{\theta}{a_1} \right) \left( 1 + \mathbf{c}_1^{(1)} + \cdots + \mathbf{c}_{n-1}^{(1)} \right).
\tag{2.3.17}
\]

Using the relation (2.3.16), the coefficient of \( x^0 \) in \( V_i(x) \), let's say \( d_i^0 \), is given by

\[
d_i^0 = a_1^n + a_1^{n-1} c_1 + \cdots + a_1 c_{n-1} c_n
\]

\[
= a_1^n a_1^{-1} \left[ a_1^{e_1^{(1)}} - \frac{\beta}{a_1} \right] a_1^{n-2} \left[ a_1^{e_2^{(1)}} - \beta e_1^{(1)} \right] + \cdots + a_1^{n-3} \left[ a_1^{e_3^{(1)}} - \beta a_1 e_2^{(1)} \right] + \cdots + a_1^{n-2} \left[ a_1^{e_n^{(1)}} - \beta a_1^{n-1} e_{n-1}^{(1)} \right]
\]

\[
= a_1^n \left( 1 - \frac{\theta}{a_1} \right) \left[ 1 + e_1^{(1)} + e_2^{(1)} + \cdots + e_{n-1}^{(1)} \right]
\]

which equals (2.3.17), thus proving the claim that \( g = h \).

**Theorem 2.2:**

Model A*: Brown forecasting procedure with fitting functions \( d(t) \), where \( f(t) = Rd(t) \).

\[d(t)\]

\( R \) is assumed to be nonsingular and \( f'(t) = [u_1^{-1}, \ldots, u_n^{-1}] \) are the fitting functions of Model A in Theorem 2.1. The coefficients \( c(t) \) of the forecast function \( c'(t)d(t) \) are estimated by discounted least squares with smoothing coefficient \( B(0 < B < 1) \).

**Model B: ARIMA model**

\[\frac{n}{i=1} \left( 1 - \frac{1}{u_i} B \right) z_t = \frac{n}{i=1} \left( 1 - B u_i B \right) a_t.\]
Then Brown's forecasting procedure using the fitting functions as specified in Model A* will provide optimal forecasts in terms of minimizing the mean squared forecast error if the underlying process follows the ARIMA Model B.

Proof:
For Model A*:

\[ b'(t)\xi(t) = b'(t)Rd(t) = c'(t)d(t) \]
where \( c(t) = R'b(t) \)

and

\[ b(t) = L'b(t-1) + h[z_t - \hat{\beta}(t-1,1)] \]

\[ R'b(t) = R'L'(R')^{-1}R'b(t-1) + h[z_t - \hat{\beta}(t-1,1)] \]

\[ c(t) = R'L'(R')^{-1}c(t-1) + R'h[z_t - \hat{\beta}(t-1,1)]. \]

For Model B:

\[ b^*(t)\xi(t) = b^*(t)Rd(t) = c^*(t)d(t) \]
where

\[ c^*(t) = R'h^*(t) \]

and

\[ b^*(t) = L'b^*(t-1) + g[z_t - \hat{\beta}(t-1,1)] \]

\[ R'h^*(t) = R'L'(R')^{-1}R'h^*(t-1) + g[z_t - \hat{\beta}(t-1,1)] \]

\[ c^*(t) = R'L'(R')^{-1}c^*(t-1) + R'g[z_t - \hat{\beta}(t-1,1)] \]

and since \( g = h \) (from Theorem 2.1) it follows that \( R'g = R'h \).

q.e.d.

Theorem 2.2 is important since sinusoidal fitting functions, which are frequently considered by Brown, can be written as linear combinations of exponential functions. For example,

\[ \hat{p}(t,1) = c_1(t) + c_2(t)\sin\omega t + c_3(t)\cos\omega t \]

can be written as

\[ \xi(t) = Rd(t) \quad (2.3.20) \]

where

\[ \begin{bmatrix} 1 \\ e^{-i\omega t} \\ e^{i\omega t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -i & 1 \\ 0 & i & 1 \end{bmatrix} \begin{bmatrix} \sin\omega t \\ \cos\omega t \end{bmatrix}. \]

Equation (2.3.20) shows that the roots of the characteristic equation for the ARIMA model which is implied by the sinusoidal fitting functions in Brown's forecasting scheme are lying on the unit circle. For the case of monthly observations with \( \omega = \frac{2\pi}{12} \), the autoregressive operator of the corresponding ARIMA model is given by \( (1 - \sqrt{3}B + B^2)(1-B) \).

Many nonstationary time series exhibit homogeneity in the sense that apart from local level and/or trend and/or periodicity one part of the series behaves very much like
the other. Stochastic processes which exhibit these
caracteristics have some roots of \( \phi(B) = 0 \) on the unit
circle. This implies that the autoregressive operator will
contain factors of the form \((1 - B), (1 - B^2), (1 - \sqrt{3B} + B^2), (1 - B^5),
\) etc., which Box and Jenkins refer to as simplifying operators.

It was shown in Table 1.1 that the \((1 - B^{12})\) operator
removes the effect of 12 equispaced roots on the unit circle.
The eventual forecast function of the ARIMA model
\[
(1 - B^{12})z_t = \theta_q(B)a_t
\]
has 12 coefficients which are updated as each new observa-
tion becomes available.

Box and Jenkins point out that for monthly data with
strong sinusoidal trend (e.g.: temperature data) an appro-
priate simplifying operator might be \((1 - \sqrt{3B} + B^2)\). For
general 12-monthly seasonal patterns a complete set of
sinusoidals can be achieved by considering the simplifying
operator \((1 - B^{12})\).

Box and Jenkins also stress the fact that over-
differencing ought to be avoided, since it leads to non
parsimonious, non invertible models. As an example we
consider the invertible model
\[
(1 - B)(1 - \sqrt{3B} + B^2)z_t = \theta_q(B)a_t.
\] (2.3.22)

The eventual forecast function of the model in (2.3.22) is
a simple 12 point sinusoidal function
\[
\hat{z}_t(t) = b_1(t) + b_2(t)\sin \frac{2\pi}{12} t + b_3(t)\cos \frac{2\pi}{12} t.
\]

Overdifferencing the series by introducing the
incorrect simplifying operator \((1 - B^{12})\) results in a non-
invertible model
\[
(1 - B^{12})z_t = \theta_q(B)(1 - B + B^2)(1 + B^2)(1 + B + B^2)(1 + \sqrt{3B} + B^2)(1 + B)a_t
\] (2.3.25)

A further discussion of this point is given by Abraham (1975).

In the following corollary to Theorem 2.1 and
Theorem 2.2 we use the fact that, if the characteristic
equation has a complex root, the conjugate complex will
be a solution too.

Corollary 2.1

Model B: We consider the ARIMA model
\[
\phi_n(B)z_t = \phi_n(B)a_t
\] (2.3.24)

where the coefficients of \( \phi_n(B) \) are real. Furthermore,
it is assumed that the roots of $\phi_n(B) = 0$ are distinct and lie on the unit circle, and that the eventual forecast function is given by

$$\hat{z}_t(\xi) = b^*_1(t)f_1(\xi) + \ldots + b^*_n(t)f_n(\xi).$$

**Model A:** Consider the Brown forecasting procedure with fitting functions

$$f'(\xi) = [f_1(\xi), \ldots, f_n(\xi)].$$

The coefficients of the forecast function

$$\hat{z}_t(\xi) = b'(t)f(\xi)$$

are fitted by discounted least squares with smoothing coefficient $\beta$, $0 < \beta < 1$.

Then the Brown forecasting procedure with fitting functions as specified in Model A will provide minimum mean squared error forecasts if the underlying stochastic process is given by the ARIMA Model B.

**Theorem 2.3:** Theorem 2.3 will relax the condition of distinct roots.

**Model B:**

$$\phi(B)z_t = \phi(B)\alpha_t$$

where the coefficients of $\phi(B)$ are real and where the roots of $\phi(B) = 0$ lie on the unit circle; furthermore, we allow the possibility of multiple roots.

The eventual forecast function is given by

$$\hat{z}_t(\xi) = b^*_1(t)f_1(\xi) + \ldots + b^*_n(t)f_n(\xi).$$

**Model A:**

Brown model with fitting functions

$$f'(\xi) = [f_1(\xi), \ldots, f_n(\xi)]$$

and the coefficients of the forecast function are estimated by discounted least squares with smoothing parameter $\beta$.

Then Model A and Model B are equivalent in terms of having

a) the same form of forecast function

b) the same recursive updating formula for the coefficients of the forecast function $b'(t)$.

**Proof:**

- **ada:** Trivially the form of the forecast function of Model A and Model B is the same.

- **adb:** We can write

$$\phi(B) = \prod_{i=1}^{s} \phi_i(B)$$

where $\phi_i(B)$, $1 \leq i \leq s$, are real valued polynomials in $B$ with distinct roots on the unit circle;
Model B can be written as

$$
\phi_1(B) \phi_2(B) \cdots \phi_s(B) z_t = \phi_1(BB) \phi_2(BB) \cdots \phi_s(BB) a_t
$$

or

$$
a_t = \frac{\phi_1(B) \phi_2(B) \cdots \phi_s(B)}{\phi_1(BB) \phi_2(BB) \cdots \phi_s(BB)} z_t.
$$

We define

$$
\omega_t(s) = \frac{\phi_s(B)}{\phi_s(BB)} z_t,
$$

and through continued application of Corollary 2.1, the claim is proved.

q.e.d.

2.4 Interpretation of the results

Summary of the equivalence theorems:

Brown's forecasting procedures with specified fitting functions and smoothing parameter $\beta$ will give minimum mean squared error forecasts if the underlying process follows the ARIMA model

$$
\phi(B) z_t = \phi(BB) a_t.
$$

(2.4.1)

The roots of $\phi(B) = 0$ lie on the unit circle and the eventual forecast function of (2.4.1) is given by the fitting functions of Brown's model.

Shortcomings of Brown's exponential smoothing methods:

In light of the above theorem the shortcomings of Brown's forecasting procedure are threefold and they are summarized below:

i) The fitting functions of Brown's method which determine the form of $\phi(B)$ are picked by unreliable identification procedures. The form of the autoregressive operator is in fact decided by the choice of the fitting functions and cannot be safely chosen by visual inspection of the time series itself. More reliable identification tools such as sample autocorrelation and sample partial autocorrelation function have to be considered.

A quadratic fitting function might well be used, and is actually used by Brown, to fit short pieces of data generated by a random walk model. If such fitting were relevant to forecasting, one might conclude that a polynomial of second degree was indicated. The random walk model, however, would actually lead to a polynomial forecast function of degree zero.

ii) The exponentially discounted weighted least squares procedure forces the moving average
operator, the right hand side of model (2.4.1),
to be of the form $\phi(B)$. It is thus automatically
determined by the autoregressive part on the left
hand side of model (2.4.1) and is a function of
the smoothing constant only.

iii) The smoothing constant $\beta$ is assumed to be known.
Brown states that the smoothing constant should
be picked between .7 and .9. Actual study of
time series, however, gives no empirical support
to this assertion and no theoretical reasons seem
to be available for discussion. The supposition
that $\beta$ ought to be picked in this range appears
strange.

The $\pi$-weights implied by Brown's model:

The $\pi$-weights for the ARIMA model (2.4.1) are derived
by equating coefficients in

$$1 - \phi_1 B - \phi_2 B^2 - \ldots - \phi_n B^n = 1 - \sum_{j=1}^{\infty} \pi_j B^j. \quad (2.4.2)$$

The weights are given by:

$$\pi_1 = (1-\beta) \phi_1$$

$$\pi_j = \beta \phi_1 \pi_{j-1} + \beta^2 \phi_2 \pi_{j-2} + \ldots + \beta^{j-1} \phi_{j-1} \pi_1 + (1-\beta^j) \phi_j \quad 2 \leq j \leq n$$

$$\pi_j = \beta^j \phi_1 \pi_{j-1} + \beta^{j+1} \phi_2 \pi_{j-2} + \ldots + \beta^n \phi_n \pi_{j-n} \quad j \geq n+1 \quad (2.4.3)$$

It is instructive to look at the $\pi$-weights since they show
how past observations are discounted to derive minimum mean
squared error forecasts. The $\pi$-weights will be plotted
for a number of examples considered in the next section.

One must ask the question: "Is there any reason to
believe that the world behaves according to this class of
restricted ARIMA models given in (2.4.1)?" Pandit (1973)
has tried to find some theoretical reasons why business,
economic, and quality control systems can be predicted by
exponential smoothing methods, giving it a "spring-
dashpoint" interpretation. The analogy seems strange and
is contradicted by many time series, which have been
modelled by the three stage iterative Box-Jenkins method.

The data themselves should determine the form of the
model and the value of its parameters. The $\pi$-weights
should depend on the underlying process which has to be
identified properly. The automatic and programmed features
of exponential smoothing procedures encourage users to
forget that they imply a particular underlying stochastic process. Canned computer programs cannot act as critics, but only as sponsors of the methods they use. It is the responsibility of the individual user to question their validity.

Computation of the forecasts:

Brown's forecasting procedures are claimed to be computationally efficient. It is easily seen, however, that the forecasts can be derived more readily directly from the difference equation of the equivalent ARIMA model (2.4.1). Thus, even if one believed in the adequacy of Brown's implied model, one should not use his method to calculate and update the forecasts.

These points are best brought out by consideration of specific examples.

2.5 Examples

Example 2.1: Daily IBM common stock closing prices

The data are given on page 526 of Box-Jenkins (1970). In Figure 2.1 the series of 369 observations is plotted. After inspection of the series, Brown (1962) argued that short pieces of the data could be represented by quadratic curves and that one, therefore, ought to consider quadratic fitting functions given by
\( \hat{p}(t, k) = b_0(t) + b_1(t)k + b_2(t)k^2. \) \hspace{1cm} (2.5.1)

He updates the coefficients of the forecast function by discounted least squares with a smoothing constant of \( \beta = .9 \).

The forecasts for this form of Brown's model are shown for several time origins and for lead times \( t = 1, 2, 3 \) in the first column of Table 2.1.

If Brown's model were to be used then it would be much more convenient to use the theory developed in this chapter and to calculate the forecasts directly from the equivalent difference equation given below

\[ (1 - B)z_t = (1 - .9B)z_{t-1}. \] \hspace{1cm} (2.5.2)

The forecasts are given in the second column of Table 2.1.

In fact, however, as was shown originally by Box and Jenkins, Brown's model seems to be totally inadequate. This is seen for example by the much larger mean squared error of the one, two, and three steps ahead forecasts given in Table 2.1.

Identification using Box and Jenkins methods leads to consider an ARIMA \((0,1,1)\) model with the moving average parameter estimated close to zero

\[ (1 - B)z_t = (1 + .087B)a_t. \] \hspace{1cm} (2.5.3)
It was noted that the model in (2.5.3) is very nearly a random walk as originally suggested by Bachelier (1900). This model implies that the best forecasts of future observations are very nearly the current value of the stock price. This is very different from Brown's model which implies that information about the next value is not only contained in the current observation but also in the observations before.

The $\pi$-weights for the models (2.5.2) and (2.5.3) are shown in the diagram in Table 2.2.

The autocorrelation of the one step ahead forecast errors for models (2.5.2) and (2.5.3) are given in Table 2.1. It can be seen that there is significant autocorrelation among the one step ahead forecast errors for the difference equation model (2.5.2) which is implied by Brown's forecasting procedure in (2.5.1). For the model (2.5.3) the autocorrelations are essentially zero.

Calculating the forecasts:

It is worth emphasizing that if forecasts were to be derived from Brown's model, one should not use Brown's method of calculating them, which is extremely laborious. It is much easier to calculate the forecasts directly.
from the equivalent difference equation. This will give
the same result, except for rounding errors, as it is
shown in Table 2.1.

The same point can be made in terms of a further
equation given by Brown.

Example 2.2: Warmdot filter sales

This series is given on page 434 of Brown (1962).
In Figure 2.2 the series of 120 observations is plotted.
Brown considers the simple 12-point sinusoidal
model

\[ p(t,k) = b_0(t) + b_1(t) \sin \frac{2\pi}{12} k + b_2(t) \cos \frac{2\pi}{12} k \]  (2.5.4)

and updates the coefficients of the forecast function by
discounted least squares with smoothing constant \( \beta = .9 \).
The forecasts for this form of Brown's model are shown in
the first column of Table 2.3.

There is no point in going through the Box-Jenkins
three stage iterative method for this series because this
appears to be an artificial series which has been manu-
factured from the model (2.5.4).

However, it is of interest to see how the theory
of this chapter applies to the method of calculating the
forecasts. The equivalent difference equation is given by
\[(1-B)(1-\sqrt{3b+e^2})z_t = (1-.9b)(1-\sqrt{(.9)B+(.9)^2b^2})a_t \quad (2.5.5)\]

and forecasts using this difference equation give the same result, except for rounding errors, as shown in the second column of Table 2.3.

The \(w\)-weights for the equivalent difference equation model are given in Table 2.4 and indicate how previous observations are used for the calculation of the forecasts.
Table 2.4: The \( \pi \)-weights implied by Brown's model (2.5.5); \((1-B)(1-\sqrt{3B+B^2})z = (1-.9B)(1-.9\sqrt{3B+.81B^2})a_t\).
APPENDIX 2.1

In this appendix we show the following lemma:

**Lemma 2.1:**

If

$$\begin{bmatrix}
1 & 1 & \ldots & 1 \\
\vdots & & & \\
a_1 & a_2 & \ldots & a_n \\
a_1 & a_2 & \ldots & a_n
\end{bmatrix}^{[n \times n]}$$

with $a_i \neq a_j$ (for $i \neq j$)

then

$$A^{-1} = (a_{ij})$$

where the $a_{ij}$ are given by the expansion in

$$P_i(x) = \frac{n}{k=1} \frac{x-a_k}{a_i-a_k} = \frac{n}{j=1} a_{ij} x^{j-1}.$$

**Proof:** The existence of the inverse of the Vandermonde matrix is shown by Hoffman and Kunze (1961). We will show that $A^{-1} A = I$.

$$A^{-1} A = \begin{bmatrix}
a_{11} & \cdots & a_{1n} \\
a_{21} & \cdots & a_{2n} \\
\vdots & & \vdots \\
a_{n1} & \cdots & a_{nn}
\end{bmatrix} \begin{bmatrix}
1 & \cdots & 1 \\
\vdots & & \\
a_1 & \cdots & a_n \\
a_1 & \cdots & a_n
\end{bmatrix} = (v_{ij})$$

where

$$v_{ij} = a_{i1} + a_j a_{i2} + \ldots + a_{ij} a_{i1} \cdot \ldots \cdot a_{in}$$

(A.2.1)

$v_{ij}$ is the coefficient of $x^0$ in the polynomial

$$P_i(x)[1+a_j x^{-1} + a_j^2 x^{-2} + \ldots + a_j^{n-1} x^{-(n-1)}]$$

(A.2.2)

since

$$P_i(x)[1+a_j x^{-1} + a_j^2 x^{-2} + \ldots + a_j^{n-1} x^{-(n-1)}]$$

$$= [a_{i1} x^0 + a_{i2} x^1 + \ldots + a_{in} x^{n-1}] [1+a_j x^{-1} + a_j^2 x^{-2} + \ldots + a_j^{n-1} x^{-(n-1)}]$$

$$= (a_{i1} + a_j a_{i2} + a_j^2 a_{i3} + \ldots + a_j^{n-1} a_{in}) x^0$$

+ other terms with $x^t (t \neq 0)$.

Furthermore,

$$P_i(x)[1+a_j x^{-1} + a_j^2 x^{-2} + \ldots + a_j^{n-1} x^{-(n-1)}]$$

$$= \frac{n}{k=1} \frac{(x-a_k)}{a_i-a_k} \frac{P_i(x)}{a_i-a_k}$$

(A.2.3)

$$= \frac{1}{a_i-a_k} \frac{P_i(x)}{a_i-a_k}$$

$$= \frac{1}{a_i-a_k} \frac{[x^{n-1} c_1(i) x^{n-2} + \ldots + c_{n-1}(i)]}{a_i-a_k}$$

$$x[1+a_j x^{-1} + \ldots + a_j^{n-1} x^{-(n-1)}].$$
where $c_1, \ldots, c_{n-1}$ are the coefficients in the expansion

$$\prod_{k \neq i} (x - a_k) = x^{n-1} c_1 x^{n-2} \cdots c_{n-1}. \tag{A.2.4}$$

The coefficient of $x^0$ in (A.2.3) is given by

$$v_{ij} = \frac{\frac{a_i^{n-1}-a_j^{n-1}}{a_i^{n-1} - a_j^{n-1}} \cdots \frac{a_i - a_j}{a_i - a_j}}{a_i^{n-1} \prod_{k \neq i} \left(1 - \frac{a_k}{a_i}\right)} \cdot \frac{u_j(j)}{s_i}. \tag{A.2.5}$$

Furthermore,

$$s_i = \frac{a_i^{n-1} \prod_{k \neq i} \left(1 - \frac{a_k}{a_i}\right)}{s_i} \quad \text{evaluated at} \quad x = 1$$

but since

$$\frac{a_i^{n-1} \prod_{k \neq i} (x - a_k)}{a_i^{n-1} \prod_{k \neq i} \left(1 - \frac{a_k}{a_i}\right)} = \frac{a_i^{n-1} - a_j^{n-1}}{a_i^{n-1} - a_j^{n-1}} \cdots \frac{a_i - a_j}{a_i - a_j}$$

with $c_i^{(1)} = a_i c_i^{(1)}$ for $1 \leq i \leq n-1$,

we get from (A.2.6)

$$s_i = \frac{a_i^{n-1} - a_j^{n-1}}{a_i^{n-1} - a_j^{n-1}} \cdots \frac{a_i - a_j}{a_i - a_j}$$

$$= \frac{a_i^{n-1} - a_j^{n-1}}{s_i} \cdots \frac{a_i - a_j}{s_i} \cdot (\frac{c_i^{(1)}}{a_i^{n-1}}) \cdot \frac{c_i^{(1)}}{a_i^{n-1}}. \tag{A.2.7}$$

For $i = j$, substituting (A.2.7) into (A.2.5), we get

$$v_{ii} = \frac{u_i(i)}{s_i} = \frac{a_i^{n-1} - a_i^{n-1} c_i^{(1)} + \cdots + a_i c_i^{(1)} + c_i^{(1)}}{s_i} = 1 \quad 1 \leq i \leq n \tag{A.2.8}$$

For $i \neq j$,

$$\prod_{k \neq i, k \neq j} (x - a_k) = x^{n-2} f_1^{(i,j)} x^{n-3} \cdots f_{n-3}^{(i,j)} x^{n-2} f_{n-2}^{(i,j)}. \tag{A.2.9}$$

From (A.2.4) it can be easily shown that

$$\begin{cases} c_1^{(1)} = f_1^{(i,j)} - a_j \\ c_2^{(1)} = f_2^{(i,j)} - a_j f_1^{(i,j)} \\ \vdots \\ c_{n-2}^{(1)} = f_{n-2}^{(i,j)} - a_j f_{n-3}^{(i,j)} \\ c_{n-1}^{(1)} = - a_j f_{n-2}^{(i,j)} \end{cases} \tag{A.2.10}$$

Thus, for $i \neq j$
\[ u_i(t) = -\sum_{j=1}^{n} u_{ij} e^{-\alpha_{ij} t} \]

where

\[ \alpha_{ij} = \frac{\gamma (1 - \sigma_{ij})}{\mu_{ij}} \]

and

\[ \gamma, \mu_{ij}, \sigma_{ij} \]

for \( i \neq j \).

\[ \frac{u_i(t)}{A} = 0 \]

for \( i = j \).

q.e.d.


Distribution of this document is unlimited.
This paper shows that exponential smoothing forecast procedures, in particular those recommended by Brown, will provide optimal (MMSE) forecasts only if the underlying process is a member of a particular restricted class of ARIMA models. Actual study of time series, however, does not give any empirical support to this restricted class of models.