GENERAL STATISTICAL PROCEDURES:
PARAMETER ESTIMATION USING
WEIBULL DISTRIBUTION, RELIABILITY
TEST OF HYPOTHESIS, AND
COMPUTATION OF EXPECTED
NUMBER OF RENEWALS

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### General Statistical Procedures: Parameter Estimation Using Weibull Distribution, Reliability Test of Hypothesis, and Computation of Expected Number of Renewals

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- Renewal rate
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- Probability theory

**ABSTRACT:**
This addendum details the statistical procedures utilized in preparing the report, Decision Risk Analysis for XM204, 105mm Howitzer, Towed Reliability/Durability Requirements (PAA-TR-1-73). A computer program was developed for this study to simulate DT/OT II testing analysis of data, the decision made based on this analysis, and the funding implications of this decision. A two-parameter Weibull family was assumed to describe durability failures. The test results were used to estimate the two parameters for each subsystem: carriage, recoil system, tube, and breech. This was followed by a test of hypothesis which tested whether sufficient information was
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available to reject the hypothesis that the required durability had been obtained. The subsystem durability was then used to compute replacement requirements over the system lifetime.
INTRODUCTION

This addendum details the statistical procedures utilized in preparing the report, Decision Risk Analysis for XM204, 105MM Howitzer, Towed Reliability/Durability Requirements (PAA-TR-1-73).¹ A computer program was developed for this study to simulate DT/OT II testing, analysis of test data, the decision made based on this analysis, and the funding implications of this decision.

A two-parameter Weibull family was assumed to describe durability failures. The test results were used to estimate the two parameters for each subsystem: carriage, recoil system, tube, and breech. This was followed by a test of hypothesis which tested whether sufficient information was available to reject the hypothesis that the required durability had been obtained. The subsystem durability was then used to compute replacement requirements over the system lifetime.

PROBLEMS

The first statistical problem, parameter estimation, was unusual in that a large number of subsystems would survive testing. This occurred as the carriage was a long-lived subsystem and the test truncation point was near the expected carriage life. Also, other failed subsystems would be replaced until one of the following occurred:

a. The carriage failed.

b. The truncation point was obtained.

c. No additional back-up subsystems were available.

Thus, for each subsystem the test data would consist of rounds-to-failure

and random truncation points. Parameter estimation was performed by maximizing the log likelihood equations.

The next problem was to use these data in a test of hypothesis. This was accomplished by using the large-sample distribution properties of maximum likelihood estimators, i.e., that the parameters have a joint normal distribution. The α-percentile confidence interval is approximately ellipsoidal; the α-percentile durability boundary was obtained by maximizing over this ellipse.

The third problem was solving for the number of renewals in a specified time interval. This was accomplished, initially, by the method of Lomnicki and later (to save computer time), by taking averages.

The purpose of this addendum is to detail these procedures.

PARAMETER ESTIMATION

The form of the density function for the two-parameter Weibull distribution used in this study is:

\[ f(t) = \lambda t^{\alpha-1} e^{-\lambda t^\alpha} \quad t > 0; \alpha > 0; \lambda > 0. \]  

Consider the following subsystem-life test. M-like systems are placed on test and each system is composed on one critical subsystem and several different noncritical subsystems. A total system configuration is required to conduct the test; however, with respect to probability of failure, each subsystem is assumed to be independent. As each non-critical subsystem failure occurs, the failure time is noted, and the failed subsystem

is replaced with an identical new subsystem. Each of the M systems con-
tinue with the test, with failed noncritical subsystems being replaced,
until either a predetermined system truncation point (for example, when
a stated number of rounds have been fired) is reached or until a critical
subsystem failure occurs. This test differs from the well-known Type I
and Type II censoring in that the subsystem truncation points are ran-
dom variables as is the number of each type of noncritical subsystems
put on test.

Let N be the total number of a particular noncritical subsystem
put on test. Let n of these subsystems fail and their failure times be
observed. The remaining m = N - n subsystems are removed from test at
the truncation points T_1, T_2, T_3,...T_m. Then the logarithmic likelihood
function lnL(α,λ), based on the above sample where (1) is the applicable
failure density function, is given by

\[ \ln L(α,λ) = n \ln α + n \ln λ + (α-1) \sum_{i=1}^{n} \ln y_i - \frac{\lambda}{\lambda} \sum_{i=1}^{n} y_i^α - \frac{λ}{j} \sum_{j=1}^{m} x_j^α \]  

(2)

where

\[ y_i = \text{an observed failure time}, \]

and

\[ x_j = \text{a system truncation point}. \]

Then

\[ \ln L(α,λ) = n \ln α + n \ln λ + (α-1) \sum_{i=1}^{n} \ln t_i - \frac{\lambda}{\lambda} \sum_{i=1}^{n} t_i^α \]

(3)
where
\[ t_i = y_i \quad i < n \]
\[ = x_i \quad n < i \leq N. \]

This function yields the following likelihood equations

\[ \frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln t_i - \sum_{i=1}^{N} \ln t_1 \frac{\alpha}{\sum_{i=1}^{n} t_i \ln t_i} = 0, \quad (4) \]

and

\[ \frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{N} \frac{\alpha}{\sum_{i=1}^{n} t_i \ln t_i} = 0. \quad (5) \]

Solving (5) in terms of \( \lambda \) and substituting into (4) yields

\[ \frac{n}{\alpha} + \sum_{i=1}^{n} \ln t_i - \frac{n}{\sum_{i=1}^{n} t_i \ln t_i} \cdot \sum_{i=1}^{N} \frac{\alpha}{\sum_{i=1}^{n} t_i \ln t_i} = 0. \quad (6) \]

Equation (6) can be solved for by a standard iterative procedure but by writing (6) in the form

\[ K(\alpha) = \frac{\sum_{i=1}^{N} \frac{\alpha}{t_i \ln t_i} - \frac{n}{\sum_{i=1}^{n} \ln t_i}}{\frac{n}{\alpha} - \frac{1}{\sum_{i=1}^{n} \ln t_i}} = 0, \quad (7) \]
it has been shown³ that if we find a $K(\alpha_1) < 0$ and a $K(\alpha_2) > 0$ and $\alpha_1$ and $\alpha_2$ are within a sufficiently narrow interval such that $\alpha_1 < \hat{\alpha} < \alpha_2$, a linear interpolation will yield the required value. While this method will eventually yield an answer, it would appear that the rate of convergence can be immensely improved by utilizing basic search techniques. In addition, the search space can be reduced by choosing an initial $\alpha_1$ that is relatively close to $\hat{\alpha}$. Dubey⁴ has suggested that the moment estimators be utilized for the initial value and Cohen³ developed a table of values relating $\alpha$ and the sample coefficient of variation. Tables relating the coefficient of variation to $\alpha$ require external storage areas (disks/tapes/drums) or, if internal to a computer program, take up precious core storage. Therefore, the following equation is utilized which yields an initial estimator based on the coefficient of variation which is within 0.1% of similar table values.

$$\alpha_1 = .64364 + .18035 \operatorname{csch}(z) - .0317523z + .000684128z^2$$
$$- .00129259 \operatorname{csch}^2(z) + .619344(e^{-z}) + .534717(e^{-z})^4$$
$$+ .54047(e^{-z})^7$$

(8)

where

$$z = \frac{S^2}{2x} = cv = \text{coefficient of variation}.$$ 

With this initial estimator, a "direct search with acceleration" is utilized to obtain the maximum likelihood estimate (MLE) of $\alpha$ or $\hat{\alpha}$.

³Cohen, A. C., Maximum Likelihood Estimation in the Weibull Distribution Based on Complete and on Censored Samples. Technometrics 7 (4), 579-588 (1965).

It has been shown\(^5\) that $\hat{\alpha}/\alpha$ is distributed independently of $\alpha$ and $\lambda$ and has the same distribution as $\hat{\alpha}$. Therefore, the bias in $\hat{\alpha}$ is independent of the true value of $\alpha$ and $\lambda$ and is only dependent on the sample size $n$. A table of unbiassing factors $B(n)$ was developed so that $E[B(n)\hat{\alpha}] = \alpha$. The values in this table are approximated by the following equation:

$$
B(N) = .06541717052 - .00006172777867n + .997393979 \tanh^{10}(n) - 220.5624312 \tanh^{7}(\ln n/n) - 1.86171021(\ln n)(\tanh(\ln n/n^2)) + .01424021769(\ln n) \tanh^{10}(n) - 39302536740000.\tanh^{8}(\ln n/n^3)
$$

Once the MLE of $\alpha$ is obtained from the search of equation (7), the appropriate unbiassing factor equation (9) is applied before solving for $\lambda$, using equation (5).

**AN ILLUSTRATIVE EXAMPLE**

Ten systems are placed on test and the truncation point, $T_i$, for each system is given (in rounds) below.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$T_1$</td>
<td>$T_2$</td>
<td>$T_3$</td>
</tr>
<tr>
<td>35,142</td>
<td>43,839</td>
<td>45,540</td>
</tr>
</tbody>
</table>

During the course of the test, 23 failures/replacement ($Y_i$) of subsystem $z$ occurred at the following time:

and at the time of each system truncation, the following subsystem times ($x_i$) were observed:

$$
\begin{align*}
  x_1 &= 1,395 \\
  x_2 &= 8,564 \\
  x_3 &= 18,354 \\
  x_4 &= 12,624 \\
  x_5 &= 25,157 \\
  x_6 &= 40,855 \\
  x_7 &= 5,863 \\
  x_8 &= 38,806 \\
  x_9 &= 14,662 \\
  x_{10} &= 8,940
\end{align*}
$$

This sample for subsystem z is from a population in which $\alpha = 1.364$ and $\lambda = .00000102336$. This data can be summarized as:

$$
\begin{align*}
  n &= 23, \quad N = 33, \quad \Sigma t_i = 499618, \quad \Sigma t_i^2 = 14396873864.
\end{align*}
$$

It follows that $\bar{x} = 21722.52$, $s^2 = 154083086.858$, and the coefficient of variation $cv = .3265$. Then from (8) the initial estimate of $\alpha = 1.81$. To obtain the maximum likelihood estimate, $\alpha$ is varied in equation (7) until $|K(\alpha)|$ is minimized. This results in $\hat{\alpha} = 1.6421$ and applying the unbiasing factor for a sample size of 33, $\hat{\alpha} = 1.5731$. 
TEST OF HYPOTHESIS

Consider the following failure density function:

\[ f(t; \alpha, \lambda) = \alpha^t \lambda^{-1} e^{-\lambda t} \quad \alpha > 0; \lambda > 0; t > 0 \]  

(10)

When the resulting sample likelihood function of a life test can be written as

\[ L = \prod_{i=1}^{n} \left( \alpha^t \lambda^{-1} e^{-\lambda t} \right)^{t_i} e^{j=n+1} \]

(11)

then, this function yields the following partial derivatives of the logarithmic likelihood equations:

\[ \frac{\partial \ln L}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \ln t_i - \lambda \sum_{i=1}^{n} t_i^\alpha \ln t_i = 0, \]  

(12)

\[ \frac{\partial \ln L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{N} t_i^\alpha = 0, \]  

(13)

\[ \frac{\partial^2 \ln L}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \lambda \sum_{i=1}^{N} t_i^\alpha (\ln t_i)^2 = 0, \]  

(14)

\[ \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} = -n \frac{\alpha}{\lambda^2} = 0, \]  

(15)

\[ \frac{\partial \ln L}{\partial \alpha \partial \lambda} = -\sum_{i=1}^{N} t_i^\alpha \ln t_i = 0. \]  

(16)
The maximum likelihood estimate (MLE) $\hat{\alpha}$ and $\hat{\lambda}$ can be obtained by solving (13) in terms of $\lambda$ and substituting into (12). Having obtained the MLE for the parameters of the Weibull law, it is then possible to determine confidence limits for meaningful parametric functions such as durability and reliability.

Durability can be defined as the probability that a randomly selected item from an infinite lot will continue to perform satisfactorily without a durability failure beyond $t_0$ and is given by

$$\text{Prob}[T > t_0] = \int_{t_0}^{\infty} f_T(t) dt = \exp(-\lambda t_0^\alpha).$$ (17)

If the parameters $\alpha$ and $\lambda$ are known, the above problem is completely solved and the exact answer is given by (17); however, usually these parameters are unknown. It is well known that the large sample MLE is approximately normally distributed about the true parameter value as a mean for large samples. This is a powerful tool and will be used to establish confidence limits on durability when the true parameters $\alpha$ and $\lambda$ are unknown. Assuming the above, it follows that

$$[\hat{\alpha} - \alpha, \hat{\lambda} - \lambda] \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} \hat{\alpha} - \alpha \\ \hat{\lambda} - \lambda \end{bmatrix} \approx \chi^2(2),$$ (18)

where

$$C_{11} = -E \left[ \frac{\partial^2 \ln L}{\partial \alpha^2} \right],$$ (19)

C_{22} = -E \left[ \frac{\partial^2 \ln L}{\partial \lambda^2} \right], \quad (20)

and

C_{12} = C_{21} = -E \left[ \frac{\partial^2 \ln L}{\partial \alpha \partial \lambda} \right]. \quad (21)

From (14) and (19)

C_{11} = E \left[ \frac{-n}{\alpha^2} + \lambda \sum_{i=1}^{N} t_i^\alpha (\ln t_i)^2 \right]. \quad (22)

Examine

I = \int_{0}^{\infty} t^\alpha (\ln t)^2 \alpha e^{-\lambda t} dt

and let

\begin{align*}
\omega &= t^\alpha, \\
\omega &= \alpha t^{\alpha-1} dt, \\
\ln \omega &= \alpha \ln t, \\
\ln t &= \frac{1}{\alpha} \ln \omega, \\
(\ln t)^2 &= \frac{1}{\alpha^2} (1n \omega)^2.
\end{align*}

then

I = \frac{\lambda}{\alpha^2} \int_{0}^{\infty} \omega (1n \omega)^2 e^{-\lambda \omega} d\omega.

From page 578 of referenced literature\textsuperscript{7}, equation 4.358(2)

\[ \int_0^\infty x^{v-1}e^{\mu x}(\ln x)^2dx = \Gamma(v)/\mu^v\{[\psi(v)-\ln(\mu)]^2 + \zeta(2,v-1)\} \]

\[ \mu > 0, \nu > 0 \]

now let

\[ x = \omega, \]
\[ \nu = 2, \]
\[ \mu = \lambda, \]

then

\[ I = \frac{\lambda}{\alpha^2} \int_0^\infty \omega(\ln \omega)^2 e^{-\lambda \omega} d\omega = \frac{\lambda \Gamma(2)}{(\alpha^2 \lambda^2)} \{[\psi(2)-\ln \lambda]^2 + \zeta(2,1)\} \]

\[ = \frac{\Gamma(2)}{\lambda \alpha^2} \{[\psi(2)+\ln \frac{1}{\lambda}]^2 + \zeta(2,1)\} \]

From page 1073 of referenced literature, equation 9.521(1) the Riemann's Zeta Function

\[ \zeta(z,q) = \sum_{N=0}^{\infty} \frac{1}{(q+N)^2} \quad z > 1 \]

\[ \therefore \zeta(2,1) = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots \]

From page 260 of referenced literature, equation 6.4.10 the Polygamma Function

\[ \psi^{(N)}(z) = (-1)^{N+1}N! \sum_{k=0}^{\infty} (z+k)^{-N-1} \quad (z \neq 0, -1, -2, \ldots) \]

\[ \psi^1(1) = (-1) \cdot 1! \sum_{k=0}^{\infty} (1+k)^{-2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \]

therefore,

\[ \zeta(2,1) = \psi^1(1) \]

and

\[ I = \frac{\Gamma(2)}{\lambda^2} \left\{ \left[ \psi(2) + \ln \frac{1}{\lambda} \right]^2 + \psi^1(1) \right\} \]

Equation (22) can now be written as

\[ c_{11} = \frac{n}{a^2} + \frac{N}{a^2} \left\{ \left[ \psi(2) + \ln \frac{1}{\lambda} \right]^2 + \psi^1(1) \right\} \quad (23) \]

where

\[ \psi(2) = .4227843351 \]

and

\[ \psi^1(1) = 1.6449340668 \]

From (16) and (21)

\[ c_{12} = E \left[ \sum_{i=1}^{N} t_i^\alpha \ln t_i \right] \quad (24) \]

Examine

\[ I = \int_0^\infty t^\alpha \ln t \alpha t^{\alpha-1} e^{-\lambda t} dt \]

and let

\[ w = t^\alpha, \]

\[ dw = \alpha t^{\alpha-1} dt, \]

\[ \ln w = \alpha \ln t, \]

\[ \ln t = \ln w / \alpha, \]
then

\[ I = \frac{\lambda}{\alpha} \int_0^\infty \omega \ln \omega \ e^{-\lambda \omega} \ d\omega. \]

From page 576 of referenced literature, equation 4.352(1)

\[ \int_0^\infty x^{v-1} e^{-\mu x} \ln x dx = \frac{1}{\mu^v} \Gamma(v)[\psi(v)-\ln(\mu)] \quad u > 0, v > 0 \]

let

\[ v = 2, \]
\[ \mu = \lambda, \]
\[ x = \omega, \]

then

\[ I = \frac{\lambda}{\alpha} \int_0^\infty \omega \ln \omega \ e^{-\lambda \omega} \ d\omega = \left(\frac{1}{\lambda}\right) \left(\frac{1}{2^2}\right) \Gamma(2)[\psi(2)-\ln\lambda] \]

\[ = \frac{1}{\alpha \lambda} \ [\psi(2)+\ln \left(\frac{1}{\lambda}\right)]. \]

Equation (24) can now be written

\[ C_{12} = \frac{N}{\lambda \alpha} \ [\psi(2)+\ln \left(\frac{1}{\lambda}\right)], \quad (25) \]

and it follows that

\[ C_{22} = \frac{N}{\lambda^2}. \quad (26) \]

As just shown, the \( C_{i,j} \) matrix is the inverse asymptotic variance-covariance matrix of \( \hat{\lambda}, \hat{\alpha} \) and is obtained by taking the negatives of the expected values of the second order derivatives of logarithms of the likelihood functions. Using (18), we can now obtain the appropriate confidence limits for the true probabilities. In the case of reliability

\[ ^7 \text{Loc. Cit.} \]
or durability, we want to insure that the true reliability or durability is above some minimum value $R_m$ or $D_m$ at a desired confidence level. From (18), then,

$$C_{11} (\hat{\alpha} - \alpha)^2 + 2C_{12} (\hat{\lambda} - \lambda)(\hat{\alpha} - \alpha) + C_{22} (\hat{\lambda} - \lambda)^2 = \chi^2(2).$$ (27)

The above equation describes the boundary of a confidence region in the parametric space $(\alpha, \lambda)$. It is approximately chi-squared distributed with two degrees of freedom and will serve to determine an ellipsoidal confidence region in the $(\alpha, \lambda)$ space. In order to obtain the upper confidence limit, all that is required is to search the parameter space, (27), varying $\alpha$ and $\lambda$ until the durability function (17) is maximized.

**A PROCEDURE FOR COMPUTING EXPECTED NUMBER OF RENEWALS**

Let $X_1, X_2, X_3, \ldots$ be a renewal process, that is, a sequence of independent, nonnegative and identically distributed random variables which are not all zero with probability one, with the probability density function $f(x)$ and the distribution function $F(x)$. $S_k = X_1 + X_2 + \ldots + X_k$ is interpreted in renewal theory as the time up to the $k$th renewal and the probability that $S_k < x$ is given by the $k$-fold convolution of $F(x)$

$$F_k(x) = \int_0^x F_{k-1}(x-t) dF(t),$$

where

$$F_0(x) = 1.$$

The primary purpose of this procedure is to calculate $N_t$ which is defined to be the maximum suffix $K$ for which $S_{k^*} < t$, subject to the
convention \( N_t = 0, \) if \( X_1 > t. \) In this application, the \( X_i \) represents successive lifetimes of the object being renewed, and \( N_t \) is the number of renewals made by time \( t, \) subject to the original object having been installed at time 0, i.e., \( N_t \) is the number of renewals in \( (0,t). \)

The usual procedure for determining various functions of Renewal Theory is to find the Laplace Transforms of these functions and then revert to the time domain. However, in the case of the Weibull distribution, this approach is not convenient and Smith and Leadbetter\(^9\) have developed an expansion of the Renewal Function into a power series of \( t^\alpha \) where \( \alpha \) is the Weibull shape parameter. White\(^10\) demonstrated a procedure for evaluating the higher moments and cumulants of the number of renewals \( N_t \) and computed the mean \( M(t) \) and the standard deviation, \( \sqrt{\text{Var} N_t} \), of \( N_t. \) Lomnicki\(^2\) developed a method to evaluate the distribution of the number of renewals which is defined by the family of functions \( W_k(t) \) where

\[
W_k(t) = F_k(t) - F_{k+1}(t) \quad (k=0,1,\ldots)
\]

and is the probability of exactly \( k \) renewals in \( (0,t). \)

In order to be able to evaluate more of the various forms of the renewal functions in the case of the Weibull renewal process, the basic method developed by Lomnicki was used.

As shown by Lomnicki, if we assume that \( W_k(t) \) is represented by the unique series

\[
W_k(t) = \sum_{s=k}^{\infty} a_k(s)t^s P_s(t^\alpha)
\]


2. Loc. Cit. 19
where
\[ P_k(t) = e^{-t} \frac{t^k}{k!} \quad (k=0,1,2,...) , \]
then
\[ W_k(t) = \sum_{s=k}^{\infty} P_s(t^a) \sum_{p=k}^{\infty} (-1)^{p+k} s^p b_k(p) \frac{b_k(p)}{\gamma(p)} , \]
and we have
\[ a_k(s) = \sum_{p=k}^{\infty} (-1)^{p+k} s^p b_k(p) \frac{b_k(p)}{\gamma(p)} \quad (k=0,1,...; s=k,k+1,...) \]
where
\[ \gamma(r) = \Gamma(ar+1)/\Gamma(r+1) \quad (r=0,1,...) \]
and
\[ b_0(s) = \gamma(s). \quad (s=0,1,...) \]
\[ b(s) = \sum_{r=k}^{s-1} b_k(r) \gamma(s-r) \quad (k=0,1,...; s=k,k+1,k+2,...) . \]

The expected number of renewals \( M(t) = E[N(t)] \) which is the renewal function is given by
\[ M(t) = \sum_{n=0}^{\infty} n P[N(t)=n] \]
\[ M(t) = \sum_{n=0}^{\infty} n W_n(t) . \]
Since
\[ W_1(t) + 2W_2(t) + 3W_3(t) + ... \]
is equal to
\[ F_1(t) - F_2(t) + 2F_2(t) - 2F_3(t) + 3F_3(t) - 3F_4(t) + ... , \]
then
\[ \sum_{n=0}^{\infty} n W_n(t) = \sum_{n=0}^{\infty} F_n(t) \]

and
\[ M(t) = \sum_{n=0}^{\infty} F_n(t) , \]

where \( F_n(t) \) is the convolution of the cumulative distribution function \( F(t) \) of the first renewal time and the \( n-1 \) subsequent renewal-time distributions and is the distribution of \( S_n \), the time of the \( n \)th renewal. Now,

\[ F_k(t) = W_k(t) + F_{k+1}(t) \]

\[ = \ldots + W_{k+1}(t) + F_{k+3}(t) \]

\[ \ldots + \sum_{r=k}^{\infty} W_r(t) \]

\[ (k=1,2,3\ldots) \]

or

\[ F_k(t) = 1 - \sum_{r=0}^{k-1} W_r(t) , \]

so we can now express \( F_k(t) \) as

\[ F_k(t) = \sum_{r=k}^{\infty} \sum_{s=r}^{\infty} a_r(s) P_s(t^g) ; \]

Substituting the Poisson cumulative function

\[ D_s(t) = \sum_{r=s}^{\infty} e^{-t} \frac{t^r}{r!} \]

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means that
\[ F_k(t) = \sum_{s=k}^{\infty} \beta_k(s) D_s(t^q) \]
where
\[ \beta_k(k) = a_k(k) , \]
and
\[ \beta_k(s) = \sum_{r=k}^{s} a_r(s) - \sum_{r=k}^{s-1} a_r(s-1) \quad (s > k) , \]
so that
\[ M(t) = \sum_{k=1}^{\infty} F_k(t) = \sum_{s=1}^{\infty} D_s(t^q) \sum_{k=1}^{s} \beta_k(s) , \]
or
\[ M(t) = \sum_{k=1}^{\infty} C(s) D_s(t^q) , \]
where
\[ C(s) = \sum_{k=1}^{s} \beta_k(s) \quad (s=1,2,...) . \]

We can now evaluate \( M_k(t) \), the Renewal Function, \( W_k(t) \); the probability of exactly \( k \) renewals; and \( F_k(t) \), the distribution of time of the \( k \)th renewal. These functions are very useful in evaluating models of reliability, inventory, and queueing process.

\[ M_k(t) = E^{(N(t))} = \sum_{n=0}^{\infty} \binom{n}{k} (F_n(t) - F_{n+1}(t)) . \]

The moments of \( N(t) \) may be derived from \( M_k(t) \) by the relationship

\[ E(N(t))^k = \sum_{n=1}^{k} \binom{k}{n} M_n(t) n! \]
where \( t_{k,n} \) are the Stirling numbers of the second kind and represent the number of ways of partitioning a set of \( k \) elements into \( n \) nonempty subsets.

In particular

\[
E\{N(t)\} = M_1(t),
\]

and

\[
E\{N(t)^2\} = M_1(t) + 2M_2(t).
\]

Or, if desired, the moments of \( N(t) \) may be derived from the definition of expected values.

\[
E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K \text{Prob}(N(t)=n),
\]

\[
E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K [F_n(t) - F_{n+1}(t)],
\]

\[
E\{N(t)^K\} = \sum_{n=0}^{\infty} n^K \hat{w}_n(t),
\]

\[
E\{N(t)^K\} = \sum_{n=1}^{\infty} n^{-(n-1)} K_{F_n(t)}.
\]


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