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TRANSFORMATIONS APPLICABLE TO MISSILE AND SATELLITE TRAJECTORY COMPUTATIONS

John J. O'Connor

RCA International Service Corporation

Prepared for:

Air Force Eastern Test Range

July 1975

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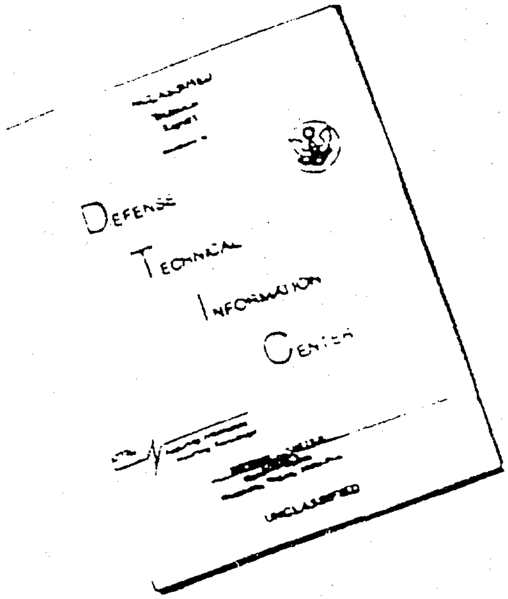

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JULY 1975

RCA INTERNATIONAL SERVICE CORPORATION
MISSILE TEST PROJECT
PATRICK AIR FORCE BASE, FLORIDA

HEADQUARTERS AIR FORCE EASTERN TEST RANGE (AFSC)
PATRICK AIR FORCE BASE, FLORIDA 32925

314

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FOREWORD

This final report was prepared by J. J. O'Connor and submitted by RCA International Service Corporation, Patrick Air Force Base, Florida 32925 under Subcontract No. 73-0030-01, with Pan American World Airways, Inc., under Prime Contract No. F08606-72-C-0030 and monitored by the Directorate of Range Operations, Systems Analysis Office, Air Force Eastern Test Range, Air Force Systems Command, Patrick Air Force Base, Florida 32925.

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This technical report has been reviewed and is approved for publication.

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This report is a self-contained treatment of transformations with applications to missile and satellite technology. Numerous examples are included in the areas of specific tracking systems, geodetic transformations, earth-fixed and inertial systems, time transformations, transformations of a statistical nature and error analysis.

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TABLE OF CONTENTS

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TABLE OF CONTENTS (Contd)

TABLE OF CONTENTS (Contd)

INTRODUCTION

Analysis of missile and satellite trajectories and tracking data involves transformations of position, velocity and acceleration time points from one system to another—systems which may be fixed or moving relative to each other. This discussion will attempt to give explicit transformation equations for the most common situations which arise and sufficient detail so that the reader can develop any cases not covered explicitly here. In all cases the reference system is that in which the coordinates of a point are known. The new or transient system is that in which the coordinates of the same point are to be determined. Originally the transient system is considered to be coincident with or superimposed upon the reference system. Then by a series of translations, rotations about its own axes and reflections of its own axes the transient system assumes the desired configuration in space. A translation is only a transfer of the origin of the transient system from (0,0,0) as expressed in the reference system to (X_o, Y_o, Z_o) as expressed in the reference system. In this preliminary discussion X, Y, Z will be used to represent the coordinates of the point in the reference X, Y, Z system. x, y, z will be used to represent the coordinates of the point in the transient x, y, z system. In a rotation the origin of the x, y, z system and one of the x, y, z coordinate axes about which the rotation takes place are unchanged. Rotations are defined as clockwise or counter clockwise and apply to an observer at the positive end of the axis of rotation looking toward the origin. A reflection is a reversal of positive and negative ends of a coordinate axis. A single reflection changes a coordinate system from right to left handed or vice versa.

I. Position Transformations in Rectangular Coordinate Systems

Translation is accomplished as follows:

$$(1)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} - \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix} = \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

A reflection of an axis is accomplished by changing the sign of the row in the identity matrix corresponding to that axis. For example, to reflect y:

$$(2)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Now consider the right handed coordinate systems in Figure 1. The x y z system has been translated from coincidence with X Y Z system to X_o Y_o Z_o and then rotated about the z axis in a counterclockwise direction through an angle θ. The point

$$P\begin{pmatrix} X, & Y, & Z \\ x, & y, & z \end{pmatrix},$$

is of course stationary throughout these transformations.

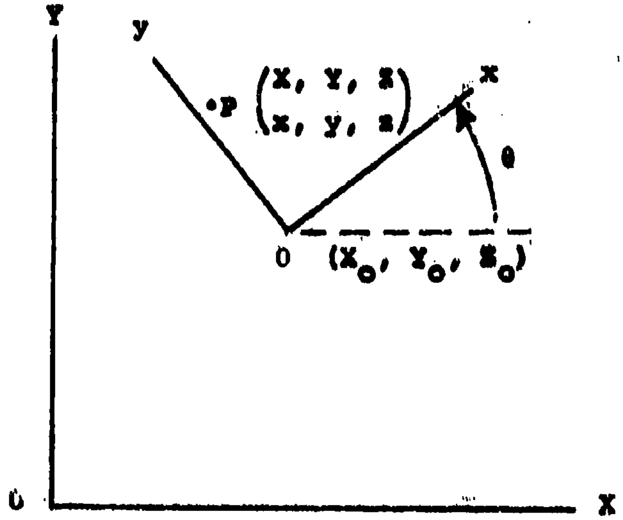


Figure 1.

It is easy to show from simple trigonometry or vector analysis that:

$$
\begin{aligned}
x &= (X - X_o)\cos\theta + (Y - Y_o)\sin\theta \\
y &= -(X - X_o)\sin\theta + (Y - Y_o)\cos\theta \qquad (3) \\
z &= (Z - Z_o)
\end{aligned}
$$

Equations (3) apply for right handed systems with rotation about z in a counterclockwise direction through an angle θ. Equations (3) may be expressed in matrix form:

$$
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix} \qquad (4)
$$

The matrix of trigonometric elements is called a transfer or transformation matrix. Clearly transfer matrix (4) applies also for left handed systems in which the rotation about z is in a clockwise direction through an angle θ.

The transfer matrix by definition is used to pre-multiply the column matrix of the coordinates to be transformed.

Now consider Figure 2 in which the direction of rotation is clockwise about z for two right handed systems.

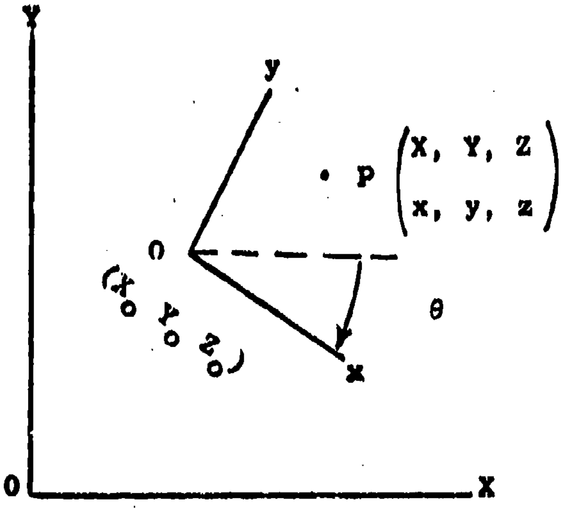


Figure 2.

The transfer matrix is the same as (4) except for a substitution of $-\theta$ for θ. Thus

$$(5)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

Therefore to reverse the direction of rotation in a given transfer matrix reverse the signs before sines.

So far only z axis has been taken as the axis of rotation. Consider the following more general summary:

Counterclockwise rotation about x for two right handed systems

$$(6)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

Counterclockwise rotation about y for two right handed systems

$$(7)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

Counterclockwise rotation about z for two right handed systems

$$(8)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

Counterclockwise rotation about x for two left handed systems

$$(9)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

Counterclockwise rotation about y for two left handed systems

$$(10)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

Counterclockwise rotation about z for two left handed systems

$$(11)\qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

Transfer matrices 6, 7, 8, 9, 10, 11 and the rules previously discussed and now summarized will permit any transformation in rectangular coordinates:

(a) Given X, Y, Z coordinates of a point P in a left handed or right handed coordinate system. Given also an x, y, z coordinate system defined relative to the reference X Y Z system. Required: the x, y, z coordinates of the point P.

(b) Locate the x y z coordinate system first to be coincident with the reference X Y Z system. Then work out a series of translations, rotations, and reflections so that finally it will be located in the proper place relative to the original reference system.

(c) Perform the required transformations in proper sequence.

(d) Since the reference system is different for each successive transformation, trouble is easily encountered in translation. The coordinates for the translation are usually known either before all the rotation transformations or after all the rotation transformations and the translation should be performed when the coordinates are known.

Illustration:

It has been found that a certain transformation involving a left handed reference system requires

(1) translation to X_o Y_o Z_o

(2) counterclockwise rotation about z through α

(3) clockwise rotation about x through β

(4) reflection of z.

The complete transformation equation may be written as follows:

$$(12)\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

Note that the sequence of the transfer matrices goes from right to left. A more practical example is given under the subsequent treatment of Euler angles.

Treatment so far has been only for rectangular coordinates. Spherical coordinate systems are generally first transformed to rectangular systems and then after necessary transformations back to spherical. Several illustrations of the technique are given in a later section treating specific actual coordinate systems.

II. Velocity Transformations in Rectangular Coordinate Systems

Equations for velocity transformations are derived directly from the corresponding position transformations by differentiation. A simple illustration should be sufficient. A common velocity transformation is between geocentric equatorial inertial and geocentric equatorial rotating, both right handed rectangular systems. In this case Z and z axes coincide always and rotation is counterclockwise.

We may differentiate equation (8) with $X_o = Y_o = Z_o = 0$ since the origins of the two systems coincide. By the usual rules for differentiating:

$$(13)\quad \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \\ \frac{dZ}{dt} \end{bmatrix} + \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X\frac{d\theta}{dt} \\ Y\frac{d\theta}{dt} \\ Z \end{bmatrix}$$

$\frac{d\theta}{dt}$ in this equation corresponds to the angular velocity of earth rotation about its polar axis. Equation 13 applies for a transformation from inertial to rotating system because θ increases counterclockwise relative to reference system. For transformation from rotating to inertial system, θ increases clockwise relative to the reference system. Thus the signs before the sines in (8) should be reversed before differentiating.

$$(14)\quad \begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{dX}{dt} \\ \frac{dY}{dt} \\ \frac{dZ}{dt} \end{bmatrix} + \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X\frac{d\theta}{dt} \\ Y\frac{d\theta}{dt} \\ Z \end{bmatrix}$$

For a point P fixed in X Y Z system, put

$$\frac{dX}{dt} = \frac{dY}{dt} = \frac{dZ}{dt} = 0.$$

III. Acceleration Transformations in Rectangular Coordinate Systems

A single illustration should be sufficient. Simply differentiate equations (13) and (14) to get acceleration transformations from (1) geocentric inertial to geocentric rotating and (2) geocentric rotating to geocentric inertial respectively. Remember that

$$\frac{d^2\theta}{dt^2} = 0.$$

$$(15)\quad \begin{bmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \\ \frac{d^2z}{dt^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{d^2X}{dt^2} \\ \frac{d^2Y}{dt^2} \\ \frac{d^2Z}{dt^2} \end{bmatrix} + 2 \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{dX}{dt}\;\frac{d\theta}{dt} \\ \frac{dY}{dt}\;\frac{d\theta}{dt} \\ \frac{dZ}{dt} \end{bmatrix}$$

$$+ \begin{bmatrix} -\cos\theta & -\sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X\left(\frac{d\theta}{dt}\right)^2 \\ Y\left(\frac{d\theta}{dt}\right)^2 \\ Z \end{bmatrix}$$

$$(16)\quad \begin{bmatrix} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \\ \frac{d^2z}{dt^2} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{d^2X}{dt^2} \\ \frac{d^2Y}{dt^2} \\ \frac{d^2Z}{dt^2} \end{bmatrix} + 2 \begin{bmatrix} -\sin\theta & -\cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{dX}{dt}\;\frac{d\theta}{dt} \\ \frac{dY}{dt}\;\frac{d\theta}{dt} \\ \frac{dZ}{dt} \end{bmatrix}$$

$$+ \begin{bmatrix} -\cos\theta & \sin\theta & 0 \\ -\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X\left(\frac{d\theta}{dt}\right)^2 \\ Y\left(\frac{d\theta}{dt}\right)^2 \\ Z \end{bmatrix}$$

For a point P fixed in X Y Z system simply put

$$\frac{dX}{dt} = \frac{d^2X}{dt^2} = 0$$

$$\frac{dY}{dt} = \frac{d^2Y}{dt^2} = 0$$

$$\frac{dZ}{dt} = \frac{d^2Z}{dt^2} = 0$$

IV. Euler's Angles (Ω, ω, i)

Consider inertial axes X Y Z in which there is a moving coordinate system x, y, z. We want to determine the coordinates of a point in the x y z system based upon the Euler angles defining the relative position of the X Y Z and x y z systems.

First let x y z and X Y Z axes coincide. Both are right handed systems. Looking downward from Z axis rotate x y z axes counter-clockwise about z through an angle Ω. Let the new system be x' y' z'. The transfer matrix is given in Equation (8).

$$(17) \qquad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

We now rotate about the x' axis through the angle i where i is measured positively above the X Y plane. Transfer matrix (6) applies.

$$(18) \qquad \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

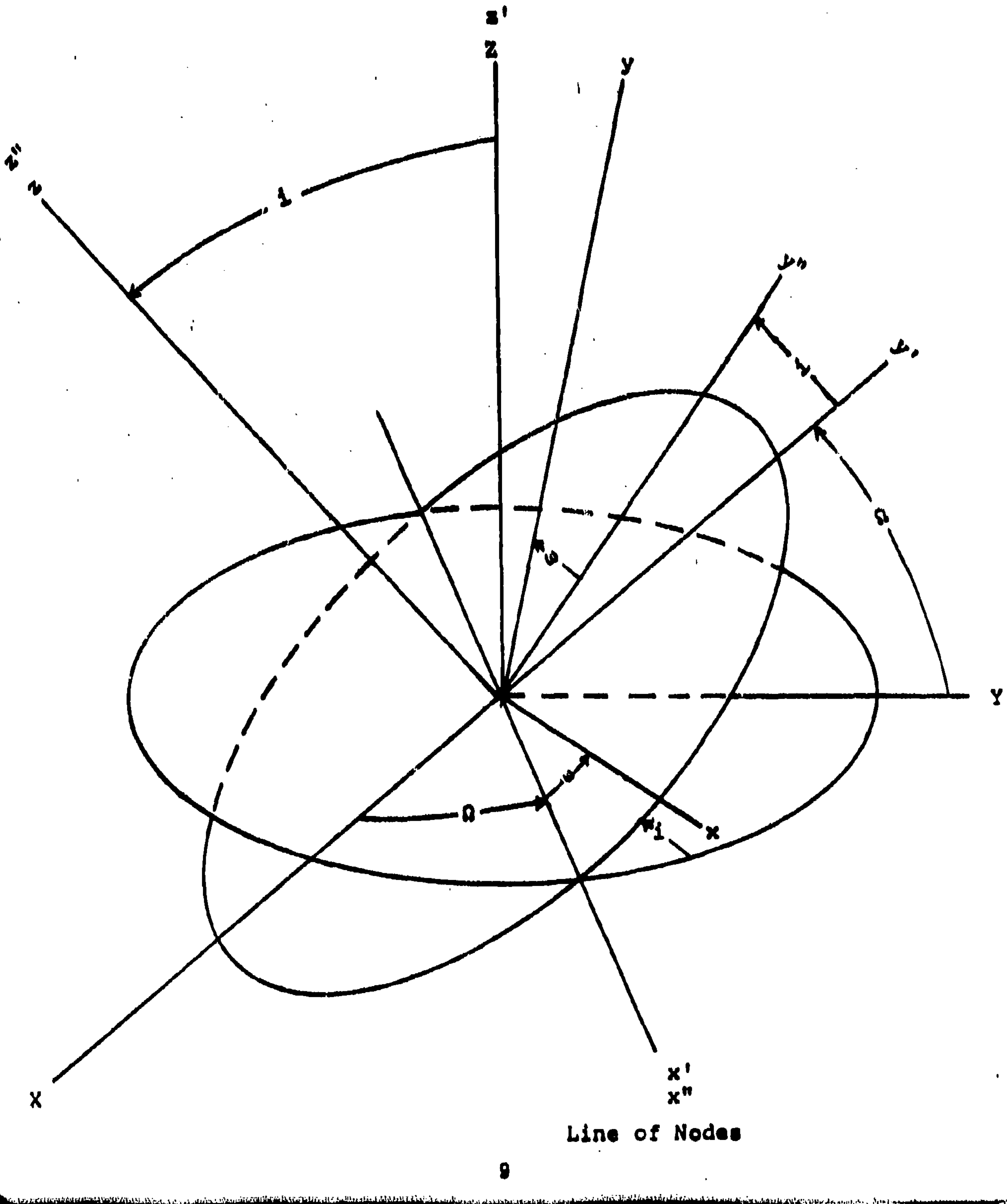
Finally rotate about the z'' axis through the angle ω. Define positions in this final system by x, y, z.

$$(19) \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix}$$

Substituting (17) and (18) into (19)

FIGURE 3

EULER ANGLES



$$(20)\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Expanding this

$$(21)\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega) & (\cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega) & (\sin\omega\sin i) \\ (-\sin\omega\cos\Omega - \cos\omega\cos i\sin\Omega) & (-\sin\omega\sin\Omega + \cos\omega\cos i\cos\Omega) & (\cos\omega\sin i) \\ (\sin i\sin\Omega) & (-\sin i\cos\Omega) & (\cos i) \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

The final transfer matrix may be inverted (only a transposition in this case) or we may proceed in inverse order like the preceding development. In any case we obtain

$$(22)\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} (\cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega) & (-\sin\omega\cos\Omega - \sin\Omega\cos i\cos\omega) & (\sin i\sin\Omega) \\ (\cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega) & (-\sin\omega\sin\Omega + \cos\Omega\cos i\cos\omega) & (-\sin i\cos\Omega) \\ (\sin i\sin\omega) & (\sin i\cos\omega) & (\cos i) \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where Equation (21) applies for transformation from the inertial system to the rotating system. Equation (22) applies for transformations from the rotating system to the inertial system, the reference system in both cases being designated in X Y Z and the transient system in x y z. X Y Z represents inertial system in Equation 21 and rotating system in Equation 22.

V. Transformation of Angular Velocities.

If the x y z system is rotating with a rotation vector $\bar{\alpha}$ it has angular velocity components α_x α_y α_z about x, y, z respectively. We wish to express α_x α_y α_z in terms of the Eulerian angles and Eulerian angular velocities.

The components of $\bar{\alpha}$ along the x", y", and z" axes are as follows

$$
\begin{array}{lcl|ccccc}
\text{due to} \longrightarrow & & & \dot{\Omega} & & \dot{\omega} & & \dot{i} \\
\hline
\alpha_{x''} & & & 0 & + & 0 & + & \dot{i} \\
\alpha_{y''} & = & & \dot{\Omega}\sin i & + & 0 & + & 0 \\
\alpha_{z''} & & & \dot{\Omega}\cos i & + & \dot{\omega} & + & 0
\end{array}
$$

These components were arrived at by inspection of Figure 3.

Now apply the transfer matrix from Equation (8) to rotate from x" y" z" system to x y z system.

$$
(23) \quad \begin{bmatrix} \alpha_x \\ \alpha_y \\ \alpha_z \end{bmatrix} = \begin{bmatrix} \cos\omega & \sin\omega & 0 \\ -\sin\omega & \cos\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha_{x''} \\ \alpha_{y''} \\ \alpha_{z''} \end{bmatrix}
$$

Expanding Equation (23)

$$
(24) \quad \begin{aligned}
\alpha_x &= \dot{\Omega} \sin i \ \sin\omega + \dot{i} \cos\omega \\
\alpha_y &= \dot{\Omega} \sin i \ \cos\omega - \dot{i} \sin\omega \\
\alpha_z &= \dot{\omega} + \dot{\Omega} \ \cos i
\end{aligned}
$$

Having three equations in three unknowns we may solve for $(\dot{\Omega}, \dot{\omega}, \dot{i})$ if we wish

$$
(25) \quad \begin{aligned}
\dot{\Omega} &= \frac{1}{\sin i} (\alpha_x \sin\omega + \alpha_y \cos\omega) \\
\dot{\omega} &= \alpha_z - \frac{\cos i}{\sin i} (\alpha_x \sin\omega + \alpha_y \cos\omega) \\
\dot{i} &= \alpha_x \cos\omega - \alpha_y \sin\omega
\end{aligned}
$$

VI. Transformations Between Two Rectangular Coordinate Systems When Cosines of Angles Between Respective Axes are Known

Many coordinate systems are stationary relative to each other. Thus it is pointless to go through a series of transformations every time data in one system is to be related to another system. For transformations of this type computer programs customarily accept the direction cosines of the axes of one system relative to the axes of the other. The Azusa system* and the geocentric rotating system form such a stationary pair.

Consider two systems x y z and X Y Z. Let the direction cosines of the x axis be ℓ_1 m_1 n_1 referred to the X Y Z system. Let the direction cosines of the y axis be ℓ_2 m_2 n_2 referred to the X Y Z system. Let the direction cosines of the z axis be ℓ_3 m_3 n_3 referred to the X Y Z system. Then it can be shown that

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \tag{26}$$

where X Y Z refer to the coordinates of the point P in the X Y Z system and x y z refer to the coordinates of the same point in the x y z system. Only three of nine direction cosines are independent.

It should furthermore be noted that the original computation of these direction cosines follows the exact series of transformations illustrated by numerous examples given previously. For example, in Equation (21), the term

$$(\cos \omega \cos \Omega - \sin \omega \cos i \sin \Omega) = \ell_1$$

where ℓ_1 is the cosine of the angle between the x axis and the X axis. Also $n_3 = \cos i$, and so forth.

VII. Definitions of Various Coordinate Systems.

It is not practical to develop direct transformation equations between any coordinate system and every other coordinate system. As an expediency all coordinate systems are related to one of three master systems.

Then the three master systems themselves are inter-related. We may develop what might be called a flow chart which will be followed for position transformations.

*Now obsolete

The three master systems are (1) geocentric earth rotating equatorial, (2) geocentric inertial equatorial and (3) heliocentric inertial equatorial.

There is always a problem in symbols when a large number of coordinate systems are being discussed. It was discovered that adherence to the standard ETR symbols for the common coordinate systems created awkward problems in other important coordinate systems. Consequently a logical system developed by Kraft Ehricke for the various geocentric and heliocentric inertial coordinate systems was adopted and the common ETR symbols were then modified as little as necessary to avoid duplication of symbols. For example, the common x y z system in use at the range is called U V W in this write-up, since x y z is used for the geocentric, inertial, equatorial, right handed system.

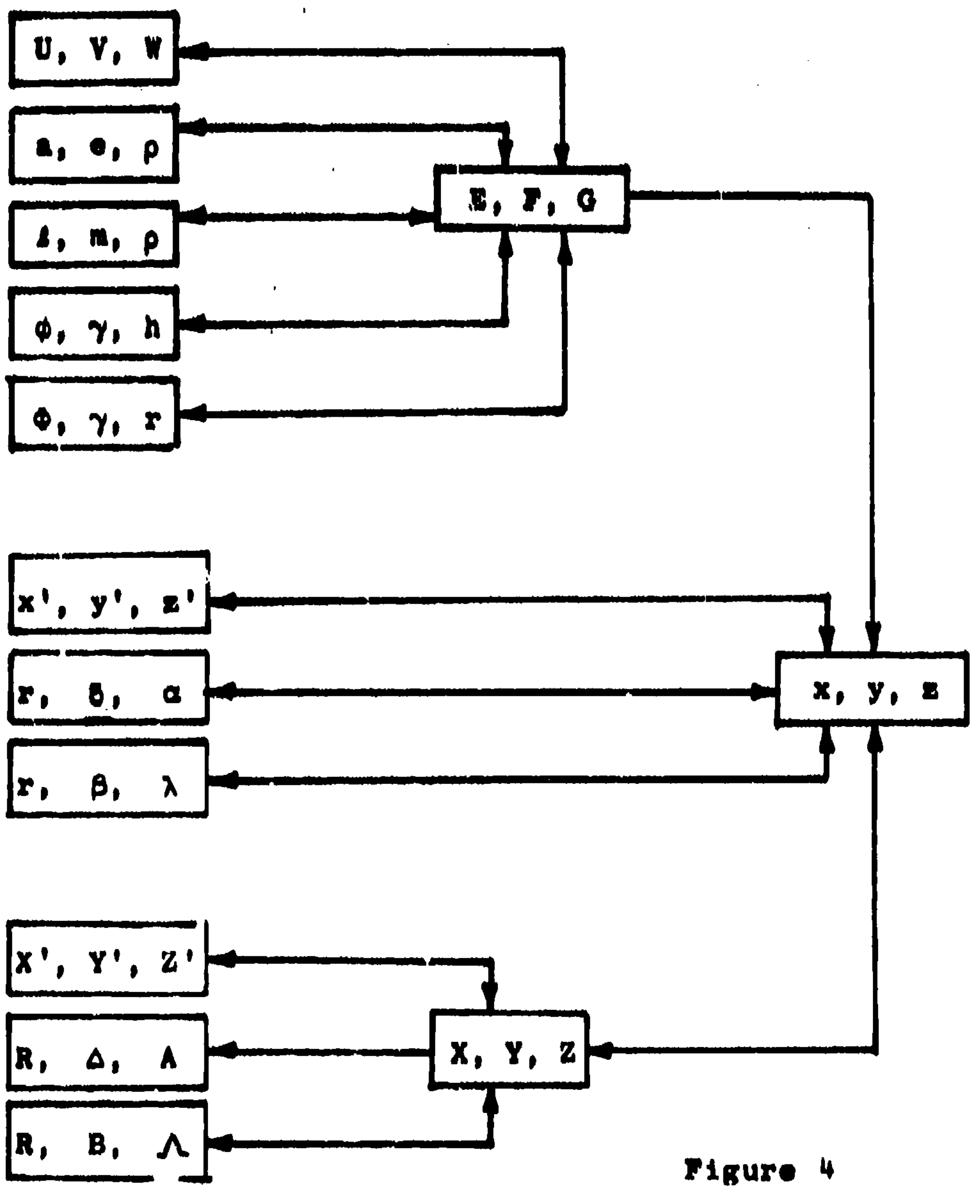


Figure 4

Any or all systems in Fig. 4 can be used to define the position of a point P in space.

A second category of transformations will be velocity transformations. The following flow chart is descriptive of the plan to be followed for velocity and also acceleration:

$$\boxed{\dot{U},\ \dot{V},\ \dot{W}} \longleftrightarrow \boxed{\dot{E},\ \dot{F},\ \dot{G}} \longleftrightarrow \boxed{\dot{x},\ \dot{y},\ \dot{z}} \longleftrightarrow \boxed{\dot{X},\ \dot{Y},\ \dot{Z}} \longleftrightarrow \boxed{\dot{X}',\ \dot{Y}',\ \dot{Z}'}$$

A final category will be those involving orbital elements and thus combined position and velocity

$$\boxed{\begin{array}{c} a,\ e,\ i,\ \Omega,\ \omega,\ u \\ \text{equatorial } \oplus \end{array}} \longleftrightarrow \boxed{\begin{array}{ccc} x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{array}}$$

$$\boxed{\begin{array}{c} a,\ e,\ i,\ \Omega,\ \omega,\ u \\ \text{equatorial } \oplus \end{array}} \longleftrightarrow \boxed{\begin{array}{ccc} X & Y & Z \\ \dot{X} & \dot{Y} & \dot{Z} \end{array}}$$

$$\boxed{\begin{array}{c} a,\ e,\ i,\ \Omega,\ \omega,\ u \\ \text{ecliptic } \oplus \end{array}} \longleftrightarrow \boxed{\begin{array}{ccc} X' & Y' & Z' \\ \dot{X}' & \dot{Y}' & \dot{Z}' \end{array}}$$

These categories will apparently satisfy most present problems in coordinate transformations involving space vehicles. The systems and nomenclature will now be defined.

<u>E, F, G</u>: (See Figure 5)

Geocentric, earth rotating*, equatorial, right handed. G is coincident with polar axis and extends in a positive direction northward. E extends positively so as to pierce the meridian of Greenwich.

<u>x, y, z</u>: (See Figure 6)

Geocentric, inertial, equatorial, right handed. z is coincident with polar axis and extends in a positive direction northward. x extends positively toward the vernal equinox.

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* Paradoxically, the words "earth rotating" and "earth fixed" are used interchangeably in the literature. Both refer to a system fixed with respect to the earth but rotating in inertial space.

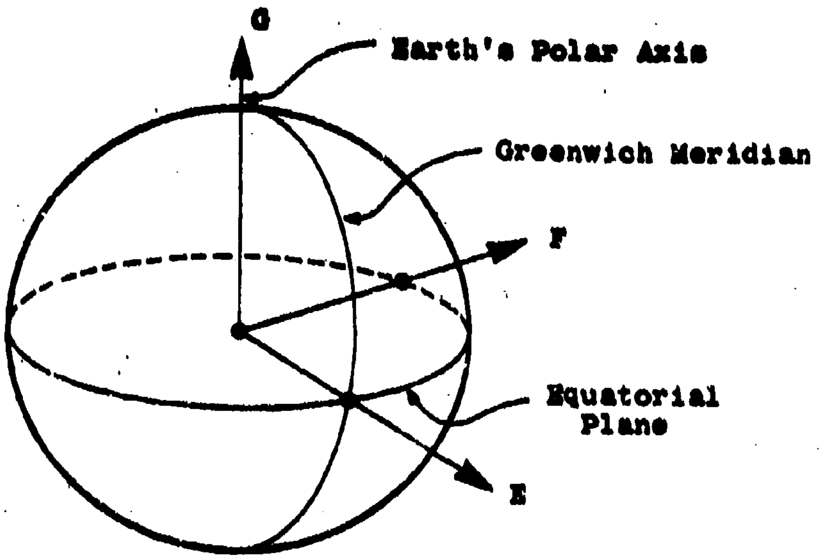


FIGURE 5 E, F, G COORDINATE SYSTEM (GEOCENTRIC)

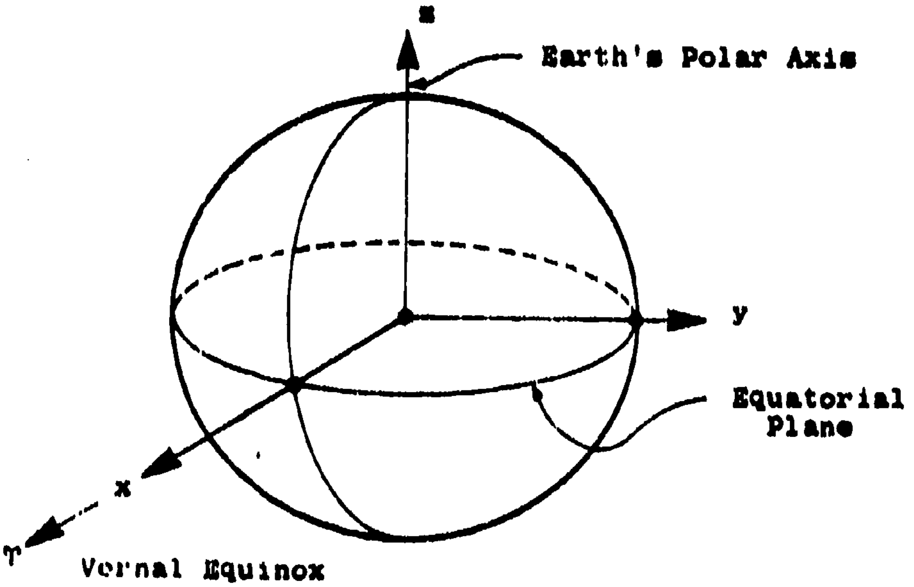


FIGURE 6 x, y, z COORDINATE SYSTEM (GEOCENTRIC)

X, Y, Z: (See Figure 7)

Heliocentric, inertial, equatorial, right handed. Z is parallel to polar axis of earth and extends in a positive direction northward. X extends positively toward vernal equinox.

U, V, W:* (See Figure 8)

Common ETR system, earth rotating, right handed. Origin somewhere near surface of spheroid at E_o, F_o, G_o. W extends in a positive direction vertically in a geodetic sense. U extends downrange at some azimuth Ψ measured east of north.

a, e, ρ:* (See Figure 9)

Common radar system, earth rotating, right handed. Origin somewhere near surface of spheroid at E_o, F_o, G_o and ϕ_o, γ_o. Through the origin and perpendicular to the geodetic vertical from the origin is an imaginary reference plane. A reference axis lies in the reference plane and originates at the origin. The reference axis points at some azimuth Ψ east of north. Azimuth a is measured in the reference plane from the reference axis in a clockwise sense as seen by an observer above the origin. The radius ρ connects the origin with the point P. The elevation e is the smaller angle between ρ and the reference plane. e is positive for points above reference plane and negative for points below reference plane. In x y z radar, with Ψ = zero: $x \rightarrow$ east,
$y \rightarrow$ north,
$z \rightarrow$ vertical.

$$0° \leq a < 360°;\ \rho \text{ is always positive};\ -90° \leq e \leq +90°$$

φ, γ, h: (See Figure 10)

Common geodetic earth rotating system. ϕ is geodetic latitude positive in northern hemisphere and negative in southern hemisphere. γ is longitude measured eastward from Greenwich. h is height above geodetic spheroid. h is positive if above spheroid and negative if below.

$$-90° \leq \phi \leq +90°;\ 0 \leq \gamma < 360°$$

The latitude and longitude of a point above the earth are determined by the intersection at the spheroidal surface of a line from the point P perpendicular to the spheroidal surface. It is convenient to disregard west longitude and consider 360° of east longitude as measured from the meridian of Greenwich.

*See note page 26.

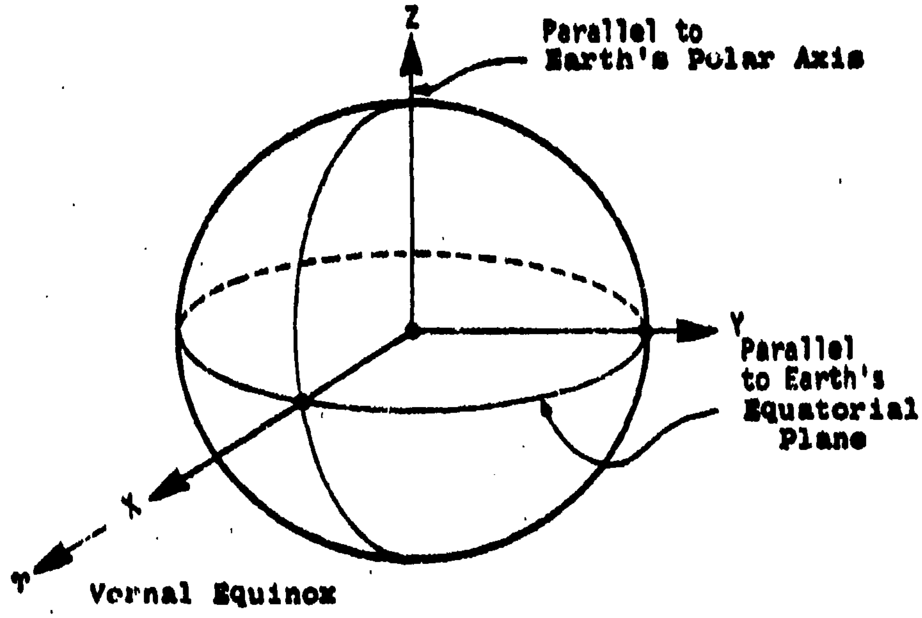


FIGURE 7 X, Y, Z COORDINATE SYSTEM (HELIOCENTRIC)

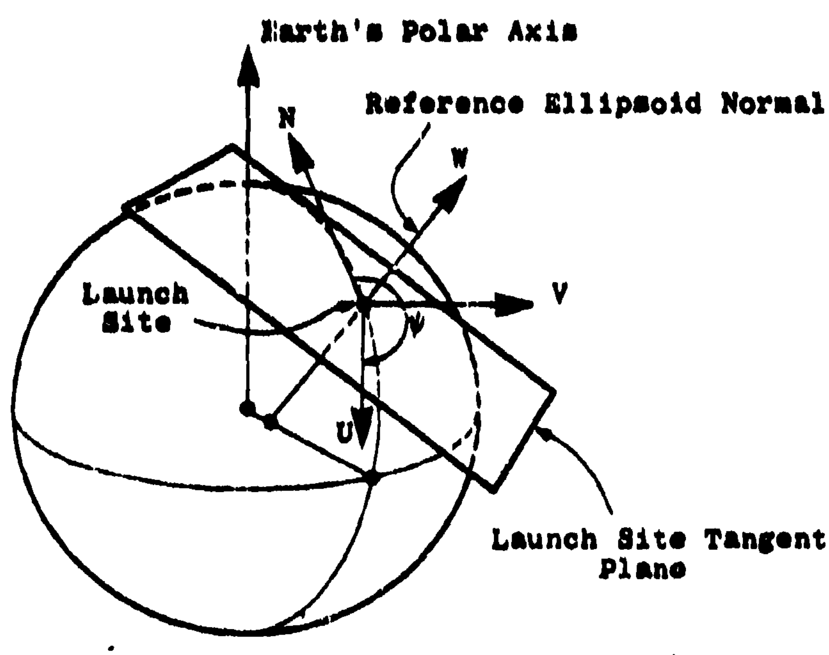


FIGURE 8 U, V, W COORDINATE SYSTEM (TOPOCENTRIC)

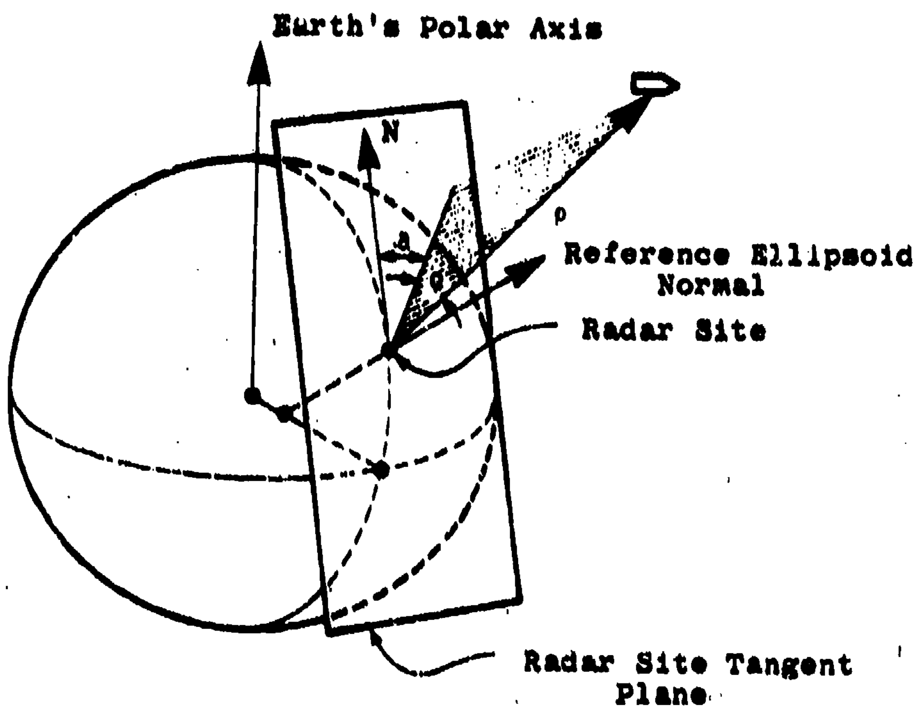


FIGURE 9 a, e, ρ COORDINATE SYSTEM (TOPOCENTRIC)

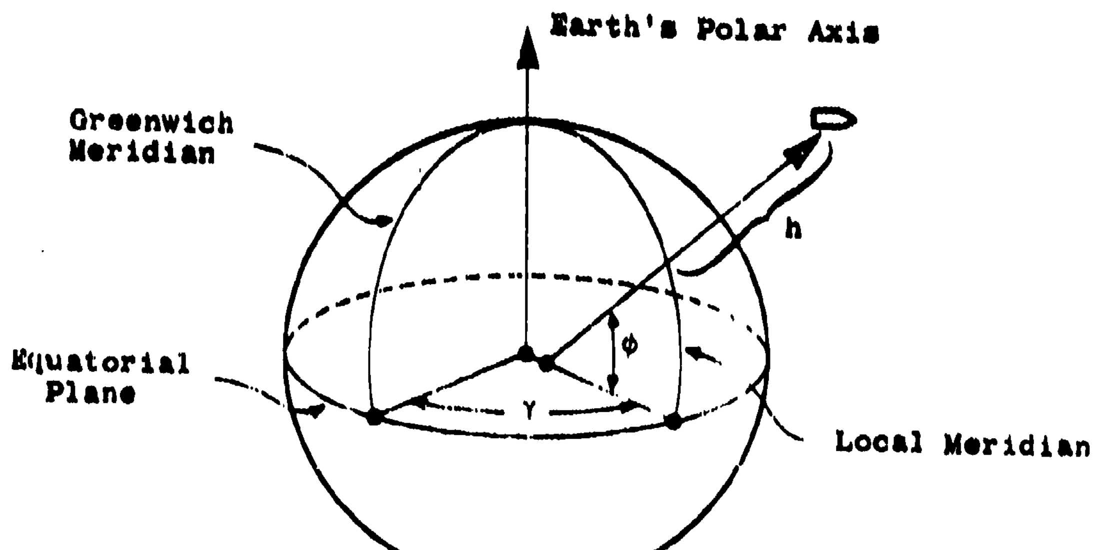


FIGURE 10 ϕ, γ, h COORDINATE SYSTEM (GEODETIC)

Φ, γ, r: (See Figure 11)

Common geocentric earth rotating, equatorial system. r is the radius connecting the center of the earth with the point P. Φ is the smaller angle between r and the equatorial plane. Φ is positive for northern hemisphere and negative for southern hemisphere. γ is longitude measured eastward from Greenwich meridian. r is always positive.

$$-90^\circ \leq \Phi \leq +90^\circ \qquad 0^\circ \leq \gamma < 360^\circ$$

ℓ, m, ρ:* (See Figure 12)

Common direction cosine system**, earth rotating. Origin somewhere near surface of earth at E_0, F_0, G_0 and ϕ_0, γ_0. Originating at the origin and perpendicular to the geodetic vertical from the origin are two reference axes intersecting at right angles. The origin is connected to the point P by a vector ρ. Direction cosines of this vector ρ with respect to the reference axes are called ℓ and m. The clockwise angle from the ℓ reference axis to the m reference axis as seen by an observer above the origin is 90°. The ℓ reference axis points at some azimuth ψ east of north. ρ is always positive.

$$-1 \leq \ell \leq +1. \qquad -1 \leq m \leq +1.$$

r, δ, α: (See Figure 13)

Geocentric, inertial, equatorial. The reference axis extends from the origin toward the vernal equinox. r connects the origin to the point P. Let r' be the vector projected upon the equatorial plane from r. The counterclockwise angle (as seen by an observer at the north celestial pole) from the reference axis to r' is the right ascension α. The smaller angle between the equatorial plane and r is called the declination δ. δ is positive in the northern hemisphere and negative in the southern hemisphere. r is always positive.

$$0 \leq \alpha < 360^\circ \qquad -90 \leq \delta \leq 90^\circ$$

x', y', z': (See Figure 14)

Geocentric, inertial, ecliptic, right handed. x' extends positively toward the vernal equinox. z' extends positively perpendicular to the ecliptic plane and toward the north pole of the ecliptic.

*See note page 26.

**Now obsolete.

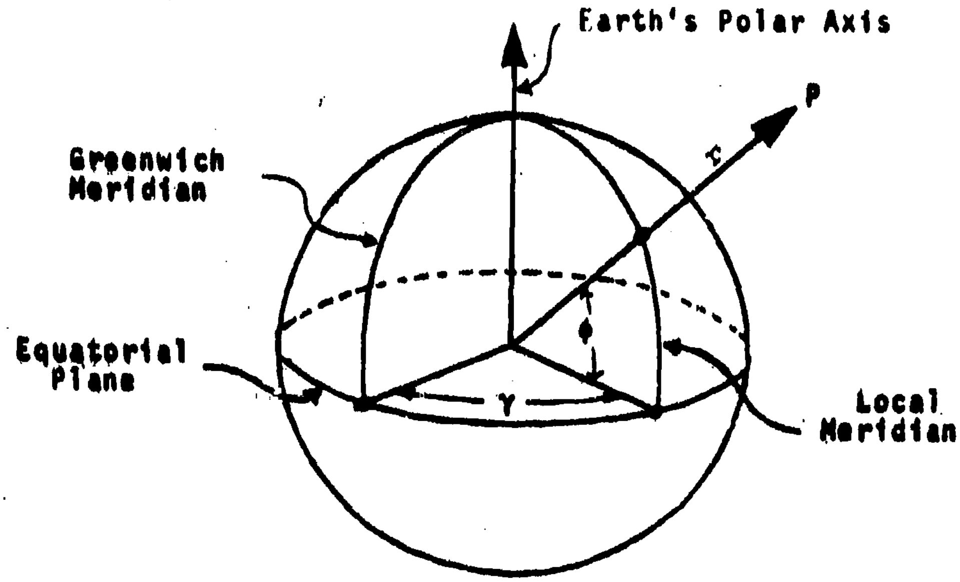


FIGURE 11 ϕ, γ, r COORDINATE SYSTEM (GEOCENTRIC)

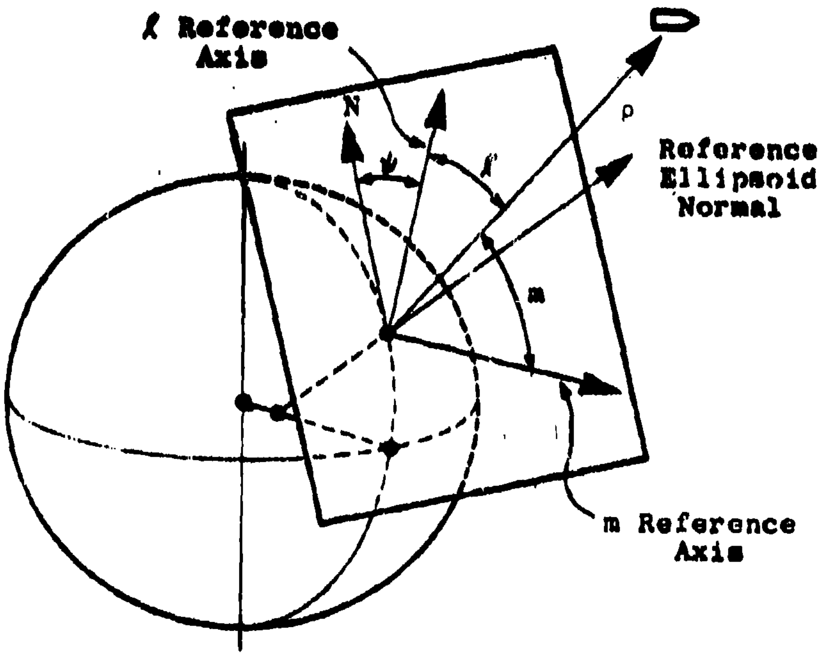


FIGURE 12 ℓ, m, ρ COORDINATE SYSTEM (TOPOCENTRIC)

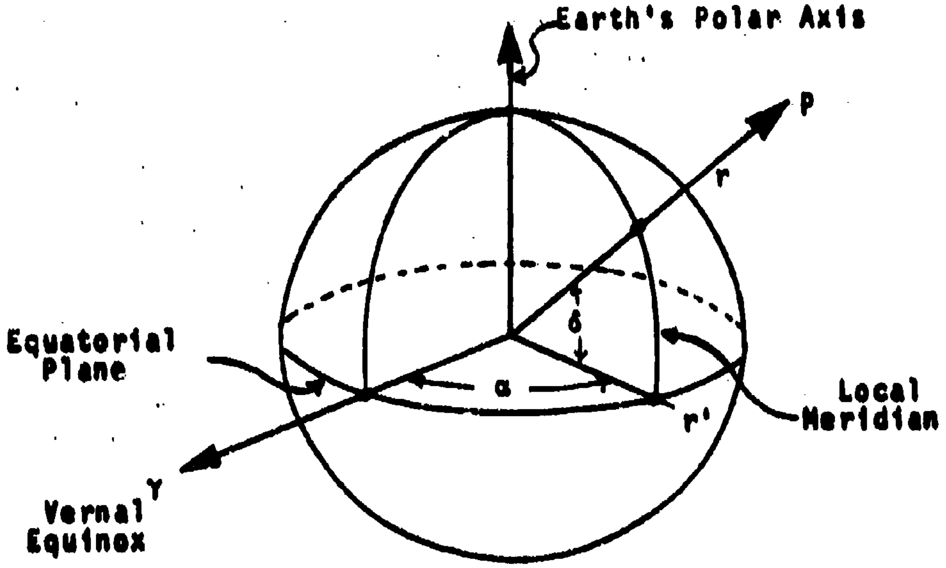


FIGURE 13 r, δ, α COORDINATE SYSTEM (GEOCENTRIC)

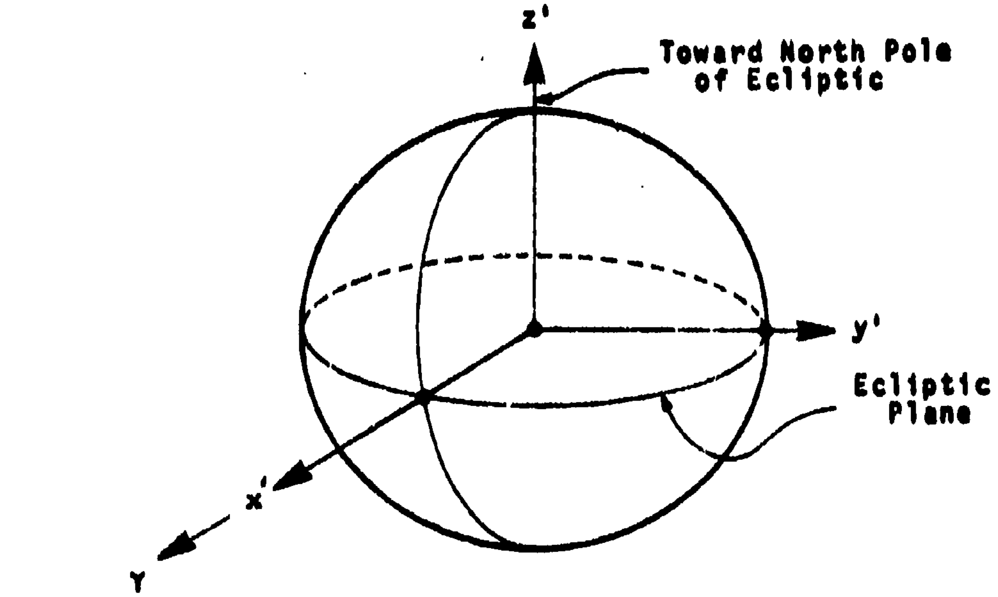


FIGURE 14 x', y', z' COORDINATE SYSTEM (GEOCENTRIC)

r, β, λ: (See Figure 15)

Geocentric, inertial, ecliptic. The reference axis extends from the origin toward the vernal equinox. r connects the origin to the point P. Let r' be the vector projected upon the ecliptic plane from r. The counterclockwise angle (as seen by an observer at the north ecliptic pole) from the reference axis to r' is the longitude λ. The smaller angle between the ecliptic and r is called the latitude β. β is positive for points on the observer's side of the ecliptic and negative on the other side of the ecliptic, the observer again being at the north ecliptic pole. r is always positive.

$$0 \leq \lambda \leq 360° \qquad -90 \leq \beta \leq 90°$$

$a, e, i, \Omega, \omega, u$ (equatorial geocentric): (See Figure 16)*

The common orbital elements. a is semi-major axis of orbit. e is eccentricity of the orbit. i is inclination of the orbit. Ω is longitude of ascending node. ω is longitude of periapsis measured from ascending node. u is argument of latitude.** The geocenter is at the principal focus of the orbit. The equatorial plane is reference plane.

R, Δ, A: (See Figure 17)

Heliocentric, inertial, equatorial. The reference axis extends from the origin toward the vernal equinox. R connects the origin to the point P. The reference plane is parallel to the equatorial plane of the earth and extends through the origin. Let R' be the vector projected upon the reference plane from R. The counterclockwise angle (as seen by an observer at the north celestial pole) from the reference axis to R' is the angle A. The smaller angle between the reference plane and R is the angle Δ. Δ is positive for points on the observer's side of the reference plane and negative on the other side of the reference plane; the observer being at the north celestial pole. R is always positive.

$$0 \leq A < 360° \qquad -90° \leq \Delta \leq +90°$$

X', Y', Z': (See Figure 18)

Heliocentric, inertial, ecliptic, rectangular. X' extends positively from the origin toward the vernal equinox. Z' extends perpendicular to the ecliptic and positively toward the north pole of the ecliptic.

---

* A duplication of symbols has been found necessary here. When an a or e is used, the text will make clear whether an orbital element or a radar coordinate is intended.

** It is quite common to see the mean anomaly M or the time of perifocal passage T used instead of u.

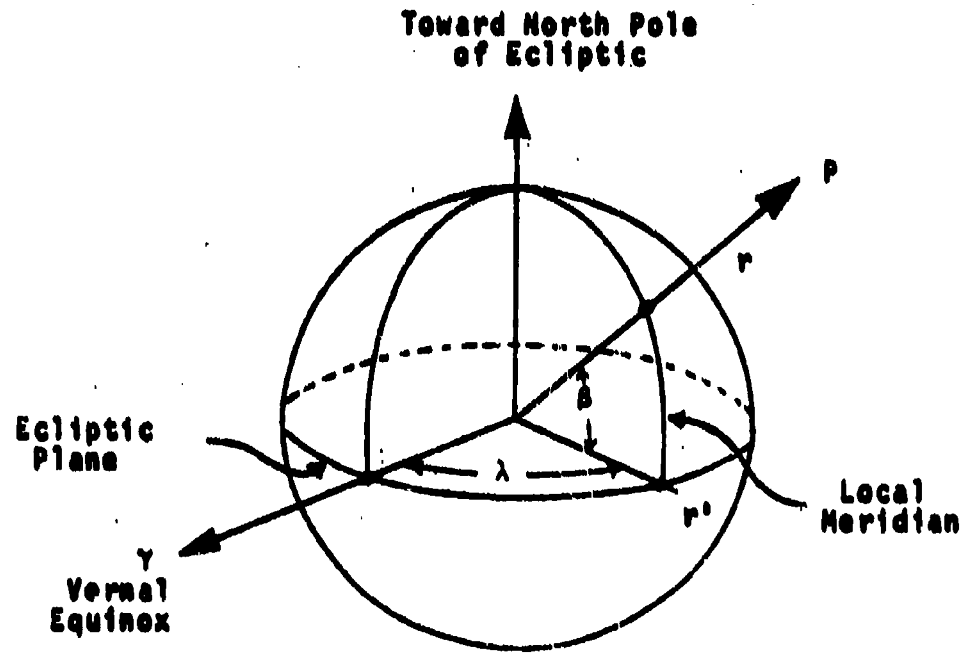
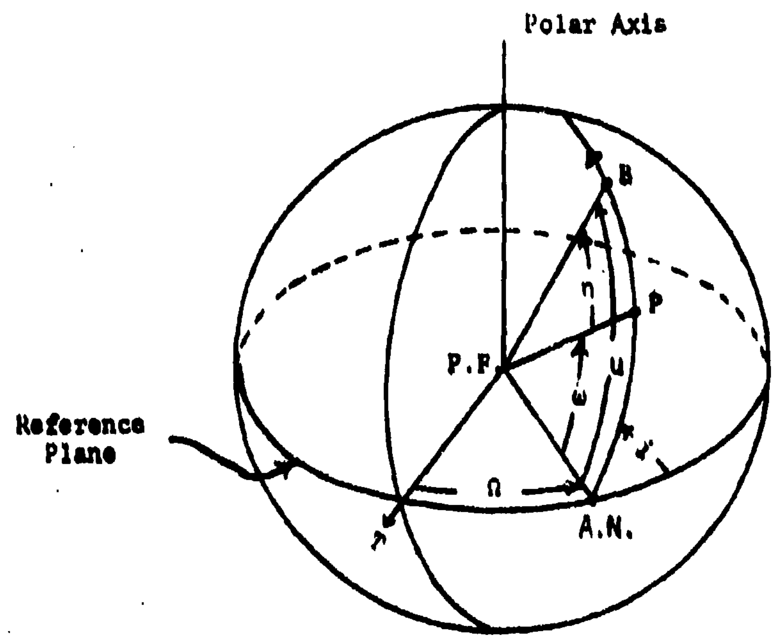
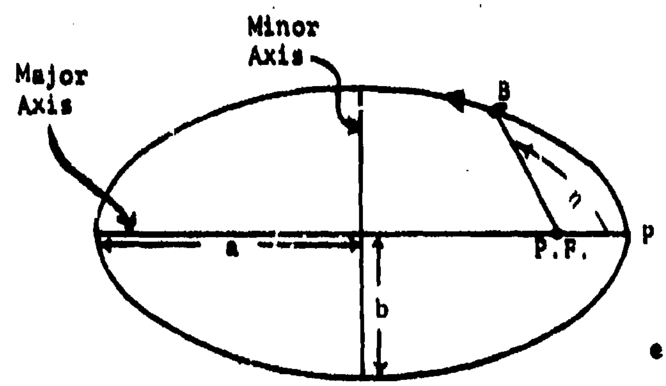


FIGURE 15 r, β, λ COORDINATE SYSTEM (GEOCENTRIC)



Orientation of Orbit



$$e = \frac{\sqrt{a^2-b^2}}{a}$$

Orbit Plane

Body is at B going in direction of arrow
P   is Periapsis
n   is True Anomaly
P.F. is Principal Focus
T   is Vernal Equinox
A.N. is Ascending Node
b   is Semiminor Axis
a,e,i,Ω ω,u are Orbital Elements

FIGURE 16   CLASSICAL ORBITAL ELEMENTS

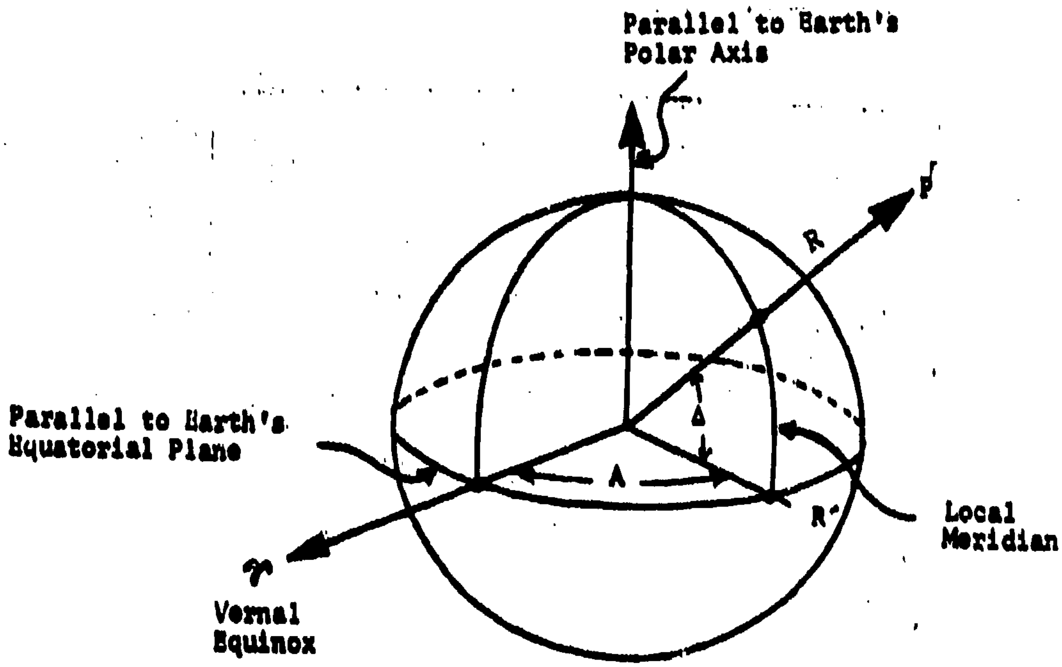


FIGURE 17 R, Δ, A COORDINATE SYSTEM (HELIOCENTRIC)

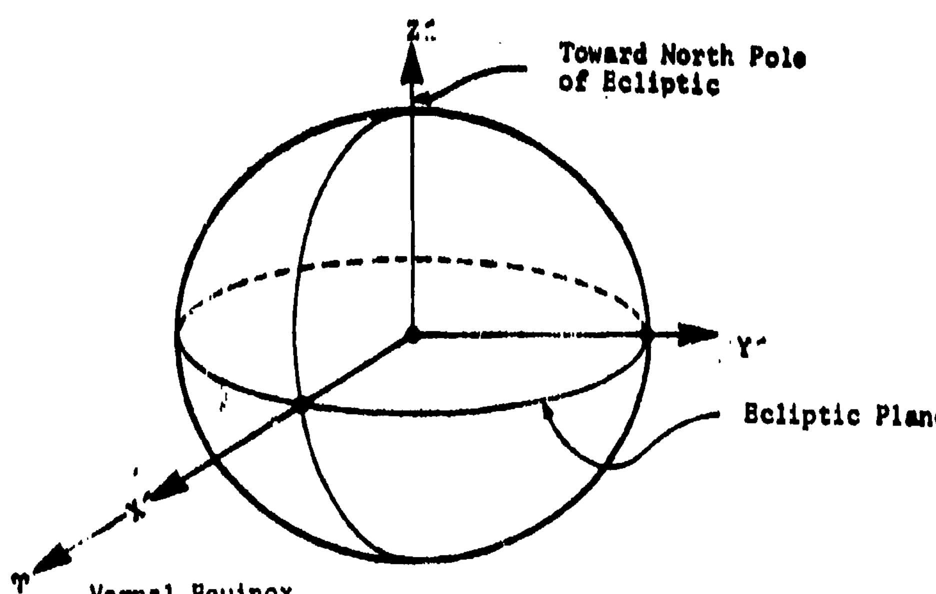


FIGURE 18 X', Y', Z' COORDINATE SYSTEM (HELIOCENTRIC)

R, B, A: (See Figure 19)

Heliocentric, inertial, ecliptic. The reference axis extends from the origin toward the vernal equinox. R connects the origin to the point P. Let R' be the vector projected upon the ecliptic plane from R. The counterclockwise angle (as seen by an observer at the north ecliptic pole) from the reference axis to R' is the longitude A. The smaller angle between the ecliptic and R is called the latitude B. B is positive for points on the observer's side of the ecliptic and negative on the other side of the ecliptic; the observer being at the north ecliptic pole. R is always positive.

$$0 \leq A < 360^\circ \qquad -90^\circ \leq B \leq +90^\circ$$

a, e, i, Ω, ω, u (equatorial heliocentric): (See Figure 16)

The common orbital elements previously described. The sun is at principal focus. The plane through center of the sun parallel to equatorial plane of earth is the reference plane.

a, e, i, Ω, ω, u (ecliptic heliocentric): (See Figure 16)

The common orbital elements still referenced to vernal equinox. The sun is at principal focus. The ecliptic plane is the reference plane.

Note on Reference Vertical:

The previous explanation refers to the geodetic vertical as a reference line for UVW, aep, imp coordinate systems, the geodetic vertical being by definition normal to some specified spheroid. Actually, it is customary for practical reasons to set up these systems using as reference the astronomic vertical determined by plumb bob or bubble level, the astronomic vertical being normal to the geoid. The astronomic and geodetic verticals usually differ at a given site by several seconds of arc and in extreme conditions up to 20 seconds of arc. The difference between the verticals at a given location is called the deflection of the vertical and depends of course upon the spheroid used. Measurements taken in an astronomic system may be corrected for deflection of the vertical before rotation to another coordinate system. The subsequent discussion of transformations (Section VIII) assumes measurements have been corrected for deflection of the vertical. See Appendix B for methods of making these corrections and for alternative transformations.

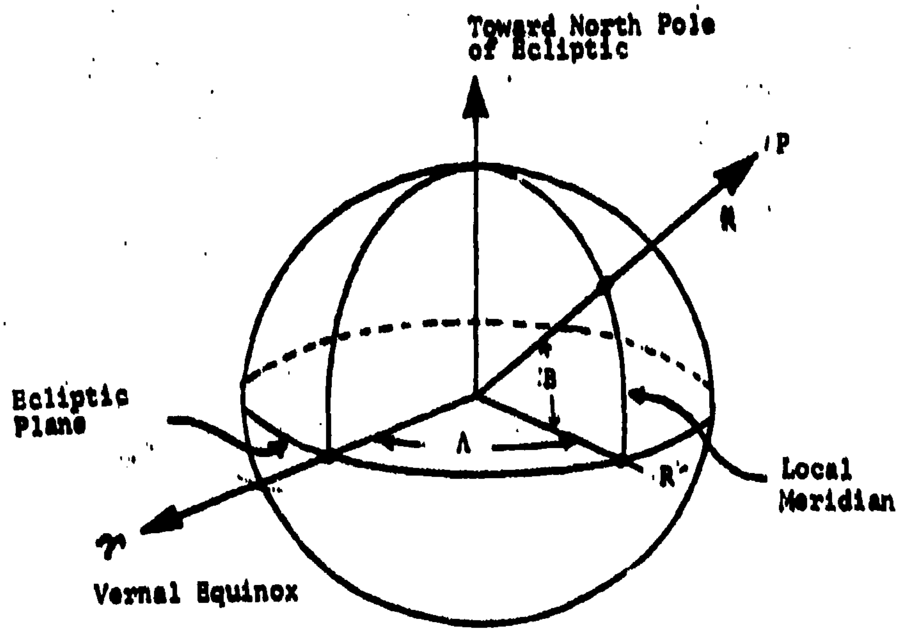


FIGURE 19 R , B, Λ COORDINATE SYSTEM (HELIOCENTRIC)

VIII. Transformation Equations Corresponding to Flow Charts.

U V W and E F G:

Consider U V W to be reference system. Rotate U V W about W to make U point directly east. Then rotate about new U axis to make W parallel to G. Then about new W axis to make U parallel to E. Thus the final position of U V W will be parallel respectively to E F G.

1st rotation: counterclockwise through angle (Ψ -90°) about W

2nd rotation: clockwise through angle (90° - ϕ_o) about U'

3rd rotation: counterclockwise through angle (270° - γ_o) about W"

$$(27)\quad \begin{pmatrix} E-E_o \\ F-F_o \\ G-G_o \end{pmatrix} = \begin{pmatrix} \cos(270-\gamma_o) & \sin(270-\gamma_o) & 0 \\ -\sin(270-\gamma_o) & \cos(270-\gamma_o) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(90-\phi_o) & -\sin(90-\phi_o) \\ 0 & \sin(90-\phi_o) & \cos(90-\phi_o) \end{pmatrix} \begin{pmatrix} \cos(\Psi-90) & \sin(\Psi-90) & 0 \\ -\sin(\Psi-90) & \cos(\Psi-90) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

or

$$(28)\quad \begin{pmatrix} E-E_o \\ F-F_o \\ G-G_o \end{pmatrix} = \begin{pmatrix} -\sin\gamma_o & -\cos\gamma_o & 0 \\ \cos\gamma_o & -\sin\gamma_c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\phi_o & -\cos\phi_o \\ 0 & \cos\phi_o & \sin\phi_o \end{pmatrix} \begin{pmatrix} \sin\Psi & -\cos\Psi & 0 \\ \cos\Psi & \sin\Psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} U \\ V \\ W \end{pmatrix}$$

or inverting

$$(29)\quad \begin{pmatrix} U \\ V \\ W \end{pmatrix} = \begin{pmatrix} \sin\Psi & \cos\Psi & 0 \\ -\cos\Psi & \sin\Psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\phi_o & \cos\phi_o \\ 0 & -\cos\phi_o & \sin\phi_o \end{pmatrix} \begin{pmatrix} -\sin\gamma_o & \cos\gamma_o & 0 \\ -\cos\gamma_o & -\sin\gamma_o & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E-E_o \\ F-F_o \\ G-G_o \end{pmatrix}$$

Origin of U V W is at E_o F_o G_o in E F G system and has geodetic coordinates (ϕ_o, γ_o).

a, e, ρ and E, F, G:

Let the reference axis from which a is measured be pointing at an azimuth of Ψ east of north. Let the origin of the a, e, ρ system be at E_o F_o G_o in the E F G system, with geodetic coordinates (ϕ_o, γ_o).

$$(30) \quad \begin{pmatrix} E-E_o \\ F-F_o \\ G-G_o \end{pmatrix} = \begin{pmatrix} -\sin\gamma_o & -\cos\gamma_o & 0 \\ \cos\gamma_o & -\sin\gamma_o & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\phi_o & -\cos\phi_o \\ 0 & \cos\phi_o & \sin\phi_o \end{pmatrix} \begin{pmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho\cos e\sin a \\ \rho\cos e\cos a \\ \rho\sin e \end{pmatrix}$$

or inverting

$$(31) \quad \begin{pmatrix} \rho\cos e\sin a \\ \rho\cos e\cos a \\ \rho\sin e \end{pmatrix} = \begin{pmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\phi_o & \cos\phi_o \\ 0 & -\cos\phi_o & \sin\phi_o \end{pmatrix} \begin{pmatrix} -\sin\gamma_o & \cos\gamma_o & 0 \\ -\cos\gamma_o & -\sin\gamma_o & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E-E_o \\ F-F_o \\ G-G_o \end{pmatrix}$$

To determine ρ, e, and a:

$$(32) \qquad \rho = \left[(E-E_o)^2 + (F-F_o)^2 + (G-G_o)^2\right]^{\frac{1}{2}}$$

Then expanding the second matrix equation gives three analytical equations. Dividing the third of these equations by ρ gives sin e, and hence e and cos e. Dividing the first and second of these equations by ρ cos e gives sin a and cos a and hence a.

ℓ, m, ρ, and E, F, G:

Let the origin of the ℓ, m, ρ system be at E_o F_o G_o relative to the E F G system, and have geodetic coordinates (ϕ_o, γ_o).

$$(33)\quad \begin{pmatrix} E-E_o \\ F-F_o \\ G-G_o \end{pmatrix} = \begin{pmatrix} -\sin\gamma_o & -\cos\gamma_o & 0 \\ \cos\gamma_o & -\sin\gamma_o & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\phi_o & -\cos\phi_o \\ 0 & \cos\phi_o & \sin\phi_o \end{pmatrix} \begin{pmatrix} \sin\psi & \cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \ell\rho \\ m\rho \\ \rho\sqrt{1-\ell^2-m^2} \end{pmatrix}$$

Inverting

$$(34)\quad \begin{pmatrix} \ell\rho \\ m\rho \\ \rho\sqrt{1-\ell^2-m^2} \end{pmatrix} = \begin{pmatrix} \sin\psi & \cos\psi & 0 \\ \cos\psi & -\sin\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\phi_o & +\cos\phi_o \\ 0 & -\cos\phi_o & \sin\phi_o \end{pmatrix} \begin{pmatrix} -\sin\gamma_o & \cos\gamma_o & 0 \\ -\cos\gamma_o & -\sin\gamma_o & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E-E_o \\ F-F_o \\ G-G_o \end{pmatrix}$$

To determine ρ, ℓ, and m:

$$(35)\qquad \rho = \left[(E-E_o)^2 + (F-F_o)^2 + (G-G_o)^2.\right]^{\frac{1}{2}}$$

Then expanding the second matrix equation gives three analytical equations. Dividing the first two equations by ρ gives ℓ and m.

Φ, γ, r and E F G:

$$(36)\quad \begin{aligned} E &= r\cos\Phi\cos\gamma & \qquad r &= (E^2 + F^2 + G^2)^{\frac{1}{2}} \\ F &= r\cos\Phi\sin\gamma & \qquad \gamma &= \tan^{-1}\left(\frac{F}{E}\right) \text{ evaluated from } 0^\circ \text{ to } 360^\circ \\ G &= r\sin\Phi & \qquad \Phi &= \sin^{-1}\frac{G}{(E^2 + F^2 + G^2)^{\frac{1}{2}}} \text{ evaluated from } -90^\circ \text{ to } +90^\circ \end{aligned}$$

ϕ, γ, h and E, F, G:

Let N be the radius of curvature in the meridian plane of the earth spheroid at the geodetic position ϕ, γ, and h.

$$N = \frac{\hat{a}}{\left(1-\hat{e}^2 \sin^2 \phi\right)^{\frac{1}{2}}}$$

where $\hat{a}$ is equatorial radius of spheroid and $\hat{e}$ is eccentricity of spheroid in meridian plane. From Appendix D numerical values for $\hat{a}$ and $\hat{e}$ can be obtained which are representative of the various spheroids.

(37)

$$E = (N + h) \cos \phi \cos \gamma$$
$$F = (N + h) \cos \phi \sin \gamma$$
$$G = [N(1 - \hat{e}^2) + h] \sin \phi$$

Now for the reverse transformation

(38) $$\gamma = \tan^{-1} \left(\frac{F}{E}\right) \text{ evaluated in region } 0^\circ \text{ to } 360^\circ.$$

Define $D = (\bar{E}^2 - \bar{F}^2)^{\frac{1}{2}}$

Compute first estimate of ϕ:

$$\phi = \tan^{-1} \frac{G}{(1-\bar{e}^2)D} \quad \text{evaluated in region} \pm 90^\circ.$$

If $\phi = 0^\circ$ or $(\pm 90^\circ)$, ϕ is established; jump to Eq. 40. Otherwise compute

$$f(\phi) = \frac{\bar{a}\,\bar{e}^2}{(1-\bar{e}^2 \sin^2 \phi)^{\frac{1}{2}}} - \frac{D}{\cos \phi} + \frac{G}{\sin \phi}$$

and

$$f'(\phi) = \frac{\bar{a}\,\bar{e}^4 \sin \phi \cos \phi}{(1-\bar{e}^2 \sin^2 \phi)^{3/2}} - \frac{D \sin \phi}{\cos^2 \phi} - \frac{G \cos \phi}{\sin^2 \phi}$$

Begin iterations with

(39) $$\phi_i = \phi_{i-1} - f(\phi_{i-1}) / f'(\phi_{i-1})$$

and continue until

$$\left| f(\phi_{i-1}) / f'(\phi_{i-1}) \right| < \text{(some tolerable error)}.$$

ϕ has now been established. Next compute

(40) $$N = \bar{a}/(1 - \bar{e}^2 \sin^2 \phi)^{\frac{1}{2}}$$

$$D_s = N \cos \phi$$

$$G_s = N (1 - \bar{e}^2) \sin \phi$$

(41) $$h = \pm \left[(D-D_s)^2 + (G-G_s)^2 \right]^{\frac{1}{2}}$$

To resolve the ambiguity in sign, compute

$$B = D_s (D-D_s) + G_s (G-G_s)$$

The sign in the computed value for B is assigned to h.

Geocentric Equatorial Earth Rotating Rectangular and Geocentric Equatorial Inertial Rectangular

Let γ_T be the instantaneous east longitude of vernal equinox, where $0 \leq \gamma_T < 360°$. These transformations involve a rotation about the G or the z axes which are coincident here.

$$\text{(42)} \quad \begin{aligned} x &= E \cos \gamma_T + F \sin \gamma_T \\ y &= -E \sin \gamma_T + F \cos \gamma_T \\ z &= G \end{aligned} \qquad \begin{aligned} E &= x \cos \gamma_T - y \sin \gamma_T \\ F &= x \sin \gamma_T + y \cos \gamma_T \\ G &= z \end{aligned}$$

Geocentric Ecliptic Spherical and Geocentric Equatorial Rectangular*

$$\text{(43)} \quad \begin{aligned} x &= r \cos \beta \cos \lambda \\ y &= r \cos \beta \sin \lambda \cos \epsilon - r \sin \beta \sin \epsilon \\ z &= r \cos \beta \sin \lambda \sin \epsilon + r \sin \beta \cos \epsilon \end{aligned} \qquad r = \sqrt{x^2 + y^2 + z^2}$$

$$\beta = \sin^{-1}\left(\frac{z \cos \epsilon - y \sin \epsilon}{\sqrt{x^2 + y^2 + z^2}}\right)$$

$$\lambda = \tan^{-1}\left(\frac{y \cos \epsilon + z \sin \epsilon}{x}\right)$$

Geocentric Ecliptic Rectangular and Geocentric Equatorial Rectangular*

$$\text{(44)} \quad \begin{aligned} x &= x' \\ y &= y' \cos \epsilon - z' \sin \epsilon \\ z &= y' \sin \epsilon + z' \cos \epsilon \end{aligned} \qquad \begin{aligned} x' &= x \\ y' &= y \cos \epsilon + z \sin \epsilon \\ z' &= z \cos \epsilon - y \sin \epsilon \end{aligned}$$

Heliocentric Equatorial Spherical and Heliocentric Equatorial Rectangular

$$\text{(45)} \quad \begin{aligned} X &= R \cos \Delta \cos A \\ Y &= R \cos \Delta \sin A \\ Z &= R \sin \Delta \end{aligned} \qquad \begin{aligned} R &= \sqrt{X^2 + Y^2 + Z^2} \\ A &= \tan^{-1} \frac{Y}{X} \\ \Delta &= \sin^{-1} \frac{Z}{\sqrt{X^2 + Y^2 + Z^2}} \end{aligned}$$

* ϵ represents the obliquity

Geocentric Equatorial Spherical and Geocentric Equatorial Rectangular

$$x = r \cos \delta \cos \alpha \qquad r = \sqrt{x^2 + y^2 + z^2}$$
$$(46) \qquad y = r \cos \delta \sin \alpha \qquad \alpha = \tan^{-1} \frac{y}{x}$$
$$z = r \sin \delta \qquad \delta = \sin^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

Heliocentric Equatorial Rectangular and Geocentric Equatorial Rectangular

Let the center of the sun have geocentric equatorial coordinates $x_\odot$ $y_\odot$ $z_\odot$.

$$x = X + x_\odot \qquad X = x - x_\odot$$
$$(47) \qquad y = Y + y_\odot \qquad Y = y - y_\odot$$
$$z = Z + z_\odot \qquad Z = z - z_\odot$$

where $x_\odot$ $y_\odot$ and $z_\odot$ are obtainable from the ephemeris for mean equator and equinox of 1950.0 or 1962.0 or some similar epoch.

Heliocentric Ecliptic Rectangular and Heliocentric Equatorial Rectangular

$$X = X' \qquad X' = X$$
$$(48) \qquad Y = Y' \cos \epsilon - Z' \sin \epsilon \qquad Y' = Y \cos \epsilon + Z \sin \epsilon$$
$$Z = Y' \sin \epsilon + Z' \cos \epsilon \qquad Z' = Z \cos \epsilon - Y \sin \epsilon$$

Helliocentric Ecliptic Spherical and Heliocentric Equatorial Rectangular

$$X = R \cos B \cos \Lambda \qquad R = \sqrt{X^2 + Y^2 + Z^2}$$

$$(49) \quad Y = R \cos B \sin \Lambda \cos \epsilon - R \sin B \sin \epsilon \qquad B = \sin^{-1}\left(\frac{Z \cos \epsilon - Y \sin \epsilon}{\sqrt{X^2 + Y^2 + Z^2}}\right)$$

$$Z = R \cos B \sin \Lambda \sin \epsilon + R \sin B \cos \epsilon$$

$$\Lambda = \tan^{-1}\left(\frac{Y \cos \epsilon + Z \sin \epsilon}{X}\right)$$

Velocities U V W System and E F G System:

$$(50) \quad \begin{pmatrix} \dot{E} \\ \dot{F} \\ \dot{G} \end{pmatrix} = \begin{pmatrix} -\sin \gamma_o & -\cos \gamma_o & 0 \\ \cos \gamma_o & -\sin \gamma_o & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \phi_o & -\cos \phi_o \\ 0 & \cos \phi_o & \sin \phi_o \end{pmatrix} \begin{pmatrix} \sin \Psi & -\cos \Psi & 0 \\ \cos \Psi & \sin \Psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{pmatrix}$$

$$(51) \quad \begin{pmatrix} \dot{U} \\ \dot{V} \\ \dot{W} \end{pmatrix} = \begin{pmatrix} \sin \Psi & \cos \Psi & 0 \\ -\cos \Psi & \sin \Psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin \phi_c & \cos \phi_o \\ 0 & -\cos \phi_o & \sin \phi_o \end{pmatrix} \begin{pmatrix} -\sin \gamma_o & \cos \gamma_o & 0 \\ -\cos \gamma_o & -\sin \gamma_o & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{E} \\ \dot{F} \\ \dot{G} \end{pmatrix}$$

Velocities: Geocentric Equatorial Inertial Rectangular to Heliocentric Equatorial Inertial Rectangular

(52)
$$\dot{x} = \dot{X} + \dot{x}_{\odot} \qquad \dot{X} = \dot{x} - \dot{x}_{\odot}$$
$$\dot{y} = \dot{Y} + \dot{y}_{\odot} \qquad \dot{Y} = \dot{y} - \dot{y}_{\odot}$$
$$\dot{z} = \dot{Z} + \dot{z}_{\odot} \qquad \dot{Z} = \dot{z} - \dot{z}_{\odot}$$

Geocentric Equatorial Earth Rotating Rectangular Velocities and Geocentric Equatorial Inertial Rectangular Velocities

Let ω_e equal rotation rate of earth (assumed positive)

(53)
$$\dot{x} = \dot{E} \cos \gamma_{\varphi} + \dot{F} \sin \gamma_{\varphi} + E\,\omega_e \sin \gamma_{\varphi} - F\,\omega_e \cos \gamma_{\varphi}$$
$$\dot{y} = -\dot{E} \sin \gamma_{\varphi} + \dot{F} \cos \gamma_{\varphi} + E\,\omega_e \cos \gamma_{\varphi} + F\,\omega_e \sin \gamma_{\varphi}$$
$$\dot{z} = \dot{G}$$

(54)
$$\dot{E} = \dot{x} \cos \gamma_{\varphi} - \dot{y} \sin \gamma_{\varphi} + x\,\omega_e \sin \gamma_{\varphi} + y\,\omega_e \cos \gamma_{\varphi}$$
$$\dot{F} = \dot{x} \sin \gamma_{\varphi} + \dot{y} \cos \gamma_{\varphi} - x\,\omega_e \cos \gamma_{\varphi} + y\,\omega_e \sin \gamma_{\varphi}$$
$$\dot{G} = \dot{z}$$

Velocities $\dot{X}$, $\dot{Y}$, $\dot{Z}$ and $\dot{X}'$, $\dot{Y}'$, $\dot{Z}'$

Differentiating the appropriate position equations:

(55)
$$\dot{X} = \dot{X}' \qquad \dot{X}' = \dot{X}$$
$$\dot{Y} = \dot{Y}' \cos \epsilon - \dot{Z}' \sin \epsilon \qquad \dot{Y}' = \dot{Y} \cos \epsilon + \dot{Z} \sin \epsilon$$
$$\dot{Z} = \dot{Y}' \sin \epsilon + \dot{Z}' \cos \epsilon \qquad \dot{Z}' = \dot{Z} \cos \epsilon - \dot{Y} \sin \epsilon$$

Acceleration: x y z and X Y Z system

$$
\begin{aligned}
\ddot{x} &= \ddot{X} + \ddot{x}_0 & \qquad \ddot{X} &= \ddot{x} - \ddot{x}_0 \\
\ddot{y} &= \ddot{Y} + \ddot{y}_0 & \qquad \ddot{Y} &= \ddot{y} - \ddot{y}_0 \\
\ddot{z} &= \ddot{Z} + \ddot{z}_0 & \qquad \ddot{Z} &= \ddot{z} - \ddot{z}_0
\end{aligned}
\tag{56}
$$

Acceleration: x y z and E F G Systems

$$\ddot{x} = \ddot{E}\cos\gamma_\varphi + \ddot{F}\sin\gamma_\varphi + 2\dot{E}\,\omega_e\sin\gamma_\varphi - 2\dot{F}\,\omega_e\cos\gamma_\varphi - E\,\omega_e^2\cos\gamma_\varphi - F\,\omega_e^2\sin\gamma_\varphi$$
$$(57)\quad \ddot{y} = -\ddot{E}\sin\gamma_\varphi + \ddot{F}\cos\gamma_\varphi + 2\dot{E}\,\omega_e\cos\gamma_\varphi + 2\dot{F}\,\omega_e\sin\gamma_\varphi + E\,\omega_e^2\sin\gamma_\varphi - F\,\omega_e^2\cos\gamma_\varphi$$
$$\ddot{z} = \ddot{G}$$

$$\ddot{E} = \ddot{x}\cos\gamma_\varphi - \ddot{y}\sin\gamma_\varphi + 2\dot{x}\,\omega_e\sin\gamma_\varphi + 2\dot{y}\,\omega_e\cos\gamma_\varphi - x\,\omega_e^2\cos\gamma_\varphi + y\,\omega_e^2\sin\gamma_\varphi$$
$$(58)\quad \ddot{F} = \ddot{x}\sin\gamma_\varphi + \ddot{y}\cos\gamma_\varphi - 2\dot{x}\,\omega_e\cos\gamma_\varphi + 2\dot{y}\,\omega_e\sin\gamma_\varphi - x\,\omega_e^2\sin\gamma_\varphi + y\,\omega_e^2\cos\gamma_\varphi$$
$$\ddot{G} = \ddot{z}$$

Accelerations U V W System and E F G System

$$(59)\quad \begin{pmatrix}\ddot{E}\\ \ddot{F}\\ \ddot{G}\end{pmatrix} = \begin{pmatrix}-\sin\gamma_o & -\cos\gamma_o & 0\\ \cos\gamma_o & -\sin\gamma_o & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \sin\phi_o & -\cos\phi_o\\ 0 & \cos\phi_o & \sin\phi_o\end{pmatrix}\begin{pmatrix}\sin\psi & -\cos\psi & 0\\ \cos\psi & \sin\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\ddot{U}\\ \ddot{V}\\ \ddot{W}\end{pmatrix}$$

$$(60)\quad \begin{pmatrix}\ddot{U}\\ \ddot{V}\\ \ddot{W}\end{pmatrix} = \begin{pmatrix}\sin\psi & \cos\psi & 0\\ -\cos\psi & \sin\psi & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\ 0 & \sin\phi_o & \cos\phi_o\\ 0 & -\cos\phi_o & \sin\phi_o\end{pmatrix}\begin{pmatrix}-\sin\gamma_o & \cos\gamma_o & 0\\ -\cos\gamma_o & -\sin\gamma_o & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\ddot{E}\\ \ddot{F}\\ \ddot{G}\end{pmatrix}$$

Accelerations ($\ddot{X}$, $\ddot{Y}$, $\ddot{Z}$ and $\ddot{X}'$, $\ddot{Y}'$, $\ddot{Z}'$)

Differentiating the appropriate velocity equations:

$$\ddot{X} = \ddot{X}' \qquad\qquad \ddot{X}' = \ddot{X}$$
$$(61)\quad \ddot{Y} = \ddot{Y}'\cos\epsilon - \ddot{Z}'\sin\epsilon \qquad\qquad \ddot{Y}' = \ddot{Y}\cos\epsilon + \ddot{Z}\sin\epsilon$$
$$\ddot{Z} = \ddot{Y}'\sin\epsilon + \ddot{Z}'\cos\epsilon \qquad\qquad \ddot{Z}' = \ddot{Z}\cos\epsilon - \ddot{Y}\sin\epsilon$$

Heliocentric Ecliptic Orbital Elements to Heliocentric Ecliptic Rectangular Coordinates

$$
\begin{aligned}
X' &= R \cos u \cos \Omega - R \sin u \cos i \sin \Omega \\
Y' &= R \cos u \sin \Omega + R \sin u \cos i \cos \Omega \qquad (62) \\
Z' &= R \sin u \sin i
\end{aligned}
$$

where $R = \dfrac{a(1-e^2)}{1+e \cos \eta}$ where $\eta = u - \omega$, $0 < \eta < 360°$

In case $\omega > u$ then $\eta = 360° + u - \omega$

Heliocentric Ecliptic Orbital Elements to Heliocentric Ecliptic Rectangular Velocities

$$
\begin{aligned}
\dot{X}' &= \dot{R} \cos u \cos \Omega - R \dot{u} \sin u \cos \Omega \\
&\quad - \dot{R} \sin u \cos i \sin \Omega - R \dot{u} \cos u \cos i \sin \Omega \\
\dot{Y}' &= \dot{R} \cos u \sin \Omega - R \dot{u} \sin u \sin \Omega \qquad (63) \\
&\quad + \dot{R} \sin u \cos i \cos \Omega + R \dot{u} \cos u \cos i \cos \Omega \\
\dot{Z}' &= \dot{R} \sin u \sin i + R \dot{u} \cos u \sin i
\end{aligned}
$$

where

$$R = \frac{a(1-e^2)}{1+e \cos \eta} \qquad \dot{R} = e \sin \eta \sqrt{\frac{K}{a(1-e^2)}}$$

$$\dot{u} = (1+e \cos \eta)^2 \sqrt{\frac{K}{a^3(1-e^2)^3}}$$

NOTE: K is used throughout this report to represent the gravitational constant of the central body.

Heliocentric Equatorial Orbital Elements to Heliocentric Equatorial Rectangular Coordinates

$$
\begin{aligned}
X &= R \cos u \cos \Omega - R \sin u \cos i \sin \Omega \\
Y &= R \cos u \sin \Omega + R \sin u \cos i \cos \Omega \\
Z &= R \sin u \sin i
\end{aligned}
\tag{64}
$$

$$\text{where } R = \frac{a(1-e^2)}{1+e \cos \eta} \qquad \text{where } \eta = u - \omega, \quad 0 \leq \eta < 360^\circ$$

In case $\omega > u$ then $\eta = 360^\circ + u - \omega$

Heliocentric Equatorial Orbital Elements to Heliocentric Equatorial Rectangular Velocities

$$
\begin{aligned}
\dot{X} &= \dot{R} \cos u \cos \Omega - R \dot{u} \sin u \cos \Omega \\
&\quad - \dot{R} \sin u \cos i \sin \Omega - R \dot{u} \cos u \cos i \sin \Omega \\
\dot{Y} &= \dot{R} \cos u \sin \Omega - R \dot{u} \sin u \sin \Omega \\
&\quad + \dot{R} \sin u \cos i \cos \Omega + R \dot{u} \cos u \cos i \cos \Omega \\
\dot{Z} &= \dot{R} \sin u \sin i + R \dot{u} \cos u \sin i
\end{aligned}
\tag{65}
$$

where

$$R = \frac{a(1-e^2)}{1+e \cos \eta} \qquad \dot{R} = e \sin \eta \sqrt{\frac{K}{a (1-e^2)}}$$

$$\dot{u} = (1+e \cos \eta)^2 \sqrt{\frac{K}{a^3 (1-e^2)^3}}$$

Geocentric Equatorial Orbital Elements to Geocentric Equatorial Rectangular Coordinates

$$
(66) \qquad
\begin{aligned}
x &= r \cos u \cos \Omega - r \sin u \cos i \sin \Omega \\
y &= r \cos u \sin \Omega + r \sin u \cos i \cos \Omega \\
z &= r \sin u \sin i
\end{aligned}
$$

where $r = \dfrac{a(1-e^2)}{1+e \cos \eta}$ where $\eta = u - \omega$, $0 \leq \eta < 360°$

In case $\omega > u$ then $\eta = 360° + u - \omega$.

Geocentric Equatorial Orbital Elements to Geocentric Equatorial Rectangular Velocities

$$
(67) \qquad
\begin{aligned}
\dot{x} &= \dot{r} \cos u \cos \Omega - r \dot{u} \sin u \cos \Omega \\
&\quad - \dot{r} \sin u \cos i \sin \Omega - r \dot{u} \cos u \cos i \sin \Omega \\
\dot{y} &= \dot{r} \cos u \sin \Omega - r \dot{u} \sin u \sin \Omega \\
&\quad + \dot{r} \sin u \cos i \cos \Omega + r \dot{u} \cos u \cos i \cos \Omega \\
\dot{z} &= \dot{r} \sin u \sin i + r \dot{u} \cos u \sin i
\end{aligned}
$$

where $r = \dfrac{a(1-e^2)}{1+e \cos \eta}$ $\qquad \dot{r} = e \sin \eta \sqrt{\dfrac{K}{a(1-e^2)}}$

$$\dot{u} = (1 + e \cos \eta)^2 \sqrt{\frac{K}{a^3(1-e^2)^3}}$$

Transformations from $x\ y\ z\ \dot{x}\ \dot{y}\ \dot{z}$ to orbital elements are given in Appendix A.

IX. Basic Epoch Transformations

East Longitude of Vernal Equinox

For transformations between earth rotating and inertial systems the east longitude of the vernal equinox has been used in this note. The following relation is convenient for computing the mean value of the geodetic east longitude of the vernal equinox at epoch T:

$$\text{(68)} \quad \text{mean } \gamma_{\Upsilon}\ (T) = 259\overset{\circ}{.}92445740 - 0\overset{\circ}{.}9856473460\ d$$

$$-(2\overset{\circ}{.}9015)\ 10^{-13}\ d^2 - \omega_e t$$

$$0 \leq \gamma_{\Upsilon}\ (T) < 360°$$

where d is number of mean solar days elapsed from 0^h January 1, 1950, Universal Time to 0^h Universal Time of epoch day; t is number of mean solar seconds elapsed from 0^h Universal Time of epoch day to epoch; ω_e is earth rotation rate relative to moving mean vernal equinox. By definition d is an integer. t is not necessarily an integer. An average value for ω_e is

$$\text{(69)} \quad \omega_e = 0.004178074622 \text{ deg/mean solar sec.}$$

The mean value of course does not include nutation effects. The true value which includes nutation effects is

$$\text{(70)} \quad \text{true } \gamma_{\Upsilon}\ (T) = \text{mean } \gamma_{\Upsilon}\ (T) - \Delta\alpha$$

where $\Delta\alpha$ is called the "nutation in right ascension" or "equation of the equinoxes". Computation of $\Delta\alpha$ is described in subsequent equations, and $\Delta\alpha$ is tabulated in the American Ephemeris. Universal Time in Eq. 68 represents UTI, and ω_e represents rotation rate of date (see Appendix J), but normally in earth satellite computations UTC and an average ω_e are used. When orbital information involving the vernal equinox is transmitted between test ranges, it is important to verify consistency in method of computation.

Nutation Transformations

Nutation effects may be of the order of 15 seconds of arc and must frequently be made. The transformation is made at some specific epoch between an inertial coordinate system representing the mean equator and equinox of epoch and an inertial coordinate system representing the true equator and equinox of epoch. Let (x y z) represent geocentric equatorial coordinates in the mean system and $(\hat{x}\ \hat{y}\ \hat{z})$ represent geocentric equatorial coordinates in the true system.

$$(71)\quad \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} \cos\Delta\lambda & -\sin\Delta\lambda\cos\bar{\epsilon} & -\sin\Delta\lambda\sin\bar{\epsilon} \\ \sin\Delta\lambda\cos\epsilon & \cos\Delta\lambda\cos\epsilon\cos\bar{\epsilon}+\sin\epsilon\sin\bar{\epsilon} & \cos\Delta\lambda\cos\epsilon\sin\bar{\epsilon}-\sin\epsilon\cos\bar{\epsilon} \\ \sin\Delta\lambda\sin\epsilon & \cos\Delta\lambda\sin\epsilon\cos\bar{\epsilon}-\cos\epsilon\sin\bar{\epsilon} & \cos\Delta\lambda\sin\epsilon\sin\bar{\epsilon}+\cos\epsilon\cos\bar{\epsilon} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

For most trajectory purposes this may be simplified to the following form:

$$(72)\quad \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = \begin{bmatrix} 1 & -\Delta\lambda\cos\bar{\epsilon} & -\Delta\lambda\sin\bar{\epsilon} \\ \Delta\lambda\cos\bar{\epsilon} & 1 & -\Delta\epsilon \\ \Delta\lambda\sin\bar{\epsilon} & \Delta\epsilon & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where $\Delta\lambda$ is nutation in longitude and ϵ is true obliquity at epoch and $\bar{\epsilon}$ is mean obliquity at epoch.

$\Delta\lambda$ may be computed if desired from the following relation. It may also be found in the nautical Almanac.

$$
\begin{aligned}
(73) \qquad \Delta\lambda = & -(47.^\circ 8927 + 0.^\circ 0482T) \times 10^{-4} \sin \Omega \\
& +0.^\circ 5800 \times 10^{-4} \sin 2\Omega - 3.^\circ 5361 \times 10^{-4} \sin 2L \\
& -0.^\circ 1378 \times 10^{-4} \sin (3L - \Gamma) \\
& +0.^\circ 0594 \times 10^{-4} \sin (L + \Gamma) \\
& +0.^\circ 0344 \times 10^{-4} \sin (2L - \Omega) \\
& +0.^\circ 0125 \times 10^{-4} \sin (2\Gamma' - \Omega) \\
& +0.^\circ 3500 \times 10^{-4} \sin (L - \Gamma) \\
& +0.^\circ 0125 \times 10^{-4} \sin (2L - 2\Gamma') \\
& -0.^\circ 5658 \times 10^{-4} \sin 2 \leftmoon \\
& -0.^\circ 0950 \times 10^{-4} \sin (2 \leftmoon - \Omega) \\
& -0.^\circ 0725 \times 10^{-4} \sin (3 \leftmoon - \Gamma') \\
& +0.^\circ 0317 \times 10^{-4} \sin (\leftmoon + \Gamma') \\
& +0.^\circ 0161 \times 10^{-4} \sin (\leftmoon - \Gamma' + \Omega) \\
& +0.^\circ 0158 \times 10^{-4} \sin (\leftmoon - \Gamma' - \Omega) \\
& -0.^\circ 0144 \times 10^{-4} \sin (3 \leftmoon + \Gamma' - 2L) \\
& -0.^\circ 0122 \times 10^{-4} \sin (3 \leftmoon - \Gamma' - \Omega) \\
& +0.^\circ 1875 \times 10^{-4} \sin (\leftmoon - \Gamma') \\
& +0.^\circ 0078 \times 10^{-4} \sin (2 \leftmoon - 2\Gamma') \\
& +0.^\circ 0414 \times 10^{-4} \sin (\leftmoon + \Gamma' - 2L) \\
& +0.^\circ 0167 \times 10^{-4} \sin (2 \leftmoon - 2L) \\
& -0.^\circ 0089 \times 10^{-4} \sin (4 \leftmoon - 2L)
\end{aligned}
$$

(74) $\Delta\epsilon = 25\overset{\circ}{.}5844 \times 10^{-4} \cos \Omega - 0\overset{\circ}{.}2511 \times 10^{-4} \cos 2\Omega$

$+1\overset{\circ}{.}5336 \times 10^{-4} \cos 2L + 0\overset{\circ}{.}0666 \times 10^{-4} \cos (3L - \Gamma)$

$-0\overset{\circ}{.}0258 \times 10^{-4} \cos (L + \Gamma)$

$-0\overset{\circ}{.}0183 \times 10^{-4} \cos (2L - \Omega)$

$-0\overset{\circ}{.}0067 \times 10^{-4} \cos (2\Gamma' - \Omega)$

$+0\overset{\circ}{.}2456 \times 10^{-4} \cos 2☾ + 0\overset{\circ}{.}0508 \times 10^{-4} \cos (2☾ - \Omega)$

$+0\overset{\circ}{.}0369 \times 10^{-4} \cos (3☾ - \Gamma')$

$-0\overset{\circ}{.}0139 \times 10^{-4} \cos (☾ + \Gamma')$

$-0\overset{\circ}{.}0086 \times 10^{-4} \cos (☾ - \Gamma' + \Omega)$

$+0\overset{\circ}{.}0083 \times 10^{-4} \cos (☾ - \Gamma' - \Omega)$

$+0\overset{\circ}{.}0061 \times 10^{-4} \cos (3☾ + \Gamma' - 2L)$

$+0\overset{\circ}{.}0064 \times 10^{-4} \cos (3☾ - \Gamma' - \Omega)$

(75) $\bar{\epsilon} = 23\overset{\circ}{.}4457587 - 0\overset{\circ}{.}013094047$

$-0\overset{\circ}{.}0088 \times 10^{-4}\ T^2 + 0\overset{\circ}{.}0050 \times 10^{-4}\ T^3$

(75a) $\epsilon = \bar{\epsilon} + \Delta\epsilon$

(76) $\Delta\alpha = \Delta\lambda \cos \bar{\epsilon}$ (see equation 70)

The various functions used in equations (73) and (74) may be defined as follows:

$\Omega = 12\overset{\circ}{.}1127902 - 0\overset{\circ}{.}0529539222d + 20\overset{\circ}{.}795 \times 10^{-4} T$
$+20\overset{\circ}{.}81 \times 10^{-4}\ T^2 + 0\overset{\circ}{.}02 \times 10^{-4}\ T^3$

$☾ = 64\overset{\circ}{.}37545167 + 13\overset{\circ}{.}1763965268d - 11\overset{\circ}{.}31575 \times 10^{-4}\ T$
$-11\overset{\circ}{.}3015 \times 10^{-4}\ T^2 + 0\overset{\circ}{.}019 \times 10^{-4}\ T^3$

$\Gamma' = 208\overset{\circ}{.}8439877 + 0\overset{\circ}{.}1114040803d - 0\overset{\circ}{.}010334\ T$
$-0\overset{\circ}{.}010343\ T^2 - 0\overset{\circ}{.}12 \times 10^{-4}\ T^3$

$L = 280\overset{\circ}{.}08121009 + 0\overset{\circ}{.}9856473354d + 3\overset{\circ}{.}03 \times 10^{-4}\ T$
$+3\overset{\circ}{.}03 \times 10^{-4}\ T^2$

$\Gamma = 282\overset{\circ}{.}08053028 + 0\overset{\circ}{.}470684 \times 10^{-4}d + 4\overset{\circ}{.}5525 \times 10^{-4}\ T$
$+4\overset{\circ}{.}575 \times 10^{-4}\ T^2 + 0\overset{\circ}{.}03 \times 10^{-4}\ T^3$

T is the number of Julian centuries of 36525 days past the epoch 0^h January 1, 1950, E. T., while d is the number of days past the same epoch. Note that the transfer matrix in (72) may be inverted for transformations from $(\hat{x}\ \hat{y}\ \hat{z})$ to $(x\ y\ z)$.

Precession Transformations

A common transformation required is that between the mean equator and equinox of 1950.0 and the mean equator and equinox of date. The first is a very useful inertial system, while the second is not an inertial system in the true sense. Let $(x\ y\ z)$ represent the geocentric equatorial coordinate system based upon the mean equator and equinox of date.

(77)
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_{1950} \\ y_{1950} \\ z_{1950} \end{bmatrix}$$

Using the standard nomenclature of the ephemeris for ζ_o, z, and θ:

(78)
$$a_{11} = -\sin \zeta_o \sin z + \cos \zeta_o \cos z \cos \theta$$

$$a_{12} = -\cos \zeta_o \sin z - \sin \zeta_o \cos z \cos \theta$$

$$a_{13} = -\cos z \sin \theta$$

$$a_{21} = \sin \zeta_o \cos z + \cos \zeta_o \sin z \cos \theta$$

$$a_{22} = \cos \zeta_o \cos z - \sin \zeta_o \sin z \cos \theta$$

$$a_{23} = -\sin z \sin \theta$$

$$a_{31} = \cos \zeta_o \sin \theta$$

$$a_{32} = -\sin \zeta_o \sin \theta$$

$$a_{33} = \cos \theta$$

where

$$(79)\qquad \zeta_o = 2304''.997T + 0''.302T^2 + 0''.0179T^3$$

$$z = 2304''.997T + 1''.093T^2 + 0''.0192T^3$$

$$\theta = 2004''.298T - 0''.426T^2 - 0''.0416T^3$$

where T is the number of Julian centuries of 36525 days past the epoch 1950.0. This epoch corresponds to Jan 0^d.923 U.T. of the year 1950.

Equation (79) may be substituted into (78) and the resulting functions may be expanded in power series of ζ_o, z and θ:

$$(80)\qquad a_{11} = 1 - 0.00029697T^2 - 0.00000013T^3$$

$$a_{12} = -a_{21} = -0.02234988T - 0.00000676T^2 + 0.00000221T^3$$

$$a_{13} = -a_{31} = -0.00971711T + 0.00000207T^2 + 0.00000096T^3$$

$$a_{22} = 1 - 0.00024976T^2 - 0.00000015T^3$$

$$a_{23} = -0.00010859T^2 - 0.00000003T^3$$

$$a_{32} = -0.00010859T^2 + 0.00000001T^3$$

$$a_{33} = 1 - 0.00004721T^2 + 0.00000002T^3 \quad .$$

The transfer matrix represented in (77) can of course be inverted for transformations from "date" to 1950.0.

To summarize, let (x,y,z) represent geocentric equatorial coordinates of a point in a system defined by mean equator and equinox of 1950.0. Let $(\hat{x},\hat{y},\hat{z})$ represent geocentric equatorial coordinates of the same point in a system defined by true equator and equinox of date. Then

$$(80a)\qquad \begin{bmatrix} \hat{x} \\ \hat{y} \\ \hat{z} \end{bmatrix} = NP \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where N is the transfer matrix in Equation (71) and P is the transfer matrix in Equation (77).

For precision space probe or high altitude satellite computations it is customary to operate in a system defined by mean equator and equinox of 1950.0. Equations of this section provide the necessary transformations. For low altitude satellites or missiles it is common to operate in a system characterized by "true equator of date and mean equinox of epoch." This uses Equation 68 and involves no nutation or precession computations. It is an inertial system in that a fixed vernal equinox is used. Sometimes, particularly in missile trajectory computations, equations of motion are expressed in a non-inertial earth rotating coordinate system with centrifugal and Coriolis terms added. This system is equivalent to "true equator of date and mean equinox of epoch" in regard to its basic accuracy and application. Finally it may be pointed out that any geocentric system involves acceleration of the geocenter itself. This acceleration becomes relatively significant at very large distances from the earth, at which time it is necessary to change over to a heliocentric system.

NOTES:

For many purposes it may be satisfactory to discard all terms with amplitude less than $0\overset{\circ}{.}3 \times 10^{-4}$ in Equations (73) and (74).

The (x y z) coordinate system discussed in this section (IX) is the common geocentric inertial equatorial right-handed system. z is coincident with polar axis and extends in a positive direction northward. x extends positively toward the vernal equinox.

Transformations similar to those given in this section (IX) can be given for the ecliptic coordinate system but at this writing no need is foreseen for these transformations.

If greater precision is needed in Equations 73 and 74, many additional terms developed by Woolard are available in the Explanatory Supplement to the American Ephemeris.

X. Orthogonal Matrices (Real)

A simple rotation of coordinate systems which does not result in a change in length of the vector is called an orthogonal transformation. The corresponding transfer matrix is an orthogonal matrix. By definition, if R is orthogonal, $R^T = R^{-1}$ and hence $RR^T = R^TR = I$, where I represents the identity matrix.

Most of the transformation matrices discussed in this report happen to be orthogonal; and hence when an inversion is called for, the simpler operation of transposing can be substituted. The necessary and sufficient conditions that a given n by n matrix be orthogonal can be derived by expanding $RR^T = I$. We obtain

$$\sum_{s=1}^{n} R_{ps} R_{qs} = \delta_{pq}$$

which must hold for every combination of values of p and q, where

$$1 \leq p \leq n$$

$$1 \leq q \leq n$$

δ_{pq} is the Kronecker delta, which is unity when $p = q$ and is zero when $p \neq q$.

Orthogonality can frequently be verified by inspection for matrices of low order. As an example, consider the transfer matrix in Equation 4:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Applying our criterion:

p	q	$\sum_{s=1}^{n} R_{ps} R_{qs}$
1	1	$\cos^2\theta + \sin^2\theta = 1$
1	2	$-\cos\theta\sin\theta + \sin\theta\cos\theta = 0$
1	3	$0 = 0$
2	2	$\sin^2\theta + \cos^2\theta = 1$
2	3	$0 = 0$
3	3	$1 = 1$

Since the elements of this matrix can be multiplied commutatively, it is not necessary to consider other combinations like (p=2, q=1). Clearly the matrix is orthogonal.

Checking for orthogonality is not only useful in avoiding inversion but is also useful in detecting errors. Applying our criterion to the more complicated matrix (Equation 78) shows that it is indeed orthogonal as should be expected. It may be of interest to apply the criterion to the approximation given in Equation 80. We must of necessity discard terms with powers of T greater than 3:

p	q	$\sum_{s=1}^{n} R_{ps} R_{qs}$
1	1	$1.00000000 + .00000000T + .00000000T^2 + .00000000T^3$
1	2	$+ .00000000T + .00000000T^2 + .00000000T^3$
1	3	$+ .00000000T + .00000000T^2 + .00000000T^3$
2	2	$1.00000000 + .00000000T + .00000000T^2 + .00000000T^3$
2	3	$+ .00000000T + .00000000T^2 + .00000000T^3$
3	3	$1.00000000 + .00000000T + .00000000T^2 + .00000000T^3$

Clearly to this degree of approximation the matrix is orthogonal.

In general, it can be stated and easily proven that:

(1) The product of orthogonal matrices is orthogonal.

(2) The inverse of an orthogonal matrix is orthogonal and is also the transpose.

(3) The transpose of an orthogonal matrix is orthogonal and is also the inverse.

(4) Orthogonal matrices are square, have inverses and may be of any size.

(5) The determinant of an orthogonal matrix is ± 1. It is $+1$ if there is no reflection of the axes or if there is an even number of reflections of the axes. It is -1 if there is an odd number of reflections of the axes.

(6) In an orthogonal transformation, the trace and determinant remain invariant.

XI. Error Analysis for Coordinate Transformations

This section deals with the propagation of errors from one coordinate system at time t_o to another coordinate system at time t_o, or from one coordinate system at time t_o to the same or another coordinate system at time t. Errors are expressed in terms of variances and covariances. The results are valid without dependence upon normality of the distribution.

The covariance matrix of a pair of vectors $y = (y_1\ y_2\ \ldots\ y_q)^T$, $z = (z_1\ z_2\ \ldots\ z_r)^T$ is defined as

$$\sigma_{yz^T} = \begin{bmatrix} \sigma_{y_1 z_1} & \sigma_{y_1 z_2} & \cdots & \sigma_{y_1 z_r} \\ \sigma_{y_2 z_1} & \sigma_{y_2 z_2} & \cdots & \sigma_{y_2 z_r} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \sigma_{y_q z_1} & \sigma_{y_q z_2} & \cdots & \sigma_{y_q z_r} \end{bmatrix}$$

The generalized law of covariance propagation states that if the elements of y and z are functions of the vector $x = (x_1\ x_2\ \ldots\ x_n)^T$ which has the covariance σ_{xx^T}, the covariance matrix of y and z is given by

$$(\sigma_{yz^T}) = (y_x)\ (\sigma_{xx^T})\ (z_x)^T$$

where

$$(y_x)^T = \left(\frac{\partial}{\partial x}\right)\ (y)^T\ ,\ (z_x)^T = \left(\frac{\partial}{\partial x}\right)\ (z)^T$$

in which $\left(\frac{\partial}{\partial x}\right)$ is the vector of partial differentiation operators

$$\left(\frac{\partial}{\partial x}\right) = \left(\frac{\partial}{\partial x_1}\ \frac{\partial}{\partial x_2}\ \cdots\ \frac{\partial}{\partial x_n}\right)^T$$

NOTE: The notation y_x is not equivalent to $\left(\frac{\partial y}{\partial x}\right)$ as normally defined but rather $\left(\frac{\partial y}{\partial x}\right)^T$. y_x has the form of the Jacobian.

Alternatively, if the inverses exist,

$$\sigma_{yz}T = (x_y)^{-1} (\sigma_{xx}T) (x_z^{-1})^T$$

or

$$\sigma_{yz}T = \left[(x_z)^T (\sigma_{xx}T)^{-1} (x_y)\right]^{-1}$$

where

$$(x_y)^T = (\frac{\partial}{\partial y}) (x)^T \qquad (x_z)^T = (\frac{\partial}{\partial z}) (x)^T$$

in which $(\frac{\partial}{\partial y})$ and $(\frac{\partial}{\partial z})$ are vectors analogous to $(\frac{\partial}{\partial x})$ already defined.

In the vast majority of practical applications, y = z and the resulting symmetric matrix is referred to as the covariance matrix of the vector y. The covariance matrix is always not only symmetric but also positive semi-definite.

It may happen that because of redundant instrumentation a set of independent column vectors $(V_1, V_2 \ldots V_n)$, each vector having an associated covariance matrix $(\Sigma_1, \Sigma_2 \ldots \Sigma_n)$, may be obtained as estimates of the true vector at some particular time point. These estimates may be combined into a single estimate $\hat{V}$ and its corresponding covariance matrix $\hat{\Sigma}$:

$$\hat{V} = \hat{\Sigma} \left[\Sigma_1^{-1} V_1 + \Sigma_2^{-1} V_2 + \ldots + \Sigma_n^{-1} V_n\right]$$

where

$$\hat{\Sigma} = \left[\Sigma_1^{-1} + \Sigma_2^{-1} + \ldots \Sigma_n^{-1}\right]^{-1} .$$

Instead of, or prior to, combining two vector estimates V_1 and V_2 and associated estimated covariance matrices Σ_1 and Σ_2, it may be desirable to test the hypothesis that they represent the same population. This has direct application, for example in detection of satellite maneuvers by comparison of two sets of orbital elements. In the application cited, the tracking data usually provide large sample sizes (greater than 30) and hence we may use the simple relation:

$$\left[V_2 - V_1\right]^T \left[\Sigma_1 + \Sigma_2\right]^{-1} \left[V_2 - V_1\right] > \chi^2$$

where χ^2 is obtained from statistical tables at some probability level, e.g. 1% or 5%, at degrees of freedom equal to number of elements making up the vector, e.g. 6 or 7.

The test is more involved for small sample sizes. First of all we must define sample size n for this test. Let N be the total number of independent measurements—in the case of radar track at one per five seconds* it would be the sum of the number of all the azimuth measurements plus the number of all the elevation measurements plus the number of all the range measurements. Suppose there are p elements in each of the vectors being compared and there are no other parameters being estimated. Then

$$n = \frac{N}{p} .$$

Therefore unit sample size in this case is the necessary and non redundant number of measurements required to define the vector. Now let the sample sizes for V_1 and V_2 respectively be n_1 and n_2.

Compute

$$\Sigma = \frac{1}{n_1+n_2-2}\left[n_1(n_1-1)\,\Sigma_1 + n_2\,(n_2-1)\,\Sigma_2\right] .$$

Now if

$$\left[V_2-V_1\right]^T \Sigma^{-1} \left[V_2-V_1\right] \geq \frac{(n_1+n_2)(n_1+n_2-2)p}{n_1 n_2 (n_1+n_2-p-1)} F_{p,\,n_1+n_2-p-1},$$

the hypothesis that $V_2=V_1$ must be rejected. F is the F statistic with p degrees of freedom for greater mean square and (n_1+n_2-p-1) degrees of freedom for lesser mean square.

NOTE: If we take the square root of each of the diagonal elements in a symmetric covariance matrix, we obtain a set of GDOPs or standard deviations in the computed parameters. GDOP is an acronym for Geometric Dilution of Precision, a term in common usage in the aerospace industry. The term is usually restricted to the computed parameters, and is not applied to the fundamental observations. GDOPs are sometimes misleading because they ignore the covariance or off-diagonal terms. See Appendix Q.

*This interval size is sufficient to justify assumption of independence.

As an example we may consider radar errors in a, e, ρ propagated into the right-handed radar xyz system, where

$$x = \rho \cos e \sin a$$
$$y = \rho \cos e \cos a$$
$$z = \rho \sin e$$

In this case we have

$$(81)\quad \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial e} & \frac{\partial x}{\partial \rho} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial e} & \frac{\partial y}{\partial \rho} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial e} & \frac{\partial z}{\partial \rho} \end{bmatrix} \begin{bmatrix} \sigma_a^2 & 0 & 0 \\ 0 & \sigma_e^2 & 0 \\ 0 & 0 & \sigma_\rho^2 \end{bmatrix} \begin{bmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial e} & \frac{\partial y}{\partial e} & \frac{\partial z}{\partial e} \\ \frac{\partial x}{\partial \rho} & \frac{\partial y}{\partial \rho} & \frac{\partial z}{\partial \rho} \end{bmatrix}$$

The first term of this expansion is

$$\sigma_x^2 = (\sigma_a^2 \ \rho^2 \cos^2 e \cos^2 a + \sigma_e^2 \rho^2 \sin^2 e \sin^2 a + \sigma_\rho^2 \cos^2 e \sin^2 a)$$

As another example take a right handed rectangular system x y z and rotate through angle Ψ about z' in a counterclockwise direction:

$$(82)\quad \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \Psi & \sin \Psi & 0 \\ -\sin \Psi & \cos \Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Then as before

$$(83)\quad \begin{bmatrix} \sigma_{x'}^2 & \sigma_{x'y'} & \sigma_{x'z'} \\ \sigma_{x'y'} & \sigma_{y'}^2 & \sigma_{y'z'} \\ \sigma_{x'z'} & \sigma_{y'z'} & \sigma_{z'}^2 \end{bmatrix} = \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_x^2 \end{bmatrix} \begin{bmatrix} \frac{\partial x'}{\partial x} & \frac{\partial y'}{\partial x} & \frac{\partial z'}{\partial x} \\ \frac{\partial x'}{\partial y} & \frac{\partial y'}{\partial y} & \frac{\partial z'}{\partial y} \\ \frac{\partial x'}{\partial z} & \frac{\partial y'}{\partial z} & \frac{\partial z'}{\partial z} \end{bmatrix}$$

But it can be readily shown that the first matrix on the right hand side of the equation above is identical to the transfer matrix relating $\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$ to $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$. Thus,

$$(84)\quad \begin{bmatrix} \sigma_{x'}^2 & \sigma_{x'y'} & \sigma_{x'z'} \\ \sigma_{x'y'} & \sigma_{y'}^2 & \sigma_{y'z'} \\ \sigma_{x'z'} & \sigma_{y'z'} & \sigma_{z'}^2 \end{bmatrix} = \begin{bmatrix} \cos\Psi & \sin\Psi & 0 \\ -\sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_z^2 \end{bmatrix} \begin{bmatrix} \cos\Psi & -\sin\Psi & 0 \\ \sin\Psi & \cos\Psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

For the general transformation between two rectangular coordinate systems defined at some particular instant by

$$(85)\quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_c \end{bmatrix} \overset{\Delta}{=} M \begin{bmatrix} X - X_o \\ Y - Y_o \\ Z - Z_o \end{bmatrix}$$

where the origin of the x y z is at $X_o\ Y_o\ Z_o$ relative to the X Y Z system, then

$$(86)\qquad \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yz} & \sigma_y^2 & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z^2 \end{bmatrix} = M \begin{bmatrix} \sigma_X^2 & \sigma_{XY} & \sigma_{XZ} \\ \sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} \\ \sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 \end{bmatrix} M^T$$

where M is defined by equation (85). <u>If the two systems are fixed relative to each other,</u> equation (86) also holds for σ matrices of the following forms. Only the right matrix will be shown but the left one is identical in form. M is again defined by Equation (85).

$$\begin{bmatrix} \sigma_{X\dot X} & \sigma_{X\dot Y} & \sigma_{X\dot Z} \\ \sigma_{Y\dot X} & \sigma_{Y\dot Y} & \sigma_{Y\dot Z} \\ \sigma_{Z\dot X} & \sigma_{Z\dot Y} & \sigma_{Z\dot Z} \end{bmatrix} \quad \text{also} \quad \begin{bmatrix} \sigma_{X\ddot X} & \sigma_{X\ddot Y} & \sigma_{X\ddot Z} \\ \sigma_{Y\ddot X} & \sigma_{Y\ddot Y} & \sigma_{Y\ddot Z} \\ \sigma_{Z\ddot X} & \sigma_{Y\ddot Y} & \sigma_{Y\ddot Z} \end{bmatrix}$$

$$\text{also} \quad \begin{bmatrix} \sigma_{\dot X X} & \sigma_{\dot X Y} & \sigma_{\dot X Z} \\ \sigma_{\dot Y X} & \sigma_{\dot Y Y} & \sigma_{\dot Y Z} \\ \sigma_{\dot Z X} & \sigma_{\dot Z Y} & \sigma_{\dot Z Z} \end{bmatrix} \quad \text{also} \quad \begin{bmatrix} \sigma_{\dot X}^2 & \sigma_{\dot X\dot Y} & \sigma_{\dot X\dot Z} \\ \sigma_{\dot Y\dot X} & \sigma_{\dot Y}^2 & \sigma_{\dot Y\dot Z} \\ \sigma_{\dot Z\dot X} & \sigma_{\dot Z\dot Y} & \sigma_{\dot Z}^2 \end{bmatrix}$$

$$\text{also} \quad \begin{bmatrix} \sigma_{\dot X\ddot X} & \sigma_{\dot X\ddot Y} & \sigma_{\dot X\ddot Z} \\ \sigma_{\dot Y\ddot X} & \sigma_{\dot Y\ddot Y} & \sigma_{\dot Y\ddot Z} \\ \sigma_{\dot Z\ddot X} & \sigma_{\dot Z\ddot Y} & \sigma_{\dot Z\ddot Z} \end{bmatrix} \quad \text{also} \quad \begin{bmatrix} \sigma_{\ddot X X} & \sigma_{\ddot X Y} & \sigma_{\ddot X Z} \\ \sigma_{\ddot Y X} & \sigma_{\ddot Y Y} & \sigma_{\ddot Y Z} \\ \sigma_{\ddot Z X} & \sigma_{\ddot Z Y} & \sigma_{\ddot Z Z} \end{bmatrix}$$

also $\begin{bmatrix} \sigma_{\ddot{X}\dot{X}} & \sigma_{\ddot{X}\dot{Y}} & \sigma_{\ddot{X}\dot{Z}} \\ \sigma_{\ddot{Y}\dot{X}} & \sigma_{\ddot{Y}\dot{Y}} & \sigma_{\ddot{Y}\dot{Z}} \\ \sigma_{\ddot{Z}\dot{X}} & \sigma_{\ddot{Z}\dot{Y}} & \sigma_{\ddot{Z}\dot{Z}} \end{bmatrix}$ and also $\begin{bmatrix} \sigma_{\ddot{X}}^2 & \sigma_{\ddot{X}\ddot{Y}} & \sigma_{\ddot{X}\ddot{Z}} \\ \sigma_{\ddot{Y}\ddot{X}} & \sigma_{\ddot{Y}}^2 & \sigma_{\ddot{Y}\ddot{Z}} \\ \sigma_{\ddot{Z}\ddot{X}} & \sigma_{\ddot{Z}\ddot{Y}} & \sigma_{\ddot{Z}}^2 \end{bmatrix}$

Considering a more general case, define the matrices A and B as follows:

$$
A = \begin{bmatrix}
\sigma_x^2 & \sigma_{xy} & \sigma_{xz} & \sigma_{x\dot{x}} & \sigma_{x\dot{y}} & \sigma_{x\dot{z}} & \sigma_{x\ddot{x}} & \sigma_{x\ddot{y}} & \sigma_{x\ddot{z}} \\
\sigma_{yx} & \sigma_y^2 & \sigma_{yz} & \sigma_{y\dot{x}} & \sigma_{y\dot{y}} & \sigma_{y\dot{z}} & \sigma_{y\ddot{x}} & \sigma_{y\ddot{y}} & \sigma_{y\ddot{z}} \\
\sigma_{zx} & \sigma_{zy} & \sigma_z^2 & \sigma_{z\dot{x}} & \sigma_{z\dot{y}} & \sigma_{z\dot{z}} & \sigma_{z\ddot{x}} & \sigma_{z\ddot{y}} & \sigma_{z\ddot{z}} \\
\sigma_{\dot{x}x} & \sigma_{\dot{x}y} & \sigma_{\dot{x}z} & \sigma_{\dot{x}}^2 & \sigma_{\dot{x}\dot{y}} & \sigma_{\dot{x}\dot{z}} & \sigma_{\dot{x}\ddot{x}} & \sigma_{\dot{x}\ddot{y}} & \sigma_{\dot{x}\ddot{z}} \\
\sigma_{\dot{y}x} & \sigma_{\dot{y}y} & \sigma_{\dot{y}z} & \sigma_{\dot{y}\dot{x}} & \sigma_{\dot{y}}^2 & \sigma_{\dot{y}\dot{z}} & \sigma_{\dot{y}\ddot{x}} & \sigma_{\dot{y}\ddot{y}} & \sigma_{\dot{y}\ddot{z}} \\
\sigma_{\dot{z}x} & \sigma_{\dot{z}y} & \sigma_{\dot{z}z} & \sigma_{\dot{z}\dot{x}} & \sigma_{\dot{z}\dot{y}} & \sigma_{\dot{z}}^2 & \sigma_{\dot{z}\ddot{x}} & \sigma_{\dot{z}\ddot{y}} & \sigma_{\dot{z}\ddot{z}} \\
\sigma_{\ddot{x}x} & \sigma_{\ddot{x}y} & \sigma_{\ddot{x}z} & \sigma_{\ddot{x}\dot{x}} & \sigma_{\ddot{x}\dot{y}} & \sigma_{\ddot{x}\dot{z}} & \sigma_{\ddot{x}}^2 & \sigma_{\ddot{x}\ddot{y}} & \sigma_{\ddot{x}\ddot{z}} \\
\sigma_{\ddot{y}x} & \sigma_{\ddot{y}y} & \sigma_{\ddot{y}z} & \sigma_{\ddot{y}\dot{x}} & \sigma_{\ddot{y}\dot{y}} & \sigma_{\ddot{y}\dot{z}} & \sigma_{\ddot{y}\ddot{x}} & \sigma_{\ddot{y}}^2 & \sigma_{\ddot{y}\ddot{z}} \\
\sigma_{\ddot{z}x} & \sigma_{\ddot{z}y} & \sigma_{\ddot{z}z} & \sigma_{\ddot{z}\dot{x}} & \sigma_{\ddot{z}\dot{y}} & \sigma_{\ddot{z}\dot{z}} & \sigma_{\ddot{z}\ddot{x}} & \sigma_{\ddot{z}\ddot{y}} & \sigma_{\ddot{z}}^2
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
\sigma_X^2 & \sigma_{XY} & \sigma_{XZ} & \sigma_{X\dot{X}} & \sigma_{X\dot{Y}} & \sigma_{X\dot{Z}} & \sigma_{X\ddot{X}} & \sigma_{X\ddot{Y}} & \sigma_{X\ddot{Z}} \\
\sigma_{YX} & \sigma_Y^2 & \sigma_{YZ} & \sigma_{Y\dot{X}} & \sigma_{Y\dot{Y}} & \sigma_{Y\dot{Z}} & \sigma_{Y\ddot{X}} & \sigma_{Y\ddot{Y}} & \sigma_{Y\ddot{Z}} \\
\sigma_{ZX} & \sigma_{ZY} & \sigma_Z^2 & \sigma_{Z\dot{X}} & \sigma_{Z\dot{Y}} & \sigma_{Z\dot{Z}} & \sigma_{Z\ddot{X}} & \sigma_{Z\ddot{Y}} & \sigma_{Z\ddot{Z}} \\
\sigma_{\dot{X}X} & \sigma_{\dot{X}Y} & \sigma_{\dot{X}Z} & \sigma_{\dot{X}}^2 & \sigma_{\dot{X}\dot{Y}} & \sigma_{\dot{X}\dot{Z}} & \sigma_{\dot{X}\ddot{X}} & \sigma_{\dot{X}\ddot{Y}} & \sigma_{\dot{X}\ddot{Z}} \\
\sigma_{\dot{Y}X} & \sigma_{\dot{Y}Y} & \sigma_{\dot{Y}Z} & \sigma_{\dot{Y}\dot{X}} & \sigma_{\dot{Y}}^2 & \sigma_{\dot{Y}\dot{Z}} & \sigma_{\dot{Y}\ddot{X}} & \sigma_{\dot{Y}\ddot{Y}} & \sigma_{\dot{Y}\ddot{Z}} \\
\sigma_{\dot{Z}X} & \sigma_{\dot{Z}Y} & \sigma_{\dot{Z}Z} & \sigma_{\dot{Z}\dot{X}} & \sigma_{\dot{Z}\dot{Y}} & \sigma_{\dot{Z}}^2 & \sigma_{\dot{Z}\ddot{X}} & \sigma_{\dot{Z}\ddot{Y}} & \sigma_{\dot{Z}\ddot{Z}} \\
\sigma_{\ddot{X}X} & \sigma_{\ddot{X}Y} & \sigma_{\ddot{X}Z} & \sigma_{\ddot{X}\dot{X}} & \sigma_{\ddot{X}\dot{Y}} & \sigma_{\ddot{X}\dot{Z}} & \sigma_{\ddot{X}}^2 & \sigma_{\ddot{X}\ddot{Y}} & \sigma_{\ddot{X}\ddot{Z}} \\
\sigma_{\ddot{Y}X} & \sigma_{\ddot{Y}Y} & \sigma_{\ddot{Y}Z} & \sigma_{\ddot{Y}\dot{X}} & \sigma_{\ddot{Y}\dot{Y}} & \sigma_{\ddot{Y}\dot{Z}} & \sigma_{\ddot{Y}\ddot{X}} & \sigma_{\ddot{Y}}^2 & \sigma_{\ddot{Y}\ddot{Z}} \\
\sigma_{\ddot{Z}X} & \sigma_{\ddot{Z}Y} & \sigma_{\ddot{Z}Z} & \sigma_{\ddot{Z}\dot{X}} & \sigma_{\ddot{Z}\dot{Y}} & \sigma_{\ddot{Z}\dot{Z}} & \sigma_{\ddot{Z}\ddot{X}} & \sigma_{\ddot{Z}\ddot{Y}} & \sigma_{\ddot{Z}}^2
\end{bmatrix}
$$

Then

$$
\begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix} \begin{bmatrix} B \end{bmatrix} \begin{bmatrix} M & 0 & 0 \\ 0 & M & 0 \\ 0 & 0 & M \end{bmatrix}^T
$$

where all four matrices are 9 x 9 and M is the same 3 x 3 previously defined. We reiterate that the two coordinate systems are assumed to be fixed relative to each other in this particular example.

To illustrate a transformation both in time and coordinate systems, we compute the covariance matrix in latitude ϕ and longitude γ at impact resulting from a covariance matrix in $\dot{x}\ \dot{y}\ \dot{z}$ at missile burn-out. Assume the following partials have been obtained numerically:

$$B = \begin{bmatrix} \frac{\partial\phi}{\partial\dot{x}} & \frac{\partial\phi}{\partial\dot{y}} & \frac{\partial\phi}{\partial\dot{z}} \\ \frac{\partial\gamma}{\partial\dot{x}} & \frac{\partial\gamma}{\partial\dot{y}} & \frac{\partial\gamma}{\partial\dot{z}} \end{bmatrix} = \begin{bmatrix} -.003 & +.005 & -.002 \\ +.017 & +.001 & +.003 \end{bmatrix}$$

where units are in degrees and feet per second.

Assume the following covariance matrix is given:

$$\Sigma_0 = \begin{bmatrix} \sigma_{\dot{x}}^2 & \sigma_{\dot{x}\dot{y}} & \sigma_{\dot{x}\dot{z}} \\ \sigma_{\dot{y}\dot{x}} & \sigma_{\dot{y}}^2 & \sigma_{\dot{y}\dot{z}} \\ \sigma_{\dot{z}\dot{x}} & \sigma_{\dot{z}\dot{y}} & \sigma_{\dot{z}}^2 \end{bmatrix} = \begin{bmatrix} +1 & +.02 & -.01 \\ +.02 & +.5 & +.03 \\ -.01 & +.03 & +.5 \end{bmatrix}$$

where units are in (ft./sec.)^2.

Therefore the covariance matrix at impact is

$$\Sigma_1 = \begin{bmatrix} \sigma_{\phi}^2 & \sigma_{\phi\gamma} \\ \sigma_{\gamma\phi} & \sigma_{\gamma}^2 \end{bmatrix} = B\ \Sigma_0\ B^T = \begin{bmatrix} +.00002218 & -.00005314 \\ -.00005314 & +.00031994 \end{bmatrix}$$

where units are in degrees squared.

By standard procedures Σ_1 can be further transformed if desired into a 95% confidence ellipse. See Appendix Q.

As a final example it may be of interest to estimate the errors in the orbital elements at time t resulting from errors in the orbital elements at time t_o. The time from t to t_o would represent a coasting phase. The errors at t can of course subsequently be transformed to errors in rectangular coordinates or look angles if desired. From the law of covariance propagation

$$\left(\sigma_{EE^T}\right)_t = E_p \left(\sigma_{EE^T}\right)_{t_o} E_p^T$$

The form of the E_p matrix involved in the equation above is shown below:

$$E_p = \begin{bmatrix} \frac{\partial a}{\partial a_o} & \frac{\partial a}{\partial e_o} & \frac{\partial a}{\partial i_o} & \frac{\partial a}{\partial \Omega_o} & \frac{\partial a}{\partial \omega_o} & \frac{\partial a}{\partial u_o} \\ \frac{\partial e}{\partial a_o} & \frac{\partial e}{\partial e_o} & \frac{\partial e}{\partial i_o} & \frac{\partial e}{\partial \Omega_o} & \frac{\partial e}{\partial \omega_o} & \frac{\partial e}{\partial u_o} \\ \frac{\partial i}{\partial a_o} & \frac{\partial i}{\partial e_o} & \frac{\partial i}{\partial i_o} & \frac{\partial i}{\partial \Omega_o} & \frac{\partial i}{\partial \omega_o} & \frac{\partial i}{\partial u_o} \\ \frac{\partial \Omega}{\partial a_o} & \frac{\partial \Omega}{\partial e_o} & \frac{\partial \Omega}{\partial i_o} & \frac{\partial \Omega}{\partial \Omega_o} & \frac{\partial \Omega}{\partial \omega_o} & \frac{\partial \Omega}{\partial u_o} \\ \frac{\partial \omega}{\partial a_o} & \frac{\partial \omega}{\partial e_o} & \frac{\partial \omega}{\partial i_o} & \frac{\partial \omega}{\partial \Omega_o} & \frac{\partial \omega}{\partial \omega_o} & \frac{\partial \omega}{\partial u_o} \\ \frac{\partial u}{\partial a_o} & \frac{\partial u}{\partial e_o} & \frac{\partial u}{\partial i_o} & \frac{\partial u}{\partial \Omega_o} & \frac{\partial u}{\partial \omega_o} & \frac{\partial u}{\partial u_o} \end{bmatrix}$$

The form of σ_{EE^T} is

$$\sigma_{EE^T} = \begin{bmatrix} \sigma_a^2 & \sigma_{ae} & \sigma_{ai} & \sigma_{a\Omega} & \sigma_{a\omega} & \sigma_{au} \\ \sigma_{ea} & \sigma_e^2 & \sigma_{ei} & \sigma_{e\Omega} & \sigma_{e\omega} & \sigma_{eu} \\ \sigma_{ia} & \sigma_{ie} & \sigma_i^2 & \sigma_{i\Omega} & \sigma_{i\omega} & \sigma_{iu} \\ \sigma_{\Omega a} & \sigma_{\Omega e} & \sigma_{\Omega i} & \sigma_\Omega^2 & \sigma_{\Omega\omega} & \sigma_{\Omega u} \\ \sigma_{\omega a} & \sigma_{\omega e} & \sigma_{\omega i} & \sigma_{\omega\Omega} & \sigma_\omega^2 & \sigma_{\omega u} \\ \sigma_{ua} & \sigma_{ue} & \sigma_{ui} & \sigma_{u\Omega} & \sigma_{u\omega} & \sigma_u^2 \end{bmatrix}$$

The derivation of the partials in E_p is beyond the scope of this report, but for an ellipse

$$E_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ \alpha & \beta & 0 & 0 & \gamma & \delta \end{bmatrix}$$

where

$$\alpha = \frac{3\sqrt{1-e_o^2}}{2\,R_t^2}\sqrt{\frac{K}{a_o}}\,(t_o-t)$$

$$\beta = \sin\eta_t\left(\frac{1}{1-e_o^2}+\frac{a_o}{R_t}\right) - \sin\eta_o\left(\frac{R_o^2}{R_t^2(1-e_o^2)}+\frac{R_o\,a_o}{R_t^2}\right)$$

$$\gamma = \left[1-\left(\frac{R_o}{R_t}\right)^2\right]$$

$$\delta = \left(\frac{R_o}{R_t}\right)^2$$

K is gravitational constant of central body
R is distance from principal focus to orbiting body
η is true anomaly
Other terms are orbital elements previously defined (page 22).

Error analyses are most efficiently performed by direct transformations of the type just discussed. However, occasionally such methods are not applicable, and Monte Carlo or simulation techniques are used. These latter types frequently require sampling from a normal distribution. If only a single independent parameter is involved or if the covariance matrix of a multi-parameter problem is diagonal, there is of course no difficulty. Such problems permit straightforward use of a table of univariate standard normal deviates. Let us consider a more complicated situation of n parameters and associated covariance matrix containing non-zero off-diagonal elements. We have the problem of sampling from a known multi-variate normal distribution described by a mean column vector m and a covariance matrix Σ. The procedure is as follows:

By standard matrix algebra, compute an orthogonal transformation matrix A such that $\left[A\ \Sigma\ A^T\right]$ is a diagonal matrix. Select a column vector $z_i = \left[z_1, z_2 \ldots z_n\right]^T$ each of whose elements is simply a number taken from a table of univariate standard normal deviates. Then the required vector of parameters is

$$x_i = m + A^T \left[A\ \Sigma\ A^T\right]^{\frac{1}{2}} z_i .$$

Additional vectors x_i may be obtained simply by substituting different z_i.

APPENDIX A

A SELECTION OF KEPLERIAN RELATIONSHIPS FOR ELLIPTICAL ORBITS

1.0 Inertial Mass Centered Rectangular Position and Velocity Components to Orbital Elements

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

By definition

$$D = x\dot{x} + y\dot{y} + z\dot{z}$$

$$\dot{r} = \frac{D}{r}$$

The flight path angle θ is the angle between the velocity vector and the local geocentric horizontal and always lies between +90° and -90°.

$$\sin\theta = \frac{\dot{r}}{v} = \frac{D}{rv}$$

Then

$$a = \frac{rK}{2K - rv^2}$$

The parameter p of an orbit is defined to be one half the latus rectum.

$$p = \frac{r^2(v^2 - \dot{r}^2)}{K}$$

By definition

$$N = \frac{rv^2 - K}{K}$$

Then

$$e = \sqrt{N^2 + (1 - N^2)\sin^2\theta}$$

The true ancmaly at epoch is found from

$$\tan\eta = \frac{(N + 1)\sin\theta\cos\theta}{N\cos^2\theta - \sin^2\theta}$$

Examination of the signs of numerator and denominator separately will reveal which of the four quadrants η is in. For a perfectly circular orbit both numerator and denominator will be zero, in which case η is set equal to zero by convention.

Recall that Kepler's constant C is equal to twice the rate at which area is being swept out by the radius vector from central body to space vehicle.

$$|C| = |\vec{r} \times \vec{v}| = \sqrt{pK}$$

Moreover

$$C_x = y\dot{z} - z\dot{y} \qquad C_y = z\dot{x} - x\dot{z} \qquad C_z = x\dot{y} - y\dot{x}$$

Now

$$\cos i = \frac{C_z}{\sqrt{pK}} \qquad 0° \leq i \leq 180°$$

$$\tan\Omega = \frac{C_x}{-C_y} \qquad 0° \leq \Omega < 360°$$

where the signs of numerator and denominator must be considered separately to determine which of four quadrants Ω is in. When both C_x and C_y are precisely zero the orbit is equatorial and Ω is defined to be zero.

$$\tan u = \frac{-x \sin \Omega \cos i + y \cos \Omega \cos i + z \sin i}{x \cos \Omega + y \sin \Omega}$$

$$0^\circ \le u < 360^\circ$$

where again the signs of numerator and denominator must be considered separately to determine quadrant of Ω.

$$\omega = u - \eta \qquad 0 \le \omega < 360^\circ$$

2.0 Additional Relationships within the Orbit Plane

Semi-major axis, a =

$$\frac{Kr}{2K-rv^2} \qquad \frac{r_A}{1+e}$$

$$\frac{r_P}{1-e} \qquad \frac{r_A+r_P}{2}$$

Semi-minor axis, b =

$$a\sqrt{1-e^2}$$

$$\sqrt{r_A r_P} \qquad (rv \cos \theta)\left(\frac{r}{2K - rv^2}\right)^{1/2}$$

Eccentric anomaly, E =

$$\sin^{-1}\left(\frac{r}{b} \sin \eta\right) \qquad 2 \tan^{-1}\left[\sqrt{\frac{1-e}{1+e}} \tan (\eta/2)\right]$$

$$\cos^{-1}\left(\frac{a-r}{ae}\right)$$

Note: $E/2$ and $\eta/2$ lie in same quadrant

Eccentricity, e =

$$\sqrt{1 - \frac{b^2}{a^2}} \qquad \sqrt{1 - \frac{r^2 v^2 \cos^2 \theta}{K^2} \frac{(2K - r v^2)}{r}}$$

$$\frac{r_A - r_P}{r_A + r_P}$$

Mean anomaly, M =

$$n\,(t - t_p)$$

$$E - e \sin E$$ (Kepler's Equation) (See Appendix D)

Radial distance from geocenter to satellite, r =

$$\frac{a\,(1-e^2)}{1 + e \cos \eta} \qquad a\,(1 - e \cos E)$$

Radial distance from geocenter to apogee, r_A =

$$a\,(1+e)$$

$$b^2/r_p \qquad r_p\left(\frac{1+e}{1-e}\right)$$

Radial distance from geocenter to perigee, r_p =

$$a\,(1-e) \qquad b^2/r_A$$

$$r_A\left(\frac{1-e}{1+e}\right)$$

True anomaly, η =

$$\cos^{-1}\left[\frac{a(1-e^2)-r}{e\,r}\right] \qquad 2\tan^{-1}\left[\sqrt{\frac{1+e}{1-e}}\,\tan(E/2)\right]$$

Note: E/2 and η/2 lie in same quadrant

Flight path angle, θ =

$$\tan^{-1}\left(\frac{e \sin \eta}{1+e \cos \eta}\right)$$, evaluated $\theta = 0^\circ \pm 90^\circ$

Path velocity, v =

$$\sqrt{K\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Mean motion, n =

$$\sqrt{\frac{K}{a^3}} \qquad \frac{2\pi}{T_K}$$

Orbital period:

Kepler, $T_K = 2\pi\sqrt{\frac{a^3}{K}}$

*Anomalistic, $T_L = T_K\left[1 + \frac{3C_{2,0}a a_e^2}{2\, r^3}(1-3\sin^2 i)\right]$

*Nodal, $T_N = T_L + T_K\left[\frac{3C_{2,0}\, a_e^2\, (2-\frac{5}{2}\sin^2 i)}{a^2\,(1-e^2)^2}\right]$

Derivatives:

$$\dot{r} = e\sin\eta\sqrt{\frac{K}{p}} = v\sin\theta$$

$$\ddot{r} = \frac{K}{r^2}\, e\cos\eta = \frac{K(p-1)}{r^3} = e\,\dot{\eta}\cos\eta\sqrt{\frac{K}{p}}$$

---

*These expressions are included here for convenience. Since they depend upon oblateness of the earth, they are not Keplerian.

$$\dot{\eta} = \frac{\sqrt{Kp}}{r^2} = \frac{v_p\, r_p}{r^2}$$

$$\ddot{\eta} = -\frac{2K}{r^3}\, e \sin \eta = -\frac{2\sqrt{Kp}}{r^3}\, \dot{r}$$

The parameter p of the orbit is equal to $a(1-e^2)$.

See Appendix N for derivatives of mean elements.

Series expansions:

$$\begin{aligned}
\eta = {} & M + \left(2e - \frac{1}{4} e^3 + \frac{5}{96} e^5 + \frac{107}{4608} e^7\right) \sin M \\
& + \left(\frac{5}{4} e^2 - \frac{11}{24} e^4 + \frac{17}{192} e^6\right) \sin 2M \\
& + \left(\frac{13}{12} e^3 - \frac{43}{64} e^5 + \frac{95}{512} e^7\right) \sin 3M \\
& + \left(\frac{103}{96} e^4 - \frac{451}{480} e^6\right) \sin 4M \\
& + \left(\frac{1097}{960} e^5 - \frac{5957}{4608} e^7\right) \sin 5M \\
& + \frac{1223}{960} e^6 \sin 6M + \frac{47273}{32256} e^7 \sin 7M \; .
\end{aligned}$$

$$\begin{aligned}
\frac{r}{a} = {} & 1 + \frac{e^2}{2} + \left[-e + \frac{3}{8} e^3 - \frac{5}{192} e^5 + \frac{7}{9216} e^7\right] \cos M \\
& + \left[-\frac{1}{2} e^2 + \frac{1}{3} e^4 - \frac{1}{16} e^6\right] \cos 2M \\
& + \left[-\frac{3}{8} e^3 + \frac{45}{128} e^5 - \frac{567}{5120} e^7\right] \cos 3M \\
& + \left[-\frac{1}{3} e^4 + \frac{2}{5} e^6\right] \cos 4M \\
& + \left[-\frac{125}{384} e^5 + \frac{4375}{9216} e^7\right] \cos 5M \\
& - \frac{27}{80} e^6 \cos 6M - \frac{16807}{46080} e^7 \sin 7M \; .
\end{aligned}$$

$$
\begin{aligned}
E = M &+ \left[e - \frac{1}{8} e^3 + \frac{1}{192} e^5 - \frac{1}{9216} e^7\right] \sin M \\
&+ \left[\frac{1}{2} e^2 - \frac{1}{6} e^4 + \frac{1}{48} e^6\right] \sin 2M \\
&+ \left[\frac{3}{8} e^3 - \frac{27}{128} e^5 + \frac{243}{5120} e^7\right] \sin 3M \\
&+ \left[\frac{1}{3} e^4 - \frac{4}{15} e^6\right] \sin 4M \\
&+ \left[\frac{125}{384} e^5 - \frac{3125}{9216} e^7\right] \sin 5M \\
&+ \frac{27}{80} e^6 \sin 6M + \frac{16807}{46080} e^7 \sin 7M.
\end{aligned}
$$

3.0 Explanation of Notation

C	is Kepler's constant vector.
C_x, C_y, C_z	are components of C in xyz coordinate system.
$C_{2,0}$	is second zonal coefficient in gravitational expression
D	is defined in text.
E	is eccentric anomaly.
K	is central gravitational parameter.
M	is mean anomaly.
N	is defined in text.
T_K	is Keplerian period.
T_A	is anomalistic period.
T_N	is nodal period.
a	is semi-major axis of orbit.
a_e	is earth equatorial radius.
b	is semi-minor axis of orbit.
e	is eccentricity.
i	is inclination.
n	is mean motion.
p	is parameter of the orbit, i.e., half the latus rectum.
r	is radial distance from geocenter to satellite.
r_A	is radial distance from geocenter to apogee.

3.0 Explanation of Notation (cont'd)

r_p is radial distance from geocenter to perigee.

t is time.

t_p is time of perigee passage.

u is argument of the latitude.

v is path velocity.

x, y, z are coordinates in geocentric rectangular inertial system.

η is true anomaly.

θ is flight path angle.

Ω is longitude of ascending node.

ω is argument of perigee.

APPENDIX B

Deflection of Vertical

The problem here is to convert coordinates from an astronomically oriented system to a geodetically oriented system.

Consider a point P near the surface of the earth. From this point P there may be erected a geodetic vertical which is perpendicular to some imaginary spheroid fitted to the geoid. The geoid is a real equipotential surface. A plumb bob at P represents a vertical perpendicular to the geoid surface and defines what is called astronomic vertical. The angular separation between the geodetic vertical and the astronomic vertical is called deflection of the vertical. Similarly the latitude (positive north) and longitude (positive east) of the point P may be determined by triangulation to obtain geodetic coordinates ϕ_G and γ_G; or the latitude and longitude of the point P may be determined astronomically to obtain astronomic coordinates ϕ_A and γ_A. The deflection of the vertical at the point P is then defined by the two components $(\phi_A - \phi_G)$ and $(\gamma_A - \gamma_G)$. In present notation ξ is used to represent $(\phi_A - \phi_G)$, the deviation in the meridian; and β is used to represent $(\gamma_A - \gamma_G)$, the deviation in longitude.*

Two rectangular coordinate systems with coincident origins both at P will now be established. Besides the point P and the upward vertical, one other reference must be established. This is a line from P pointing toward true north as defined by the stars. This line (call it N) from the point P toward true north is parallel to the axis of rotation of the earth and is not dependent upon whether the system is geodetic or astronomic. In addition, each coordinate system has a horizontal reference plane through P perpendicular to its respective upward vertical.

Define the right-handed coordinate axes of the astronomic system by (x_A, y_A, z_A) and the right-handed coordinate axes of the

*Not to be confused with deviation in prime vertical defined by $\eta = (\gamma_A - \gamma_G) \cos \phi_G$.

geodetic system by (x_G, y_G, z_G). z_A is the astronomic vertical with positive upward and z_G is geodetic vertical with positive upward. x_A and y_A lie in the astronomic horizontal plane with y_A lying along the intersection of that plane with the plane defined by z_A and N. y_A thus points northward and x_A points eastward in the astronomic plane. x_G and y_G lie in the geodetic horizontal plane with y_G lying along the intersection of that plane with the plane defined by z_G and N. y_G thus points northward and x_G points eastward in the geodetic plane.

For present purposes it is not necessary to define the position of P more completely, but in general a third coordinate (height) is needed to define P. In the astronomic system this is usually height above mean sea level and in the geodetic system it is height above the spheroid.

Now consider some point P" different from P and assume that its coordinates are (x_A, y_A, z_A) in the astronomic system. Given the deflection of the vertical, it is desired to compute its coordinates (x_G, y_G, z_G) in the corresponding geodetic system. The complete derivation will not be developed here but follows along the same lines as discussed in the body of this report. According to Equation (26) in body of this report:

$$\begin{bmatrix} x_G \\ y_G \\ z_G \end{bmatrix} = \begin{bmatrix} \ell_1 & m_1 & n_1 \\ \ell_2 & m_2 & n_2 \\ \ell_3 & m_3 & n_3 \end{bmatrix} \begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix}$$

where direction cosines of x_G axis in (x_A, y_A, z_A) system are $\ell_1\ m_1\ n_1$; direction cosines of y_G axis in (x_A, y_A, z_A) system are $\ell_2\ m_2\ n_2$; direction cosines of z_G axis in (x_A, y_A, z_A) system are $\ell_3\ m_3\ n_3$. The following direction cosines are exact:

$$\ell_1 = +\cos\beta$$

$$m_1 = -\sin\phi_A \sin\beta$$

$$n_1 = +\cos\phi_A \sin\beta$$

$$\ell_2 = +\sin\phi_G \sin\beta$$

$$m_2 = +\cos\xi - \sin\phi_A \sin\phi_G(1 - \cos\beta)$$

$$n_2 = +\sin\xi + \sin\phi_G \cos\phi_A(1 - \cos\beta)$$

$$\ell_3 = -\cos\phi_G \sin\beta$$

$$m_3 = -\sin\xi + \sin\phi_A \cos\phi_G(1 - \cos\beta)$$

$$n_3 = +\cos\xi - \cos\phi_A \cos\phi_G(1 - \cos\beta)$$

The latitudes and longitudes occurring in these direction cosines represent the common origin of the two rectangular coordinate systems. If the latitude and longitude is known in one system and if the deflection of the vertical is known, then the latitude and longitude in the other coordinate system is known, or if the latitude and longitude are known in both systems, then the deflection of the vertical is also known. As soon as (x_G, y_G, z_G) have been obtained, it is possible to transform to any other system by the equations given in the body of this report.

In case there is a reason for obtaining astronomic coordinates from geodetic coordinates,

$$
\begin{bmatrix} x_A \\ y_A \\ z_A \end{bmatrix} = \begin{bmatrix} \ell_1 & \ell_2 & \ell_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix} \begin{bmatrix} x_G \\ y_G \\ z_G \end{bmatrix}
$$

where the elements in the transformation matrix are the same as defined earlier.

The following approximations are accurate to first degree in small angles, (angles in radians). ϕ may be either ϕ_G or ϕ_A without further impairment of accuracy. In the alternative formulation η represents deviation in prime vertical.

$$\ell_1 = +1$$
$$m_1 = -\beta \sin \phi = -\eta \tan \phi$$
$$n_1 = +\beta \cos \phi = +\eta$$
$$\ell_2 = +\beta \sin \phi = +\eta \tan \phi$$
$$m_2 = +1$$
$$n_2 = +\xi$$
$$\ell_3 = -\beta \cos \phi \quad = -\eta$$
$$m_3 = -\xi$$
$$n_3 = +1$$

The following relations are accurate to first degree in small angles:

$$\Delta e = e_A - e_G = + \eta \sin a + \xi \cos a$$
$$\Delta a = a_A - a_G = + \eta \tan \phi - \eta \tan e \cos a + \xi \tan e \sin a$$

where a represents azimuth measured clockwise from north and e represents elevation measured upward from horizontal plane.

A rather obvious point in star calibration of a radar is that a station with vertical defined by a plumb bob or bubble level and with latitude and longitude defined astronomically should observe azimuth and elevation of a star computed without any explicit consideration of deflection of the vertical.

ALTERNATIVE TRANSFORMATIONS

The preceding discussion in this appendix gives the transformation from astronomic to geodetic system with coincident origins. Subsequent transformations to other systems at other origins may then take place. Sometimes a different procedure is more convenient. The alternative procedure is dependent upon the fact that the direction of the gravity vector as determined by astronomic observation is parallel to the ellipsoid normal at the astronomical position. For example, in the case of an astronomically oriented radar system, we can immediately apply Equation 30, Page 29, to obtain true (E,F,G) if we use the astronomic coordinates ϕ_A and γ_A (in place of geodetic ϕ_0 and γ_0) for the station in the first two matrices on the right-hand side of the equation and if we use values for (E_0,F_0,G_0) computed from the geodetic coordinates of the station. Similarly, using Equation 31, we can transform from EFG to actual radar observations by supplying astronomic coordinates ϕ_A and γ_A and true geodetic (E_0,F_0,G_0) for the radar site. Thus the explicit application of the direction cosines developed in this appendix is not always required.

APPENDIX C

STAR POSITIONS

INTRODUCTION

For precision optical measurements and various star calibration exercises it is necessary to make transformations of star positions. The word "position" as used in this appendix represents only direction cosines or angular coordinates. The object of this section is to arrive at a computational method for deriving look angles (azimuth and elevation) for the stars based on information from star catalogs, local geodetic position of the observer and time of observation. Some definitions of star places follow. Note that none of these computed places contains any correction for the travel time of light from star to observer. Also there is some confusion in nomenclature so that the following definitions are not universally accepted:

true place: barycentric (practically heliocentric) actual exact geometric position referred to instantaneous true earth equator and equinox.

mean place: barycentric (practically heliocentric) exact geometric position referred to mean equator and equinox usually at the beginning of some specified Besselian year. At this epoch it differs from true place by effect of nutation.

apparent place: geocentric position referred to instantaneous true equator and equinox including effects of annual aberration and annual parallax but not refraction, diurnal aberration or geocentric parallax. (Note: geocentric parallax is entirely negligible for star positions and will not be mentioned again.)

observed place: topocentric position referred to instantaneous equator and equinox including effects of annual and diurnal aberration, annual parallax, and refraction.

The problem then is to compute the observed place, given the mean place. This is an ancient problem and many methods have been developed. Modern computers permit rigorous and efficient transformations. The rigorous method presented here is in use at U.S. Naval Observatory and represents optimization for computer application.

MEAN TO APPARENT PLACE

The general transformation from mean to apparent place is as follows:

$$P = R_6 R_7 + R_5 \left[R_3 (M + R_1 R_2) + R_4 \right]$$

where M is vector of direction cosines for the mean place and P is vector of direction cosines for the apparent place. The various stages are as follows:

$(M + R_1 R_2)$ is a correction for proper motion

R_3 transforms to mean equinox at beginning of Besselian year nearest date of observation

R_4 corrects for annual aberration

R_5 corrects for nutation and precession to instant of observation

$R_6 R_7$ correct for parallax

Explanation of the various matrices follows:

M

$$M = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_0$$

where x_0, y_0, z_0 are direction cosines defining the star position referred to mean equator and epoch of star catalog.

$$x_0 = \cos\alpha_0 \cos\delta_0$$

$$y_0 = \sin\alpha_0 \cos\delta_0$$

$$z_0 = \sin\delta_0$$

where α_0 is tabulated right ascension and δ_0 is tabulated declination. The subscript (0) is used to represent the equinox and epoch of star catalog.

$\underline{R_1}$

$$R_1 = \begin{bmatrix} \mu_x & \dot{\mu}_x \\ \mu_y & \dot{\mu}_y \\ \mu_z & \dot{\mu}_z \end{bmatrix}$$

where μ_x μ_y μ_z $\dot{\mu}_x$ $\dot{\mu}_y$ $\dot{\mu}_z$ represent direction cosines of proper motion and direction cosines of proper motion derivative obtainable from the table values for μ_α and μ_δ .

$$\mu_x = - \mu_\alpha \sin\alpha_0 \cos\delta_0 - \mu_\delta \cos\alpha_0 \sin\delta_0$$

$$\mu_y = + \mu_\alpha \cos\alpha_0 \cos\delta_0 - \mu_\delta \sin\alpha_0 \sin\delta_0$$

$$\mu_z = + \mu_\delta \cos\delta_0$$

$$\dot{\mu}_x = - x_0\mu^2 - 0.000205 \, \pi \, V \, \mu_x$$

$$\dot{\mu}_y = - y_0\mu^2 - 0.000205 \, \pi \, V \, \mu_y$$

$$\dot{\mu}_z = - z_0\mu^2 - 0.000205 \, \pi \, V \, \mu_z$$

where

$$\mu^2 = \mu_\alpha^2 \cos^2\delta_0 + \mu_\delta^2$$

μ, μ_α and μ_δ are expressed in radians per tropical year. x_0, y_0, z_0 values are obtained from matrix M. π is parallax in seconds of arc.

V is radial velocity of star in kilometers per second. Times associated with proper motion, proper motion components and components of derivative of proper motion are tropical centuries.

$\underline{R_2}$

$$R_2 = \begin{bmatrix} T \\ \frac{1}{2}T^2 \\ 0 \end{bmatrix}$$

where T is the time in tropical centuries from the initial mean equinox of star catalog to the instant of observation.

$\underline{R_3}$

$$R_3 = \begin{bmatrix} X_x & Y_x & Z_x \\ X_y & Y_y & Z_y \\ X_z & Y_z & Z_z \end{bmatrix}$$

where

$$X_x = \cos \zeta_0 \cos \theta \cos z^* - \sin \zeta_0 \sin z^*$$

$$Y_x = - \sin \zeta_0 \cos \theta \cos z^* - \cos \zeta_0 \sin z^*$$

$$Z_x = - \sin \theta \cos z^*$$

$$X_y = \cos \zeta_0 \cos \theta \sin z^* + \sin \zeta_0 \cos z^*$$

$$Y_y = - \sin \zeta_0 \cos \theta \sin z^* + \cos \zeta_0 \cos z^*$$

$$Z_y = - \sin \theta \sin z^*$$

$$X_z = \cos \zeta_0 \sin \theta$$

$$Y_z = - \sin \zeta_0 \sin \theta$$

$$Z_z = \cos \theta$$

where

$$\zeta_0 = (2304''.250+1''.396T_0)T' + 0''.302T'^2 + 0''.018T'^3$$

$$z^* = \zeta_0 + 0''.791T'^2$$

$$\theta = (2004''.682-0''.853T_0)T' - 0''.426T'^2 - 0''.042T'^3$$

where T_0 is time in tropical centuries from 1900.0 to initial mean equinox of star catalog. T' is time in tropical centuries from $(1900.0+T_0)$ to beginning of Besselian year nearest date of observation.

$\underline{R_4}$

$$R_4 = \begin{bmatrix} -D \\ +C \\ +C \tan \varepsilon \end{bmatrix}$$

where C and D are Besselian Day numbers expressed in radians and interpolated to instant of observation by use of first and second differences. ε is mean obliquity of the ecliptic.

$$\varepsilon = 23°27'08''.26 - 46''.845(T_0+T) - 0''.0059(T_0+T)^2 + 0''.00181(T_0+T)^3$$

The notation (T_0+T) has been used properly earlier to represent tropical centuries from 1900.0 to instant of observation. In computing ε it is somewhat more proper to interpret it as Julian centuries from 1900 January 0.5 ET to instant of observation. However the difference is slight and the earlier definition can be used here also if more convenient.

A strict treatment of annual aberration requires the consideration of elliptic terms but these are not justified in terms of overall accuracy and are here neglected.

$\underline{R_5}$

$$R_5 = \begin{bmatrix} 1 & -f & -(A+Bf) \\ f & 1 & (B-Af) \\ A & -B & 1 \end{bmatrix}$$

where A and B are Besselian Day numbers and f is an Independent Day number. All are expressed in radians and interpolated to the instant of observation by use of first and second differences from a daily tabulation provided in American Ephemeris.

$\underline{R_6\ R_7}$

$$R_6\ R_7 = \begin{bmatrix} -C \sec \varepsilon \\ -D \cos \varepsilon \\ -D \sin \varepsilon \end{bmatrix} \left[\pi/k\right]$$

where C, D and ε are defined for matrix R_4; π is annual parallax in arc seconds and k is constant of aberration, 20".47.

$\underline{P}$

$$P = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_T$$

where x_T, y_T, z_T are direction cosines defining the apparent star position at instant of observation. The subscript (T) is used to represent the equinox and epoch at instant of observation. At this instant

$$x_T = \cos \alpha_T \cos \delta_T$$

$$y_T = \sin \alpha_T \cos \delta_T$$

$$z_T = \sin \delta_T$$

$$\alpha_T = \arctan (y_T/x_T)$$

$$\delta_T = \arctan\left[z_T/(x_T^2+y_T^2)^{\frac{1}{2}}\right]$$

APPARENT PLACE TO OBSERVED PLACE

P may be corrected for diurnal aberration using either (x_T y_T z_T) coordinates or (α_T δ_T), but it is somewhat simpler to use (α_T δ_T):

$$\alpha = \alpha_T + 0''.3198 \text{ r} \cos \phi \cos h \sec \delta_T$$

$$\delta = \delta_T + 0''.3198 \text{ r} \cos \phi \sin h \sin \delta_T$$

where

α and δ represented corrected values; i.e., including effects of diurnal aberration

r is geocentric distance of observer in equatorial earth radius units

ϕ is geocentric latitude of observer

h is hour angle of star

Transform (α, δ) to (x, y, z)

$$x = \cos \alpha \cos \delta$$

$$y = \sin \alpha \cos \delta$$

$$z = \sin \delta$$

These values of (x, y, z) may be transformed to a topocentric system centered at the observer with x' and y' in horizon plane, x' east and y' north. z' is up.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R_8 \, R_9 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

where

$$R_8 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \sin\phi & \cos\phi \\ 0 & -\cos\phi & \sin\phi \end{bmatrix}$$

$$R_9 = \begin{bmatrix} -\sin\alpha_1 & \cos\alpha_1 & 0 \\ -\cos\alpha_1 & -\sin\alpha_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where ϕ is geodetic latitude of observer and α_1 is instantaneous right ascension of observer. The azimuth and elevation are

$$a = \arctan(x'/y')$$

$$e = \arcsin z'$$

The correction remaining is that for refraction.

Normal refraction does not affect the azimuth of a celestial object. The observed elevation is greater than the geometric elevation; but the observed azimuth is the same as the actual azimuth. Modern numerical ray tracing offers the most accurate approach to refraction correction and is applicable to all elevation angles, in contrast to older methods which are limited to elevation angles generally above 15°. Ray tracing requires an a priori index of refraction profile, determined for example from Rawinsonde data. (This profile incidentally is the principal source of error in the computation.) The index of refraction is assumed to be a function only of altitude above mean sea level for all latitudes and longitudes in the vicinity of the observer. The method described here is presented more fully in Item 7 of the bibliography.

The elevation correction is given rigorously by

$$\Delta e = b \cos e_o \int_0^1 \frac{d\beta}{(1-b\beta)\left[(1-b\beta)^2(1+s/r_o)^2 - \cos^2 e_o\right]^{1/2}},$$

where

$$b = \frac{n_o - 1}{n_o} \quad \text{and} \quad \beta = \frac{n_o - n}{n_o - 1}$$

and where

e_o is apparent elevation angle

n_o is index of refraction at camera altitude

r_o is the mean radius of the earth

$\left.\begin{matrix} s \\ n \end{matrix}\right\}$ represent the height and index of refraction at points between camera and celestial object.

The integral in the expression for Δe is conveniently evaluated using Gauss' method of approximate quadratures, which states that

$$\int_0^1 f(\beta)\, d\beta \approx \sum_{j=0}^{p} K_j\, f(X_j) \quad .$$

For $p = 8$, the values in the summation are

j	K_j	X_j
0	0.04063719418	0.0159198801
1	0.09032408035	0.0819844462
2	0.1303053482	0.1933142835
3	0.1561735385	0.3378732882
4	0.1651196775	0.5000000000
5	0.1561735385	0.6621267117
6	0.1303053482	0.8066857164
7	0.09032408035	0.9180155537
8	0.04063719418	0.9840801198

For each tabular value of X_j one computes

$$f(X_j) = \frac{1}{(1-bX_j)\left[(1-bX_j)^2\,(1+s_j/r_o)^2 - \cos^2 e_o\right]^{1/2}} \quad .$$

Evaluation of $f(X_j)$ requires a value for s_j.

The value for s_j is obtained as follows:

Compute

$$n_j = n_o - X_j(n_o - 1)$$

Let n_j fall between two successive entries in the index of refraction profile as shown here.

s	n
s_o	n_o
s_1	n_1
.	.
.	.
.	.
s_i	n_i
(s_j)	(n_j)
s_{i+1}	n_{i+1}
.	.
.	.
.	.
s_t	1.000 000 000

The entry s_o corresponds to the height of the camera.

The entry s_t corresponds to the height of the celestial object.

The value of s_j is obtained by logarithmic interpolation in the following procedure:

Compute

$$g_k = \left(\frac{1}{s_{i+1} - s_i}\right) \log \left(\frac{n_i - 1}{n_{i+1} - 1}\right)$$

and then

$$s_j = s_i + \frac{1}{g_k} \log \left(\frac{n_i - 1}{n_j - 1}\right) \quad .$$

Each of the values of $f(X_j)$ is then multiplied by the corresponding K_j, and the products are summed to obtain the value of the integral in the expression for Δe.

It should be recognized that the index of refraction profile varies from color to color and hence the index should be chosen after consideration of (1) the spectral characteristics of the particular star, (2) film sensitivity to various colors and (3) transmission characteristics of any filters used.

The index of refraction profile may be computed from a density profile by means of the following standard formula:

$$n = 1 + \left[0.000000222 + \frac{0.00000000126}{\lambda^2}\right]\rho$$

where

ρ is the air density in grams per cubic meter

λ is wavelength of light in microns.

APPENDIX D

Conversion of Geodetic Coordinates from One Datum to Another

Historically the parameters defining a geodetic datum consist of latitude and longitude of an initial point (origin), an azimuth of a line (direction), equatorial radius and flattening of associated spheroid, and the geoid separation at the origin. A geodetic datum is properly oriented if the Laplace equation has been satisfied at the datum origin and at all other Laplace stations and if all other reductions have been made rigorously. The datum origin serves as a reference for a network of geodetic measurements over that particular vicinity of the earth's surface. It is somewhat more realistic and practical (though less precise) to define a datum in terms of the geodetic coordinates of this network of stations associated with the datum, because these networks are used to establish the relationships between datums.

Clearly each datum has a directly calculable spheroid center which can serve as the origin for a particular geocentric EFG* coordinate system—referred to in Table I simply as "origin of coordinate system." These various origins of coordinate systems have no meaning in the absolute sense, but are always given in a relative sense. The relative origins of the various coordinate systems are determined in a least squares process using a large number of stations whose coordinates have been determined on two or more of the various datums and are of course subject to some error. The relative origins given in Table I are not defined values or directly transformable from defined values. Properly oriented datums have geocentric rectangular coordinate systems with respective axes parallel.

The more recent datums are based on world-wide satellite observations and the corresponding datum origins are given simply as "geocentric." These datums are extremely accurate and precise and provide a means for determining the relationships among the

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*See Figure 5, p.15

older surface-referenced datums. The sensitivity of the satellite datums is such that there are indications (still somewhat inconclusive) that the older geodetic datums are not perfectly oriented but are slightly rotated and/or tilted by about a second of arc. These possible discrepancies are considered to apply to the reference surface only and not to the reference spheroid. At this time it seems somewhat premature to introduce these rotations and tilts into the geodetic transformations and they will not be further discussed in this report.

Table I is provided for illustrative purposes rather than for reference. In a report of this nature it will not be practical to maintain an updated table.

Some of the spheroids are defined by semimajor axis ($\hat{a}$) and semiminor axis (b), while others are defined by semimajor axis ($\hat{a}$) and flattening (f). Some computer programs require eccentricity ($\hat{e}$). Simple exact relationships between these parameters are

$$\frac{1}{f} = \frac{\hat{a}}{\hat{a}-b}$$

$$f = 1 - \sqrt{1-\hat{e}^2}$$

$$\hat{e} = \sqrt{2f-f^2} = \sqrt{\hat{a}^2-b^2}\,/\hat{a}$$

"Semimajor axis" is a synonym for equatorial radius; and "semiminor axis" is a synonym for polar radius.*

Table I includes a unified set of datum shifts, i.e., EFG values referenced to WGS-72. Many of these shifts were published with the WGS-72 model. The remaining shifts have been related to WGS-72 by a common published NAD-27 tie point. These shifts do not enjoy official status at AFETR; however, they are accurate,

*Not to be confused with polar radius of curvature, which is equal to ($\hat{a}^2$/b).

eliminate considerable ambiguity and provide a definite numerical value in many instances where no official shift exists.

The WGS-72 spheroid and accompanying shifts are identical to the NWL-10F. The International spheroid is also known as the Hayford.

Now consider the basic problem of transforming from geodetic latitude (ϕ), longitude (γ) and height (h) on one datum to the corresponding coordinates on another datum, or:

> Given $(\phi, \gamma, h)_1$ representing a point on datum No. 1, what is $(\phi, \gamma, h)_2$ representing the same point on datum No. 2?

The standard solution to this problem uses the short or long version of Molodenskiy's formula. This formula is presented in any textbook of geodesy. It has the advantage of providing satisfactory accuracy without necessity of handling numbers with many digits. With present electronic computers and long word lengths this advantage has disappeared. A more convenient and potentially more precise transformation is currently in use at ETR:

> Transform from $(\phi, \gamma, h)_1$ to $(E, F, G)_1$ using Equation 37. Translate from $(E, F, G)_1$ $(E, F, G)_2$ using increments from Table 1. Transform from $(E, F, G)_2$ to $(\phi, \gamma, h)_2$ using Equations 38, 39, 40, 41.

As an example, transform from (28°, 280°, 30 m) on NAD-27 to Mercury-60. The EFG coordinates on the first datum are respectively: +978655.76 m, -5550232.62 m, and +2976353.57 m. From Table I the geocentric origin of Mercury-60 relative to that of NAD-27 is (-3, -111, -225). Therefore, the EFG coordinates of the point on Mercury-60 are respectively: +978658.76 m, -5550121.62 m, and +2976578.57 m. These transform to 28.000455°, 280.000226°, and 27.02 m.

TABLE I

GEODETIC CONSTANTS

Datum	Spheroid	Equatorial Radius (Meters)	Polar Radius (Meters)	1/f	Origin of Coordinate System (Meters) E	F	G
Adindan	Clarke-1880	6378249.145	---	293.465	-152	- 26	+212
Ascension Astro-58	International	6378388	---	297.	-214	+ 91	+ 48
Australian	Australian Nat.	6378160	---	298.25	-122	- 41	+146
Canton Island Astro-66	International	6378338	---	297.	+295	-285	-382
European (ED)	International	6378388	---	297.	- 84	-103	-127
Great Britain (1936)	Airy	6377563.4	---	299.325	+368	-120	+425
Guam (1963)	Clarke 1866	6378206.4	6356583.3	---	- 89	-235	+254
Indian (ID)	Everest	6377276.3	6356075.4	---	+183	+728	+250
Johnston Island Astro-61	International	6378388	---	297.	+192	- 59	-211
Kaula-61	Kaula-61	6378165	---	298.3	+ 1	+ 15	- 20
Mercury-60	Fischer-60	6378166	---	298.3	- 25	+ 46	- 49
Mercury-68	Fischer-68	6378150	---	298.3	- 4	+ 12	- 7
Nanking-60	International	6378388	---	297.	-113	-349	- 23
North American (NAD-27)	Clarke-1866	6378206.4	6356583.8	---	- 22	+157	+176
NWL-8E	NWL-8E	6378145	---	298.25	+ 1	- 2	- 9
GEM-6	GEM-6	6378144	---	298.257	0	- 2	- 11
NWL-9C	NWL-9C	6378145	---	298.25	+ 8	+ 1	- 2
NWL-9D	NWL-9D	6378145	---	298.25	+ 2	+ 7	- 3
Old Hawaiian (OHD)	Clarke-1866	6378206.4	6356583.8	---	+ 56	-270	-189
SAO-66	SAO-66	6378165	---	298.25	+ 8	+ 5	0
SAO-67	SAO-67	6378142	---	298.255	+ 4	+ 2	- 9
SAO-69	SAO-69	6378140	---	298.25	+ 5	+ 4	- 5
SAO-73	SAO-73	6378140	---	298.256	+ 9	+ 3	0
South African (ARC) 1950	Clarke-1880	6378249.145	---	293.465	-129	-131	-282
South American 1969	South American 1969	6378160	---	298.25	- 77	+ 3	- 45
South Asia	Fischer-1960	6378166	---	298.3	+ 21	- 61	- 15
Tokyo (TD)	Bessel-1841	6377397.2	6356079.0	---	-140	+516	+673
Vanguard	Hough	6378270	---	297.	- 21	+162	- 11
Wake Island Astro-52	International	6378388	---	297.	+283	- 44	+141
Wake-Eniwetok 1960	Hough	6378270	---	297.	+112	+ 68	- 44
WGS-60	WGS-60	6378165	---	298.3	+ 28	+ 29	- 35
WGS-66	WGS-66	6378145	---	298.25	+ 2	+ 3	- 12
WGS-72	WGS-72	6378135	---	298.26	0	0	0

APPENDIX E

Geoid Height Computations

For locations on, or referenced to, the surface of the ocean it is sometimes desirable to calculate the geoidal height from the expression for the geopotential and its associated spheroid. This is possible because the free ocean surface is an equipotential surface defining the geoid. The accuracy of the computation is of the order of ten meters and hence tidal effects are negligible. The accuracy is limited by the present knowledge of the geopotential.

In 1961, the IAU adopted a standard form for the general case of the expression for the earth's external gravity potential:

$$(1) \qquad U = \frac{GM}{r}\left[1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{\hat{a}}{r}\right)^{n} P_{nm} (\sin \phi) \left\{C_{nm} \cos m\gamma + S_{nm} \sin m\gamma\right\}\right] ,$$

where

r = radius from center of coordinate system

ϕ = geocentric latitude

γ = geographic longitude (Positive east of Greenwich)

P_{nm} = associated Legendre functions

GM = geocentric gravitational constant of Earth

$\hat{a}$ = equatorial radius of spheroid

n, m = indices representing degree and order, respectively

C, S = numerical coefficients experimentally measured.

Since the coefficients are obtained from satellite observations, the center of coordinates is the dynamical center of mass of the earth and the first degree harmonics are zero. The mean surface of the free oceans is an equipotential surface (the geoid) resulting from the gravitational and rotational potential:

$$(2) \qquad W = U + \frac{1}{2} \omega_e^2 r^2 \cos^2 \phi$$

where ω_e is earth rotation rate.

The geoid is approximated by an ellipsoid of revolution. The potential for this ellipsoid is described by

$$(3) \qquad U^* = \frac{GM}{r} \left[1 + C_2 \left(\frac{\hat{a}}{r} \right)^2 P_2 (\sin \phi) + C_4 \left(\frac{\hat{a}}{r} \right)^4 P_4 (\sin \phi) \right]$$

with an equipotential surface resulting from gravitational and rotational potential represented by

$$(4) \qquad W^* = U^* + \frac{1}{2} \omega_e^2 r^2 \cos^2 \phi$$

The difference between the geoid potential and that of the reference ellipsoid is

$$T = W - W^*$$

If H denotes the height of the geoid above the ellipsoid, then

$$T = \int g d H$$

where g is gravity. To a good approximation

$$H = \frac{T}{g_0}$$

where g_0 is local gravity at surface of spheroid.

$$g_0 = -\frac{\partial W^*}{\partial r}$$

$$g_0 = \frac{GM}{r^2}\left[1 + 3C_2\left(\frac{\hat{a}}{r}\right)^2 P_2\ (\sin\phi) + 5C_4\left(\frac{\hat{a}}{r}\right)^4 P_4\ (\sin\phi)\right]$$

$$- \omega_e^2 r \cos^2\phi$$

For simplified computation and with only trivial error, r is set equal to $\hat{a}$ in the expression for T and also in the expression for g. Then the geoid height formula becomes (dropping the argument (sin ϕ) in the Legendre expression to simplify notation):

$$H = \frac{\hat{a}\left\{C_3P_3 + \sum_{n=5}^{\infty} C_nP_n + \sum_{n=2}^{\infty}\sum_{m=1}^{n} P_{nm}\ (C_{nm}\cos m\gamma + S_{nm}\sin m\gamma)\right\}}{1 + 3C_2P_2 + 5C_4P_4 - (\omega\ \hat{a}\cos\phi)^2\ \frac{\hat{a}}{GM}}$$

The errors resulting from the approximations used in deriving formula for H are in general quite insignificant in comparison with errors introduced into the H computation by the uncertainties in the numerical coefficients.

If gravity is described by GM, C_2, C_4 and point masses m_i, then

$$H = \frac{\hat{a}\sum_i m_i\left[1 - 2\left(\frac{r_i}{\hat{a}}\right)\cos\beta_i + \left(\frac{r_i}{\hat{a}}\right)^2\right]^{-\frac{1}{2}}}{1 + 3C_2P_2 + 5C_4P_4 - (\omega\ \hat{a}\cos\phi)^2\ \frac{\hat{a}}{GM}}$$

where r_i is geocentric distance of m_i

$\cos\beta_i = \sin\phi_i\sin\phi + \cos\phi_i\cos\phi\cos(\gamma - \gamma_i)$

where (ϕ, γ) is latitude and longitude of point where we are computing H

(ϕ_i, γ_i) is latitude and longitude of m_i

NOTE: m_i may be either positive or negative, and is expressed in earth mass units.

APPENDIX F

(PART A)

Sodano's Noniterative Solution of The Inverse Geodetic Problem

(from GIMRADA Research Note No. 11)

INTRODUCTION

A geodesic is defined as the curve of minimum length between two points on the surface of a spheroid. The problem is to compute the length S of the geodesic between two points P_1 and P_2 on a spheroid, the forward azimuth α_{1-2} of the geodesic at P_1, and the back azimuth α_{2-1} of the geodesic at P_2. Azimuth is measured positively clockwise (eastward) from north. Longitude is measured positively eastward from meridian of Greenwich and latitude is measured positively northward from equator.

Sodano's original algorithm is somewhat awkward and incompatible with standard subroutines. Consequently, the algorithm presented in this appendix represents a modification introduced by RCA Data Processing. With 12-digit floating point computations the error is less than a foot in geodesics of 6000 nautical miles and less than a thousandth of a foot in a geodesic of 1 nautical mile. All angle errors are less than a hundredth of a second.

It is important that the arctan function used in the following computations examine signs of numerator and denominator and locate the angle unambiguously in the proper quadrant.

COMPUTATION FORM

Input:

ϕ_1, γ_1 = Geodetic latitude and longitude of Point P_1

ϕ_2, γ_2 = Geodetic latitude and longitude of Point P_2

a, b = Semimajor and semiminor axes of spheroid

Compute:

$$f = \text{Spheroidal flattening} = 1 - \frac{b}{\hat{a}}$$

$$L = \gamma_2 - \gamma_1$$

$$\beta_1 = \tan^{-1}\left[(b \sin \phi_1)/(\hat{a} \cos \phi_1)\right]$$

$$\beta_2 = \tan^{-1}\left[(b \sin \phi_2)/(\hat{a} \cos \phi_2)\right]$$

$$A = \sin \beta_1 \sin \beta_2$$

$$B = \cos \beta_1 \cos \beta_2$$

$$\cos \delta = A + B \cos L$$

$$n = (\hat{a}-b)/(\hat{a}+b)$$

$$(\beta_2-\beta_1) = (\phi_2-\phi_1) + 2\left[A(n+n^2+n^3) - B\,(n-n^2+n^3)\right] \sin\,(\phi_2-\phi_1) \text{ radians}$$

$$\sin \delta = +\left\{\left[\sin L \cos \beta_2\right]^2 + \left[\sin\,(\beta_2-\beta_1) + 2 \cos \beta_2 \sin \beta_1 \sin^2\,(L/2)\right]^2\right\}^{1/2}$$

$$\delta = \tan^{-1}\,(\sin \delta/\cos \delta) \text{ evaluated in positive radians} \leq \pi$$

$$c = B \sin L/\sin \delta$$

$$m = 1 - c^2$$

$$S = b\,\left\{\left[1+f+f^2\right]\delta + A\left[(f+f^2) \sin \delta - f^2\delta^2/(2 \sin \delta)\right] - \left[m/2\right]\left[(f+f^2)\,(\delta + \sin \delta \cos \delta) - f^2\delta^2/\tan \delta\right]\right.$$

$$- \left[\lambda^2 f^2/2\right] \sin \delta \cos \delta$$

$$+ \left[f^2 m^2/16\right] \left[\delta + \sin \delta \cos \delta - 2 \sin \delta \cos^3 \delta - 8 \delta^2/\tan \delta\right]$$

$$+ \left[\lambda m\ f^2/2\right] \left[\sin \delta \cos^2 \delta + \delta^2/\sin \delta\right]\Big\}\ \text{in same units}$$

$\hat{a}$ and b

$$\lambda = L + c \left\{ \left[f+f^2\right] \delta - \left[\lambda\ f^2/2\right] \left[\sin \delta + 2\delta^2/\sin \delta\right]\right.$$

$$\left. + \left[mf^2/4\right] \left[\sin \delta \cos \delta - 5\delta + 4\delta^2/\tan \delta\right]\right\}\ \text{radians}$$

$$\alpha_{1-2} = \tan^{-1} \left\{ \left[\cos \beta_2 \sin \lambda\right] \Big/ \left[\sin (\beta_2-\beta_1)\right.\right.$$

$$\left.\left. + 2 \cos \beta_2 \sin \beta_1 \sin^2 (\lambda/2)\right]\right\}$$

$$\alpha_{2-1} = \tan^{-1} \left\{ \left[-\cos \beta_1 \sin \lambda\right] \Big/ \left[2 \cos \beta_1 \sin \beta_2 \sin^2 (\lambda/2)\right.\right.$$

$$\left.\left. - \sin (\beta_2 - \beta_1)\right]\right\}$$

(PART B)

Sodano's Noniterative Solution of The Direct Geodetic Problem

(From GIMRADA Research Note No. 11)

INTRODUCTION

The latitude and longitude of a point P_1, the length of a geodesic, and the forward azimuth determine the latitude and longitude of of the point P_2 and the back azimuth. Sign conventions and definitions are the same as given in Introduction to Part A of this appendix. This algorithm also represents a modification introduced by RCA Data Processing. The accuracy is equivalent to the algorithm previously given for the inverse method. The special requirement for the arctan function exists here also.

COMPUTATION FORM

Input:

ϕ_1, γ_1 = Geodetic latitude and longitude of point P_1

S = Length of geodesic

α_{1-2} = Forward azimuth

$\hat{a}, b$ = Semimajor and semiminor axes of spheroid

Compute:

$$f = \text{Spheroid flattening} = 1 - \frac{b}{\hat{a}}$$

$$\varepsilon^2 = \text{Second eccentricity squared} = (\hat{a}^2 - b^2)/b^2$$

$$\theta = S/b \text{ radians}$$

$$\beta_1 = \tan^{-1}\left[(b \sin \phi_1)/(\hat{a} \cos \phi_1)\right]$$

$$g = \cos \beta_1 \cos \alpha_{1-2}$$

$$h = \cos \beta_1 \sin \alpha_{1-2}$$

$$m = \left[1 + (\epsilon^2/2) \sin^2 \beta_1\right] \left[1-h^2\right] \Big/ 2$$

$$n = \left[1 + (\epsilon^2/2) \sin^2 \beta_1\right] \left[\sin^2 \beta_1 \cos \theta + g \sin \beta_1 \sin \theta\right] \Big/ 2$$

$$L = h \left[-f \theta + 3f^2 n \sin \theta + 3f^2 m (\theta - \sin \theta \cos \theta) \Big/ 2\right] \text{ radians}$$

$$M = m \epsilon^2$$

$$N = n \epsilon^2$$

$$\delta = \theta - N \sin \theta + (M/2) (\sin \theta \cos \theta - \theta)$$
$$+ (5/2) N^2 \sin \theta \cos \theta$$
$$+ (M^2/16) (11 \theta - 13 \sin \theta \cos \theta - 8 \theta \cos^2 \theta$$
$$+ 10 \sin \theta \cos^3 \theta) + (MN/2) (3 \sin \theta$$
$$+ 2 \theta \cos \theta - 5 \sin \theta \cos^2 \theta) \text{ radians}$$

$$\sin \beta_2 = \sin \beta_1 \cos \delta + g \sin \delta$$

$$\cos \beta_2 = + \left[h^2 + (g \cos \delta - \sin \beta_1 \sin \delta)^2\right]^{1/2}$$

$$\phi_2 = \tan^{-1}\left[(\hat{a} \sin \beta_2) \Big/ (b \cos \beta_2)\right]$$

$$\lambda = \tan^{-1}\Big[(\sin \delta \sin \alpha_{1-2}) \Big/ (\cos \beta_1 \cos \delta$$

$$- \sin \beta_1 \sin \delta \cos \alpha_{1-2})\Big]$$

$$\gamma_2 = \gamma_1 + \lambda + L$$

$$\alpha_{2-1} = \tan^{-1}\left[- h/(\sin \beta_1 \sin \delta - g \cos \delta)\right]$$

(PART C)

Test Case

$\hat{a} = 6378388.00000$ m $\qquad b = 6356911.94600$ m

$\phi_1 = +20°$ $\qquad \phi_2 = +45°$

$\gamma_1 = 0°$ $\qquad \gamma_2 = +106°$

$\alpha_{1-2} = 42.9416764031°$ $\qquad \alpha_{2-1} = 295.288499970°$

$S = 9649412.50618$ m

APPENDIX G

KALMAN, $\alpha\beta$ and $\alpha\beta\gamma$ FILTERS

1.0 KALMAN FILTER

Conventional least-squares differential correction (Appendix S) is generally used in batch data processing. In applications where statistical estimations are required only after a batch of observations has accumulated, this process is considered to be more efficient than the Kalman filter process. When a new statistical estimation of the parameters is required after each observation, then the Kalman filter process is usually more efficient. The two methods in general are equivalent in accuracy. It is not the purpose of this appendix to compare these processes further.

The basic Kalman filter process is represented by the following five equations, which constitute a single update:

$$\hat{X}_i = \Phi_{ij} \bar{X}_j$$

$$\hat{P}_i = \Phi_{ij} \bar{P}_j \Phi_{ij}^T + \Phi_{ij} G_j Q_j G_j^T \Phi_{ij}^T$$

$$K_i = \hat{P}_i \Phi_{ki}^T M_k^T \left[M_k \Phi_{ki} \hat{P}_i \Phi_{ki}^T M_k^T + R_k\right]^{-1}$$

$$\bar{X}_i = \hat{X}_i + K_i \left[Y_k - M_k \Phi_{ki} \hat{X}_i\right]$$

$$\bar{P}_i = \hat{P}_i - K_i M_k \Phi_{ki} \hat{P}_i = (I - K_i M_k \Phi_{ki}) \hat{P}_i$$

where

- i — represents the known epoch of the state vector which is being estimated in present update,
- j — represents the known epoch of the state vector which was estimated in the update just prior to present update,
- k — represents the time of the most recent observation,
- $\underset{n\times 1}{X}$ — is state vector,

Φ_{ij} $n \times n$	is a matrix of partial derivatives of the state vector components at time i with respect to state vector components at time j and is called a state transition matrix,
Φ_{ki} $n \times n$	is a matrix of partial derivatives of the state vector components at time k with respect to state vector components at time i and is called a state transition matrix,
P $n \times n$	is covariance matrix of the state vector,
G $n \times s$	is the matrix of partial derivatives of the state vector components with respect to the unadjusted parameters,
Q $s \times s$	is covariance matrix of unadjusted parameters,
Y $m \times 1$	is vector of observations,
M $m \times n$	is the matrix of partial derivatives of the observation components with respect to the state vector components,
R $m \times m$	is the covariance matrix of the observations Y,
K $n \times m$	is the Kalman filter,
n	is number of elements (components) in state vector,
s	is number of elements (components) in vector of unadjusted parameters,
m	is number of elements (components) in an observation,
$(\hat{\ })$	represents a preliminary estimate based upon a series of all prior observations not including the most recent observation,
$(\tilde{\ })$	represents a final estimate based upon the combination of the preliminary estimate and the most recent observation,
I $n \times n$	represents an identity matrix.

The only partial derivative matrices which represent a transition over time are the Φ matrices. In the formalism just presented i, j and k are kept distinct. Normally two of these will be coincident. For example, in an end-point filter application i and k increment at each update but they remain coincident; and $j = i-1 = k-1$. In an earth-to-moon trajectory the equations just presented may be used to compute a series of continually improving vectors having a fixed common epoch previously selected for a midcourse maneuver. In this latter case i will be constant and equal to j; and k will be a variable representing times earlier than $i = j$. It should also be pointed out that in the actual application of the filter equations, all of the matrices shown in the algorithm here may not be explicitly developed. It may, for example, be preferable to compute a single matrix which represents a combination of two shown in this algorithm. Some sort of equations of motion (or equivalent) with starting conditions are assumed and needed in connection with the algorithm just presented.

To illustrate the application, a simple one-dimensional numerical example will be given of an end-point filter. Let the equation of motion be

$$x_i = x_j + \dot{x}_j \Delta t + \frac{1}{2} \ddot{x}_j (\Delta t)^2 + \frac{1}{6} \dddot{x}_j (\Delta t)^3$$

where

$$i = j + 1 = k, \text{ and}$$

$$\Delta t = \text{time interval between observations.}$$

The state vector is $X = [x, \dot{x}, \ddot{x}]^T$. The observation vector is $Y = [x]$. Φ_{ki} reduces to an identity matrix. The coefficient in the last term on the right-hand side of the equation of motion is defined to be a random variable with zero mean. Time units will be defined such that Δt is unity. This definition is always possible if observations are taken at equal time intervals. In the discussion which follows, the letter (c) located at the lower right-hand corner of a matrix indicates that it is constant over all time points.

The following numerical values are given:

$$X_j = \begin{bmatrix} 1.0600 \\ 0.4400 \\ 0.0550 \end{bmatrix} \qquad P_j = \begin{bmatrix} 0.0070 & 0 & 0 \\ 0 & 0.0030 & 0 \\ 0 & 0 & 0.0001 \end{bmatrix}$$

$$Q_j = [0.000036]_c \qquad R_k = [0.01]_c \qquad Y_k = [1.4273]$$

We now evaluate the following matrices of partial derivatives:

$$\Phi_{ij} \; G_j = \begin{bmatrix} \frac{1}{6} \\ \frac{1}{2} \\ 1 \end{bmatrix}_c$$

$$\Phi_{ij} \, G_j \, Q_j \, G_j^T \, \Phi_{ij}^T = \begin{bmatrix} 0.000001 & 0.000003 & 0.000006 \\ 0.000003 & 0.000009 & 0.000018 \\ 0.000006 & 0.000018 & 0.000036 \end{bmatrix}_c$$

$$\Phi_{ij} = \begin{bmatrix} 1 & 1 & \frac{1}{2} \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad \Phi_{ki} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_c$$

$$M_k = [1 \quad 0 \quad 0]_c$$

Following through the Kalman algorithm, we obtain:

$$\hat{X}_i = \begin{bmatrix} 1.5275 \\ 0.\ 950 \\ 0.0550 \end{bmatrix}$$

$$\hat{P}_i = \begin{bmatrix} 0.010019 & 0.003037 & 0.000034 \\ 0.003037 & 0.003071 & 0.000062 \\ 0.000034 & 0.000062 & 0.000044 \end{bmatrix}$$

$$K_1 = \begin{bmatrix} 0.50047454 \\ 0.15170588 \\ 0.00169839 \end{bmatrix}$$

$$\bar{X}_1 = \begin{bmatrix} 1.477352 \\ 0.479799 \\ 0.054830 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 0.005005 & 0.001517 & 0.000017 \\ 0.001517 & 0.002610 & 0.000057 \\ 0.000017 & 0.000057 & 0.000044 \end{bmatrix}$$

This completes one pass through the algorithm and the estimation of a vector and covariance matrix for one particular point. We are now ready to accept a new observation for a time point (Δt) later and compute a vector and covariance matrix at this new time point, and so on.

Some ingenuity must sometimes be employed to arrive at the state transition matrix Φ. Φ is a deterministic and known relationship and can, in fact, be a numerical integration step. As another specific example, in a two-body force field Lagrange's f and g series may be used over distances that are not too large. The f and g series are very practical and versatile tools frequently used in astrodynamics. In special applications even the aspherical earth, atmospheric drag and the influence of other bodies can be approximately accounted for in the f and g series. Also in some instances one can use a closed form in place of the infinite series. To illustrate the basic form of the f and g series, equations will be given for a satellite traveling outside the atmosphere around a spherical earth. Canonical units (Appendix O) will be used. Rectangular coordinates are expressed in a geocentric inertial system. The equations of motion are simply

$$\underline{r} = f\,\underline{r}_0 + g\,\dot{\underline{r}}_0$$

$$\dot{\underline{r}} = \dot{f}\,\underline{r}_0 + \dot{g}\,\dot{\underline{r}}_0$$

where the underline indicates a vector, $\underline{r}$ is position vector, $\dot{\underline{r}}$ is velocity vector, the zero subscript indicates epoch conditions; the non-subscripted vectors represent vectors at some time t distant from epoch. The first few terms in the f and g series are

$$f = 1 - \frac{1}{2}\,ut^2 + \frac{1}{2}\,upt^3 + \frac{1}{24}\,(3\,uq - 15\,up^2 + u^2)\,t^4 + \frac{1}{8}\,(7\,up^3 - 3\,upq - u^2\,p)\,t^5 + \ldots$$

$$g = t - \frac{1}{6}\,ut^3 + \frac{1}{4}\,upt^4 + \frac{1}{120}\,(9\,uq - 45\,up^2 + u^2)\,t^5 + \ldots$$

where

$$u \equiv 1/r_0^3 ,$$

$$p \equiv \underline{r}_0 \cdot \dot{\underline{r}}_0/r_0^2 ,$$

$$q \equiv (V_0^2 - r_0^2\,u)/r_0^2 ,$$

where r_0 is the magnitude of the position vector at epoch and V_0 is the magnitude of the velocity vector at epoch. In obtaining $\dot{f}$ and $\dot{g}$ by taking derivatives of f and g, remember that u, p and q are constants.

As another specific example in two-body mechanics, reference is made to Page 62 of this report.

In precise orbital applications employing all the known perturbations, the Kalman filter operates quite satisfactorily in concert with the Encke prediction techniques.

2.0 THE $\alpha\beta$ FILTER

The $\alpha\beta$ filter is a simplified version of the Kalman filter. When covariances between the observation channels are negligible, the $\alpha\beta$ filter can be mathematically equivalent to the Kalman filter. As general rules of thumb, the $\alpha\beta$ filter is inferior to Kalman in noise reduction, superior in maneuver-following capability, requires about one-fortieth of the computer processing time needed by Kalman and requires much less computer storage.

Equations for computing a smooth line through the observations from a single channel may be written:

$$\bar{X}_n = X_{n,p} + \alpha_n (X_n - X_{n,p}); \text{ position end-point estimate}$$

$$\bar{\dot{X}}_n = \dot{X}_{n,p} + \frac{\beta_n}{T} (X_n - X_{n,p}); \text{ velocity estimate}$$

$$X_{n+1,p} = \bar{X}_n + T\,\bar{\dot{X}}_n \text{ ; predicted position}$$

$$\dot{X}_{n+1,p} = \bar{\dot{X}}_n \text{ ; predicted velocity}$$

where

X_n	is measured position at n^{th} time point
$X_{n,p}$	is predicted position for n^{th} time point
$\dot{X}_{n,p}$	is predicted velocity for n^{th} time point
T	is time interval between samples
α_n	is the position damping factor for n^{th} point
β_n	is the velocity damping factor for n^{th} point
n	is the number of the measurement, the first measurement being X_1.

Initiation of the $\alpha\beta$ filter requires the prior acquisition of X_1 and X_2.

The factors α_n and β_n normally lie between zero and unity for obvious reasons. For least squares smoothing with an $\alpha\beta$ filter,

$$\alpha_n = \frac{2(2n-1)}{n(n+1)} \quad \text{and}$$

$$\beta_n = \frac{6}{n(n+1)} \quad .$$

In practice these two equations are used for computing α_n and β_n for $1 \leq n \leq 15$. For $n > 15$, α_n is fixed at 0.242, and β_n is fixed at 0.025; otherwise the late observations tend to be ignored.

The simplest type of $\alpha\beta$ filter is one in which α and β are fixed constants. For ramp type maneuvers it can be shown that optimally α and β (fixed) are related as follows:

$$\beta = \frac{\alpha^2}{2-\alpha}$$

3.0 THE $\alpha\beta\gamma$ FILTER

The $\alpha\beta$ concept may be extended to an $\alpha\beta\gamma$ filter:

$$\bar{X}_n = X_{n,p} + \alpha_n (X_n - X_{n,p}); \text{ position end-point estimate}$$

$$\bar{\dot{X}}_n = \dot{X}_{n,p} + \frac{\beta_n}{T} (X_n - X_{n,p}); \text{ velocity estimate}$$

$$\bar{\ddot{X}}_n = \ddot{X}_{n,p} + \frac{\gamma_n}{T^2} (X_n - X_{n,p}); \text{ acceleration estimate}$$

$$X_{n+1,p} = \bar{X}_n + T\,\bar{\dot{X}}_n + \frac{T^2}{2}\,\bar{\ddot{X}}_n \quad ; \text{ predicted position}$$

$$\dot{X}_{n+1,p} = \bar{\dot{X}}_n + T\,\bar{\ddot{X}}_n \quad ; \text{ predicted velocity}$$

$$\ddot{X}_{n+1,p} = \bar{\ddot{X}}_n \quad ; \text{ predicted acceleration}$$

where

$\ddot{X}_{n,p}$ is predicted acceleration for n^{th} time point,

γ_n is acceleration damping factor for n^{th} point,

and other symbols are defined in the $\alpha\beta$ algorithm.

Initiation of the $\alpha\beta\gamma$ filter requires the prior acquisition of X_1, X_2 and X_3.

The $\alpha\beta\gamma$ filter is superior to the $\alpha\beta$ filter in manuever-following capability but obviously requires more computer storage and processing time. The $\alpha\beta$ (or $\alpha\beta\gamma$) filter is readily applicable to adaptive filtering, where the numerical values for the damping factors vary as a function of certain specialized conditions. It is also possible to substitute more sophisticated equations of motion for the simple prediction equations shown in this discussion.

<u>Note:</u>

To initialize the $\alpha\beta$ filter assuming prior acquisition of X_1, X_2:

$$\bar{X}_2 = X_2$$

$$\bar{\dot{X}}_2 = (X_2 - X_1)/T$$

To initialize the $\alpha\beta\gamma$ filter assuming prior acquisition of X_1, X_2, X_3:

$$\bar{X}_3 = X_3$$

$$\bar{\dot{X}}_3 = (X_3 - X_2)/T$$

$$\bar{\ddot{X}}_3 = (X_3 - 2X_2 + X_1)/T^2$$

APPENDIX H

POLAR MOTION

Polar motion is a displacement of the Earth's crust with respect to the spin axis. It is thus distinct from precession and nutation which describe the motion of the spin axis in inertial space. The International Polar Motion Service (IPMS) observes and issues data on the polar motion. The effect is 15 meters or less with respect to the mean pole of 1903.0. Since present survey methods from satellites are somewhat better than 15 meters, it is desirable to refer all survey data to the mean pole of 1903.0 for comparison and standardization. In actual application the standard survey can be transformed to latitude and longitude of date.

The coordinate system used to describe polar motion is a rectangular grid (X,Y) centered on the mean pole of 1903.0.* The X-Y plane is tangent to the earth at the north pole, with the +X along the Greenwich meridian, and +Y along the 90 west longitude meridian.

If the north latitude and east longitude of a point related to the standard system are ϕ_s and γ_s respectively and the instantaneous latitude and longitude of the same point as affected by polar motion are ϕ_i and γ_i, then the following simple transformations apply:

$$\gamma_i = \gamma_s + \tan \phi \, (X \sin \gamma + Y \cos \gamma)$$

$$\phi_i = \phi_s + X \cos \gamma - Y \sin \gamma$$

where the unsubscripted ϕ's and γ's may be either standard or instantaneous with no loss in accuracy.

Azimuth measurements (a) as made from a point located at position (ϕ,γ) are also affected by polar motion as follows:

$$a_i = a_s + \sec \phi \, (X \sin \gamma + Y \cos \gamma)$$

where a is measured positively in a direction eastward from north in a local horizontal plane.

The IPMS values of X and Y are correct to the order of 0.5 meter. Changes of almost 5 meters in a month may be observed in X and/or Y. Monthly bulletins are issued by IPMS giving current data. At present there is no reliable prediction service from IPMS or other sources valid for more than two weeks. IPMS actually supplies X and Y in terms of seconds of geocentric great circle arc and these values may be substituted directly into equations for γ_i, ϕ_i and a_i.

*Presently called the Conventional International Origin (CIO).

To illustrate the application, let (x,y,z) designate the inertial, right-handed, rectangular, geocentric, equatorial coordinates of a point in a system defined by mean equator and equinox of epoch and CIO, where the z axis extends positively northward along the spin axis, and the x axis extends positively toward the vernal equinox. Let $(\hat{E},\hat{F},\hat{G})$ designate the earth-fixed, right-handed, rectangular, geocentric, equatorial coordinates of a point in a system defined by equator of date as affected by pole wander, where the $\hat{G}$ axis extends positively northward along the spin axis, and the $\hat{E}$ axis extends positively toward the meridian of Greenwich. Then

$$\begin{bmatrix} \hat{E} \\ \hat{F} \\ \hat{G} \end{bmatrix} = DSNP \begin{bmatrix} x \\ y \\ z \end{bmatrix} ,$$

where P transforms from mean equator and equinox of epoch to mean equator and equinox of date (Eq. 77); N transforms from mean equator and equinox of date to true equator and equinox of date (Eq. 71); S transforms from inertial system to earth-fixed system defined by CIO (Eq. 42); D transforms from CIO to pole of date.

P and N have been explicitly presented. S is implicitly given in Equation 42.

$$S = \begin{bmatrix} \cos \gamma_{\Upsilon} & -\sin \gamma_{\Upsilon} & 0 \\ \sin \gamma_{\Upsilon} & \cos \gamma_{\Upsilon} & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$

where γ_{Υ} is true geodetic east longitude of vernal equinox of date (Equation 70).

$$D = \begin{bmatrix} 1 & 0 & X \\ 0 & 1 & -Y \\ -X & Y & 1 \end{bmatrix} ,$$

where X and Y have been defined in this appendix and must be expressed here in radian units.

The transformation illustrated here is numerically simpler than the use of the equation previously given in this appendix for γ_i and ϕ_i. Earth-fixed site locations are normally maintained in the rectangular or polar coordinates consistent with CIO.

APPENDIX I

VEHICLE-CENTERED COORDINATE SYSTEMS

1. PYT COORDINATE SYSTEM

Reentering missiles are subject to large aerodynamic forces. It is customary in aerodynamics to resolve these forces into components parallel to and perpendicular to the local velocity vector. The force component parallel to the velocity vector is usually identified with drag. The force component perpendicular to the local velocity vector is known by many names, one of which is lift. Lift has two components: one along a coordinate in the trajectory plane and the other along a coordinate perpendicular to the trajectory plane. In the analysis of reentry trajectories it is convenient to have the residuals expressed in a coordinate system consistent with these forces.

Let

$\underline{V}^1$ represent the earth fixed unit relative* velocity vector in computed trajectory at some time t.

$\underline{J}^1$ represent a unit vector from missile directed toward the center of the earth.

Define

$\underline{T}^1 = \underline{V}^1$ identified with a unit vector directed along the trajectory at time t.

$\underline{Y}^1 = \dfrac{\underline{J}^1 \times \underline{V}^1}{\left|\underline{J}^1 \times \underline{V}^1\right|}$ identified with a unit vector perpendicular to plane of trajectory, or more precisely, unit vector opposite to angular momentum vector at time t.

$\underline{P}^1 = \underline{Y}^1 \times \underline{V}^1$ identified with a unit vector in plane of trajectory, perpendicular to velocity vector and in practical cases with an upward component at time t.

Let (E, F, G) represent the usual right handed rectangular earth fixed geocentric coordinate system with E and F in equatorial plane, E through Greenwich meridian, F through longitude 90° east, and G polar.

---

*Relative to the local atmosphere, i.e., corrected for wind.

Then if $\underline{V}$ and $\underline{J}$ are expressed in the EFG system:

$\underline{P}^1$ has components P_e^1, P_f^1, P_g^1

$\underline{Y}^1$ has components Y_e^1, Y_f^1, Y_g^1

$\underline{T}^1$ has components T_e^1, T_f^1, T_g^1

These components represent direction cosines of the unit vectors in the EFG coordinate system. Consequently, the residuals at any time t may be transformed from the EFG system to the PYT system as follows:

$$\begin{bmatrix} P \\ Y \\ T \end{bmatrix} = \begin{bmatrix} P_e^1 & P_f^1 & P_g^1 \\ Y_e^1 & Y_f^1 & Y_g^1 \\ T_e^1 & T_f^1 & T_g^1 \end{bmatrix} \begin{bmatrix} E_o - E_c \\ F_o - F_c \\ G_o - G_c \end{bmatrix}$$

where subscript (o) refers to the observed quantity and subscript (c) refers to the computed quantity. In practice the residuals are not usually computed in the EFG system but in some tracking system coordinates. In that case a standard transformation must be made ahead of the one indicated.

2. HCL COORDINATE SYSTEM

This coordinate system is moving in inertial space and has its origin at the instantaneous position of the satellite in the computed or reference orbit. The H coordinate extends outward from the origin along an extension of the reference radius vector, which originates at the mass center of the earth and terminates at the origin. The cross track coordinate C is measured in the direction of the vector cross product of reference satellite inertial velocity vector and the reference radius vector. Thus the direction of C is opposite to the angular momentum vector. The L coordinate is directed so as to make the HCL a right handed orthogonal coordinate system. Thus L has precisely the direction of the reference inertial velocity vector if the orbit happens to be circular. Note that the HCL coordinates of an actual satellite position represent the differences in the positions of the actual and reference satellites in the sense (actual - reference).

The coordinate transformations are as follows. Assume that the coordinates of test object and the position and velocity components of reference object are known at some time t in a geocentric inertial rectangular right handed coordinate system with X and Y axes in equatorial plane and Z northward. The longitude of ascending node is measured from X axis. The X axis may for example be directed through vernal equinox or perhaps through meridian of Greenwich at $t = 0$. The final HCL coordinates are independent of this choice of direction.

Let

(X_t, Y_t, Z_t) = inertial position coordinates of test object at some time t

(X, Y, Z) = inertial position coordinates of reference object at time t

$(\dot{X}, \dot{Y}, \dot{Z})$ = inertial velocity coordinates of reference object at time t

u = argument of the latitude of reference object at time t

i = Inclination angle of reference object at time t

Ω = Longitude of ascending node of reference orbit at time t

$$C_1 = Y\dot{Z} - Z\dot{Y}$$

$$C_2 = Z\dot{X} - X\dot{Z}$$

$$C_3 = X\dot{Y} - Y\dot{X}$$

$$h = \left(C_1^2 + C_2^2 + C_3^2\right)^{1/2}$$

$$R = \left(X^2 + Y^2 + Z^2\right)^{1/2}$$

$$\sin i = \frac{\left(C_1^2 + C_2^2\right)^{1/2}}{h}$$

$$\cos i = \frac{C_3}{h}$$

If $\sin i = 0$, set $\sin \Omega = 0$ and $\cos \Omega = 1$

If $\sin i \neq 0$,

$$\sin \Omega = \frac{C_1}{h \sin i}$$

$$\cos \Omega = \frac{-C_2}{h \sin i}$$

Then

$$\sin u = \left[\cos i(-X \sin \Omega + Y \cos \Omega) + Z \sin i\right]/R$$

$$\cos u = \left[X \cos \Omega + Y \sin \Omega\right]/R$$

$$\begin{bmatrix} H \\ C \\ L \end{bmatrix} = \begin{bmatrix} (\cos u \cos \Omega - \sin u \cos i \sin \Omega) & (\cos u \sin \Omega + \sin u \cos i \cos \Omega) & (\sin u \sin i) \\ (-\sin i \sin \Omega) & (\sin i \cos \Omega) & (-\cos i) \\ (-\sin u \cos \Omega - \cos u \cos i \sin \Omega) & (-\sin u \sin \Omega + \cos u \cos i \cos \Omega) & (\cos u \sin i) \end{bmatrix} \begin{bmatrix} X_t - X \\ Y_t - Y \\ Z_t - Z \end{bmatrix}$$

The preceding development uses classical orbital elements in the final formulation. In many applications it may be preferable to use position and velocity components.

Let:

$\underline{V}$ represent inertial velocity in computed trajectory at some time t.

$\underline{R}$ represent position vector in computed trajectory at some time t.

$\underline{H}^1 = \dfrac{\underline{R}}{|\underline{R}|}$ represent the unit vector directed along $\underline{R}$.

$\underline{C}^1 = \dfrac{\underline{V} \times \underline{R}}{|\underline{V} \times \underline{R}|}$ represent the unit vector perpendicular to plane of computed trajectory, or more precisely, unit vector opposite to computed angular momentum vector at time t.

$\underline{L}^1 = \underline{H}^1 \times \underline{C}^1$ represent the unit vector in plane of computed trajectory, perpendicular to position vector and lying generally along the velocity vector but usually with some angular separation from the velocity vector.

Then

$$\begin{bmatrix} H \\ C \\ L \end{bmatrix} = \begin{bmatrix} H_X^1 & H_Y^1 & H_Z^1 \\ C_X^1 & C_Y^1 & C_Z^1 \\ L_X^1 & L_Y^1 & L_Z^1 \end{bmatrix} \begin{bmatrix} X_t - X \\ Y_t - Y \\ Z_t - Z \end{bmatrix}$$

where subscripts X, Y, Z are used to denote components along X, Y, Z axes.

Note: The symbol $|\ |$ which encloses a vector and a vector cross product in the denominator of two of the equations above represents the positive square root of the sum of the squares of the coefficients of the orthogonal unit vectors.

TIME ERROR

Having the HCL coordinates, it is a simple step to obtain the corresponding time error in a computed satellite position.

$$\alpha = \sin^{-1} \frac{L}{R}$$

$$\dot{\eta} = \frac{h}{R^2}$$

$$\Delta t = \frac{\alpha}{\dot{\eta}}$$

APPENDIX J

TIME TRANSFORMATIONS

In this section, time units will be specified where they are important. In general, sidereal time will always be expressed in sidereal time units. Universal time will always be expressed in mean solar time units with the exception of UTC, which is expressed in atomic time units. Ephemeris and atomic times will be expressed in their respective time units.

It should be emphasized that all equations giving the time in one system corresponding to the time in another system assume the same instant of time in both systems. This instant is called epoch. The day in which epoch occurs is called epoch day.

A. Sidereal Time

Sidereal time is the basis for all time systems which depend upon earth rotation to define the time intervals. Apparent sidereal time is determined by the rotation of the earth relative to the stars and is defined as the hour angle of the vernal equinox, reckoned along the celestial equator westward from the instantaneous local celestial meridian to the hour circle through the celestial pole and the true equinox at the instant. Thus a star transits the meridian at a sidereal time equal to its right ascension. In conformity with its definition, local sidereal time is determined directly from star observations using a zenith tube. Sidereal time referenced to the true equinox of date is called apparent sidereal time. Sidereal time referenced to the mean equinox of date is called mean sidereal time. The relationship between them is

Mean Sidereal Time = Apparent Sidereal Time - Equation of Equinoxes

The equation of the equinoxes is defined as the right ascension of the mean equinox referred to the true equator and equinox. This can be computed from equations given in Section IX of this report or obtained from tables in the American Ephemeris.

Greenwich sidereal time (GST) is related to local sidereal time (LST) as follows:

GST (seconds) = LST (seconds) + (1/15) (Local West Longitude in arc seconds)

One day should be subtracted if necessary to maintain $0 \leq GST < 1$ day.

This same relationship applies for either apparent or mean sidereal time, although generally the apparent time is changed to mean time immediately. In the preceding equation the units could just as well be (minutes and arc minutes) or (hours and degrees).

The right ascensions of stars are given by fundamental star catalogs. By decision of the International Astronomical Union in 1961 the FK4 catalog is designated for sidereal time observations. Sidereal time is not a uniform time scale because it is tied directly to the variable rotation rate of the earth.

The sidereal second is defined as the $(1/86400)^{th}$ part of the time interval between two successive transits of the mean vernal equinox on the international meridian. The sidereal day has 86400 sidereal seconds. The sidereal month is the true period of revolution of the moon about the earth with respect to inertial space, not with respect to vernal equinox of date. The sidereal month is approximately 27.396465 sidereal days. The sidereal year is defined as the period of one complete revolution of the earth about the sun with respect to inertial space. The sidereal year is approximately 366.256399 sidereal days. Sidereal time can be determined to an accuracy of about 2 milliseconds.

B. Universal Time UTO

There are three other kinds of Universal Time (UT) besides UTO. They will be discussed in subsequent sections. UTO and Greenwich mean sidereal time (GMST) have a mathematically defined relationship between them.

$$UTO = (0.997269566414) \left[GMST - 23925\overset{s}{.}836 - 8640184\overset{s}{.}542T - 0.0929T^2 \right]^*$$

where

$$T = \frac{(\text{Julian Date at } 0^h \text{ UT of epoch day}) - 24105020}{36525}$$

Some care must be experienced in applying the equation for UTO, since occasionally there are two possible answers only one of which is valid. The term in brackets should always be normalized to fall in the region (0 to +86636.555 secs) by adding sufficient increments of 86400 seconds to the raw value first calculated. A typical ambiguous case is (86500 seconds or 100 seconds), both of which lie between (0 and +86636.555 seconds). The correct choice must be made on the basis of a priori knowledge of UTO. Since there is almost 24 hours difference in the two values of UTO, there is generally no question as to which answer is valid. The above equation for UTO can be used to solve for GMST in case UTO is known — no ambiguity in this application. Consequently, it should be clear that sidereal time and UTO are equivalent pieces of information.

UTO and the equivalent ST are both contaminated by variations in the position of the pole of rotation (polar motion) and by variations in the earth rate of rotation. Both of these effects are small and not recognized until this century. However, at present they are quite significant and are not usually neglected.

*Tabulated under "Universal and Sidereal Times" in American Ephemeris.

C. Universal Time (UT1)

Refer to Appendix H of this report for discussion of Polar Motion. X and Y supplied by IPMS may be substituted into the following equation to obtain UT1:

$$UT1 = UT0 - (1/15)(X \sin \lambda + Y \cos \lambda) \tan \phi$$

where λ is east longitude from Greenwich; ϕ is north latitude. If X and Y are in arc seconds, then the second term on the right is in seconds of time. Typically, UT1 and UT0 differ by less than 30 milliseconds. The above equation can of course also be used to solve for UT0 if UT1 is known.

D. Universal Time (UT2)

UT2 is obtained by making seasonal earth rotation rate corrections to UT1.

$$UT2 = UT1 + a \sin 2\pi t + b \cos 2\pi t + c \sin 4\pi t + d \cos 4\pi t$$

where t is fraction of tropical year from beginning of Besselian year and (a, b, c, d) are empirical constants issued by Bureau International de l'Heure (BIH) in Paris under direction of International Astronomical Union.

For the period 1962-68 these were:

a	b	c	d
$+0\overset{s}{.}022$	$-0\overset{s}{.}012$	$-0\overset{s}{.}006$	$+0\overset{s}{.}007$

The seasonal variation between UT2 and UT1 is $\pm$ 30 milliseconds. UT2 is still subject to some long period variation plus secular variation. However, it is closer to uniform time than any other system based on earth rotation. The equation above can, of course, also be used to solve for UT1 if UT2 is known. The arguments ($2\pi t$ and $4\pi t$) in the equation above are in radians.

E. Atomic Time (AT)

This system depends upon the atomic resonance corresponding to transition between the two hyperfine levels of the ground state of cesium 133. The frequency has been measured to be 9192631770 Hz per ephemeris second.

A1 is the system operated by the U.S. Naval Observatory. NBS-A is the system maintained by the US Bureau of Standards. International atomic time (IAT) is maintained by Bureau International de l'Heure (BIH) in Paris, France.

The various atomic time systems have different epochs. The most prestigious system in this country is A1. The epoch for A1 is $0^h\ 0^m\ 0^s$ UT2 on January 1, 1958, at which instant A1 was $0^h\ 0^m\ 0^s$.

Atomic time is very close to uniform time. It is slightly affected by local magnetic fields and by relativity perturbations.

F. Coordinate Time

Coordinate time is atomic time with added periodic relativistic corrections for diurnal and orbital positions of the clock site. The orbital correction is about a thousand times that of the diurnal correction. The total correction over the complete period of a year falls between $\pm$ 2 milliseconds. The expression for the correction is

$$\begin{aligned} \Delta t &= 1\overset{s}{.}658 \times 10^{-3}\ [\sin E_\odot + 0.0368] \\ &+ 2\overset{s}{.}03 \times 10^{-6} \cos\phi\ [\sin(UT + \lambda) - \sin\lambda] \\ &+ \text{smaller terms} \end{aligned}$$

where

$E_\odot$ = eccentric anomaly of the sun

UT = universal time

ϕ, λ = latitude and east longitude of clock.

G. Ephemeris Time (ET)

This time system is independent of earth rotation. It is a gravitational system based on the geometric longitude of the sun--specifically Newcomb's Tables of the Sun. It is theoretically a strictly uniform measure of time in the Newtonian sense. Logically, ephemeris time could be determined from the relative orbital motion of any of the heavenly bodies; and to obtain maximum precision it is usually determined from the motion of the moon as described by Brown's lunar theory. Ephemeris time deduced from the comparison of observations with the lunar ephemerides j = 0, j = 1 and j = 2 (reference numbers specified in the recommendations of the International Astronomical Commission 4, Prague, 1967) is denoted by ET0, ET1 and ET2, respectively, where the suffixes 0, 1, 2 reflect differences in systems of astronomical constants and not differences in definition. ET0 is official for Years 1960 - 1967; ET1 for Years 1968 - 1971; ET2 for 1972 onwards; but any of the three may be projected outside its official region using information in the American Ephemeris. Differences among ET0, ET1 and ET2 at corresponding Universal times are generally less than 300 milliseconds and the precision of each is of the order of 100 milliseconds. At the present time when a suffix is not used with ET, then ET2 is implied. The following simple relation holds:

$$ET = UT2 + \Delta T$$

ΔT is given in the American Ephemeris and is presently of the order of 43 seconds. Sometimes a subscript (0, 1, 2) is attached to ΔT to indicate that when added to UT2 one obtains ET0, ET1, ET2, respectively. Also as a matter of practical convenience ΔT is sometimes tabulated such that when added to UTC one obtains ET.

The standard epoch of ephemeris time is 1900 January $0^{\mathrm{h}}.5$ ET. The instant to which this designation is assigned is near the beginning of calendar year 1900 when the geometric mean longitude of the sun referred to mean equinox of date was precisely 279° 41' 48".04. The primary unit of ephemeris time is the length of the tropical century at the standard epoch. This ephemeris second is (1/31556925.9747) part of the primary unit and almost identical to the mean solar second (to better than 1 part in 10^8).

Ephemeris time is uniform within the limits set by the theory and constants used in its determination. Agreement with atomic time is good. If we compare ET with the function ($32^{\mathrm{s}}.15$ + A1) over a ten year interval we obtain an average disagreement of only 10 milliseconds with no discernible trends or periodic variations. Since ET is not known accurately for several years after the fact, it is convenient to estimate it from

$$ET = 32\overset{s}{.}15 + A1$$

or alternatively from

$$ET = 32\overset{s}{.}18 + I\ AT.$$

Ephemeris time agreed with Universal Time at some instant around 1903, and the ephemeris day is approximately the average of the mean solar days during the nineteenth century.

11. Broadcast Time (UTC) (Coordinated Universal Time)

This is the time system in use at AFETR. UTC and AT have a mathematically defined relationship between them. They are equivalent pieces of information. The time unit in each is the international atomic second. Time expressed as AT (BIH) and time expressed as UTC (BIH) differ by an integral number of international atomic seconds. This difference is adjustable by plus or minus one second at the exact beginning of each calendar month in order to insure that the disagreement between UTC and UT2 will not exceed 0.7 seconds. This adjustment will normally take place no oftener than once every eight or ten months and will be announced eight weeks in advance.

Time Service Publications Series 7 and 11, published by the U. S. Naval Observatory (USNO), and BIH circular D (USNO Series 15) provide information for converting UTC to UT1, UT2, A1, IAT, and ET. This information is available weekly and projects ahead two weeks. Also provided are measured and two weeks extrapolation of the instantaneous pole.

Information in these bulletins can be used to compute the instantaneous angular velocity of the earth. Time Service Bulletin Series 7 provides a current numerical value for

$$DUT1 = UT1 - UTC$$

and also the time derivative of DUT1 which here will be designated by $\dot{\overline{DUT1}}$. With this dimensionless constant then the inertial angular velocity of earth over period covered by the bulletin is (in units of radians per international atomic second):

$$\omega_I = .0000729211514667\ (1 + \dot{\overline{DUT1}})$$

The corresponding angular velocity of the earth relative to mean vernal equinox is (in units of radians per international atomic second):

$$\omega_T = (.0000729211585468 + .428 \times 10^{-14}T)\ (1 + \dot{\overline{DUT1}})$$

where T is the number of Julian centuries of 36525 U.T. days elapsed since noon UT on January 0, 1900.

Present day precision orbit work takes account of this variable rotation rate.

I. Solar Time

The definition of Greenwich mean solar time is not identical with the definition of Universal Time (UT2). However in practice no distinction is made numerically; and Greenwich mean solar time, or simply Greenwich mean time, or simply GMT, is set equal to UT2. The following relationship applies between GMT and local mean solar time (LMT).

$$\text{GMT (secs)} = \text{LMT (secs)} + \frac{\text{Local W. Long. (arc secs)}}{15}$$

In this equation the units could just as well be (minutes and arc minutes) or (hours and degrees). The letter S is never used to abbreviate solar since it is already used to represent sidereal. Mean solar time differs from apparent solar time by the "Equation of Time".

$$\text{Mean solar time} = \text{apparent solar time} - \text{Equation of Time}$$

The Equation of Time is no longer included in the American Ephemeris but is listed in the Nautical Almanac.

J. Zone Time (ZT)

The world is divided into twenty-four zones each having a width of 15° (one hour) of longitude, in each of which the same standard time is kept. (Sometimes the border of a zone is slightly modified to make it correspond to certain geographic boundaries.) The meridian of Greenwich is taken as the center of the system and of Zone 0. Zones eastward from Greenwich are numbered $\Delta Z = -1, -2$, etc., progressively and those to the west, $\Delta Z = +1, +2$, etc., according to the number of hours to be added to ZT to obtain UT. Thus

$$UT = ZT + \Delta Z$$

where ΔZ is called the zonal correction or zonal description. The twelfth zone is divided into two parts by the date line, that to the west being $\Delta Z = +12$ and that to the east $\Delta Z = -12$. When crossing the line on a westerly course, the date must be advanced one day.

In the center of each zone is a standard meridian. The zone time corresponds to the mean solar time at that meridian. For example, in the Eastern Zone ($\Delta Z = +5$) the standard meridian is 75° west longitude.

In practice, zone time is usually regulated according to UTC.

K. Julian Dates

The system of Julian day numbers is a continuous sequential numbering of days from an epoch so remote that all astronomical events of historical record will be assigned positive Julian dates. The epoch of the Julian cycle is 4713 BC, January 1, Greenwich mean noon on the Julian proleptic calendar, at which time the Julian date was exactly zero.

Julian Date (JD) is measured in days of UT. Julian Ephemeris Date (JED) is measured in days of ET. Either date is specified by the day number followed by the decimal portion of a day elapsed since 12^h UT or ET. A table of day numbers is given in the American Ephemeris. (JD) and (JED) are related by

$$JED = JD + \Delta T$$

where ΔT has been previously discussed on Page J-6 and here must be supplied in fractional days.

There is a Modified Julian Date (anathema to most astronomers) defined by

MJD = JD - 2400000.5.

L. Besselian Date

The Besselian year begins at the instant of time at which the right ascension of Newcomb's fictitious mean sun, affected by aberration and referred to the mean equinox of date is precisely $18^h\ 40^m$. This instant, designated by the notation (.0) after the year (e.g., 1950.0) always falls near the beginning of the Gregorian calendar year. 1900.0 is the basic epoch, which corresponds to JED 2415020.31352 = 1900 January 0.81352 ET. The JED for the beginning of any other Besselian year may be computed from

$$JED\ (1900.0 + \tau) = JED\ (1900.0) + (365.24219879 - 0.856 \times 10^{-8}\ \tau)\ \tau$$

where τ is integer years.

The length of the Besselian year differs only a fraction of a second from the length of the tropical year. The length of any particular Besselian year is

$$\text{Besselian year} = \text{tropical year} - 0\overset{s}{.}148\ T$$

where T is measured in tropical centuries from 1900.

Every usage of decimals with years implies that Besselian years are intended.

M. Constants

It is not the purpose of this appendix to provide constants, but as a matter of convenience the following approximate values are listed:

Time Intervals:

Earth inertial rotation period	=	86164.0989041	mean solar secs.
Earth sidereal rotation period	=	86164.09054	mean solar secs.
1 Tropical year	=	365.2422	mean solar days
1 Sidereal year	=	365.2564	mean solar days
1 Julian year	=	365.25	mean solar days
1 Sidereal (sec, hour, day)	=	.997269566414	mean solar (sec, hour, day)
1 Universal time unit	=	1 mean solar time unit	
1 mean solar second	=	1 ephemeris second to about 1 part in 10^8	
1 atomic second	=	1 ephemeris second to about 1 part in 10^9	
1 ephemeris second	=	1/31556925.9747 of length of tropical year 1900.0	

In each of the time systems (sidereal, universal, ephemeris, mean solar) there are 60 seconds per minute, 60 minutes per hour, and 24 hours per day.

Standard Epochs

1900 Jan 0.5 UT	=	1900 Jan 0 Greenwich mean noon
	=	1899 Dec 31 Greenwich mean noon
	=	JD 2415020.0
	=	JED 2415019.99995
1950 Jan 0.5 UT	=	1950 Jan 0 Greenwich mean noon
	=	1949 Dec 31 Greenwich mean noon
	=	JD 2433282.0
	=	JED 2433282.00034

Standard Epochs (continued)

Besselian Date 1900.0	=	JD 2415020.31357
	=	JED 2415020.31352
	=	1900 Jan 0.81352 ET
Besselian Date 1950.0	=	JD 2433282.42309
	=	JED 2433282.42343
	=	1950 Jan 0.92343 ET

N. Timing Polynomials

The primary times used in trajectory and orbit determination work are ET, AT, UT1 and UTC. At corresponding instants these various times have been tabulated by the U.S. Naval Observatory. As we have seen, discontinuities exist in these relationships, but continuity is maintained over discrete time periods of the order of months. JPL, SAO and others have fit these tabulated data over discrete time periods and interpolate over them. The difference between ET and AT is described by a simple constant. The difference between ET and UTC is described by a simple constant. The difference between ET and UT1 is described by a second degree time polynomial. The difference between UT1 and UTC is described by a second degree time polynomial. The errors associated with the polynomials are about one millisecond.

APPENDIX K

EQUATIONS OF MISSILE MOTION FOR THE TEST RANGE

This entire discussion is from the viewpoint of the missile test range performing its primary function of determining the trajectory of a test missile from radar tracking data. Consequently, we will consider only three-degree-of-freedom motion.

The basic acceleration equations in an earth-fixed coordinate system are:

$$\ddot{e} = G_e + S_e + R_e + D_e + L_e + T_e \quad ,$$

$$\ddot{f} = G_f + S_f + R_f + D_f + L_f + T_f \quad \text{and}$$

$$\ddot{g} = G_g + S_g + R_g + D_g + L_g + T_g \quad , \text{ where}$$

e, f and g represent coordinates in a geocentric rectangular earth-fixed coordinate system, with the e axis in equatorial plane through Greenwich meridian, f in equatorial plane through 90° east longitude, and g polar northward. The subscripts on the right-hand side refer to components along e, f and g axes. G represents acceleration due to gravity. S represents Coriolis acceleration. R represents centrifugal acceleration. D represents drag acceleration. L represents lift acceleration. Wind accelerations are included in lift and drag. T represents powered thrust.

Appendix T treats in some detail the acceleration due to gravity. The Coriolis and centrifugal terms are, respectively

$$S_e = 2\omega\dot{f} \quad ,$$

$$S_f = -2\omega\dot{e} \quad ,$$

$$S_g = 0 \quad ,$$

$$R_e = \omega^2 e \quad ,$$

$$R_f = \omega^2 f \quad \text{and}$$

$$R_g = 0 \quad ,$$

where ω is earth rotation rate relative to inertial coordinate system.

Drag, by definition, is the component of the aerodynamic forces directed along the negative relative velocity vector ($-\underline{V}$). The relative velocity vector components and total magnitude are

$$V_e = \dot{e} - W_e \quad ,$$

$$V_f = \dot{f} - W_f \quad ,$$

$$V_g = \dot{g} - W_g \quad \text{and}$$

$$V = \left(V_e^2 + V_f^2 + V_g^2\right)^{1/2} \quad .$$

W represents the wind velocity measured in the rotating coordinate system. The drag components are

$$D_e = -\rho V V_e \left(\frac{AC_D}{2w}\right) \quad ,$$

$$D_f = -\rho V V_f \left(\frac{AC_D}{2w}\right) \quad \text{and}$$

$$D_g = -\rho V V_g \left(\frac{AC_D}{2w}\right) \quad .$$

In these equations ρ represents air density; A represents cross-sectional area of missile; C_D represents drag coefficient; w represents mass of missile; W and ρ are, of course, functions of altitude and are obtained from Rawinsonde data.

$\left(\frac{AC_D}{2w}\right)$ may be expressed as a function of various parameters.

Options required are a table of C_D vs time, C_D vs Mach number, and C_D vs altitude with up to 300 entries. Also required is a fifth degree polynomial to be used either as a table multiplier or independently. A choice of either time or altitude should be available as independent variable in the polynomial. It is quite practical to solve for all six polynomial coefficients in a least squares adjustment.

The lift acceleration $\underline{L}$ acts in a plane perpendicular to the relative velocity vector $\underline{V}$. It also lies in a plane containing the relative velocity vector and the longitudinal axis of the missile. The lift acceleration has a magnitude

$$L = \rho V^2 \left(\frac{AC_L}{2W}\right)$$

where C_L is lift coefficient. C_L varies strongly with angle of attack and is in fact zero for zero angle of attack.* C_L also varies with Mach and Reynolds numbers. In order to visualize the direction of the lift vector, imagine a plane perpendicular to the instantaneous relative velocity vector and passing through the center of gravity of the missile. Let the center of gravity of the missile define the origin of a rectangular coordinate system in this plane with the P axis lying in a vertical plane and pointing positively in an upward direction, and with the Y axis pointing positively in a horizontal direction to the right as viewed by an observer looking in the direction of missile travel. The lift vector lies in the P-Y plane and its direction is described by an angle θ measured positively clockwise from P by an observer looking in the direction of missile travel.

Options required are a table of C_L and θ vs time, C_L and θ vs altitude, $C_L \cos\theta$ and $C_L \sin\theta$ vs time, and $C_L \cos\theta$ and $C_L \sin\theta$ vs altitude with provision for up to 300 entries.** Also required is a pair of fifth degree polynomials to be used either as table multipliers or independently. A choice of either time or altitude should be available as independent variable in the polynomials.

*The angle of attack is the angle between the relative velocity vector and the longitudinal axis of the missile.

**$C_L \cos\theta$ is sometimes called coefficient of lift. $C_L \sin\theta$ is sometimes called coefficient of side.

Attempts at ETR to use pulse radar track data to determine separate precessional and nutational components in an epicyclic description of the motion of a spinning missile have been unsuccessful. Hence, the simpler treatment described here has been adopted as standard practice and is adequate for trajectory computations.

The P-Y components of the lift acceleration are simply

$$\ddot{P} = L \cos \theta \quad \text{and}$$

$$\ddot{Y} = L \sin \theta$$

where C_L, θ, $C_L \cos \theta$ or $C_L \sin \theta$ may be represented by a polynomial or the product of a polynomial and a table.

The lift acceleration will now be resolved along the efg coordinate axes. Let $\underline{J}^1$ represent a unit vector from missile directed toward the center of the earth and let $\underline{V}^1$ represent unit relative velocity vector. Then let

$$\underline{Y}^1 = \frac{\underline{J}^1 \times \underline{V}^1}{\left| \underline{J}^1 \times \underline{V}^1 \right|}$$

and

$$\underline{P}^1 = \underline{Y}^1 \times \underline{V}^1$$

where the superscript (1) is used to denote a unit vector. $\underline{P}^1$ has components $P_e^{\,1}$, $P_f^{\,1}$, $P_g^{\,1}$. $\underline{Y}^1$ has components $Y_e^{\,1}$, $Y_f^{\,1}$, $Y_g^{\,1}$. Since these components are nothing more than direction cosines,

$$L_e = \ddot{P} P_e^{1} + \ddot{Y} Y_e^{1} \quad ,$$

$$L_f = \ddot{P} P_f^{1} + \ddot{Y} Y_f^{1} \quad \text{and}$$

$$L_g = \ddot{P} P_g^{1} + \ddot{Y} Y_g^{1} \quad .$$

Instead of C_D and C_L, the missile contractor is likely to give C_X (the axial force coefficient sometimes also designated C_A) and C_N (the normal force coefficient). C_X and C_N are referenced to the body axes of the missile instead of to the velocity vector. C_D and C_L may be obtained from C_X and C_N as follows:

$$C_D = C_X \cos \alpha_T + C_N \sin \alpha_T \quad ,$$

$$C_L = C_N \cos \alpha_T - C_X \sin \alpha_T \quad ,$$

where α_T is total angle of attack.

The total angle of attack α_T is the angle between the relative velocity vector and the missile longitudinal axis. It has components α_p and α_Y in the pitch and yaw planes respectively. The angles α_T, α_p and α_Y are related as follows:

$$\tan^2 \alpha_T = \tan^2 \alpha_Y + \tan^2 \alpha_p \quad .$$

Figure 1 shows the missile body axes, the velocity vector with its components, and the total angle of attack with its components.

The origin of the XYZ coordinate system is at the center of gravity of the missile. X extends positively forward along the missile longitudinal axis. Y extends positively horizontal and to the right. Z completes the rectangular right-handed system. If we define the aerodynamic accelerations acting along the negative missile axes (-X, -Y, -Z) by AX, AY and AZ respectively,

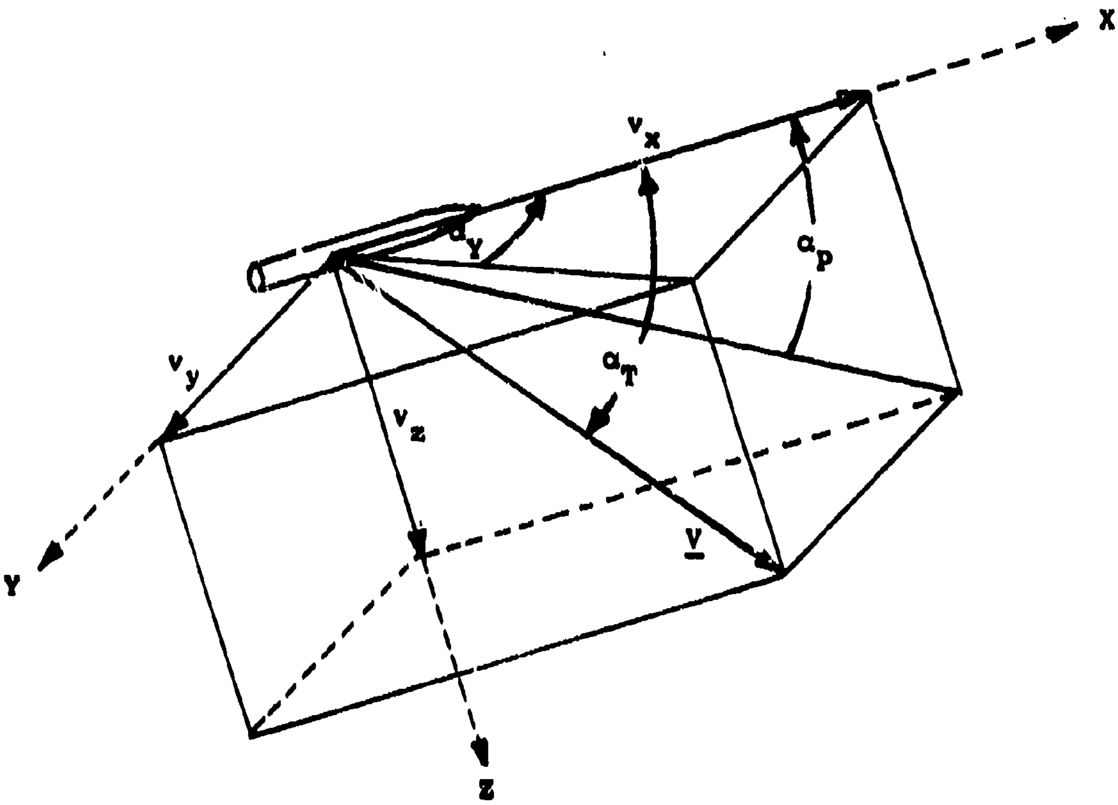


FIGURE 1. MISSILE BODY SYSTEM

then

$$AX = \rho V^2 \left(\frac{AC_X}{2W}\right) ,$$

$$AY = \rho V^2 \left(\frac{AC_Y}{2W}\right) \quad \text{and}$$

$$AZ = \rho V^2 \left(\frac{AC_Z}{2W}\right)$$

where C_Y and C_Z are orthogonal components of the normal force coefficient and have the relationship

$$C_N{}^2 = C_Y{}^2 + C_Z{}^2 .$$

The magnitude of the resultant aerodynamic acceleration vector is given by any of the following alternative expressions:

$$\left[(AX)^2 + (AY)^2 + (AZ)^2\right]^{1/2}$$

$$\frac{\rho V^2 A}{2W} \left(C_X{}^2 + C_Y{}^2 + C_Z{}^2\right)^{1/2}$$

$$\frac{\rho V^2 A}{2W} \left(C_X{}^2 + C_N{}^2\right)^{1/2}$$

$$\frac{\rho V^2 A}{2W} \left(C_D{}^2 + C_L{}^2\right)^{1/2}$$

Again referring to Figure 1, we note that the velocity vector $\underline{V}$ defines a rectangular solid with sides v_x, v_y and v_z. The total angle of attack is inherently positive. Its components α_p and α_Y extend positively as shown. The orientation angle ϕ of the normal force is measured in a plane perpendicular to the missile axis and is positive clockwise from the intersection of the vertical plane through the missile axis as viewed by an observer looking forward along the missile axis. We have simply

$$\tan \phi = \frac{-\tan \alpha_Y}{\tan \alpha_p} .$$

The lift orientation angle θ (defined previously) is given by

$$\tan \theta = \frac{\tan \beta \sin E \cos \alpha_Y - \sin \alpha_Y}{\tan \beta \sin \alpha_Y + \sin E \cos \alpha_Y}$$

where

$$\sin E = \tan \alpha_P \cos \alpha_T \quad ,$$

$$\tan \beta = \frac{\tan \phi \sin \alpha_Y}{\tan \phi \sin E \cos \alpha_Y + \cos E} \quad \text{and}$$

$$\tan \phi = \frac{\cos \alpha_Y (\tan F + \tan E)}{\cos^2 \alpha_Y - \tan E \tan F} \quad .$$

In these equations the angles α_Y, α_P, α_T, β, E and ϕ all lie inside the region $-90° + 90°$. The angles ϕ and θ lie in the quadrants determined by the signs in the numerator and denominator in the expressions for $\tan \phi$ and $\tan \theta$. The angle ϕ is measured from the horizontal positively upward to the vector extending forward along the missile axis. The angle F is the flight path angle measured from the horizontal positively upward to the velocity vector.

The remaining terms in the equations of motion are associated with the powered thrust. Since powered-flight trajectories on the test range are estimated on a point-by-point basis, equations of powered flight are used very little. However, there are two formulations of powered flight which are useful and will be presented here. The first option is residual thrust after burnout:

$$\underline{T} = T_o \exp\left[-\lambda (t-t_o)\right] \frac{\underline{V}}{V}$$

This thrust vector acts along the velocity vector $\underline{V}$ with an original value of T_o at time t_o and decaying with a decay constant λ. It is practical to estimate T_o and λ in a least squares adjustment. $\underline{T}$ can be readily transformed to the efg coordinate system as was demonstrated for the lift acceleration.

The second option uses three sets of fifth degree time polynomials — one set for each of three burn periods. Each set consists of three polynomials — one polynomial for each of three rectangular coordinates. The rectangular coordinate system used is efg. In this powered flight option provision should be made for several intervals with corresponding start and stop times.

In the event that the same computer program is to be used for precision satellite and space probe work as well as missile trajectories, then the earth-fixed coordinate system used here should be replaced by an inertial coordinate system.

DISCUSSION OF OPTIONS

It is impossible to describe exact procedures followed in all trajectory problems. A few general and somewhat heterogenous comments will be offered.

The mathematics presented is useful for either spinning or non-spinning projectiles and even aerodynamically guided missiles. For free-flight trajectories entirely above 300,000 feet, the choice of epoch is purely one of convenience or program limitation and has no influence on the accuracy of the computed trajectories. In the computation of reentry trajectories, it is nearly always good practice to set epoch for the position – velocity vector at reentry – generally defined as 300,000 feet altitude. The reason for this is that track data above 300,000 feet altitude usually provide highly accurate estimates for the values of position and velocity at reentry and these estimates are relatively unimproved by the track data at lower altitudes. These estimates are given heavy weight in a reentry computation and can, if necessary, be enforced so as to assure continuity with the midcourse trajectory. Reference time for the drag polynomial is usually best established at impact time, which is ordinarily known from the time of loss of telemetry signal. The reason for this choice is that one can set the a priori value of the constant term in the drag polynomial to correspond to theoretical drag, and by putting an a priori small standard deviation on this value one can ensure that the final drag curve in the extrapolated region near impact will not depart too far from theoretical. Because lift may be known to be zero over the last 10 or 15 seconds before impact, it is desirable to have a start time and stop time capability in the program. The reference time for the lift polynomial may be set, for example, at 15 seconds prior to impact, and the a priori value of the constant term in the lift polynomial may be chosen to correspond to zero lift with an a priori small standard deviation. Lift would be set to terminate 15 seconds before impact.

Time is most generally used as the independent variable in the aerodynamic tables and polynomials, but there are occasions where other independent variables are preferable. When the drag vs Mach number is not double-valued, such a table can conveniently be used for a number of trajectories associated with the same missile. Also it can be set up prior to launch. When impact time is not known or when it is desirable to set up the program prior to launch, it may be desirable to use altitude in the aerodynamic tables and polynomials instead of time.

When theoretical aerodynamic tables are available, they should always be used because their use reduces the required number of adjusted parameters and computer running time. Tables are never absolutely necessary, however, if there are sufficient track data. Low-degree independent polynomials representing $C_L \cos \theta$ and $C_L \sin \theta$ cannot be used realistically for processing re-entering missiles. If independent polynomials are used in this case, they should represent C_L and θ — and convergence is generally agonizingly slow. When lift tables are used along with multiplying polynomials, the best form of the lift table is $C_L \cos \theta$ and $C_L \sin \theta$. In tables of C_L and θ, θ is not usually monotonic, and hence a polynomial multiplier of θ cannot be used, and a polynomial multiplier of C_L alone would not provide sufficient flexibility.

SINGULARITY CONSIDERATIONS

It is well known that the coefficients in ordinary power series polynomials are highly correlated in a fitting process, and this correlation becomes progressively worse with higher degree polynomials — a situation conducive to loss of precision in the adjustment process. This problem is frequently relieved by resorting to orthogonal polynomials. It has in fact become a rule of thumb in numerical analysis to use orthogonal polynomials sometimes for sixth degree and always for degrees of seven or more. Below sixth degree the benefit from orthogonal polynomials is marginal. Since in estimation of aerodynamic parameters we are concerned with polynomials usually no higher than third degree and never higher than fifth degree, orthogonal polynomials are not presently used for this application at this test range. When singularity problems are encountered with the polynomial coefficients, the analyst may reduce the degree of the polynomial, set smaller standard deviations on some of the a priori values, or resort to ridge regression. (See Appendix AA.)

APPENDIX L

Transformations of Pairs of Angular Coordinates

This report has discussed primarily transformations of coordinates defining precisely a point in space. Many times there may be only a pair of angular coordinates defining the direction of the point from some origin, perhaps topocentric right ascension and declination (α, δ) or radar azimuth and elevation (a, ϵ). The problem is to transform from one to the other set of angular coordinates.

The rotations are performed exactly as for a point in space by assuming a point along the line of sight to be located at unit distance from the origin. For example, consider a topocenter at (ϕ, γ), where ϕ represents geodetic latitude and γ represents east longitude from Greenwich. At this origin erect two right-handed rectangular coordinate systems. The first is the usual radar system with x east, y north, and z up. The second is an equatorial system with X and Y parallel to equatorial plane and X toward vernal equinox. Z is parallel to polar axis and pointing northward.

Two rotations are required to get from XYZ to xyz:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

where

$$\theta = (90° - \phi)$$

$$\psi = (90° + \gamma + \text{geodetic east longitude of vernal equinox})$$

If we represent the elements of the combined transfer matrix by b_{ij} and transform the rectangular coordinates to their angular equivalents,

$$\begin{bmatrix} \cos e \cos a \\ \cos e \sin a \\ \sin e \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} \cos \delta \cos \alpha \\ \cos \delta \sin \alpha \\ \sin \delta \end{bmatrix}$$

Hence

$$\tan a = \frac{b_{21} \cos \delta \cos \alpha + b_{22} \cos \delta \sin \alpha + b_{23} \sin \delta}{b_{11} \cos \delta \cos \alpha + b_{12} \cos \delta \sin \alpha + b_{13} \sin \delta}$$

$$\sin e = b_{31} \cos \delta \cos \alpha + b_{32} \cos \delta \sin \alpha + b_{33} \sin \delta$$

"a" will be located unambiguously by considering signs of numerator and denominator. The sign of sin e will determine either first or fourth quadrant.

Clearly in transformations of pairs of angular coordinates it is not possible to translate the origin.

APPENDIX M
The ADBARV Orbital Elements

These elements are defined in an inertial frame and represent respectively right ascension (α), declination (δ), flight path angle (β), azimuth of velocity vector (A), distance from earth mass center (r), and scalar velocity (v). They explicitly define position and velocity vectors at some particular instant or epoch.

The position vector originates at the center of the earth and terminates at the orbiting body. It has magnitude (r) and orientation defined by (α) and (δ). (α) and (δ) are described on page 19 of this report. See also Figure 20.

The velocity vector with magnitude (v) originates at the terminus of the position vector and extends in a direction defined by (β) and (A). (β) is the angle between the velocity vector and the extension of the position vector. For example, (β) is zero for an object traveling outward along the position vector and is 90° for an object traveling perpendicular to the position vector. In order to describe (A), we erect a local horizontal plane perpendicular to the position vector and located so as to include the point defining the terminus of the position vector and the origin of the velocity vector. From this point and lying in the horizontal plane lies a northward reference direction line N. The projection of the velocity vector upon this horizontal plane defines a direction line V. (A) is the angle measured eastward from N to V. Thus if V points north, then (A) is zero. If V points southwest, then (A) is 225°. (A) and (α) are defined in region 0° to 360°. (δ) is defined in region $\pm$ 90°. β is defined in region 0° to 180°.

It is useful to relate the ADBARV elements to the (xyz) geocentric inertial equatorial right-handed rectangular coordinate system. x extends positively toward vernal equinox and z is coincident with polar axis and extends in a positive direction northward. The relations follow:

$$x = r \cos \alpha \cos \delta$$

$$y = r \sin \alpha \cos \delta$$

$$z = r \sin \delta$$

$$\dot{x} = v\left[(\cos \beta \cos \delta - \cos A \sin \beta \sin \delta) \cos \alpha - \sin A \sin \alpha \sin \beta\right]$$

$$\dot{y} = v\left[(\cos \beta \cos \delta - \cos A \sin \beta \sin \delta) \sin \alpha + \sin A \cos \alpha \sin \beta\right]$$

$$\dot{z} = v\left[\cos A \sin \beta \cos \delta + \cos \beta \sin \delta\right]$$

$$r = (x^2 + y^2 + z^2)^{1/2}$$

$$v = (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)^{1/2}$$

$$\alpha = \tan^{-1} \left(\frac{y}{x}\right)$$

$$\delta = \sin^{-1} \left(\frac{z}{r}\right)$$

$$\beta = \cos^{-1} \left(\frac{x\dot{x} + y\dot{y} + z\dot{z}}{rv}\right)$$

$$A = \tan^{-1} \left[\frac{r(x\dot{y} - y\dot{x})}{y(y\dot{z} - z\dot{y}) - x(z\dot{x} - x\dot{z})}\right]$$

In order to locate (α) or (A) unambiguously in one of four quadrants, the signs of numerator and denominator must be examined separately. (δ) is located ($\pm$) depending upon the sign of z. (β) is located in first or second quadrant depending upon the sign of numerator.

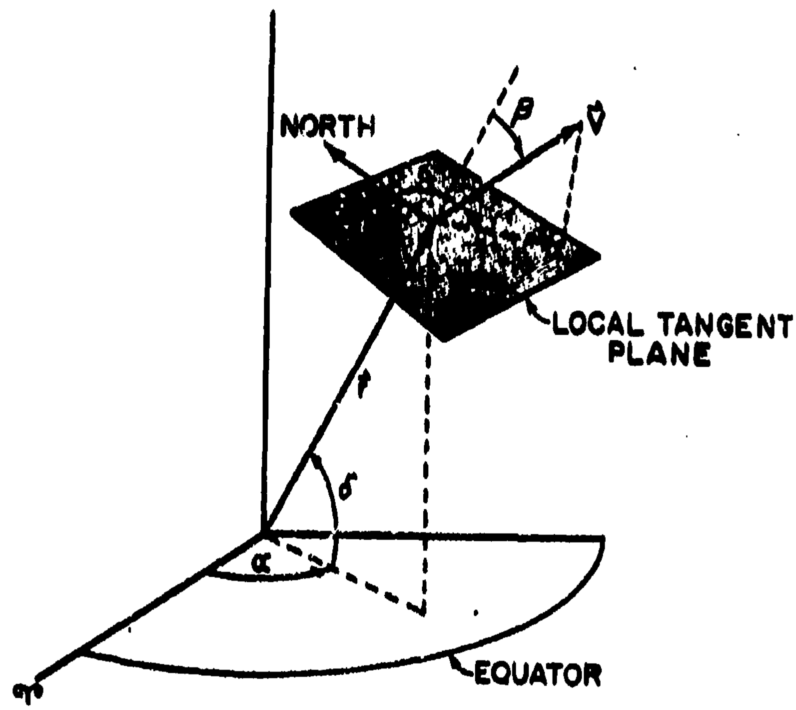


FIGURE 20 ADBARV ORBITAL ELEMENTS

APPENDIX N

MEAN ELEMENTS

INTRODUCTION

In this report the classical elements a, e, i, Ω, ω and u have been discussed along with equations relating them to rectangular position and velocity components. The equations presented relate osculating orbital elements to actual position and velocity rectangular components; and they also relate mean orbital elements to mean position and velocity components. The purpose of this appendix is to relate osculating elements and mean elements.

The concept of mean elements requires a few words of explanation. If a central force field were the only force acting on an object, the classical Keplerian elements describing the size, shape and orientation of the orbit would remain constant in time. Because of perturbative forces the classical elements vary significantly in time. Two of the most influential perturbative forces are associated with the second and third degree zonal harmonic terms (J_2 and J_3) in the geopotential. These forces cause secular, short period and long period variations in the classical elements of an earth satellite. Short periods are of the order of time of one complete passage of the satellite around the earth. Long periods are of the order of time of one complete passage of perigee around the earth. The elements as affected by J_2 and J_3 are called osculating elements. It is possible mathematically to remove the periodic effects of J_2 and J_3 and arrive at new elements called mean elements. (If only the short period effects are removed the new elements are called quasi-mean.) Mean elements will still vary with time but the variations can be described very closely by third degree time polynomials. Consequently the mean elements lend themselves to what is called general perturbations--a field of immense value in orbit determination and ephemeris generation over relatively long periods of elapsed time. In summary, the osculating elements represent the actual position and velocity of the object but are poorly behaved in time. The mean elements do not represent the actual position and velocity but are well behaved in time. The ability to transform between mean and osculating elements provides the capability of knowing the actual position and velocity associated with mean elements. The name "mean element" was given to the modified element because it represents a smooth "average" value through the cyclic behavior characteristic of the osculating element.

A typical application in orbit determination is as follows: First, each pass of tracking data is reduced to a state vector (i.e., position and velocity components) using some form of filter. Second, all such vectors are transformed to sets of mean Keplerian elements, each set corresponding to the time of that particular pass. Third, polynomials in time are passed in a least squares sense through individual mean elements. The resulting polynomials then represent the multi-pass solution and may be used to compute a set of mean elements at any particular time of interest, perhaps much later than the observations. At this time of interest then the computed mean elements can be transformed to osculating elements and thence perhaps to look angles for some sensor.

The use of general perturbations and mean elements avoids the time consuming numerical integration associated with special perturbations and also the round off and truncation errors; and consequently over long periods of elapsed time general perturbations are sometimes preferable to special perturbations.

At AFETR the Kozai formulation has been applied successfully in many routine calculations. The Fraser formulation has also been used and here is considered superior in several respects to the Kozai method. Algorithms for both transformations will be given. In addition, a self-contained description of ephemeris generation (recommended by Aerospace Defense Command) using simplified general perturbations is given in Section V of this appendix.

I. Kozai Algorithm - Quasi-Mean to Osculating

Given quasi-mean elements $\bar{a}$, $\bar{e}$, $\bar{I}$, $\bar{\Omega}$, $\bar{\omega}$ and $\bar{M}$, compute corresponding osculating elements a, e, i, Ω, ω and M, where

$$a = \bar{a} + \delta a_s$$

$$e = \bar{e} + \delta e_s$$

$$i = \bar{I} + \delta i_s$$

$$\Omega = \bar{\Omega} + \delta\Omega_s$$

$$\omega = \bar{\omega} + \delta\omega_s$$

$$M = \bar{M} + \delta M_s$$

First compute iteratively the eccentric anomaly $\bar{E}$ from Kepler's equation:

$$\bar{E}_{i+1} = \bar{E}_i - \frac{\bar{E}_i - \bar{e}\sin\bar{E}_i - \bar{M}}{1 - \bar{e}\cos\bar{E}_i}$$

where $\bar{E}_i = \bar{M}$ can be used to start the process. Iterations continue until successive differences are negligibly small. Then compute the true anomaly $\bar{\eta}$.

$$\bar{\eta} = 2\tan^{-1}\left[\left[\frac{1+\bar{e}}{1-\bar{e}}\right]^{\frac{1}{2}} \frac{\sin\left(\frac{\bar{E}}{2}\right)}{\cos\left(\frac{\bar{E}}{2}\right)}\right]$$

The mean radius from the center of the earth to the object is given by

$$\bar{r} = \bar{a}(1 - \bar{e}\cos\bar{E})$$

and the mean semi-latus rectum by

$$\bar{p} = \bar{a}(1 - \bar{e}^2)$$

Then

$$\delta a_s = \frac{\frac{\bar{a}J}{\bar{p}^2}\left[1 - \frac{3}{2}\sin^2\bar{I}\right]\left[1 - \bar{e}^2\right]^{\frac{1}{2}}}{1 - \frac{J}{\bar{p}^2}\left[1 - \frac{3}{2}\sin^2\bar{I}\right]\left[1 - \bar{e}^2\right]^{\frac{1}{2}}}$$

$$+ \frac{J}{\bar{a}}\left\{\frac{2}{3}\left[1 - \frac{3}{2}\sin^2\bar{I}\right]\left[\left(\frac{\bar{a}}{\bar{r}}\right)^3 - (1 - \bar{e}^2)^{-\frac{3}{2}}\right]\right.$$

$$\left. + \left(\frac{\bar{a}}{\bar{r}}\right)^3 \sin^2\bar{I}\cos 2(\bar{\eta} + \bar{\omega})\right\}$$

$$\delta e_s = \frac{1 - \bar{e}^2}{\bar{e}}\left(\frac{J}{\bar{a}^2}\right)\left\{\frac{1}{3}\left[1 - \frac{3}{2}\sin^2\bar{I}\right]\left[\left(\frac{\bar{a}}{\bar{r}}\right)^3 - (1 - \bar{e}^2)^{-\frac{3}{2}}\right]\right.$$

$$\left. + \frac{1}{2}\left(\frac{\bar{a}}{\bar{r}}\right)^3 \sin^2\bar{I}\cos 2(\bar{\eta} + \bar{\omega})\right\}$$

$$- \frac{\sin^2\bar{I}}{2\bar{e}}\left(\frac{J}{\bar{a}\,\bar{p}}\right)\left\{\cos 2(\bar{\eta} + \bar{\omega}) + \bar{e}\cos(\bar{\eta} + 2\bar{\omega})\right.$$

$$\left. + \frac{1}{3}\bar{e}\cos(3\bar{\eta} + 2\bar{\omega})\right\}$$

$$\delta i_s = \frac{J}{4\bar{p}^2}\sin 2\bar{I}\left[\cos 2(\bar{\eta} + \bar{\omega}) + \bar{e}\cos(\bar{\eta} + 2\bar{\omega})\right.$$

$$\left. + \frac{\bar{e}}{3}\cos(3\bar{\eta} + 2\bar{\omega})\right]$$

$$
\begin{aligned}
\delta\omega_s = \frac{J}{\bar{p}^2\bar{e}}\Bigg\{ & \bar{e}\left(2 - \frac{5}{2}\sin^2\bar{I}\right)(\bar{\eta} - \bar{M} + \bar{e}\sin\bar{\eta}) \\
& + \left(1 - \frac{3}{2}\sin^2\bar{I}\right)\left[\left(1 - \frac{\bar{e}^2}{4}\right)\sin\bar{\eta} + \frac{\bar{e}}{2}\sin 2\bar{\eta} + \frac{\bar{e}^2}{12}\sin 3\bar{\eta}\right] \\
& - \sin(\bar{\eta} + 2\bar{\omega})\left[\frac{1}{4}\sin^2\bar{I} + \bar{e}^2\left(\frac{1}{2} - \frac{15}{16}\sin^2\bar{I}\right)\right] \\
& + \frac{\bar{e}^2}{16}\sin^2\bar{I}\sin(\bar{\eta} - 2\bar{\omega}) - \frac{\bar{e}}{2}\left(1 - \frac{5}{2}\sin^2\bar{I}\right)\sin 2(\bar{\eta} + \bar{\omega}) \\
& + \sin(3\bar{\eta} + 2\bar{\omega})\left[\frac{7}{12}\sin^2\bar{I} - \frac{\bar{e}^2}{8}\left(1 - \frac{19}{8}\sin^2\bar{I}\right)\right] \\
& + \frac{3\bar{e}}{8}\sin^2\bar{I}\sin(4\bar{\eta} + 2\bar{\omega}) + \frac{\bar{e}^2}{16}\sin^2\bar{I}\sin(5\bar{\eta} + 2\bar{\omega})\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
\delta\Omega_s = -\frac{J}{\bar{p}^2}\cos\bar{I}\Bigg[& (\bar{\eta} - \bar{M} + \bar{e}\sin\bar{\eta}) - \frac{1}{2}\sin 2(\bar{\eta} + \bar{\omega}) \\
& - \frac{\bar{e}}{2}\sin(\bar{\eta} + 2\bar{\omega}) - \frac{\bar{e}}{6}\sin(3\bar{\eta} + 2\bar{\omega})\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
\delta M_s = \frac{J}{\bar{p}^2\bar{e}}(1 - \bar{e}^2)^{\frac{1}{2}}\Bigg\{ & -\left(1 - \frac{3}{2}\sin^2\bar{I}\right)\left[\left(1 - \frac{\bar{e}^2}{4}\right)\sin\bar{\eta} \right. \\
& \left. + \frac{\bar{e}}{2}\sin 2\bar{\eta} + \frac{\bar{e}^2}{12}\sin 3\bar{\eta}\right] \\
& + \sin^2\bar{I}\left[\frac{1}{4}\left(1 + \frac{5\bar{e}^2}{4}\right)\sin(\bar{\eta} + 2\bar{\omega}) - \frac{\bar{e}^2}{16}\sin(\bar{\eta} - 2\bar{\omega}) \right. \\
& - \frac{7}{12}\left(1 - \frac{\bar{e}^2}{28}\right)\sin(3\bar{\eta} + 2\bar{\omega}) - \frac{3\bar{e}}{8}\sin(4\bar{\eta} + 2\bar{\omega}) \\
& \left. - \frac{\bar{e}^2}{16}\sin(5\bar{\eta} + 2\bar{\omega})\right]\Bigg\}
\end{aligned}
$$

where $J = \frac{3}{2} J_2 a_e^2$

and a_e is equatorial earth radius.

For $\bar{e}$ equal to or very nearly zero the Kozai method is not applicable because of choice of elements.

II. Frazer Algorithm - Mean to Osculating

Preliminary Discussion

Given mean elements $\bar{a}$, $\bar{e}$, $\bar{I}$, $\bar{\Omega}$, $\bar{\omega}$, $\bar{M}$ compute corresponding elements a, e, i, Ω, ω, M. The first step is to develop unit vectors $\vec{U}$, $\vec{V}$, $\vec{W}$. The reference plane is the orbit plane and contains $\vec{U}$ and $\vec{V}$. $\vec{W}$ is perpendicular to the reference plane. These vectors are expressed as components in the usual inertial geocentric equatorial rectangular system x y z (page 14):

$$\vec{U} = \begin{bmatrix} U_x \\ U_y \\ U_z \end{bmatrix} = \begin{bmatrix} \cos\bar{u} \cos\bar{\Omega} - \sin\bar{u} \sin\bar{\Omega} \cos\bar{I} \\ \cos\bar{u} \sin\bar{\Omega} + \sin\bar{u} \cos\bar{\Omega} \cos\bar{I} \\ \sin\bar{u} \sin\bar{I} \end{bmatrix}$$

$$\vec{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \begin{bmatrix} -\sin\bar{u} \cos\bar{\Omega} - \cos\bar{u} \sin\bar{\Omega} \cos\bar{I} \\ -\sin\bar{u} \sin\bar{\Omega} + \cos\bar{u} \cos\bar{\Omega} \cos\bar{I} \\ \cos\bar{u} \sin\bar{I} \end{bmatrix}$$

$$\vec{W} = \begin{bmatrix} W_x \\ W_y \\ W_z \end{bmatrix} = \begin{bmatrix} \sin\bar{\Omega} \sin\bar{I} \\ -\cos\bar{\Omega} \sin\bar{I} \\ \cos\bar{I} \end{bmatrix}$$

Then compute $\bar{\eta}$ as in Section I of this appendix. Next

$$\bar{p} = \bar{a}(1 - \bar{e}^2)$$

$$\bar{r} = \frac{\bar{p}}{1 + \bar{e}\cos\bar{\eta}}$$

$$\bar{\dot{r}} = \left[\frac{K}{\bar{p}}\right]^{\frac{1}{2}} \bar{e}\sin\bar{\eta}$$

$$\bar{r}\,\bar{\dot{\eta}} = \left[\frac{K}{\bar{p}}\right]^{\frac{1}{2}} (1 + \bar{e}\cos\bar{\eta})$$

where K is earth gravitational constant.

The mean state vector is

$$\vec{\bar{r}} = \begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix} = \bar{r}\,\vec{U}$$

$$\vec{\bar{\dot{r}}} = \begin{bmatrix} \bar{\dot{x}} \\ \bar{\dot{y}} \\ \bar{\dot{z}} \end{bmatrix} = \bar{\dot{r}}\,\vec{U} + \bar{r}\,\bar{\dot{\eta}}\,\vec{V}$$

The osculating state vector* is

$$\vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \bar{\vec{r}} + \delta\vec{r}$$

$$\dot{\vec{r}} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \bar{\dot{\vec{r}}} + \delta\dot{\vec{r}}$$

It is convenient to break $\delta\vec{r}$ and $\delta\dot{\vec{r}}$ down into short and long period variations

$$\delta\vec{r} = \delta\vec{r}_s + \delta\vec{r}_\ell$$

$$\delta\dot{\vec{r}} = \delta\dot{\vec{r}}_s + \delta\dot{\vec{r}}_\ell$$

Computation of Short Period Variations $\delta\vec{r}_s$ and $\delta\dot{\vec{r}}_s$

$$\delta\vec{r}_s = \delta r_s \vec{U} + \bar{r}(\delta u_s + \delta\Omega_s \cos\bar{I})\vec{V}$$

$$+ \bar{r}(\delta i_s \sin\bar{u} - \delta\Omega_s \cos\bar{u} \sin\bar{I})\vec{W}$$

$$\delta\dot{\vec{r}}_s = [\delta\dot{r}_s - \bar{r}\,\bar{\dot{\eta}}(\delta u_s + \delta\Omega_s \cos\bar{I})]\vec{U}$$

$$+ [\delta r\dot{\eta}_s + \bar{\dot{r}}(\delta u_s + \delta\Omega_s \cos\bar{I})]\vec{V}$$

$$+ [\bar{\dot{r}}(\delta i_s \sin\bar{u} - \delta\Omega_s \cos\bar{u} \sin\bar{I})$$

$$+ \bar{r}\,\bar{\dot{\eta}}\,(\delta i_s \cos\bar{u} + \delta\Omega_s \sin\bar{u} \sin\bar{I})]\vec{W}$$

*The classical osculating elements may be derived from this state vector by mathematics of Appendix A.

where

$$\delta r_s = a\bar{p}\left\{\sin^2\bar{I}\cos 2\bar{u} + (1 - 3\cos^2\bar{I})\left[1 - \frac{(1-\bar{e}^2)^{\frac{1}{2}}}{1+\bar{e}\cos\bar{\eta}} + \frac{\bar{e}\cos\bar{\eta}}{1+(1-\bar{e}^2)^{\frac{1}{2}}}\right]\right\}$$

$$\delta\dot{r}_s = -a_1\left\{2\sin^2\bar{I}(1+\bar{e}\cos\bar{\eta})^2\sin 2\bar{u} + (1 - 3\cos^2\bar{I})\bar{e}\sin\bar{\eta}\left[-\frac{(1-\bar{e}^2)^{\frac{1}{2}}}{2} + \frac{(1+\bar{e}\cos\bar{\eta})^2}{1+(1-e^2)^{\frac{1}{2}}}\right]\right\}$$

$$\delta r\dot{\eta}_s = a_1\left\{\sin^2\bar{I}\left[2\cos 2\bar{u} + 2\bar{e}\cos(2\bar{u}-\bar{\eta}) + \bar{e}\cos\bar{\eta}\cos 2\bar{u}\right](1+\bar{e}\cos\bar{\eta}) - (1+\bar{e}\cos\bar{\eta})(1-3\cos^2\bar{I})\left[\frac{3}{2} + \bar{e}\cos\bar{\eta}\,\frac{2+(1-\bar{e}^2)^{\frac{1}{2}}}{1+(1-\bar{e}^2)^{\frac{1}{2}}} + \frac{\bar{e}^2 - 2(e\sin\bar{\eta})^2}{2[1+(1-\bar{e}^2)^{\frac{1}{2}}]}\right]\right\}$$

$$\delta i_s = a\sin\bar{I}\cos\bar{I}\left\{3\left[\cos 2\bar{u} + \bar{e}\cos(2\bar{u}-\bar{\eta})\right] + \bar{e}\cos(2\bar{u}+\bar{\eta})\right\}$$

$$\delta\Omega_s = -a\cos\bar{I}\left\{6(\bar{\eta} - \bar{M} + \bar{e}\sin\bar{\eta}) - 3\left[\sin 2\bar{u} + \bar{e}\sin(2\bar{u}-\bar{\eta})\right] - \bar{e}\sin(2\bar{u}+\bar{\eta})\right\}$$

$$\delta u_c = -\frac{a}{2}\left\{6(1 - 5\cos^2\bar{I})(\bar{\eta} - \bar{M}) + 4\left[1 - 6\cos^2\bar{I} + \frac{1 - 3\cos^2\bar{I}}{1+(1-\bar{e}^2)^{\frac{1}{2}}}\right]\bar{e}\sin\bar{\eta}\right.$$

$$+ (1 - 3\cos^2\bar{I})\,\frac{2(\bar{e}\sin\bar{\eta})(\bar{e}\cos\bar{\eta})}{1 + (1 - \bar{e}^2)^{\frac{1}{2}}}$$

$$+ 2(5\cos^2\bar{I} - 2)\,\bar{e}\sin(2\bar{u} - \bar{\eta})$$

$$+ (7\cos^2\bar{I} - 1)\sin 2\bar{u} + 2\cos^2\bar{I}\,\bar{e}\sin(2\bar{u} - \bar{\eta})\Big\}$$

where

$$\alpha = \frac{J_2\, a_e^2}{4\bar{p}^2}$$

$$\alpha_1 = \alpha\left[\frac{K}{\bar{p}}\right]^{\frac{1}{2}}$$

For orbits of low eccentricity it is desirable to compute $(\bar{\eta} - \bar{M})$ where it appears above by

$$\bar{\eta} - \bar{M} = \sin^{-1}\left[\left(\frac{\bar{e}\sin\bar{\eta}}{1 + \bar{e}\cos\bar{\eta}}\right)\left(\frac{(1 - \bar{e}^2)^{\frac{1}{2}} + 1 + \bar{e}\cos\bar{\eta}}{1 + (1 + \bar{e}^2)^{\frac{1}{2}}}\right)\right]$$

$$+ \frac{(1 - \bar{e}^2)^{\frac{1}{2}}\,\bar{e}\sin\bar{\eta}}{1 + \bar{e}\cos\bar{\eta}}$$

Computation of Long Period Variations $\delta\vec{r}_\ell$ and $\delta\dot{\vec{r}}_\ell$

$$\delta\vec{r}_\ell = \alpha_2\,\bar{r}\big[\sin\bar{I}(1 + \bar{e}\cos\bar{\eta})\sin\bar{u}\,\vec{U}$$

$$+ \sin\bar{I}(2 + \bar{e}\cos\bar{\eta})\cos\bar{u}\,\vec{V} + \cos\bar{I}\,\bar{e}\cos\bar{\eta}\,\vec{W}\big]$$

$$\delta\dot{\vec{r}}_\ell = -\alpha_2\left(\frac{K}{\bar{p}}\right)^{\frac{1}{2}}\big[\sin\bar{I}(1 + \bar{e}\cos\bar{\eta})\cos\bar{u}\,\vec{U}$$

$$+ \sin\bar{I}(\sin\bar{u} + \bar{e}\sin\bar{\omega})\vec{V} + \cos\bar{I}\,\bar{e}\sin\bar{\eta}\,\vec{W}\big]$$

where

$$\alpha_2 = \frac{J_3\, a_e}{2J_2\bar{p}}$$

III. Transformation from Osculating to Mean Elements

In this and the subsequent section the use of the word "mean" is intended to include not only mean but also quasi-mean. This transformation cannot be made directly, but must be done iteratively, each iteration requiring about as much computation as an ordinary direct computation. In the case of the Frazer elements three iterations are nearly always sufficient. In the case of the Kozai elements sometimes many more iterations are required and special techniques beyond the scope of this report may be necessary. The approach given here applies explicitly to Frazer elements but the same general approach may be applied to any elements.

Given a, e, i, Ω, ω, M, compute $\bar{a}$, $\bar{e}$, $\bar{I}$, $\bar{\Omega}$, $\bar{\omega}$ and $\bar{M}$. As a first step these osculating elements are transformed to the true osculating state vector:

$$\begin{bmatrix} \vec{r} \\ \dot{\vec{r}} \end{bmatrix}_T$$

This vector and the accompanying osculating elements are taken as the first estimate of the mean elements in the following iterative process:

$$\begin{bmatrix} \vec{r} \\ \dot{\vec{r}} \end{bmatrix}_i = \begin{bmatrix} \bar{\vec{r}} \\ \dot{\bar{\vec{r}}} \end{bmatrix}_i + \begin{bmatrix} \delta\vec{r} \\ \delta\dot{\vec{r}} \end{bmatrix}_i$$

$$D_i = \begin{bmatrix} \vec{r} \\ \dot{\vec{r}} \end{bmatrix}_T - \begin{bmatrix} \vec{r} \\ \dot{\vec{r}} \end{bmatrix}_i$$

$$\begin{bmatrix} \bar{\vec{r}} \\ \dot{\bar{\vec{r}}} \end{bmatrix}_{i+1} = D_i + \begin{bmatrix} \bar{\vec{r}} \\ \dot{\bar{\vec{r}}} \end{bmatrix}_i$$

The three equations included in the dashed line represent one iteration. The result from the third equation is used as direct input to the first equation and also to compute a new estimate of the δ vector for the first equation. When D meets convergence criteria of 1.0 foot and .001 ft/sec, the process is said to have converged and the associated mean elements from the following third equation are accepted.

IV. Transformation of Mean Elements from Epoch at t_0 to Time t

The equations for mean element update are simply MacLaurin's series expansion

$$\Delta = t - t_0$$

$$\bar{a} = \bar{a}_0 + \dot{a}\Delta + \frac{\ddot{a}}{2}\Delta^2$$

$$\bar{e} = \bar{e}_0 + \dot{e}\Delta + \frac{\ddot{e}}{2}\Delta^2$$

$$\bar{I} = \bar{I}_0 + \dot{i}\Delta$$

$$\bar{\omega} = \bar{\omega}_0 + \dot{\omega}\Delta + \frac{\ddot{\omega}}{2}\Delta^2$$

$$\bar{\Omega} = \bar{\Omega}_0 + \dot{\Omega}\Delta + \frac{\ddot{\Omega}}{2}\Delta^2$$

$$\bar{M} = \bar{M}_0 + \bar{n}\Delta + \frac{\dot{n}}{2}\Delta^2 + \frac{\ddot{n}}{6}\Delta^3$$

In the equations of this section bars will be omitted but implied over the time derivatives of the mean elements. Thus $\dot{a}$ means the first time derivative of $\bar{a}$. Subscript (0) is used to represent epoch conditions. $\bar{n}$ represents the mean mean motion and is equivalent to $\dot{M}$. As implied in the introduction to this appendix these derivatives are preferably estimated from empirical fitting of tracking data. There are occasions however where analytical expressions are useful. One particular application is the use of known analytical relationships to constrain the experimentally derived derivatives. Consequently analytical relationships for the mean element derivatives will be presented here. We will need

$$\bar{n} = \left\{ \frac{K}{\bar{a}^3} \left[1 - J \frac{(1 - \bar{e}^2)^{\frac{1}{2}}}{\bar{p}^2} (1 - \frac{3}{2} \sin^2 \bar{I}) \right] \right\}^{\frac{1}{2}} .$$

In order to compute $\dot{\bar{a}}$ it is first necessary to compute the rate of apogee decay ($\dot{A}$) and the rate of perigee decay ($\dot{P}$) using McNair and Boykin's method:

$$\dot{A} = - \frac{86400B}{2\pi} \left[\frac{\bar{a}K(1 + \bar{e})}{(1 - \bar{e})^3} \right]^{\frac{1}{2}} \int_0^{2\pi} \rho (1 + \bar{e}^2 + 2\bar{e} \cos\bar{\eta})^{\frac{1}{2}} (1 + \cos\bar{\eta}) d\bar{\eta}$$

$$\dot{P} = - \frac{86400B}{2\pi} \left[\frac{\bar{a}K(1 - \bar{e})}{(1 + \bar{e})^3} \right]^{\frac{1}{2}} \int_0^{2\pi} \rho (1 + \bar{e}^2 + 2\bar{e} \cos\bar{\eta})^{\frac{1}{2}} (1 - \cos\bar{\eta}) d\bar{\eta}$$

where B = ballistic coefficient = $\frac{C_D A}{W}$

where C_D = drag coefficient

A = reference area

W = mass

$\bar{a}$ = mean semi-major axis assumed constant over one revolution

$\bar{e}$ = mean eccentricity assumed constant over one revolution

ρ = atmospheric density at each integration step

$\bar{\eta}$ = true anomaly at each step

Numerical integration is accomplished at 10° steps in $\bar{\eta}$ over one complete satellite revolution. ρ is obtained from some model atmosphere.

Then

$$\dot{a} = \frac{1}{2} (\dot{A} + \dot{P})$$

$$\frac{\dot{n}}{2} = - \frac{3\bar{n}\dot{a}}{4\bar{a}}$$

$$\frac{\ddot{a}}{2} = \bar{a} \left(- 2d + \frac{20}{9} c^3\right)$$

where

$$c = \frac{\dot{n}}{2} \frac{1}{\bar{n}}$$

$$d = Qc^2 \left[1 + \frac{n_0}{3(n_0 - \bar{n})}\right]$$

where

$$n_0 = 16.667 \text{ revs/day}$$

$$Q = 0, \text{ if } \bar{e} \geq 0.06$$

$$Q = 4, \text{ if } \bar{e} < 0.06 \text{ and } \bar{n} \leq 16.204$$

$$Q = 13, \text{ if } \bar{e} < 0.06 \text{ and } \bar{n} > 16.204 \quad .$$

Further derivatives are

$$\frac{\ddot{n}}{6} = - \frac{3\bar{n} \left(\frac{\ddot{a}}{2}\right) + 5\dot{a} \left(\frac{\dot{n}}{2}\right)}{6\bar{a}}$$

$$\dot{e} = \frac{\dot{a}}{\bar{a}} (1 - \bar{e})$$

$$\frac{\ddot{e}}{2} = \left(\frac{\ddot{a}}{2}\right) \frac{1 + \bar{e}}{\bar{a}}$$

$$\dot{i} = 0$$

$$\dot{\omega} = \frac{\bar{n}J(5\cos^2 \bar{I} - 1)}{2\bar{p}^2}$$

$$\frac{\ddot{\omega}}{2} = \frac{\dot{\omega}}{\bar{n}}\left(\frac{\dot{n}}{2}\right)\left[1 + \frac{4}{3}\left(\frac{1 - \bar{e}}{1 + \bar{e}}\right)\right]$$

$$\dot{\Omega} = -\frac{\bar{n}J\cos \bar{I}}{\bar{p}^2}$$

$$\frac{\ddot{\Omega}}{2} = \frac{\dot{\Omega}}{\bar{n}}\left(\frac{\dot{n}}{2}\right)\left[1 + \frac{4}{3}\left(\frac{1 - \bar{e}}{1 + \bar{e}}\right)\right]$$

Other relationships used in some algorithms are

$$\bar{a} = \left[\frac{K}{\bar{n}^2}\right]^{\frac{1}{3}}\left[1 + \frac{\delta}{3} - \frac{\delta^2}{3}\right], \text{ where}$$

$$\delta = -J\left(\frac{\bar{n}^2}{K}\right)^{2/3}(1 - \bar{e}^2)^{-\frac{3}{2}}\left(1 - \frac{3}{2}\sin^2 \bar{I}\right) \quad .$$

$$\dot{a} = (-4/3)(\bar{a}/\bar{n})(\dot{n}/2) \quad .$$

$$\frac{\ddot{a}}{2} = (2\bar{a}/\bar{n})\left[\left(\frac{10}{9\bar{n}}\right)\left(\frac{\dot{n}}{2}\right)^2 - \frac{\ddot{n}}{6}\right] \quad .$$

$$\dot{e} = \left(-\frac{4}{3}\right)\left(\frac{1 - \bar{e}}{\bar{n}}\right)\left(\frac{\dot{n}}{2}\right) \quad .$$

$$\frac{\ddot{e}}{2} = \left(\frac{2(1 - \bar{e})}{\bar{n}}\right)\left(\frac{2}{9\bar{n}}\right)\left(\frac{\dot{n}}{2}\right)^2 - \frac{\ddot{n}}{6} \quad .$$

V. EPHEMERIS GENERATION USING SIMPLIFIED GENERAL PERTURBATIONS WITH NORAD TWO-LINE ELEMENT SET

NOTE: For simplicity, the notation in this section is not exactly the same as in Sections I to IV of this appendix.

1.0 PHYSICAL CONSTANTS

The following are needed for the computation of the updated earth-fixed geocentric position and velocity vectors:

$K = 11467.874$ (earth radii)^3/(mean solar day)^2*

$J_2 = 1082.549 \times 10^{-6}$

$J_3 = -\ 2.435 \times 10^{-6}$

$\omega_I = 6.30038749$ radians/day

This discussion carries the computation only as far as geocentric, earth-fixed position and velocity vectors. In order to compute look angles, one needs along with the geodetic position of the station the NORAD spheroid constants:*

Earth radius = 6378145 meters

Earth flattening = 1/298.25

*See note Page N-23.

2.0 COORDINATE SYSTEM

The adopted coordinate system is that of the mean equinox of epoch and true equator of date. The following relation is used for computing the value in radians of the geodetic east longitude of the mean vernal equinox at epoch (t_o):

$$\gamma_{\Upsilon} \text{ at } t_o = 4.5365375881 - 0.0172027914 51d$$
$$-5.0641 \times 10^{-15} d^2 - 6.30038810F$$
$$0 \leq \gamma_{\Upsilon} \text{ at epoch} < 2\pi$$

where d is number of mean solar days elapsed from 0^h January 1, 1950, Universal Time to 0^h Universal Time of epoch day; F is the fractional part of a day from 0^h Universal Time of epoch day to epoch.

3.0 TIME

Included in the Two-Line Element Set is epoch year, epoch day and the fractional day in UTC. UTC is used throughout all computations; and if a discontinuity occurs in UTC between epoch and time of interest, then this offset should be taken into account.

4.0 OUTLINE OF COMPUTATIONAL PROCEDURE

The Two-Line Element Set includes eight mean elements. Three additional mean elements are needed and must be computed by the element user. These eleven mean elements constitute the initial conditions. To these initial conditions we first add secular effects, then effects due to atmospheric drag, then long periodic effects. We then solve Kepler's Equation and compute some polar coordinates of the satellite. We then add in some short periodic effects to obtain osculating parameters. These osculating parameters are then transformed into explicit inertial position and velocity vectors. A final transformation is made to a geocentric, rectangular, earth-fixed coordinate system, from which look angles may be readily calculated. To avoid a multiplicity of subscripts, the subscript associated with only the most recent update or correction is attached to a parameter.

5.0 EIGHT MEAN ELEMENTS AT EPOCH INCLUDED IN TWO-LINE ELEMENT SET

M_o = mean anomaly (degrees)

Ω_o = right ascension of node (degrees)

ω_o = argument of perigee (degrees)

e_o = eccentricity (dimensionless)

i_o = inclination (degrees)

n_o = mean motion [rev/(mean solar day)]

$\dot{n}_o$ = time derivative of n_o [rev/(mean solar day)^2]

$\ddot{n}_o$ = second time derivative of n_o [rev/(mean solar day)^3]

NOTE: As soon as these elements are input to the computer, all angle measurements are transformed to radians. Computation will proceed using earth radius units, radians and mean solar days.

6.0 THREE ADDITIONAL MEAN ELEMENTS AT EPOCH TO BE COMPUTED BY ELEMENT USER

a_o = semi-major axis [earth radii]

$\dot{\Omega}_o$ = time derivative of Ω_o [radians/(mean solar day)]

$\dot{\omega}_o$ = time derivative of ω_o [radians/(mean solar day)]

These elements are computed as follows:

$$a' = [K/n_o^2]^{\frac{1}{3}}$$

$$\delta = -\frac{3}{2} J_2 (a')^{-2}(1-e_o^2)^{-\frac{3}{2}} (1-\frac{3}{2}\sin^2 i_o)$$

$$a_o = a'[1+\frac{1}{3}\delta - \frac{1}{3}\delta^2]$$

$$TEMP = \frac{3}{2} J_2 \left[a_o(1-e_o^2)\right]^{-2} n_o$$

$$\dot{\Omega}_o = -TEMP \cos i_o$$

$$\dot{\omega}_o = TEMP\ (2-2.5 \sin^2 i_o)$$

It is convenient to include in this section on initial conditions the mean orbital longitude L_o and the mean perigee distance q_o .

$$L_o = \omega_o + M_o + \Omega_o$$

$$q_o = a_o\ (1-e_o)$$

<u>NOTE</u>: All the preceding computations are to be performed only once per element set. All the remaining computations must be performed once for each time point in the ephemeris.

7.0 <u>COMPUTE SECULAR PERTURBATIONS IN MEAN ELEMENTS AT TIME $(t-t_o)$ FROM EPOCH</u>

$$L_s = L_o + (n_o + \dot{\omega}_o + \dot{\Omega}_o)(t-t_o)$$

$$\Omega_s = \Omega_o + \dot{\Omega}_o\ (t-t_o)$$

$$\omega_s = \omega_o + \dot{\omega}_o\ (t-t_o)$$

8.0 <u>ADD IN ATMOSPHERIC DRAG PERTURBATIONS</u>

$$L_A = L_s + \frac{1}{2}\ \dot{n}_o(t-t_o)^2 + \frac{1}{3}\ \ddot{n}_o(t-t_o)^3$$

$$n_A = n_o + \dot{n}_o(t-t_o) + \frac{1}{2}\ \ddot{n}_o(t-t_o)^2$$

$$a_A = a_o\ \left(\frac{n_o}{n_A}\right)^{2/3}$$

$$e_A = 1 - \frac{q_o}{a_A} \text{ if } a_A \geq q_o \text{ ; otherwise } e_A = 0.$$

$$p_A = a_A\,(1-e_A{}^2)$$

$$a_{xA} = e_A \cos \omega_s$$

$$a_{yA} = e_A \sin \omega_s$$

9.0 ADD IN LONG PERIODIC PERTURBATIONS

$$L_\ell = L_A - \frac{1}{4}\,\frac{J_3}{J_2}\,\frac{a_{xA}}{p_A}\sin i_o\left(\frac{3+5\cos i_o}{1+\cos i_o}\right)$$

$$0 \le L_\ell < 2\pi$$

$$a_{y\ell} = a_{yA} - \frac{1}{2}\,\frac{J_3}{J_2}\,\frac{\sin i_o}{p_A}$$

$$\omega_\ell = \tan^{-1}\frac{a_{y\ell}}{a_{xA}} \qquad 0 \le \omega_\ell < 2\pi$$

$$e_\ell = \left(a_{xA}{}^2 + a_{y\ell}{}^2\right)^{\frac{1}{2}}$$

$$p_\ell = a_A\,(1-e_\ell{}^2)$$

$$M_\ell = L_\ell-\omega_\ell-\Omega_s \qquad 0 \le M_\ell < 2\pi$$

$$M_\ell = E_\ell-e_\ell \sin E_\ell \qquad \text{(Kepler's Equation)}$$

The solution of Kepler's Equation for E_ℓ by straightforward iteration is as follows:

We drop the subscript (ℓ) temporarily and introduce a subscript (k) to indicate the sequence of estimates starting with $k = 0$:

$$E_{k+1} = M + e \sin E_k \;.$$

Initiate the process with $\sin E_o = \sin M$ to obtain the first estimate E_1 and continue until a convergence criterion is met:

$$\left|E_{k+1} - E_k\right| < \epsilon \;.$$

Having the eccentric anomaly E_ℓ, we compute the corresponding true anomaly v_ℓ:

$$\frac{v_\ell}{2} = \tan^{-1} \left(\frac{1+e_\ell}{1-e_\ell}\right)^{\frac{1}{2}} \tan \frac{E_\ell}{2}$$

where $\frac{v_\ell}{2}$ is evaluated in same quadrant as $\frac{E_\ell}{2}$.

$$u_\ell = v_\ell + \omega_\ell \qquad 0 \leq u < 2\pi$$

$$r_\ell = a_A \left(1-e_\ell \cos E_\ell\right)$$

$$\dot{r}_\ell = (K a_A)^{\frac{1}{2}} \; \frac{e_\ell \sin E_\ell}{r_\ell}$$

$$r_\ell \, \dot{v}_\ell = (K p_\ell)^{\frac{1}{2}} / r_\ell$$

10.0 ADD IN SHORT PERIODIC PERTURBATIONS

$$r_h = r_\ell + \frac{1}{4} \frac{J_2}{p_\ell} \sin^2 i_o \cos 2 u_\ell$$

$$u_h = u_\ell - \frac{1}{8} \frac{J_2}{p_\ell^2} \left(6 - 7 \sin^2 i_o\right) \sin 2 u_\ell$$

$$\Omega_h = \Omega_s + \frac{3}{4} \frac{J_2}{p_\ell^2} \cos i_o \sin 2 u_\ell$$

$$i_h = i_o + \frac{3}{4} \frac{J_2}{p_\ell^2} \sin i_o \cos i_o \cos 2 u_\ell$$

11.0 TRANSFORM TO INERTIAL POSITION AND VELOCITY VECTORS

$$\begin{bmatrix} M \end{bmatrix} = \begin{bmatrix} -\sin \Omega_h \cos i_h \\ \cos \Omega_h \cos i_h \\ \sin i_h \end{bmatrix}$$

$$[N] = \begin{bmatrix} \cos \Omega_h \\ \sin \Omega_h \\ 0 \end{bmatrix}$$

$$[U] = [N] \cos u_h + [M] \sin u_h$$

$$[V] = -[N] \sin u_h + [M] \cos u_h$$

$$[r] = r_h \, [U]$$

$$[\dot{r}] = \dot{r}_\ell \, [U] + r_\ell \dot{v}_\ell \, [V]$$

12. TRANSFORM TO GEOCENTRIC POSITION AND VELOCITY VECTORS

$$\begin{bmatrix} E \\ F \\ G \end{bmatrix} = \begin{bmatrix} \cos \gamma_\Upsilon & -\sin \gamma_\Upsilon & 0 \\ \sin \gamma_\Upsilon & \cos \gamma_\Upsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} [r]$$

$$\begin{bmatrix} \dot{E} \\ \dot{F} \\ \dot{G} \end{bmatrix} = \begin{bmatrix} \cos \gamma_\Upsilon & -\sin \gamma_\Upsilon & 0 \\ \sin \gamma_\Upsilon & \cos \gamma_\Upsilon & 0 \\ 0 & 0 & 1 \end{bmatrix} [\dot{r}]$$

$$+ \, \omega_I \begin{bmatrix} \sin \gamma_\Upsilon & \cos \gamma_\Upsilon & 0 \\ -\cos \gamma_\Upsilon & \sin \gamma_\Upsilon & 0 \\ 0 & 0 & 0 \end{bmatrix} [r]$$

where

$$\gamma_\Upsilon = (\gamma_\Upsilon \text{ at } t_o) - \omega_I (t-t_o)$$

and where

the EFG coordinate system is geocentric, earth-fixed, equatorial, rectangular and right-handed. G is coincident with earth polar axis and extends in a positive direction northward. E extends positively so as to pierce the meridian of Greenwich. The units are earth radii and mean solar days.

NOTE: It may be necessary to reference the stations to some spheroid other than the NORAD spheroid. Ideal alternatives are NWL-9C, NWL-8E and NWL-8 because they are identical in size and shape with the NORAD spheroid and have centers sufficiently close to the NORAD spheroid. Other acceptable spheroids are Kaula-61, Mercury Fischer-68, SAO-66, SAO-67 and SAO-69. The dimensions of each of these last five differ somewhat from the NORAD spheroid and one may question whether these changes necessitate changes in any of the physical constants in the NORAD algorithm. The answer is no, with the possible exception of the central gravitational parameter K.

For the NORAD, NWL-9C, NWL-8E and NWL-8 spheroids 11467.874 (earth radii)^3/(mean solar day)^2 is equivalent to 398601.2 km^3/sec^2, a standard value derived from space probe trajectory analysis. For the other five acceptable spheroids these two numbers are not equivalent. For these last five spheroids there is a slight advantage in maintaining K = 11467.874 (earth radii)^3/(mean solar day)^2 for near earth satellites with semi-major axes less than about two earth radii, and there is increasing advantage in maintaining K = 398601.2 km^3/sec^2 — transformed to (earth radii)^3/(mean solar day)^2 — as the orbit semi-major axis increases beyond two earth radii.

APPENDIX O

CANONICAL UNITS AND MISCELLANEOUS CONSTANTS

1.0 CANONICAL OR CHARACTERISTIC UNITS

It is sometimes desirable from considerations of computational accuracy, computational speed and/or analytical simplicity to use canonical units in orbital mechanics. These units are always defined numerically on the basis of a circular orbit under the influence of a central force alone, with no drag or other perturbations; but they may be freely used in any kind of orbit with perturbations of all kinds, and such use does not represent an approximation or source of error in the orbital computations. When these units are used, the central gravitational parameter K becomes unity and does not appear explicitly in the equations of motion. Let

CUL = canonical unit of length
CUA = canonical unit of acceleration
CUV = canonical unit of velocity
CUT = canonical unit of time

By definition a satellite in circular orbit at one CUL from center of central body is subject to one CUA, travels at rate of one CUV and traverses one radian in one CUT, based on central force theory with no drag or other perturbations. CUL may be chosen arbitrarily but is usually selected to be mean equatorial radius of planet for planetocentric orbits and one astronomical unit (i.e., semi-major axis of earth's orbit) for heliocentric orbits. After CUL is chosen, then CUA, CUV and CUT are automatically determined:

$$\mathrm{CUA} = K/(\mathrm{CUL})^2$$

$$\mathrm{CUV} = [K/(\mathrm{CUL})]^{1/2}$$

$$\mathrm{CUT} = [(\mathrm{CUL})^3/K]^{1/2}$$

As an example, for $K = 398601\ km^3/sec^2$ and NWL - 8E spheroid

$$CUL = 6378145\ m$$

$$CUA = 9.79827462102\ m/sec^2$$

$$CUV = 7905.36629655\ m/sec$$

$$CUT = 806.812076850\ sec$$

2.0 SOME MISCELLANEOUS CONSTANTS AND CONVERSION FACTORS

π	=	3.141592653589793
e	=	2.718281828459045
M	=	0.4342944819032518
1 foot (international)	=	0.3048 meters*
1 foot (American Survey)	=	(1200/3937) meters*
	=	0.3048006096012192 meters
1 nautical mile	=	1852 meters*
	=	6076.115485564304 international feet
1 statute mile	=	1609.344 meters*
	=	5280 international feet*
1 radian	=	$(180/\pi)$ degrees*
	=	57.29577951308232 degrees
1 degree	=	(6400/360) mils*
1 degree	=	17.77777777777778 mils
1 degree	=	17.45329251994329 milliradians
1 knot	=	1.687809857101196 international feet/second
1 pound	=	0.45359237 kilograms*
1 slug	=	32.17404855643045 pounds
1 millibar	=	1.01971621297 grams/centimeter^2
Speed of light in vacuum	=	299792456.2 $\pm$ 1.1 meters/second**
Earth gravitational parameter	=	398600.8 $\pm$ 0.4 kilometers^3/second^2**
One astronomical unit	=	149597893.0 $\pm$ 5 kilometers**
1 milligal	=	0.001 centimeter/second^2*

NOTE: Seconds as used here represent international atomic seconds. (See Appendix J)

* Exact by definition

** Experimental and approximate

APPENDIX P

TRANSFORMATION OF MEAN ANOMALY TO ECCENTRIC ANOMALY

This involves the solution of Kepler's Equation:

$$M = E - e \sin E,$$

where M is mean anomaly, E is eccentric anomaly, and e is eccentricity. This was the first transcendental equation to engage the attention of mathematicians and hence there are well over a hundred treatments of this problem in the literature. Present practice on electronic computers is usually a simple straightforward iteration using

$$E_{k+1} = M + e \sin E_k ,$$

initiating the process with

$$E_o = M$$

and continuing until a convergence criterion is met:

$$\left| E_{k+1} - E_k \right| < \epsilon .$$

This method has advantages of simplicity and small storage requirements. However, it may be unnecessarily wasteful of computer time. Eight-place accuracy (radians) may required 36 iterations if e is large.

A method which is always as fast and sometimes twice as fast is the first-order differential correction process:

$$E_{k+1} = E_k + \frac{M - E_k + e \sin E_k}{1 - e \cos E_k} ,$$

where

$$E_o = M .$$

Seven iterations or less will generally give eight-place accuracy (radians). If eccentricities are less than .01, then the straightforward iteration is just as fast as the first-order differential correction process.

For eccentricities less than 0.6, additional computer time can be saved by using a series expansion to improve the estimate of E prior to either one of the iteration processes described:

$$E_0 = M + e \sin M + (e^2/_2) \sin 2M + (e^3/_8)(3 \sin 3M - \sin M)$$
$$+ (e^4/_6)(2 \sin 4M - \sin 2M) + (e^5/_{384})(125 \sin 5M - 81 \sin 3M$$
$$+ 2 \sin M) + \ldots \quad .$$

No time will be saved using this series unless a special routine is used to develop sin 2M, sin 3M, etc. functions from the values for sin M and cos M; thus

$$\sin 2M = 2 \sin M \cos M$$
$$\sin 3M = 3 \sin M - 4 \sin^3 M$$
$$\sin 4M = 4 \cos M (\sin M - 2 \sin^3 M)$$
$$\sin 5M = 16 \sin^5 M - 20 \sin^3 M + 5 \sin M \quad .$$

The expansion for E_0 is generally equivalent in accuracy to about five straightforward iterations and in computer time to about half that number.

APPENDIX Q

Transformation of Covariance Matrix to Probability Region

The boundary of a confidence, tolerance or prediction region can be constructed on the basis of a random sample of N observations transformed to a mean position column vector $\underline{u}$ and a sample covariance matrix S. A confidence region is one which has a preassigned probability of including a parameter (e.g., the mean) of the population sampled. A tolerance region will on the average contain exactly some specified proportion of the population.* A prediction region has a specified probability of containing the next observation (or perhaps the mean of the next h observations). The regions discussed here are all centered about the mean of the sample and may be one, two or three dimensional, specified by $p = 1, 2, 3$, respectively. Conditions assumed are: optimum (i.e., minimum) size probability region, random normal distribution of observations, mean and covariance matrix of population estimated from the observations. The probability region will be a straight line ($p = 1$), an ellipse ($p = 2$), or an ellipsoid ($p = 3$).

In general, for an x y z rectangular coordinate system, let

$$\underline{u} = \begin{bmatrix} \underline{x} \\ \underline{y} \\ \underline{z} \end{bmatrix} \qquad u = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad S = \begin{bmatrix} S_x^2 & S_{xy} & S_{xz} \\ S_{yx} & S_y^2 & S_{yz} \\ S_{zx} & S_{zy} & S_z^2 \end{bmatrix}$$

In particular, for a two-dimensional or one-dimensional region suitably oriented, one or two of the coordinate axes may be omitted and the order of S reduced accordingly.

The equation of the probability region is

*This is a commonly accepted definition but not the only one.

$$\left[u - \underline{u}\right]^T S^{-1} \left[u - \underline{u}\right] = Q ,$$

where

$$Q = \frac{p}{N} \frac{(N-1)}{(N-p)} F_{p,\, N-p}$$ for confidence region for the population mean,

$$Q = \frac{(N^2-1)\, p}{N\, (N-p)} F_{p,\, N-p}$$ for tolerance region,

$$Q = \left(\frac{1}{h} + \frac{1}{N}\right) \frac{p\,(N-1)}{N-p} F_{p,\, N-p}$$ for prediction region for mean of next h observations,

and where F is the tabulated F statistic for p and (N-p) degrees of freedom. p is degrees of freedom for greater mean square (numerator), and (N-p) is degrees of freedom for lesser mean square (denominator) in F tables.

NOTE: The probability regions are generally computed using the covariance matrix of the mean vector output from a prior error analysis. This covariance matrix does not correspond to S but rather to S/N or $S\left(\frac{1}{h} + \frac{1}{N}\right)$. Hence the covariance matrix of the mean vector may be substituted for S in all of the equations of this appendix if Q is modified accordingly by removing the factors $\left(\frac{1}{N}\right)$ or $\left(\frac{1}{h} + \frac{1}{N}\right)$.

For some applications, it is more convenient to consider the equation of the probability region in the form

$$R^T S^{-1} R = Q ,$$

where

$$R = \begin{bmatrix} r\,l \\ r\,m \\ r\,n \end{bmatrix} = \left[u - \underline{u}\right] ,$$

and r is magnitude of radius vector from center of probability region to the boundary, and (l, m, n) represent direction cosines of r relative to the x y z coordinate system. If (l, m, n) are given, it is then an easy matter to solve for r. (l, m, n) may, for example, be chosen to represent the direction cosines of the instantaneous velocity vector of a missile; and then $\pm$ r would define the in-track position probability interval at that instant.

For some purposes, it is useful to know the lengths of the semi-axes of the probability region. Form the matrix

$$A = [S - dI] ,$$

set its determinant equal to zero and solve for d. There will be one, two or three non-zero roots. All roots are real and non-negative. Let d_i represent the value of the i^{th} root. Corresponding to d_i there is a semi-axis a_i with length equal to $\sqrt{Q\, d_i}$, where Q is computed using p equal to the number of non-zero roots (unless a different p is known a priori). I in the A matrix is the identity matrix with order equal to order of S.

It is also frequently desirable to know the orientation of the semi-axes. Let the order of S be j. If j equals one, there is only one root d_1 and only one semi-axis a_1, and it lies along the single coordinate axis in the problem. For j > 1, the orientation of the semi-axes may be determined as follows, where we assume for the moment that all the roots d_i are distinct:

Set d = d_i in the A matrix. There will be at least one row s in the resulting matrix for which all the cofactors ($\tilde{A}_{st}$, t = 1 to j) do not vanish. Then the unit vector k_i describing the orientation of the semi-axis a_i has the following components along the coordinate axes:

$$k_i = \left\{ (\tilde{A}_{s1})^2 + \ldots + (\tilde{A}_{sj})^2 \right\}^{-\frac{1}{2}} \begin{bmatrix} \tilde{A}_{s1} \\ \cdot \\ \cdot \\ \cdot \\ \tilde{A}_{sj} \end{bmatrix} .$$

The components of k_1 are direction cosines of the semi-axis a_1. The procedure may be repeated for all the distinct roots d_1. It may be noted that

$$\left| S - dI \right| = 0$$

constitutes what is called the characteristic equation of the matrix S. The roots d_1 are called eigenvalues and the unit vectors k_1 are called normalized eigenvectors. ("Latent roots" and "characteristics roots" are synonymous with "eigenvalues".)
If two or more roots d_1 are identical, the orientation of all semi-axes can be computed; but the very nature of the problem precludes a unique solution. The only problem of possible interest to us here is that in which there is one distinct root and two non-zero repeated roots. A non-unique solution to this problem of possible interest can be obtained as follows:

Proceed as before to determine the three semi-axes and also the unit vector k_1 for the semi-axis corresponding to the distinct root d_1. d_2 and d_3 will then be identical, and a_2 and a_3 will also be identical. Form the matrix

$$[S - d_2\, I]$$

where d_2 is the numerical value of the repeated root of characteristic equation, and let some non-zero row of this matrix be denoted by (e f g). Then a possible value for k_2, if both g and f are not zero is

$$k_2 = \left\{ g^2 + f^2 \right\}^{-\frac{1}{2}} \begin{bmatrix} 0 \\ g \\ -f \end{bmatrix}$$

or if both f and g are zero,

$$k_2 = \frac{1}{e} \begin{bmatrix} 0 \\ -e \\ 0 \end{bmatrix}$$

The corresponding value for k_3 may be computed from the vector cross product

$$k_3 = k_1 \times k_2 .$$

The discussion so far has treated position data, but the mathematics applies equally well to velocity data or acceleration data. $\underline{u}$, u and S may be defined in terms of velocity, for example, instead of position. Then a_1 will represent a semi-axis of velocity and k_1 will represent a unit vector defining the direction of a_1.

A final word of caution should be given in regard to possibly biased observations. The particular "population" being estimated may or may not require an assumption of unbiased data. If such an assumption is required and the observations are known to have a bias uncertainty, it may be possible to augment the covariance matrix to take care of the problem. Each problem should be carefully analyzed before proceeding with the routine of calculating a probability region.

<u>Numerical Example</u>

Given:

$$\begin{bmatrix} s_x^2 & s_{xy} & s_{xz} \\ s_{yx} & s_y^2 & s_{yz} \\ s_{zx} & s_{zy} & s_z^2 \end{bmatrix} = \begin{bmatrix} +5.09375 & -.487139 & +.937500 \\ -.487139 & +3.53125 & -1.40729 \\ +.937500 & -1.40729 & +3.37500 \end{bmatrix}$$

$$\bar{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{aligned} N &= 20 \\ p &= 3 \end{aligned}$$

Compute:

Center of probability region, lengths of semi-axes, and orientation of semi-axes for a 95% confidence ellipsoid for population mean, a 95% tolerance ellipsoid, and a 95% prediction ellipsoid for the next observation. Assume unbiased observations.

Solution:

a. The specified probability regions are all centered at x = 1, [illegible]. The semi-axes are computed starting with the matrix.

$$A = \begin{bmatrix} (+5.09375-d) & -.487139 & +.937500 \\ -.487139 & (+3.53125-d) & -1.40729 \\ +.937500 & -1.40729 & (+3.37500-d) \end{bmatrix}$$

Setting the determinant equal to zero, we obtain

$$d^3 - 12d^2 + 44d - 48 = 0 \ .$$

The roots of this equation are

$$d_1 = +2 \ ,$$
$$d_2 = +4 \ ,$$
$$d_3 = +6 \ .^*$$

Now $Q = \dfrac{3\,(20-1)}{20(20-3)}\,(3,20) = 0.5365$ for confidence ellipsoid.

$Q = \dfrac{(20^2-1)\,3}{20(20-3)}\,(3,20) = 11.2659$ for tolerance ellipsoid.

$Q = (1 + \dfrac{1}{20})\,\dfrac{3(20-1)}{20-3}\,(3,20) = 11.2659$ for prediction ellipsoid.

Therefore

		Confidence	Tolerance	Prediction
a_1	=	1.0359	4.7468	4.7468
a_2	=	1.4650	6.7130	6.7130
a_3	=	1.7942	8.22170	8.22170

$$k_1 = \begin{bmatrix} -.1250 \\ +.6495 \\ +.7500 \end{bmatrix} \qquad k_2 = \begin{bmatrix} +.6495 \\ +.6250 \\ -.4330 \end{bmatrix} \qquad k_3 = \begin{bmatrix} +.7500 \\ -.4330 \\ +.5000 \end{bmatrix}$$

* The integer roots are not fortuitous but were contrived for purposes of simplicity.

CLOSED FORM SOLUTION OF CUBIC HAVING THREE REAL ROOTS

We are given

$$Ax^3 + Bx^2 + Cx + D = 0 \quad .$$

Divide through by A, so that

$$x^3 + Px^2 + Qx + R = 0 \quad .$$

Compute

$$a = (P^2 - 3Q)/3 \quad \text{and}$$

$$b = (9PQ - 2P^3 - 27R)/27 \quad .$$

Evaluate ϕ, where $0 \leq \phi \leq \pi$, from

$$\cos\phi = (3/2)(b/a)\sqrt{3/a} \quad .$$

Then the roots are

$$x_1 = 2\sqrt{a/3}\,\cos\left[\phi/3\right] - P/3 \quad ,$$

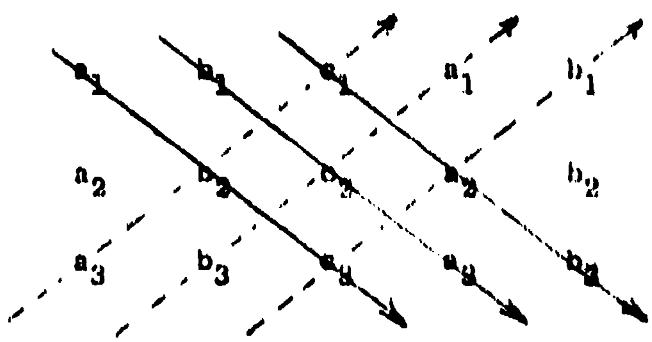
$$x_2 = 2\sqrt{a/3}\,\cos\left[\phi/3 + 120^\circ\right] - P/3 \quad \text{and}$$

$$x_3 = 2\sqrt{a/3}\,\cos\left[\phi/3 + 240^\circ\right] - P/3 \quad .$$

EVALUATION OF DETERMINANT OF ORDER 3

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Set up



Take products of elements on descending diagonal lines with the sign + and take products of elements on ascending diagonal lines with the sign − . The aggregate of the six products thus obtained

$$a_1\, b_2\, c_3 + b_1\, c_2\, a_3 + c_1\, a_2\, b_3 - a_3\, b_2\, c_1$$
$$- b_3\, c_2\, a_1 - c_3\, a_2\, b_1$$

will be the value of the determinant.

APPENDIX B

RADAR MEASUREMENT EQUATIONS

The following equations are useful in adjustment processes and simulation exercises:

$$\underbrace{R}_{\text{measurement}} = \underbrace{R_t}_{\text{true}} + \underbrace{r_1}_{\text{zero set}} + \underbrace{r_2 R_t}_{\text{scale factor}} + \underbrace{r_3 \dot{R}_t}_{\text{velocity lag and timing}}$$

$$+ \underbrace{r_4 \ddot{R}_t}_{\text{acceleration lag}} + \underbrace{r_5 \csc E_t}_{\text{residual refraction}} + \underbrace{\epsilon_R}_{\text{random error}}$$

$$\underbrace{A}_{\text{measurement}} = \underbrace{A_t}_{\text{true}} + \underbrace{a_1}_{\text{zero set}} + \underbrace{a_2 \dot{A}_t}_{\text{velocity lag and timing}} + \underbrace{a_3 \ddot{A}_t}_{\text{acceleration lag}}$$

$$+ \underbrace{a_4 \tan E_t}_{\text{non-orthogonality}} + \underbrace{a_5 \sec E_t}_{\text{collimation}} + \underbrace{a_6 \sin A_t + a_7 \cos A_t}_{\text{encoder eccentricity}}$$

$$+ \underbrace{u \sin A_t \tan E_t - v \cos A_t \tan E_t}_{\text{mislevel}} + \underbrace{\epsilon_A}_{\text{random error}}$$

$$\underbrace{E}_{\text{measurement}} = \underbrace{E_t}_{\text{true}} + \underbrace{e_1}_{\text{zero set}} + \underbrace{e_2 \dot{E}_t}_{\text{velocity lag and timing}} + \underbrace{e_3 \ddot{E}_t}_{\text{acceleration lag}}$$

$$+ \underbrace{e_4 \operatorname{ctn} E_t}_{\text{residual refraction}} + \underbrace{e_5 \sin E_t}_{\text{encoder eccentricity}} + \underbrace{e_6 \cos E_t}_{\text{encoder eccentricity and droop}}$$

$$+ \underbrace{u \cos A_t + v \sin A_t}_{\text{mislevel}} + \underbrace{\epsilon_E}_{\text{random error}}$$

* Elevation collimation errors are included in e_1. In these equations: R represents range; A, azimuth; E, elevation.

The zero-set errors are constant bias or off-set values. Scale factor represents the range error resulting from an error in the oscillator frequency or in velocity of propagation. Non-orthogonality represents the lack of perpendicularity between the azimuth and elevation axes. Collimation represents the lack of perpendicularity between the R-F beam and the mechanical elevation axis. Mislevel represents a tilt of the azimuth plane — u being the northward component and v being the eastward component. This tilt is measured with respect to the local horizontal to the geodetic spheroid. Velocity and acceleration lags are dynamic radar errors which become significant in azimuth at high elevations and sometimes during missile reentry at lower elevations, and which become significant in elevation in missile reentry. Droop represents the sag of the R-F axis. Timing errors refer to errors in the time tag on the data. The random errors represent noise in the data with zero means. Residual refraction and encoder eccentricity require no explanation. The numerical coefficients to be adjusted are r_1, r_2, r_3, r_4, r_5, a_1, a_2, a_3, a_4, a_5, a_6, a_7, e_1, e_2, e_3, e_4, e_5, e_6, u and v. Some approximate a priori one-sigma values for these coefficients in the case of MIPIR radars are shown in Table I.

NOTE: In ship radar calibration it is frequently preferable to express the mislevel in amplitude M and phase angle F, where amplitude is positive upward from local horizontal and phase angle is measured positive eastward from north. The mislevel term for azimuth is then

$$M \left[\sin(A_t - F) \right] \tan E_t$$

and the mislevel term for elevation is

$$M \left[\cos(A_t - F) \right] \quad .$$

TABLE I

A PRIORI ONE-SIGMA VALUES FOR MIPIR ERROR COEFFICIENTS

Error Coefficients	Values
r_1	20 feet*
r_2	0.0000005
r_3 (velocity lag)	0.0001 seconds
r_3 (timing)	0.00001 seconds
r_4	0.0005 (seconds)^2
r_5	0.1 foot
a_1	0.00005 radians
a_2 (velocity lag)	0.003 seconds
a_2 (timing)	0.00001 seconds
a_3	0.06 (seconds)^2
a_4	0.00002 radians
a_5	0.00004 radians
a_6	zero
a_7	zero
e_1	0.00005 radians
e_2 (velocity lag)	0.003 seconds
e_2 (timing)	0.00001 seconds
e_3	0.06 (seconds
e_4	0.00002 radians
e_5	zero
e_6 (encoder)	zero
e_6 (droop)	0.00004 radians
u	0.00004 radians
v	0.00004 radians

*in echo track. May be larger in beacon track.

APPENDIX S

Differential Correction

Transformations treated in the other sections and appendices of this report are direct and precisely determined. Some of the most important transformations in missile and satellite technology are not of this type; but instead they are iterative because of assumed linearity in the partial derivatives, and statistical because of redundancy in the data. It is the purpose of this appendix to sketch out the solution for problems of this latter type. It is beyond the scope of this report to treat this subject in other than the barest elements. It is a complex subject in both theory and application. Complexity in the applications is associated with the required partial derivatives, matrix partitioning, constraint matrices, bounding of corrections, logical sequence of iterations, a priori parameter weighting, data weighting, selection of parameters to be adjusted, and effects from errors in parameters not adjusted. The bibliography contains some references which may be helpful for further study.

Four types of applications will be considered here:

(1) One in which the observations are connected by some functional relationship; e.g., a missile or satellite in free or ballistic flight described by equations of motion.

(2) One in which the observations are not connected by a functional relationship; e.g., a missile in powered flight.

(3) A variation of (2) in which, although the actual observations are not connected by a functional relationship, the observations are commonly affected by some error such as survey or zero set biases.

(4) A simple non-statistical type of problem using non-redundant data.

These four types will be treated in order. An explanation of all notation is given at the end of this appendix. For various reasons the notation may not conform with notation elsewhere in this report.

Type (1)

This can be most easily explained by illustration using a simple orbit determination and associated estimation of radar error coefficients. We assume all observations to be made from radars, although the extension to other type sensors is simple and obvious. The radar measurement equations referenced in this discussion are treated in Appendix R.

The observations at the various time points are functionally connected by the following simple equations of motion which are satisfactory for most low altitude satellite work:

$$\ddot{X} = \frac{\partial U}{\partial X} + \omega^2 X + 2\omega\dot{Y} - D\rho V\dot{X}$$

$$\ddot{Y} = \frac{\partial U}{\partial Y} + \omega^2 Y - 2\omega\dot{X} - D\rho V\dot{Y}$$

$$\ddot{Z} = \frac{\partial U}{\partial Z} - D\rho V\dot{Z}$$

In these equations the acceleration is defined in terms of a geocentric earth-fixed right-handed rectangular coordinate system. (See Glossary.) The first term on the right of each equation represents acceleration due to gravity. The last term on the right of each equation represents acceleration due to aerodynamic drag. The other terms on the right sides of the equations of motion are the usual ones describing Coriolis and centripetal accelerations in a rotating coordinate system. U represents the Earth's gravitational potential and is defined by

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{n(max)} \sum_{m=0}^{n} \left(\frac{\hat{a}}{r}\right)^n P_{nm}(\sin\phi)(C_{nm}\cos m\gamma + S_{nm}\sin m\gamma)\right].$$

The gravitational model is defined by the experimentally determined values for C_{nm} and S_{nm}. Any of the newer models issued by Naval Weapons Laboratory, Smithsonian Astrophysical Observatory, Applied Physics Laboratory or NASA may be used. Any of the newer atmospheric models issued by NASA, Smithsonian Astrophysical Observatory or U. S. Air Force may be used to describe the air density ρ. Consequently, given a vector $(X\ Y\ Z\ \dot{X}\ \dot{Y}\ \dot{Z})$ and the various models and constants entering into the equations of motion, one can numerically integrate the equations of motion and thereby generate a satellite orbit for as many revolutions as necessary.

The orbital parameters to be estimated are the constant D and the components of the vector $(X\ Y\ Z\ \dot{X}\ \dot{Y}\ \dot{Z})$ at an epoch corresponding to first track point.

The object is to make a simultaneous estimate of the orbital parameters and the radar error coefficients. The mathematical procedure is iterative and is based upon the criterion of minimization of the sum of squares of the weighted measurement residuals. A weighted measurement residual (dimensionless) is simply a measurement residual divided by the a priori estimate of standard deviation in random error in that particular measurement. Measurement residuals are obtained as follows:

An initial estimate of a vector $(X\ Y\ Z\ \dot{X}\ \dot{Y}\ \dot{Z})$ at the first track point may be obtained by a simple curve fitting and numerical differentiation process on a short span of actual RAE track data transformed to XYZ. An orbit is generated by numerical integration of equations of motion, and then by a coordinate transformation the resulting XYZ time points are transformed to theoretical RAE measurements for the radar sites. These theoretical measurements are used in place of the true but unknown measurements on the right side of the radar measurement equations to produce "computed" measurements. Random errors are never included in the computed measurements, and on the first iteration all other error coefficients are set to zero since they are unknown and have expectations of zero. After the first iteration there are non-zero estimates of the radar error coefficients to use in producing computed measurements. The computed measurements are differenced with the actual measurements to obtain measurement residuals.

The measurement residuals from the first computation will generally be much larger than the random errors in the actual radar measurements and thus indicate that one or more of the adjusted parameters have been poorly estimated. Corrections are therefore computed and added to the original estimates of all the adjustable parameters. Corrections are computed using a basic least squares iterative equation satisfying the original minimization criterion:*

$$F = \left[\sum_{i=1}^{p} (Q_i^T M_i^{-1} Q_i) + J^{-1} \right]^{-1} \left[\sum_{i=1}^{p} (Q_i^T M_i^{-1} q_i) + J^{-1} H \right]$$

F is a column vector of corrections for the adjusted parameters, including both orbital parameters and radar error coefficients. This equation is used together with the equations of motion and the radar measurement equations like any other set of iterative equations. After a number of iterations, reductions in the sum of squares of the weighted measurement residuals become negligibly small, at which time the adjustment process is said to have converged and final estimations of all parameters have been obtained.

This type of adjustment is also known as "orbital constraint." When the time points are functionally related as they are here, then observations at all time points add information and strength to the solution for any particular time point. The covariance matrix of all the adjusted parameters is given by the first factor on the right-hand side of the equation for F. However, if there are any unadjusted erroneous constants or parameters contributing to error in the solution, then a more accurate estimate of the covariance matrix of the adjusted parameters is given by:

$$\left[\sum_{i=1}^{p} (Q_i^T M_i^{-1} Q_i) + J^{-1} \right]^{-1} + B N B^T .$$

The subject of effects from errors in unadjusted parameters is a very controversial one and beyond the scope of this report. It

* Assumes no serial correlation in the observations.
See Appendix AA.

may be mentioned that in some circumstances an unbiased and better solution is obtained by adding the unadjusted parameter effects directly into M_1, in which case the first factor in the equation for F will be the covariance matrix of adjusted parameters including the effects from unadjusted parameters. Numerical as well as theoretical difficulties are reduced appreciably if unadjusted parameters can be avoided by adjusting for all influential parameters.

A problem frequently encountered in least squares adjustment is high correlation among errors in the adjusted parameters. Methods for treating non-orthogonal problems of this type have been investigated at length at this test range. Ridge regression has been found to be a satisfactory solution to the problem of correlated parameters and is discussed in Appendix AA.

Finally, it should be pointed out that random errors in the observations have a harmful influence on the accuracy of the adjustments and should be controlled as far as possible.

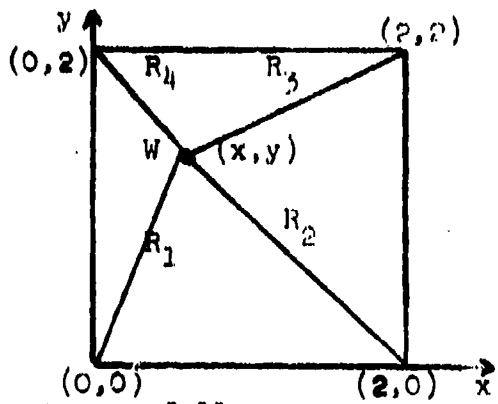
Type (2)

This type is similar to Type (1) and uses the same basic equation, but has one important difference. The observations entering into each least squares adjustment must all be at the same time point, and of course no least squares adjustment can be performed at any time point for which redundant data are not available. In a powered-flight missile trajectory, for example, an independent adjustment of missile position is made at each time point. A simple example will demonstrate the application.

EXAMPLE:

Consider an array of 4 transponders, one at each corner of a square, two miles on a side. Assume a splash occurs at some point W, and at the instant of splash each transponder observes the range from the transponder site to the splash. Designate these range observations R_i. Let the corresponding standard deviations and variances in the observations be S_i and M_i, respectively.

Adopt an x y coordinate system as follows:



Numerical data are as follows:

Transponders	R_i	S_i	M_i
1	1.7	0.1	0.01
2	1.9	0.2	0.04
3	1.9	0.3	0.09
4	0.3	0.4	0.16

Data errors are uncorrelated. There are no other sources of error, and there is no a priori knowledge of splash location. Start with initial estimates: $x = 1$ and $y = 1$.

We have the following relations for the R's in terms of x and y:

$$R_1^2 = x^2 + y^2$$

$$R_2^2 = (x-2)^2 + y^2$$

$$R_3^2 = (x-2)^2 + (y-2)^2$$

$$R_4^2 = x^2 + (y-2)^2$$

From the initial estimates of x and y we compute the following values for the ranges:

$$R_1 = 1.4142$$
$$R_2 = 1.4142$$
$$R_3 = 1.4142$$
$$R_4 = 1.4142$$

Subtracting these from the observed values, we obtain residuals

$$G_1 = +.2858$$
$$G_2 = +.4858$$
$$G_3 = +.4858$$
$$G_4 = -1.1142$$

The weighted rms residuals are therefore

$$\tfrac{1}{2}\left\{\left(\frac{.2858}{.1}\right)^2 + \left(\frac{.4858}{.2}\right)^2 + \left(\frac{.4858}{.3}\right)^2 + \left(\frac{1.1142}{.4}\right)^2\right\}^{\frac{1}{2}} = 2.472 .$$

We now seek to reduce the weighted rms residuals by improving the estimates of x and y, using the iterative relation

$$F = \left[\sum_{i=1}^{4} Q_i^T M_i^{-1} Q_i\right]^{-1} \left[\sum_{i=1}^{4} Q_i M_i^{-1} G_i\right] .$$

From the definition of Q_i and from the expressions for R_i as a function of x and y,

$$Q_1 = \begin{bmatrix} \frac{\partial R_1}{\partial x} & \frac{\partial R_1}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{R_1} & \frac{y}{R_1} \end{bmatrix}$$

$$Q_2 = \begin{bmatrix} \frac{\partial R_2}{\partial x} & \frac{\partial R_2}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x-2}{R_2} & \frac{y}{R_2} \end{bmatrix}$$

$$Q_3 = \begin{bmatrix} \frac{\partial R_3}{\partial x} & \frac{\partial R_3}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x-2}{R_3} & \frac{y-2}{R_3} \end{bmatrix}$$

$$Q_4 = \begin{bmatrix} \frac{\partial R_4}{\partial x} & \frac{\partial R_4}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{x}{R_4} & \frac{y-2}{R_4} \end{bmatrix}$$

In numerically evaluating the partials, the most recent computed values for R_i are used, along with the most recent estimates for x and y. We now evaluate $Q_i^T M_i^{-1} Q_i$ and sum them:

$$\frac{1}{R_1 M_1}\begin{bmatrix} x \\ y \end{bmatrix} Q_1 = \begin{bmatrix} +20.21 \\ +20.21 \end{bmatrix}$$

$$\frac{1}{R_2 M_2}\begin{bmatrix} x-2 \\ y \end{bmatrix} Q_2 = \begin{bmatrix} -\ 8.59 \\ +\ 8.59 \end{bmatrix}$$

$$\frac{1}{R_3 M_3}\begin{bmatrix} x-2 \\ y-2 \end{bmatrix} Q_3 = \begin{bmatrix} -\ 3.82 \\ -\ 3.82 \end{bmatrix}$$

$$\frac{1}{R_4 M_4}\begin{bmatrix} x \\ y-2 \end{bmatrix} Q_4 = \begin{bmatrix} -\ 4.93 \\ +\ 4.93 \end{bmatrix}$$

$$\sum_{i=1}^{4} Q_i M_i^{-1} Q_i = \begin{bmatrix} +\ 2.87 \\ +29.91 \end{bmatrix}$$

Similarly $Q_i^T M^{-1} Q_i$ may be evaluated and summed:

$$\frac{1}{R_1^2 M_1}\begin{bmatrix} x^2 & xy \\ xy & y^2 \end{bmatrix} = \begin{bmatrix} +50.001 & +50.001 \\ +50.001 & +50.001 \end{bmatrix}$$

$$\frac{1}{R_2^2 M_2}\begin{bmatrix} (x-2)^2 & y(x-2) \\ y(x-2) & y^2 \end{bmatrix} = \begin{bmatrix} +12.500 & -12.500 \\ -12.500 & +12.500 \end{bmatrix}$$

$$\frac{1}{R_3^2 M_3}\begin{bmatrix} (x-2)^2 & (x-2)(y-2) \\ (x-2)(y-2) & (y-2)^2 \end{bmatrix} = \begin{bmatrix} +5.556 & +5.556 \\ +5.556 & +5.556 \end{bmatrix}$$

$$\frac{1}{R_4^2 M_4}\begin{bmatrix} x^2 & x(y-2) \\ x(y-2) & (y-2)^2 \end{bmatrix} = \begin{bmatrix} +3.125 & -3.125 \\ -3.125 & +3.125 \end{bmatrix}$$

$$\sum_{i=1}^{4} Q_i^T M_i^{-1} Q_i = \begin{bmatrix} +71.182 & +39.932 \\ +39.932 & +71.182 \end{bmatrix}$$

The inverse is

$$\left[\sum_{i=1}^{4} Q_i^T M_i^{-1} Q_i\right]^{-1} = \begin{bmatrix} +.020500 & -.011500 \\ -.011500 & +.020500 \end{bmatrix}$$

Therefore

$$F = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} +.020500 & -.011500 \\ -.011500 & +.020500 \end{bmatrix} \begin{bmatrix} +2.87 \\ +29.91 \end{bmatrix} = \begin{bmatrix} -.285 \\ +.580 \end{bmatrix}$$

The new estimates for x and y are

$$x = 1 - .285 = +.715$$
$$y = 1 + .580 = +1.580$$

This completes the first iteration and we start the second iteration. With these latest estimates of x and y, the new values for R_i are

$$R_1 = 1.734$$

$$R_2 = 2.037$$

$$R_3 = 1.352$$

$$R_4 = .829$$

The residuals are

$$G_1 = -.034$$

$$G_2 = -.137$$

$$G_3 = +.548$$

$$G_4 = -.529$$

The weighted rms is now 1.191, down from the previous 2.472. The partials are reevaluated and it is found that

$$\sum_{i=1}^{4} Q_i^T M_i^{-1} G_i = \begin{bmatrix} -7.831 \\ -5.972 \end{bmatrix}$$

and

$$\left[\sum_{i=1}^{4} Q_i^T M_i^{-1} Q_i\right]^{-1} = \begin{bmatrix} +.030792 & -.009286 \\ -.009286 & +.012726 \end{bmatrix}$$

from which

$$F = \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} -.187 \\ -.003 \end{bmatrix}$$

The new estimates for x and y are then

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} +.715 - .187 \\ +1.580 - .003 \end{bmatrix} = \begin{bmatrix} + .528 \\ +1.577 \end{bmatrix}$$

We will only start the third iteration to obtain the weighted rms residuals. Computed ranges and residuals are

$$R_1 = 1.663 \qquad G_1 = + .037$$

$$R_2 = 2.157 \qquad G_2 = - .257$$

$$R_3 = 1.532 \qquad G_3 = + .368$$

$$R_4 = .677 \qquad G_4 = - .377$$

These give a weighted rms of 1.022 down from 1.191. In a real situation these iterations would continue until the reduction in weighted rms becomes trivial. Normally the final weighted rms will be approximately unity. When convergence has finally been achieved, final estimates for x and y will be obtained, and the matrix

$$\left[\sum_{i=1}^{4} Q_i^T M_i^{-1} Q_i\right]^{-1}$$

will represent the covariance matrix for x and y if there are no influential erroneous unadjusted parameters.

Type (3)
A powered-flight missile trajectory computation is seldom as simple as Type (2). The type (2) adjustment represents what is commonly called an "inner" iteration. Having made a trial estimate of the trajectory by an inner iteration, one can then temporarily fix these time point positions and then perform an "outer" iteration to estimate parameters such as radar error model coefficients and survey errors which are common to two or more of the trajectory points. This outer iteration represents a type (1) adjustment over all time points with the important exception that Q does not contain partial derivatives of computed measurements with respect to orbital parameters, and consequently F includes only corrections for survey and radar error coefficients. In the adjustment process inner iterations and outer iterations are alternated optimally until convergence is reached. The final inner iteration gives the covariance matrix for each position time point (usually in a topocentric x y z system). The final outer iteration gives the covariance matrix for the adjusted parameters.

Type (4)
This uses the same basic equation as the first three types, but here it becomes considerably simplified because the covariance matrix of the observations is omitted. We no longer need weighted observations or weighted residuals since all observations can be fitted to any desired degree of precision, and residuals can be driven as near zero as desired. J^{-1} is also set to zero since it will not influence the final solution and it will reduce the rate of convergence. Also omitting the summations, which are no longer pertinent, and realizing that Q will be square and presumably have an inverse, we have simply

$$F = \left[Q^T Q\right]^{-1} \left[Q^T G\right] = Q^{-1} G .$$

This simple and rather obvious iterative relationship can be quite useful.

For example, given a position point $(X, Y, Z)_0$ at time t_0 and another point $(X, Y, Z)_1$ at time t_1, it is possible to derive a free-fall trajectory between the points in a few iterations. The solution involves repetitive corrections of the estimated velocity vector $(\dot{X}, \dot{Y}, \dot{Z})_o$ at time t_o. In the process the partial derivative matrix Q must be numerically evaluated, where

$$Q = \begin{bmatrix} \frac{\partial X_1}{\partial \dot{X}_o} & \frac{\partial X_1}{\partial \dot{Y}_o} & \frac{\partial X_1}{\partial \dot{Z}_o} \\ \frac{\partial Y_1}{\partial \dot{X}_o} & \frac{\partial Y_1}{\partial \dot{Y}_o} & \frac{\partial Y_1}{\partial \dot{Z}_o} \\ \frac{\partial Z_1}{\partial \dot{X}_o} & \frac{\partial Z_1}{\partial \dot{Y}_o} & \frac{\partial Z_1}{\partial \dot{Z}_o} \end{bmatrix}$$

These derivatives are obtained by incrementing $\dot{X}_o$, $\dot{Y}_o$ and $\dot{Z}_o$ individually and integrating to time t_1 and differencing the observed (X,Y,Z) at time t_1 with the computed (X,Y,Z) at time t_1. The resulting Q matrix together with residual matrix G obtained by straight integration of most recent estimate of vector at t_o gives a correction F to the velocity components at t_o.

Another application occurs in reentry trajectory computations. Here we are given a vector $(X,Y,Z,\dot{X},\dot{Y},\dot{Z})$ at reentry and an impact latitude ϕ and longitude ψ at time t. The object is to adjust the lift components and the drag factor so that the resulting computed trajectory will go through the a priori impact point at the right time and thus establish a mean trajectory.

The drag is a vector tangent to the trajectory and is equal to the product of dynamic pressure and the drag parameter D. The lift is a vector perpendicular to the trajectory and is equal to the product of dynamic pressure and the lift parameter L. An orientation angle θ defines the direction of the lift in the plane perpendicular to the drag vector. We assume L and θ to be constant. We assume an a priori D curve to be relatively correct and we will compute a constant factor D_f such that the corrected D is equal to the product of D_f and a priori D. The problem is to compute D_f, L and θ. The solution converges better, however, if we adjust D_f, L sin θ and L cos θ. The iterative form is then

$$\begin{bmatrix} D_f \\ L\cos\theta \\ L\sin\theta \end{bmatrix}_{k+1} = \begin{bmatrix} D_f \\ L\cos\theta \\ L\sin\theta \end{bmatrix}_k + \begin{bmatrix} \frac{\partial\phi}{\partial D_f} & \frac{\partial\phi}{\partial(L\cos\theta)} & \frac{\partial\phi}{\partial(L\sin\theta)} \\ \frac{\partial\gamma}{\partial D_f} & \frac{\partial\gamma}{\partial(L\cos\theta)} & \frac{\partial\gamma}{\partial(L\sin\theta)} \\ \frac{\partial t}{\partial D_f} & \frac{\partial t}{\partial(L\cos\theta)} & \frac{\partial t}{\partial(L\sin\theta)} \end{bmatrix}_k^{-1} \begin{bmatrix} \phi_T - \phi_k \\ \gamma_T - \gamma_k \\ t_T - t_k \end{bmatrix}$$

In this formula (ϕ_T, γ_T, t_T) represent the true latitude, longitude and time of impact. The subscript k denotes the iteration number. Given the results from iteration k, the formula gives a new estimate for the (k+1) iteration. The partials are computed numerically by integrating separately with incremental values of D_f, L cos θ, and L sin θ and observing the incremental changes in ϕ, γ, t. Conditions are very nearly linear and convergence is rapid.

In the simple examples given in this appendix the partial derivatives have been obtained analytically or by finite difference. In more complex situations it is common practice to use variational equations. These will be discussed briefly in the next section.

VARIATIONAL EQUATIONS

Variational equations are not a substitute for analytical partial derivatives. Analytical partials are always preferable when mathematically feasible, and they are feasible for such parameters as radar error model coefficients and survey errors. On the other hand it is not mathematically possible to derive analytical partials for parameters such as orbital elements, aerodynamic coefficients and gravity anomalies. For parameters of this latter type one can use finite difference methods or variational equations. There are advantages and disadvantages associated with the finite difference method, but modern complex adjustment programs almost without exception use variational equations instead of finite difference methods to obtain the required partial derivatives. Variational equations are ordinary differential equations containing as dependent variables the partial derivatives themselves. In the usual application the variational equations are numerically integrated simultaneously with the differential equations of motion.

The variational equations are nothing more than the partial derivatives of the total acceleration equation with respect to the adjustable parameters. Normally there will be three second order (corresponding to six first order) differential equations for each scalar parameter to be adjusted. Let the total acceleration be expressed functionally by

$$\ddot{\underline{r}} = \ddot{\underline{r}}\,(\underline{r}, \dot{\underline{r}}, \underline{\psi})$$

where $\underline{r}$ is a position vector with first and second derivatives $\dot{\underline{r}}$ and $\ddot{\underline{r}}$ respectively and where $\underline{\psi}$ is a vector of explicit and/or implicit parameters in the equations of motion. In general, $\underline{r}$ and its first derivative are also functions of $\underline{\psi}$. The variational equation for some parameter ψ of $\underline{\psi}$ is expressed as

$$\frac{\partial \ddot{\underline{r}}}{\partial \psi} = \left[\frac{\partial \ddot{\underline{r}}}{\partial \underline{r}}\right] \frac{\partial \underline{r}}{\partial \psi} + \left[\frac{\partial \ddot{\underline{r}}}{\partial \dot{\underline{r}}}\right] \frac{\partial \dot{\underline{r}}}{\partial \psi} + \left[\frac{\partial \ddot{\underline{r}}\,(\underline{r}, \dot{\underline{r}}, \underline{\psi})}{\partial \psi}\right] .$$

The terms in brackets are determined analytically. The last term, called the non-homogeneous term, is usually the most difficult to compute. Because ψ is independent of time and because the derivatives are continuous, the order of differentiation in this latest equation may be interchanged to obtain

$$\frac{\partial^2}{\partial t^2}\left(\frac{\partial \underline{r}}{\partial \psi}\right) = \left[\frac{\partial \ddot{\underline{r}}}{\partial \underline{r}}\right]\left(\frac{\partial \underline{r}}{\partial \psi}\right) + \left[\frac{\partial \ddot{\underline{r}}}{\partial \dot{\underline{r}}}\right]\frac{\partial}{\partial t}\left(\frac{\partial \underline{r}}{\partial \psi}\right) + \left[\frac{\partial \ddot{\underline{r}}(\underline{r},\dot{\underline{r}},\underline{\psi})}{\partial \psi}\right].$$

A double integration of this equation with respect to time then yields the trajectory partial derivatives

$$\frac{\partial \underline{r}}{\partial \psi} ,$$

which express the sensitivity of the computed position points to variation in the parameter ψ .

GLOSSARY FOR APPENDIX S

A	radar azimuth angle. Defined by two vectors lying in the local horizontal plane and originating at the radar gimbals. The reference vector points northward in the plane and the other vector is a projection of the radius vector extending toward the tracked object. Angle is positive eastward from north.
B	matrix of partial derivatives of adjusted parameters with respect to unadjusted parameters.
C_{nm}	dimensionless spherical harmonic coefficient in geopotential expression having degree n and order m.
D	drag parameter of satellite or missile equal to product of drag coefficient and cross sectional area divided by twice the mass.
D_f	drag factor.
E	radar elevation angle. Measured positively in vertical plane from local horizontal plane upward to radius vector to tracked object.
$\underset{j\times 1}{F}$	column vector of corrections to be added to most recent estimate of adjusted parameters.
$\underset{3\times 1}{G_i}$	column vector of measurement residuals, defined as actual minus computed, for the i^{th} observation.
$\underset{j\times 1}{H}$	column vector of differences between a priori and current estimates of the parameters, defined as a priori minus current.
$\underset{j\times j}{J}$	a priori covariance matrix of parameters to be adjusted.
L	lift parameter of missile, equal to product of lift coefficient and cross sectional area divided by twice the mass.
$\underset{3\times 3}{M_i}$	covariance matrix of measurements at i^{th} observation.
N	covariance matrix of unadjusted parameters.
$P_{nm}(\sin \phi)$	associated Legendre polynomial of degree n and order m with argument $\sin \phi$.
$\underset{3\times j}{Q_i}$	matrix of partial derivatives of the computed measurements with respect to the adjusted parameters for the i^{th} observation.

R	range. Distance from radar or other sensor to tracked object.
S	estimated standard deviation in observation.
S_{nm}	dimensionless spherical harmonic coefficient in geopotential expression having degree n and order m.
T	matrix transpose if used as superscript. "True" if subscript.
U	earth gravitational potential.
V	satellite or missile velocity (scalar).
W	splash point.
X, Y, Z	position coordinates of tracked object in a geocentric earth fixed rectangular right handed coordinate system with X and Y axes in equatorial plane, X extending through meridian of Greenwich and Y through 90° east longitude. Z points northward along spin axis.
$\dot{X}, \dot{Y}, \dot{Z}$	velocity components of missile or satellite in XYZ coordinate system.
$\ddot{X}, \ddot{Y}, \ddot{Z}$	acceleration components of missile or satellite in XYZ coordinate system.
$\hat{a}$	earth equatorial radius.
i	observation index.
j	number of adjusted parameters including those which describe the trajectory and the radar error coefficients.
k	iteration number.
n, m	degree and order respectively in spherical harmonic expression for geopotential.
p	total number of observations in type (1) adjustment. Total number of observations at a particular time point in type (2) adjustment.
r	distance from center of earth to missile or satellite.
t	time.
x, y, z	topocentric rectangular coordinate system.
Σ	summation.
λ	east longitude from Greenwich.

μ	earth gravitational constant, equal to product of Newton's constant and earth mass.
ρ	atmosphere density.
ϕ	geocentric latitude.
θ	orientation angle defining direction of lift in plane perpendicular to velocity vector.
ω	earth rotation rate.
Δ	increment.
(-1)	used as superscript to denote matrix inverse.
ψ	adjustable parameter.
(-)	underline. Vector.

APPENDIX T

NOTES ON THE EARTH'S POTENTIAL FUNCTION

1.0 DEVELOPMENT OF ALTERNATIVE FORMS

From Newton's Universal Law of Gravitation it is possible to show that gravity represents a conservative field of force, and hence the work done by the gravitational field on a particle moving from one point to another is independent of the path taken. In astronomical convention the total work done by the gravitational force while a particle of unit mass moves from infinity in to some point P is the gravitational potential at P. Potential is a scalar quantity. The gradient of the potential is a vector representing the gravitational force per unit mass or acceleration. The potential outside the earth satisfies Laplace's Equation. Laplace's Equation is a second order linear homogeneous differential equation. Particular solutions of Laplace's Equation are known as spherical harmonics, and the most general solution can be reached by their addition. In mathematical treatment of the gravity potential (also called gravipotential or geopotential) it is convenient to use spherical coordinates r, β, and γ, where the origin is at the mass center; r is distance from geocenter to P; β is complement of geocentric latitude of P; γ is east longitude from Greenwich. In spherical coordinates Laplace's Equation takes the form:

$$r \frac{\partial^2 (rU)}{\partial r^2} + \frac{1}{\sin\beta}\frac{\partial}{\partial\beta}\left(\sin\beta \frac{\partial U}{\partial\beta}\right) + \frac{1}{\sin^2\beta}\frac{\partial^2 U}{\partial\gamma^2} = 0 \quad ,$$

where U represents the geopotential. The general solution to this equation expressed in the form recommended by the International Astronomical Union in 1962 is

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left(\frac{\hat{a}}{r}\right)^{n} P_{n,m}(\sin\phi)\left(C_{n,m}\cos m\gamma + S_{n,m}\sin m\gamma\right)\right]$$

where μ represents the central gravitational parameter; n and m are non-negative indices representing degree and order respectively; $\hat{a}$ is earth equatorial radius; ϕ is geocentric latitude;

$C_{n,m}$ and $S_{n,m}$ are experimentally determined coefficients; $P_{n,m}$ $(\sin\phi)$ is an associated Legendre polynomial. A simple Legendre polynomial $P_n(x)$, where x is equal to $\sin \Phi$ in this case, may be defined by Rodriques' formula:

$$P_n(x) = \frac{1}{2^n\ n!} \frac{d^n}{dx^n} (x^2-1)^n .$$

The corresponding associated Legendre polynomial of m^{th} order and n^{th} degree is

$$P_{n,m}(x) = (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_n(x).$$

Note that for $m = 0$, $P_{n,m}$ (x) reduces to P_n (x). Also note that for $m > n$, $P_{n,m}$ (x) reduces to zero. See Section 3.0 for recursive methods in computing Legendre associated polynomials.

These functions are incidentally related to Gegenbauer polynomials:

$$T^m_{n-m} (x) = \frac{d^m}{dx^m} P_n (x)$$

and

$$T_n^0 (x) = P_n (x) .$$

If we attempt to evaluate U or the accelerations associated with U, we note that although the products of the associated Legendre polynomials and their corresponding numerical coefficients decline only moderately with increasing degree and order, the associated Legendre polynomials increase greatly in value and the numerical coefficients decrease greatly. This circumstance may lead to loss in computational accuracy and also to a misinterpretation of the importance of individual numerical coefficients. To improve this situation, it has become standard practice to multiply the numerical coefficients by normalizing factors such that the resulting normalized coefficients have values approximately proportional to their "effects"; the corresponding associated Legendre polynomials are of course divided by these same factors in order to satisfy the equation for U.

Consider the following four products extracted from the equation for U:

$$P_{n,m}\ (\sin\phi)\ [C_{n,m}\ \cos m\ \gamma]\ \text{with } m \neq 0,$$

$$P_{n,m}\ (\sin\phi)\ [S_{n,m}\ \sin m\ \gamma]\ \text{with } m \neq 0,$$

$$P_{n,m}\ (\sin\phi)\ [C_{n,m}\ \cos m\ \gamma]\ \text{with } m = 0,$$

$$P_{n,m}\ (\sin\phi)\ [S_{n,m}\ \sin m\ \gamma]\ \text{with } m = 0.$$

The fourth product is clearly zero at all points on the earth's surface and hence is of no interest in regard to normalization. If we compute the root mean square over a unit sphere for each of the other three products—considering this to be a measure of their effects—we obtain respectively:

$$N\ C_{n,m}\ ,$$

$$N\ S_{n,m}\ ,$$

$$N\ C_{n,m}\ ,$$

where

$$N = \sqrt{\frac{(n+m)!}{(n-m)!\,(2n+1)\,\chi}}$$

and where $\chi = 1$ for $m = 0$, and $\chi = 2$ for $m \neq 0$. Consequently, this factor N is commonly used for normalization. To change conventional $C_{n,m}$ and $S_{n,m}$ to normalized $\bar{C}_{n,m}$ and $\bar{S}_{n,m}$:

$$\bar{C}_{n,m} = N\ C_{n,m}\ ,$$

$$\bar{S}_{n,m} = N\ S_{n,m}\ .$$

This change necessitates a corresponding and desirable change in $P_{n,m}$ (sinϕ):

$$\bar{P}_{n,m}\ (\sin\phi) = (1/N)\ P_{n,m}\ (\sin\phi).$$

As a consequence, $\overline{P}_{n,m}$, $\overline{C}_{n,m}$ and $\overline{S}_{n,m}$ do not show the strong trends observed in $P_{n,m}$, $C_{n,m}$ and $S_{n,m}$; and the values of $\overline{C}_{n,m}$ and $\overline{S}_{n,m}$ are approximately proportional to their effects.

The type of normalization just described is called "full normalization" or "Kaula normalization." There are alternative forms used for simplicity. One such form uses

$$N = \sqrt{\frac{(n+m)!}{(n-m)!\chi}} ,$$

which is called simply "normalization".

Another form uses

$$N = \sqrt{\frac{(n+m)!}{(n-m)!}} ,$$

which is called APL normalization.

The literature does not always specify which form of normalization is used, and the reader may be required to take a given numerical normalized coefficient and unnormalize it by various methods to compare with a known unnormalized value.

Table I gives the Legendre polynomials and associated Legendre polynomials up through $(n,m = 6, 6)$. Table II gives the corresponding normalization factors. $P_{n,m}(x)$ is called a zonal harmonic when $(n > m = 0)$. $P_{n,m}(x)$ is called a sectorial harmonic when $(n = m > 0)$. $P_{n,m}(x)$ is called a tesseral harmonic when $(n > m > 0)$.

For our applications

$$P_{n,m}(x) \text{ is equivalent to } P_{n,m}(\sin\phi).$$

Therefore, for example, in Table I

$$P_{1,1}(x) = P_{1,1}(\sin\phi) = \cos\phi$$

$$P_{3,2}(x) = P_{3,2}(\sin\phi) = 15 \sin\phi \cos^2\phi$$

It should be noted that to avoid the use of a comma, many writers use

$$P_n^m\ (x) \text{ instead of } P_{n,m}\ (x).$$

Also where there is no risk of confusion, many writers use

$$P_{nm}\ (x) \text{ instead of } P_{n,m}\ (x).$$

These variations simply represent differences in notation.

For purposes of this discussion, the geopotential form recommended by the International Astronomical Union in 1962 will be referred to as the standard form and all others will be compared to it. An obvious variation of the standard form is

$$U = \frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n P_{n,m}\ (\sin\phi)(C_{n,m} \cos m\,\gamma + S_{n,m} \sin m\,\gamma)\ .$$

In this form $C_{0,0}$ is defined to be unity.* All other symbols retain the same meaning and have the same values that they have in the standard form. Another obvious variation of the standard form is

$$U = \frac{\mu}{r} \left\{ 1 + \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \left[C_n\ P_n\ (\sin\phi) + \sum_{m=1}^{n} (C_{n,m} \cos m\,\gamma + S_{n,m} \sin m\,\gamma)\ P_{n,m}\ (\sin\phi) \right] \right\},$$

where the first summation is over zonals only.

Some publications prefer the (J, K) notation in place of the (C, S) notation. The equations for U are the same as those previously given with the exception that a negative sign replaces

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*In 1963 Kaula recommended that μ be defined as 0.3986032×10^{21} cm^3/sec^2 and that $C_{0,0}$ be permitted to depart from unity. This recommendation has not gained general acceptance.

TABLE I

LEGENDRE POLYNOMIALS

AND

ASSOCIATED LEGENDRE POLYNOMIALS

n	m	$P_n^m(x)$
0	0	1
1	0	x
1	1	$(1-x^2)^{1/2}$
2	0	$(3x^2-1)/2$
2	1	$3x(1-x^2)^{1/2}$
2	2	$3(1-x^2)$
3	0	$(5x^3-3x)/2$
3	1	$(1-x^2)^{1/2}\ (15x^2-3)/2$
3	2	$15x(1-x^2)$
3	3	$15(1-x^2)^{3/2}$
4	0	$(35x^4-30x^2+3)/8$
4	1	$(1-x^2)^{1/2}\ (35x^3-15x)/2$
4	2	$(1-x^2)\ (105x^2-15)/2$
4	3	$105x(1-x^2)^{3/2}$
4	4	$105(1-x^2)^2$
5	0	$(63x^5-70x^3+15x)/8$
5	1	$(1-x^2)^{1/2}\ (315x^4-210x^2+15)/8$
5	2	$(1-x^2)\ (315x^3-105x)/2$
5	3	$(1-x^2)^{3/2}\ (945x^2-105)/2$
5	4	$945x(1-x^2)^2$
5	5	$945(1-x^2)^{5/2}$

TABLE I (Contd)

n	m	$P_n^m(x)$
6	0	$(231x^6-315x^4+105x^2-5)/16$
6	1	$(1-x^2)^{1/2}\ (693x^5-630x^3+105x)/8$
6	2	$(1-x^2)\ (3465x^4-1890x^2+105)/8$
6	3	$(1-x^2)^{3/2}\ (3465x^3-945x)/2$
6	4	$(1-x^2)^2\ (10395x^2-945)/2$
6	5	$10395x(1-x^2)^{5/2}$
6	6	$10395(1-x^2)^3$

TABLE II

VARIOUS NORMALIZATION FACTORS

n	m	APL Normalization	Normalization	Full Normalization
0	0	1.000000	1.000000	1.000000
1	0	1.000000	1.000000	0.5773503
1	1	1.414214	1.000000	0.5773503
2	0	1.000000	1.000000	0.4472136
2	1	2.449490	1.732051	0.7745967
2	2	4.898979	3.464101	1.549193
3	0	1.000000	1.000000	0.3779645
3	1	3.464102	2.449490	0.9258202
3	2	10.95445	7.745966	2.927700
3	3	26.83282	18.97367	7.171373
4	0	1.000000	1.000000	0.3333333
4	1	4.472136	3.162278	1.054093
4	2	18.97367	13.41641	4.472137
4	3	70.99296	50.19960	16.73320
4	4	200.7984	141.9859	47.32863
5	0	1.000000	1.000000	0.3015113
5	1	5.477226	3.872984	1.167749
5	2	28.98275	20.49390	6.179143
5	3	141.9859	100.3992	30.27150
5	4	602.3952	425.9577	128.4311
5	5	1904.941	1346.997	406.1349
6	0	1.000000	1.000000	0.2773501
6	1	6.480741	4.582576	1.270978
6	2	40.98780	28.98275	8.038369
6	3	245.9268	173.8965	48.23021
6	4	1346.997	952.4707	264.1678
6	5	6317.974	4467.482	1239.057
6	6	21886.11	15475.82	4292.220

the positive sign ahead of the first summation. The last mentioned equation thus transforms to

$$U = \frac{\mu}{r}\left\{1 - \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \left[J_n P_n(\sin\phi) + \sum_{m=1}^{n} (J_{n,m} \cos m\,\gamma + K_{n,m} \sin m\,\gamma)\, P_{n,m}(\sin\phi)\right]\right\}.$$

As a result

$$C_{n,m} = -J_{n,m} \qquad \text{and} \qquad S_{n,m} = -K_{n,m} \;;$$

also $C_n = -J_n$.

A form frequently used in geodesy is

$$U = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{1}{r}\right)^{n+1} (A_{n,m} \cos m\,\gamma + B_{n,m} \sin m\,\gamma)\, P_{n,m}(\sin\phi) .$$

For this relation

$$C_{n,m} = A_{n,m}/(\mu\, a^n)$$

and $S_{n,m} = B_{n,m}/(\mu\, a^n)$.

Another form used by geodesists is

$$U = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^{n+1} (a_{n,m} \cos m\,\gamma + b_{n,m} \sin m\,\gamma)\, P_{n,m}(\sin\phi),$$

where

$$C_{n,m} = \left(\frac{a}{\mu}\right) a_{n,m}$$

and $S_{n,m} = \left(\frac{a}{\mu}\right) b_{n,m}$.

A form sometimes used by JPL is

$$U = \frac{\mu}{r}\left[1 - \frac{2}{3} J \left(\frac{a}{r}\right)^2 P_2(\sin\phi) - \frac{2}{5} H \left(\frac{a}{r}\right)^3 P_3(\sin\phi) + \frac{8}{35} D \left(\frac{a}{r}\right)^4 P_4(\sin\phi)\right] .$$

The conversion relations are

$$C_{2,0} = -\frac{2}{3} J$$

$$C_{3,0} = -\frac{2}{5} H$$

$$C_{4,0} = +\frac{8}{35} D$$

The coefficients in front of J, H, and D are actually combined with the Legendre polynomial, which is not explicitly given in their expression for U.

A form used by Sterne, Baker, Herrick and others is as follows:

$$U = \frac{\mu}{r}\left[1 - \frac{2}{3} J\left(\frac{a}{r}\right)^2 P_2(\sin\phi) - \frac{2}{5} H\left(\frac{a}{r}\right)^3 P_3(\sin\phi) + \frac{4}{15} K\left(\frac{a}{r}\right)^4 P_4(\sin\phi)\right] .$$

The conversion relations are

$$C_{2,0} = -\frac{2}{3} J ,$$

$$C_{3,0} = -\frac{2}{5} H ,$$

$$C_{4,0} = +\frac{4}{15} K .$$

The coefficients in front of J, H, and K are actually combined with the Legendre polynomial, which is not given explicitly in their expression for U.

A form used in an older ETR computer program is

$$U = \frac{\mu}{r}\left[1 - \frac{\alpha}{3r^2} P_2(\sin\phi) + \frac{\beta}{5r^4} P_4(\sin\phi)\right] .$$

The conversion relations are

$$C_{2,0} = -\alpha/(3a^2)$$

$$C_{4,0} = +\beta/(5r^4)$$

Again the constant coefficients in the denominators of the terms containing α and β are actually combined with the Legendre polynomial, which is not explicitly given in the original expression for U. Note that α and β are not dimensionless.

We now consider forms which make use of amplitude and phase angle. A form sometimes used by Smithsonian and STL is as follows:

$$U = \frac{\mu}{r}\left\{1 + \sum_{n=1}^{\infty} \sum_{m=0}^{n} \left(\frac{a}{r}\right)^n J_{n,m} P_{n,m} (\sin\phi) \cos\left[m(\gamma - \gamma_{n,m})\right]\right\} .$$

The conversion relations are

$$C_{n,m} = J_{n,m} \cos (m\,\gamma_{n,m}) ,$$

$$S_{n,m} = J_{n,m} \sin (m\,\gamma_{n,m}) .$$

Conversely, for $m > 0$,

$$J_{n,m} = +\sqrt{C_{n,m}^{\ 2} + S_{n,m}^{\ 2}}$$

$$\gamma_{n,m} = \frac{1}{m} \tan^{-1} \left(\frac{S_{n,m}}{C_{n,m}}\right) ,$$

where $\tan^{-1}$ is evaluated in region $0^\circ \pm 180^\circ$.

For m = 0,

$$J_{n,0} = C_{n,0} .$$

A form used by JPL, SCF and also by ETR and recommended by International Geophysical Union is

$$U = \frac{\mu}{r} \left\{ 1 - \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^n \left[J_n P_n (\sin\phi) \right.\right.$$

$$\left.\left. - \sum_{m=1}^{n} J_{n,m} P_{n,m} (\sin\phi) \cos\left\{ m (\gamma - \gamma_{n,m}) \right\} \right] \right\} .$$

The conversion relations for $(m > 0)$ are

$$C_{n,m} = J_{n,m} \cos (m \gamma_{n,m}) ,$$

$$S_{n,m} = J_{n,m} \sin (m \gamma_{n,m}) ,$$

$$J_{n,m} = +\sqrt{C_{n,m}^{2} + S_{n,m}^{2}} ,$$

$$\gamma_{n,m} = \frac{1}{m} \tan^{-1} \left(\frac{S_{n,m}}{C_{n,m}}\right) ,$$

where $\tan^{-1}$ is evaluated in region 0° ± 180° .

For m = 0

$$J_{n,0} = -C_{n,0} .$$

All these various forms clearly lead to confusion. Since none of the variant forms seems to have any advantage over the standard form, the standard form is becoming almost universally accepted. The same normalization formulas discussed in connection with the standard form apply also to the variant forms. Wherever the index m is omitted from a coefficient or polynomial in the previous discussion, m is zero. Also when m is zero, $S_{n,m}$, $K_{n,m}$ and $\gamma_{n,m}$ may be set to zero.

A great deal of confusion exists over the inter-relationships between the geopotential model and the earth spheroid. The geopotential U describes the gravitational potential outside the earth and is referenced to a coordinate system with origin at the dynamical or mass center. The geoid is an equipotential surface described by the combination of gravitational and rotational potential. This combined potential W is given by

$$W = U + \frac{1}{2} \omega_e^2 r^2 \cos^2 \phi ,$$

where ω_e is earth rotation rate and where W is chosen such that the corresponding equipotential surface corresponds most closely to the mean surface of the free oceans. The geoid is approximated by a prolate spheroid which most closely fits the geoid and whose center coincides with the mass center of the earth. Spheroids determined since 1966 have centers agreeing within 15 meters of each other and semi-axes agreeing within about 10 meters. It is common practice to assume that all spheroids have parallel rectangular axes. Possible small tilts in the datums are presently ignored at AFETR.

When a laboratory establishes a geopotential model and accompanying spheroid, it simultaneously arrives at the coordinates for all the tracking sites used in the adjustment. These site locations, the geopotential model and the spheroid constitute a compatible set. Furthermore, for purposes of compatibility, shifts are determined which permit the transfer of sites located on other spheroids to the spheroid associated with the geopotential model. (See Appendix D.)

Consequently an ideal situation, and one which is generally attainable, is to use a spheroid consistent with the geopotential model and express all site locations on that spheroid. Sometimes, however, for some reason it is desirable to use a spheroid different from that associated with the geopotential model. This practice is generally satisfactory if the extraneous spheroid has a center within 15 meters of that associated with the geopotential model and if the values of R agree to 1 ppm. If the spheroid differences lie outside these limits, errors develop which usually cannot be ignored but which can in large part be corrected. The largest source of error and one which most computer programs can correct is in the gravitational potential for an off-center spheroid. It is necessary to add appropriate values of $C_{1,0}$, $C_{1,1}$ and $S_{1,1}$ to the geopotential model as described in Section 2.0. Inclusion of these terms takes care of perhaps 90% of the total error, but sometimes the remaining 10% is still intolerably large. The remaining error (generally incorrectible) arises from false assumptions in regard to rotational characteristics of the earth and its atmosphere and from an erroneous reference surface for the atmosphere. Most of the datums and spheroids shown in Table I of Appendix D are unsatisfactory for trajectory equations of motion even with geopotential models modified by appropriate values of $C_{1,0}$, $C_{1,1}$ and $S_{1,1}$. For example, the use of NAD-27 on Clarke-1866 is not recommended for use with trajectory equations of motion.

Some of the older geopotential models have poor values for μ. It is good practice always to use $\mu = 398600.8 \pm 0.1\ km^3/sec^2$. This is the most accurate determination at present. Some trajectories and calibration exercises are extremely sensitive to the value used for μ.

One further point of possible concern is the earth's atmosphere. Strictly speaking, geopotential models determined from satellite data apply only outside the atmosphere. Even the use of Laplace's Equation inside the atmosphere is not strictly correct. Actually, at present accuracy levels, the effect of the atmosphere upon all the coefficients except μ is negligible. The atmosphere is responsible for 1 ppm of the published values for μ. It is hardly even worthwhile to correct μ for trajectories inside the atmosphere, however, since drag uncertainties overwhelm this small correction.

A typical modern* geopotential model is included in this appendix simply as an example. This model was developed at Goddard by a simple averaging of coefficients from many published models. It bears the ETR catalog number of SA-22. Evaluations here at ETR show it to be one of the more accurate unclassified models. This model is incidentally a truncation of a larger model. Many analysts have been reluctant to truncate geopotential models because of the correlations among the coefficients — correlations of perhaps 0.2. Recent studies by the author indicate that although truncation of a larger model is not an ideal method for obtaining a smaller model, it is acceptable as a practical measure. A simple and somewhat better method is to transform the larger model to a smaller model. Probably the simplest suitable transformation procedure is the following two-step process:
(1) Using an accurate, large geopotential model and the EFG coordinate system (Section VII), generate a series of trajectories of only those types to be used subsequently with the small model;
(2) Treating the EFG time points as observations and using an a priori small geopotential model obtained by truncation, perform a least-squares, multiple-arc fitting process, adjusting the geopotential coefficients in the small model. Unless the trajectories used in the data generation are uniformly distributed over the entire earth, the resulting small model will be biased by the transformation and must be used only over the pre-selected region used in the generation.

The bibliography contains a reference describing the accuracy of the coefficients in modern large geopotential models. The approach discussed there has been used to update error estimates to the 1973-75 era, and these updated estimates are shown in Table IV at the end of this appendix.

*Jan. 1970

2.0 MODIFICATION OF GEOPOTENTIAL MODELS FOR APPLICATION WITH OFF-CENTER EARTH SPHEROIDS

Occasionally a missile test program stipulates the use of an old spheroid whose center is considerably different from the center of mass of the earth. In powered-flight trajectory computations this stipulation presents no problem whatever. In free-flight trajectory computations using equations of motion, some corrective measures may be necessary.

An obvious approach is to transform the sites to a modern spheroid, compute the trajectory, and then transform the trajectory back to the old spheroid. An alternative approach, which has advantages in some circumstances, is to work entirely with the old spheroid but to modify the geopotential model to account for the discrepancy between the center of mass of the earth and the center of the coordinate system. This alternative approach will be described here.

Modern geopotential models do not contain terms in $C_{1,0}$, $C_{1,1}$ and $S_{1,1}$ because with coordinate systems centered at the mass center these terms are zero. These three coefficients have the following meaning: $C_{1,0}$ is the number of e.r.u.* from an equatorial plane through the center of mass to the origin of coordinates, positive in a southerly direction. $C_{1,1}$ is the number of e.r.u. from a meridional plane through 90° east and west longitudes and the center of mass to the origin of coordinates, positive on the side away from Greenwich. $S_{1,1}$ is number of e.r.u. from a meridional plane through Greenwich including the center of mass to the origin of coordinates, positive on west side of Greenwich. An example will be presented based on data given in Appendix D.

Assume the center of SAO-67 spheroid is exactly at the mass center of the earth. Then if Mercury-Fischer 1960 is used, the origin of the coordinate system relative to the center of mass is

*e.r.u. represents earth radius units.

(Table I, Appendix D):

$$E = -29 \text{ m}$$
$$F = +44 \text{ m}$$
$$G = -40 \text{ m}$$

Expressed in e.r.u., these are

$$E = -4.55 \times 10^{-6}$$
$$F = +6.90 \times 10^{-6}$$
$$G = -3.27 \times 10^{-6}$$

These correspond therefore to

$$C_{1,0} = +6.27 \times 10^{-6}$$
$$C_{1,1} = +4.55 \times 10^{-6}$$
$$S_{1,1} = -6.90 \times 10^{-6}$$

Corrective measures of this type in the geopotential are hardly ever worth making for discrepancies less than 15 meters between center of mass and center of coordinate system. Beyond 15 meters the error becomes increasingly important. An uncorrected error in center of coordinate system shows up as cyclic errors in the computed trajectory with amplitudes of the order of the missing $C_{1,0}$, $C_{1,1}$ and $S_{1,1}$, and with frequencies associated with the orbital periods and the rotation rate of the earth. In addition, there may be a secular error whose magnitude is dependent upon the size of the missing $C_{1,0}$, $C_{1,1}$ and $S_{1,1}$, and also upon the length of the trajectory covered by the tracking instrumentation.

As discussed earlier in this appendix, there are other errors associated with off-center spheroids, some of which are not readily correctable.

3.0 COMPUTATION OF ACCELERATION COMPONENTS

One of the most time-consuming operations in orbit determinations is computation of gravitational accelerations due to central body. Consequently, a great deal of experimentation has gone into optimizing the computations. Evaluations at Aerospace, ESSA, Data Dynamics, ETR and elsewhere generally agree that for small geopotential models (degree no higher than 4) it is preferable to program the acceleration components directly in rectangular coordinates. For larger models it is preferable to use recursive methods to arrive at acceleration components in an east-north-radial coordinate system. From here on there are minor differences in methods. Probably the simplest way to complete the transformation is to perform a rotation to the body centered rectangular system. This is the method presently used by Aerospace and ETR and is the one presented here.

The acceleration components in the geocentric earth fixed EFG coordinate system* are

$$\begin{bmatrix} \ddot{E} \\ \ddot{F} \\ \ddot{G} \end{bmatrix} = \begin{bmatrix} -\sin\gamma & -\sin\phi\cos\gamma & \cos\phi\cos\gamma \\ \cos\gamma & -\sin\phi\sin\gamma & \cos\phi\sin\gamma \\ 0 & \cos\phi & \sin\phi \end{bmatrix} \begin{bmatrix} \frac{1}{r\cos\phi}\frac{\partial U}{\partial\gamma} \\ \frac{1}{r}\frac{\partial U}{\partial\phi} \\ \frac{\partial U}{\partial r} \end{bmatrix}$$

*See Page 11 for description of EFG system and Page 10 for $\phi\gamma r$ system.

where

$$\frac{1}{r\cos\phi}\frac{\partial U}{\partial\gamma} = \frac{-\mu}{r^2\cos\phi}$$

$$\cdot\left[\sum_{n=1}^{n(max)}\left(\frac{\hat{a}}{r}\right)^n \sum_{m=0}^{n} m\,P_{nm}(\sin\phi)(C_{nm}\sin m\gamma - S_{nm}\cos m\gamma)\right]$$

$$\frac{1}{r}\frac{\partial U}{\partial\phi} = \frac{\mu}{r^2}$$

$$\cdot\left[\sum_{n=1}^{n(max)}\left(\frac{\hat{a}}{r}\right)^n \sum_{m=0}^{n} P'_{nm}(\sin\phi)\cos\phi\,(C_{nm}\cos m\gamma + S_{nm}\sin m\gamma)\right]$$

$$\frac{\partial U}{\partial r} = \frac{-\mu}{r^2}$$

$$\cdot\left[1 + \sum_{n=1}^{n(max)}(n+1)\left(\frac{\hat{a}}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\phi)(C_{nm}\cos m\gamma + S_{nm}\sin m\gamma)\right]$$

and where $P'_{nm}(\sin\phi)$ represents the derivative of the Legendre function with respect to $\sin\phi$.

Recursive formulas used in computing Legendre associated functions and their derivatives are as follows (where the argument $\sin\phi$ has been omitted but should be understood):

For m=0

$$P_n = \left[(2n-1)\sin\phi\, P_{n-1} - (n-1)\, P_{n-2}\right]/n$$

$$P'_n\cos\phi = \sin\phi\, P'_{n-1}\cos\phi + n\,P_{n-1}\cos\phi$$

For initial values use $P_0 = P'_1 = 1$; and $P_1 = \sin\phi$.

For $m \neq 0$

$$P_{nm}/\cos\phi = \left[(2n-1)\sin\phi\, P_{n-1,m}/\cos\phi - (n+m-1)P_{n-2,m}/\cos\phi\right]/(n-m)$$

$$P'_{nm}\cos\phi = (n+m)P_{n-1,m}/\cos\phi - n\sin\phi\, P_{nm}/\cos\phi$$

For initial values use $P_{m-1,m} = 0$ and

$$P_{m,m}/\cos\phi = 1\cdot 3 \cdots (2m-1)\cos^{m-1}\phi$$

1.0 RESONANCE EFFECTS

Resonance effects, primarily in-track, are observed in long-arc trajectories. They are most pronounced in polar or near-polar orbits and may show amplitudes up to a thousand feet or so depending upon the degree of resonance. As an example, a satellite with period of 6150 seconds will display resonance associated with the (17,14), (16,14), (15,14) and (14,14) geopotential coefficients; and numerical values for these co-efficients should be included in the geopotential model used with this satellite even though no other terms higher than (8,8) are included. Resonant periods and corresponding order of the harmonic coefficients are listed in Table III. Although every non-zonal term is resonant at some period, it is common practice to define "resonance terms" as those with $m \geq 12$. This parlance is implied in Table IV.

TABLE III

SATELLITE RESONANT PERIODS AND THE CORRESPONDING ORDER OF THE HARMONIC COEFFICIENTS

Period (Sec)			m	Period (Sec)			m
88760	to	83560	1	9860	to	9290	9
44380	to	41780	2	8880	to	8360	10
29590	to	27850	3	8070	to	7600	11
22190	to	20890	4	7400	to	6960	12
17750	to	16710	5	6830	to	6428	13
14790	to	13930	6	6341	to	5970	14
12680	to	11940	7	5917	to	5571	15
11100	to	10450	8	5148	to	5222	16

MODEL SA-22

(Multiply all $\bar{C}$ and $\bar{S}$ values by 10^{-6})

n	m	$\bar{C}$	$\bar{S}$
2	0	-484.1733	-
2	2	2.434	-1.398
3	0	0.9623	-
3	1	1.891	0.219
3	2	0.834	-.653
3	3	0.726	1.371
4	0	0.5497	-
4	1	-.550	- .452
4	2	0.316	0.604
4	3	0.915	- .121
4	4	-.125	0.212
5	0	0.0633	-
5	1	-.037	- .061
5	2	0.553	- .248
5	3	-.357	- .012
5	4	-.142	0.117
5	5	0.040	- .501
6	0	-.1792	-
6	1	-.086	0.056
6	2	0.027	- .329
6	3	0.051	0.079
6	4	-.090	- .459
6	5	-.220	- .501
6	6	-.072	- .261
7	0	0.0860	-
7	1	0.159	0.039
7	2	0.324	0.089
7	3	0.206	- .094
7	4	-.224	- .043
7	5	0.055	- .040
7	6	-.266	0.115
7	7	0.070	0.043

MODEL SA-22 (Contd)

n	m	$\bar{C}$	$\bar{S}$
8	0	0.0655	-
8	1	-.047	0.028
8	2	0.059	0.020
8	3	-.012	0.081
8	4	-.094	0.025
8	5	-.050	0.021
8	6	-.038	0.259
8	7	0.040	0.025
8	8	-.142	0.020
12	12	-.031	0.001
13	12	-.070	0.068
13	13	-.063	0.059
14	12	0.003	-.025
14	13	0.008	0.089
14	14	-.040	-.009
15	12	-.075	-.010
15	13	-.038	-.019
15	14	0.008	-.013
15	15	-.005	-.010
17	13	0.006	0.011

TABLE IV

AVERAGE ONE-SIGMA ERROR ESTIMATES FOR FULLY NORMALIZED GEOPOTENTIAL COEFFICIENTS

(σ VALUES SCALED BY 10^8)

Degree	Zonals	Non-Resonant Tesserals & Sectorials (Order: < 12)	Resonant Tesserals & Sectorials (Order: 12, 13, 14)
0	1 ppm	---	---
1	28.	17.	---
2	0.6	1.4	---
3	0.7	3.0	---
4	0.8	1.9	---
5	1.0	3.5	---
6	1.1	2.3	---
7	1.2	3.9	---
8	1.4	2.8	---
9	1.5	4.2	---
10	1.6	3.3	---
11	1.8	4.4	---
12	1.9	3.5	1.9
13	2.0	4.5	2.0
14	2.2	4.0	2.2
15	2.3	4.6	2.3
16	2.4	4.3	2.4
17	2.5	---	2.6
18	2.7	---	2.7
19	2.8	---	2.8
20	2.9	---	3.0
21	3.1	---	3.1
22	3.2	---	3.2

APPENDIX U

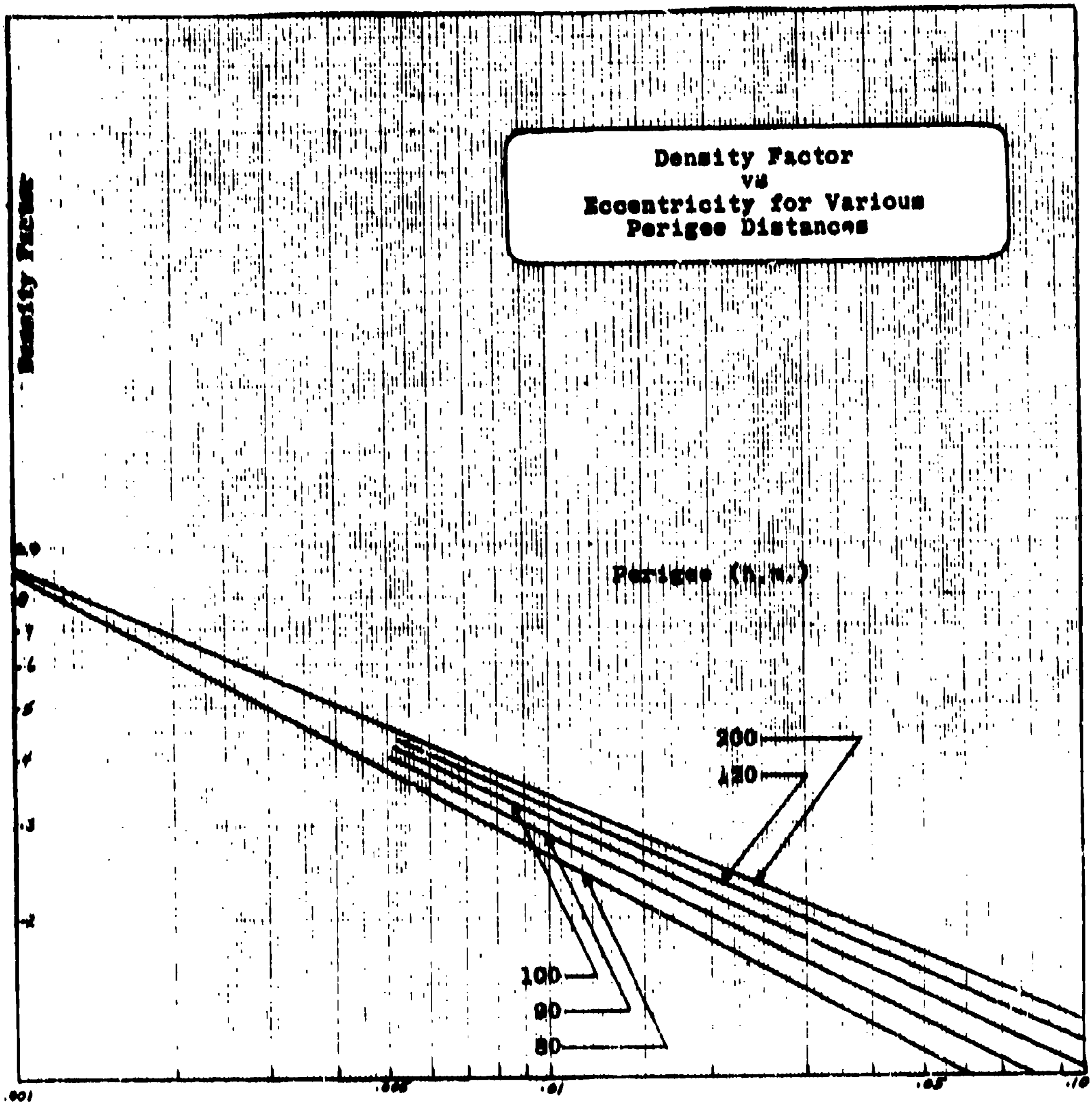
Differential Expressions Relating Errors in Initial Conditions to Ephemeris Errors

In examining the uncertainties associated with orbital parameters after an orbit determination, an analyst may have difficulty in estimating the effect of these small uncertainties upon the subsequent ephemeris. The following equations provide accurate quantitative answers to such questions. Abbreviations and symbols are explained at the end of this appendix.

$$
\begin{aligned}
H = \; & + \Delta a \left\{ (1-e\cos f) - \left(\tfrac{3}{2}\, ne \sin f\right)(t-\tau) \right\} \\
& - \Delta e \left\{ a \cos f + 2ae \sin^2 f \right\} \\
& - \Delta \tau \left\{ nae \sin f \right\} \\
& + \Delta b \Big\{ \rho a^2 n \Big[-2(t-t_o) + \tfrac{3}{2}\, ne (t-t_o)^2 \sin f \\
& \qquad - \tfrac{9}{8} (\Delta b) \rho a n^3 (t-t_o)^4 \Big] \Big\} \\
& + \Delta K \left\{ \left(\frac{e \sin f}{2\, na^2} \right) (t-\tau) \right\}
\end{aligned}
$$

$$
\begin{aligned}
C = \; & - \Delta i \left\{ a(1-e\cos f) \sin u \right\} \\
& + \Delta \Omega \left\{ a(1-e\cos f) \sin i \cos u \right\} \\
& + \Delta b \left\{ \tfrac{1}{2}\, \rho a^2\, \omega_\Omega (\sin i \sin u)(t-t_o)(1-e\cos f) \right\}
\end{aligned}
$$

$$
\begin{aligned}
L = \; & - \Delta a \left\{ \frac{3n}{2} (1+e\cos f)(t-\tau) \right\} \\
& + \Delta e \left\{ a(2-e\cos f) \sin f \right\} \\
& + \Delta \Omega \left\{ a(1-e\cos f) \cos i \right\} \\
& + \Delta \omega \left\{ a(1-e\cos f) \right\} \\
& - \Delta \tau \left\{ na(1+e\cos f) \right\} \\
& + \Delta b \left\{ -\tfrac{3}{2}\, \rho\, a^2 n^2 (1+e\cos f)(t-t_o)^2 \right\} \\
& + \Delta K \left\{ \left(\frac{1+e\cos f}{2na^2} \right) (t-\tau) \right\}
\end{aligned}
$$



Eccentricity

Figure 1 Density Factor vs Eccentricity for Various Perigee Distances

ABBREVIATIONS AND SYMBOLS FOR APPENDIX U

Symbol	Definition
(H,C,L)	Coordinates of test object in a moving rectangular coordinate system with origin at instantaneous position of reference object. H extends outward along radius vector. L is in the orbit plane, perpendicular to H and in the direction of motion of the reference object. C is perpendicular to orbit plane and opposite to angular momentum vector.
$(a,e,i,\Omega,\omega,\tau)$	Classical orbital elements. a is semi-major axis. e is eccentricity. i is inclination angle. Ω is longitude of ascending node. ω is argument of perigee. τ is time of perigee passage.
f	True anomaly.
$u = \omega + f$	Argument of the latitude.
$n = \sqrt{\frac{K}{a^3}}$	Mean motion.
$b = \frac{C_D A}{2m}$	Ballistic coefficient or drag parameter.
(C_D, A, m)	C_D is dimensionless drag coefficient. A is frontal area of satellite. m is mass of satellite.
ρ	Effective atmospheric density. It is some fraction of the density at perigee. See Figure 1.
K	Gravitational parameter of Earth. Equals $GM_\oplus$ where G is Newton's gravitational constant and $M_\oplus$ is mass of Earth.
$(\Delta a,\Delta e,\Delta i,\Delta\omega,\Delta\Omega,\Delta\tau,\Delta b,\Delta K)$	Errors in the orbital elements, drag parameter and gravitational parameter at epoch. Or alternatively, characteristics of test orbit minus characteristics of reference orbit.

t_o Time of epoch.

t Time.

ω_3 Rotational rate of atmosphere about earth axis, assumed to be equal to rotational rate of earth. Also represents rotational rate of earth.

APPENDIX V

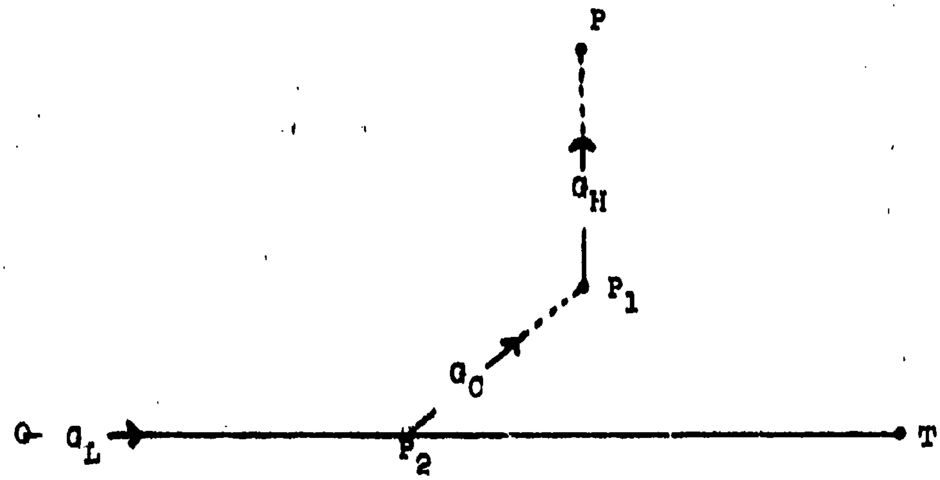
Orthogonal Geodesic Coordinate System

For this discussion a geodesic is defined as a curve lying on the surface of the earth spheroid and representing the minimum path length between two points located on that surface. A spheroid trace is defined as the path on the earth spheroid surface traced out by a perpendicular to the earth spheroid surface from some moving vehicle — ship, aircraft or missile. Of all the simple mathematical curves, the geodesic frequently agrees most closely with the actual spheroid trace and hence serves as a useful reference coordinate axis. The geodesic may connect the origin (O) at the point of departure with the terminus (T) at the point of arrival, or in the case of a missile at the intended point of arrival. This reference coordinate axis will be called G_L.

Another coordinate axis G_C is defined as follows:

At some particular time there is a point P_1 established on the spheroid surface by a perpendicular from the vehicle P to the spheroid surface. At that same instant there is some other point P_2 located on the coordinate axis G_L such that a geodesic connecting P_1 and P_2 will intersect perpendicularly with G_L. This geodesic connecting points P_1 and P_2 defines a moving coordinate axis G_C. The coordinates of the point P_1 are therefore: $G_L = \overline{OP_2}$; $G_C = \overline{P_2P_1}$. Distance is measured positively from O to T along the G_L axis. If an imaginary observer traveling from O toward T arrives at P_2 and sees P_1 off to the left, then the distance along the geodesic from P_2 to P_1 is positive; and if P_1 is off to the right, then the distance from P_2 to P_1 is negative. The axes G_L and G_C clearly form a curvilinear coordinate system, and distances $\overline{OP_2}$ and $\overline{P_2P_1}$ are not straight line distances but are distances measured along geodesics.

The third coordinate required to define the vehicle position P is the height: $G_H = \overline{P_1P}$. $\overline{P_1P}$ is a straight line distance. G_H is positive if P is above the spheroid surface and negative if P is below the spheroid surface. G_L G_C G_H form a right handed system:



The problem to be considered is development of transformation equations between the G_L G_C G_H coordinate system described here and the ϕ γ h coordinate system described on Page 16. We see immediately that G_H = h and hence we may devote all our attention to curves on the surface of spheroid involving only ϕ, γ, G_L and G_C.

If one is given the coordinates G_L and G_C of the point P_1 and either the geodetic position (ϕ_O, γ_O) of O and (ϕ_T, γ_T) of T or the geodetic position of O and the forward azimuth A_{OT} at O, then one can straightforwardly obtain the latitude and longitude (ϕ_1, γ_1) of P_1 by Sodano's inverse and direct methods. (See Appendix F.) Consequently, the only problem remaining to be discussed is the transformation from (ϕ_1, γ_1) to (G_L, G_C). The procedure is an iterative one:

(1) Determine the length g_{O1} of geodesic from (ϕ_O, γ_O) to (ϕ_1, γ_1) by Sodano's inverse method to use as first estimate of the geodesic g_{O2} from O to P_2.

(2) Obtain an estimate of coordinates of P_2 (ϕ_2, γ_2) by applying Sodano's direct method using known values

of ϕ_0, γ_0, A_{0T} and the estimated value for g_{02}.

(3) Determine back azimuth A_{20} at P_2 corresponding to geodesic g_{02}.

(4) Determine geodesic g_{21} from P_2 to P_1 and corresponding forward azimuth A_{21}.

(5) Form $\cos (A_{21}-A_{20}) = \cos \alpha$, the cosine of angle formed by intersection of g_{02} and g_{21} at P_2.

(6) Decrement g_{02} by some small value and repeat steps 2 - 5.

(7) Two passes through Steps 2 - 5 with different values of g_{02} give a numerical partial:

$$\frac{\Delta (\cos \alpha)}{\Delta g_{02}}$$

(8) The iterative equation to reduce $\cos \alpha$ to zero and arrive at a final value for g_{02} is

$$\left[g_{02}\right]_{i+1} = \left[g_{02}\right]_{i} - \left[\frac{\cos \alpha}{\frac{\Delta(\cos \alpha)}{\Delta g_{02}}}\right]_{i}$$

(9) Convergence is rapid and iteration is terminated when successive differences in estimates of g_{02} become less than some predetermined value.

(10) The final value of g_{02} and the final value of g_{21} represent the coordinates G_L and G_C respectively of the point P_1.

(11) For many applications one pass through Steps 1, 2 and 4 is sufficiently accurate, because g_{02} is usually many times larger than g_{21}. For almost all applications one pass through Steps 1 - 7 and a second pass through Steps 2 - 4 will be sufficient.

EXAMPLE:

Let $\phi_0 = 28°$

$\gamma_0 = 280°$

$\Lambda_{OT} = 110°$

$\phi_1 = 27.3948998495°$

$\gamma_1 = -78.0894076694°$

$\hat{a} = 6378140$ meters

$1/f = 298.25$

Solve for G_L and G_C

(The answer is known to be: $G_L = 200000$ m; $G_C = 3000$ m.)

SOLUTION:

(Numbers in parentheses represent step numbers in previous discussion.)

(1) $g_{01} = 200022.443269 = \text{est } g_{02}$

(2) $\phi_2 = 27.3695330078$

$\gamma_2 = -78.1000076956$

(3) $A_{20} = 290.882826201$

(4) $g_{21} = 3000.08397063$

$\Lambda_{21} = 20.4541998564$

(5) $A_{21}-A_{20}$ = 89.571373635 = α

$\cos\alpha$ = .00748087121

(6) Take $s_{02} - 5$ = 200017.443269

(2) ϕ_2 = 27.3695490919

γ_2 = -78.1000549159

(3) A_{20} = 290.882804492

(4) s_{21} = 3000.05074054

A_{21} = 20.549664577

(5) $A_{21}-A_{20}$ = 89.666860085 = α

$\cos\alpha$ = .00581435563

(7) $\Delta \cos\alpha$ = -.00166651558

Δs_{02} = - 5

$$\frac{\Delta (\cos\alpha)}{\Delta s_{02}} = .000333303116$$

(8) $$s_{02} = 200017.443269 - \frac{.00581435563}{.000333303116}$$

$$= 199999.998621$$

(2) ϕ_2 = 27.3696052091

γ_2 = -78.1002196697

(3) A_{20} = 290.88272875

(4) s_{21} = 2999.99995976

At this stage the computed values for G_L and G_C are

G_L = 199999.98621 meters

G_C = 2999.99995976 meters

Comparison with the known answer shows the errors to be:

G_L : .05 inch

G_C : .002 inch

Since we are down to the precision level of the program, further iterations here would not be useful. In a normal operation when the answer is not known, one further iteration might be made to be sure of convergence.

NOTE: Orthogonal geodesic coordinates (also known as Laborde Projections) form the basis for conformal grid maps such as Universal Transverse Mercator (Appendix W). Steps (1) and (2) on pp. V-2 and V-3, used to provide a first estimate for point P_2, can cause difficulties in the event that the coordinate G_C is large relative to G_L. This is a highly unlikely circumstance in present missile and satellite applications. If such a situation is anticipated in some particular application, then a spherical solution should be substituted for making the first estimate.

APPENDIX W

UNIVERSAL TRANSVERSE MERCATOR (UTM) COORDINATE SYSTEM

UTM is a conformal coordinate system used for artillery and short range missile maps. UTM describes the earth spheroidal surface from 80° south latitude to 80° north latitude, with 600 zones in the southern hemisphere and 600 zones in the northern hemisphere. Each zone covers 6° in longitude and 8° in latitude. The longitudinal boundaries of the zones begin at 180° from Greenwich and continue eastward with 186° E, 192° E, 198° E, etc., the longitudinal regions being numbered 1, 2, 3 ... 60. The latitudinal boundaries of the zones begin at 80° S and continue northward with 72° S, 64° S, 56° S, etc., the latitudinal regions being lettered CDEFGHJKLMNPQRSTUVWX. Thus the point at 189° E longitude and 4° N latitude is at the center of zone 2M. Each zone is characterized by an interior rectangular grid system and by a central meridian. The central meridian for zone 1P, for example, is at 183° E longitude. Grid coordinates (northing and easting) describe the position of a point within a zone.

Transformation equations presented here were obtained from 1st Geodetic Survey Squadron, Warren AFB, Wyoming, and represent modifications of those given in U. S. Army TM5-241-8. These modifications are minor and do not represent an impairment of accuracy.

ALGORITHM:

All longitudes are measured positive east from Greenwich. Latitudes are measured positive north of the equator and negative south of equator. In conformance with normal usage N will be used to denote north latitude and also northing. Also E will be used to denote east longitude and also easting. The context will prevent ambiguity. Twelve digit floating point computations are adequate. The following notation is used:

ϕ, λ	=	geodetic latitude and longitude
N, E	=	northing and easting
N_0, E_0	=	false northing and false easting
λ_0	=	longitude of central meridian
k	=	scale factor
k_0	=	scale factor at central meridian
N'	=	$(N-N_0)/k_0$, true northing
E'	=	$(E-E_0)/k_0$, true easting
γ	=	grid convergence (geodetic azimuth minus grid azimuth, approximately)
a, b	=	major and minor semiaxes of the spheroid
p	=	a^2/b, polar radius of curvature
e^2	=	$(a^2-b^2)/a^2$, first eccentricity squared
ϵ	=	$e^2/(1-e^2)$, second eccentricity squared
n	=	$(a-b)/(a+b)$
ω	=	rectifying latitude
Z	=	zone number

Spheroid constants are computed as follows:

$$A = 1 + 3n\left[-1 + (7/4)\, n\, (1-n/0.679)\right]$$

$$B = 0.4 \times 10^{-10} + (A-1)/A$$

$$C = -2.5\, n\, (1+n/0.539)$$

$$D = 1.234\, C$$

$$F = 1 - A + 0.14 \times 10^{-9}$$

$$G = 3.5\, n\, (1-n/0.3269)$$

$$H = 1.388\, G$$

The following constants are used in UTM for all spheroids:

$$N_0 = 0 \text{ for northern hemisphere}$$
$$= 10\,000\,000 \text{ for southern hemisphere}$$

$$E_0 = 500\,000$$

$$k_0 = 0.9996$$

$$\lambda_0(\text{deg}) = 6Z-183$$

The direct problem is defined as the computation of grid coordinates (northing and easting), scale factor, and convergence, given geodetic coordinates and central meridian value. The following equations apply, in which all angles are in radians:

$$g^2 = \epsilon \cos^2 \phi$$

$$V^2 = 1 + g^2$$

$$t = \tan \phi$$

$$\omega = \phi + B\, t \cos^2 \phi \left[1 + C \cos^2 \phi \,(1 + D \cos^2 \phi)\right]$$

$$N_f' = \omega\, p\, A \qquad \text{NOTE: } p \text{ must be in meters.}$$

$$\Delta L = (\lambda-\lambda_0) \cos \phi$$

$$N' = N_f' + t\,(p/2V)(\Delta L)^2 \Big\{1+(1/12)(\Delta L)^2 \Big[5-t^2 + g^2(9+4g^2) + 2(\Delta L)^2 \big\{1+t^2\,[-1+(1/60)t^2]\big\}\Big]\Big\}$$

$$N = N' k_0 + N_0$$

$$E' = (p/V)\,\Delta L \Big\{1 + (1/6)(\Delta L)^2 \Big[1-t^2 + g^2 + (1/20)(\Delta L)^2 \big\{5 + t^2(-18+t^2) + 15 g^2 [1 - 4t^2]\big\}\Big]\Big\}$$

$$E = E' k_0 + E_0$$

$$P = (E'/p)\, V^2$$

$$k = k_0 \left\{ 1 + (1/2)\ P^2\ [1 + (1/12)\ P^2] \right\}$$

$$\gamma = t\ \Delta L \left\{ 1 + (1/3)(\Delta L)^2 \left[1 + g^2\ (3 + 2\ g^2) + (1/5)\ (\Delta L)^2\ (2 - t^2) \right] \right\}$$

This completes the direct problem. The reverse problem is defined as the computation of geodetic coordinates, given universal transverse mercator grid coordinates and central meridian value. The following equations apply, in which all angles are in radians:

$$N' = (N - N_0)/k_0$$

$$E' = (E - E_0)/k_0$$

$$\omega = N'/(A\ p)$$

$$\phi_f = \omega + F \cos^2 \omega \tan \omega\ [1 + G \cos^2 \omega (1 + H \cos^2 \omega)]$$

$$g^2 = \epsilon \cos^2 \phi_f$$

$$V^2 = 1 + g^2$$

$$Q = (V/p)\ E'$$

$$t = \tan \phi_f$$

$$\phi = \phi_f + (1/2)\ t\ Q^2 \left\{ -1 - g^2 + (1/12)\ Q^2 \left[5 + 3 \left(t^2 [1 - g^2 (2 + 3g^2)] + g^2\ (2 - g^2) \right) - (1/2)\ Q^2 \left(4 + 3t^2\ (2 + t^2) \right) \right] \right\}$$

$$\Delta\lambda = (Q/\cos \phi_f) \left\{ 1 - (1/6)\ Q^2 \left[1 + 2\ t^2 + g^2 - (1/20)\ Q^2 \left(5.05 + 4t^2\ (7 + 6t^2) \right) \right] \right\}$$

$$\lambda = \lambda_0 + \Delta\lambda$$

This completes the reverse problem.

EXAMPLE (UTM, Clarke 1866)

Latitude	:	34° 15' 34".742 N
Longitude	:	96° 2' 43".158 W
Central Meridian	:	99° W
Northing	:	3794702.172 meters
Easting	:	772075.812 meters
Scale Factor	:	1.00051259
Convergence	:	+ 1° 39' 51".627

APPENDIX X

EARTH SHADOW

It is sometimes desirable, particularly in photography, to know when a satellite is in full sunlight and when in the shadow of the earth. Figure 1 shows the geometry of the situation. In this figure

$$\beta = 0.5283 \text{ deg}$$

$$a_s = 6.9598 \times 10^5 \text{ km}$$

$$a_e = 6.37814 \times 10^3 \text{ km}$$

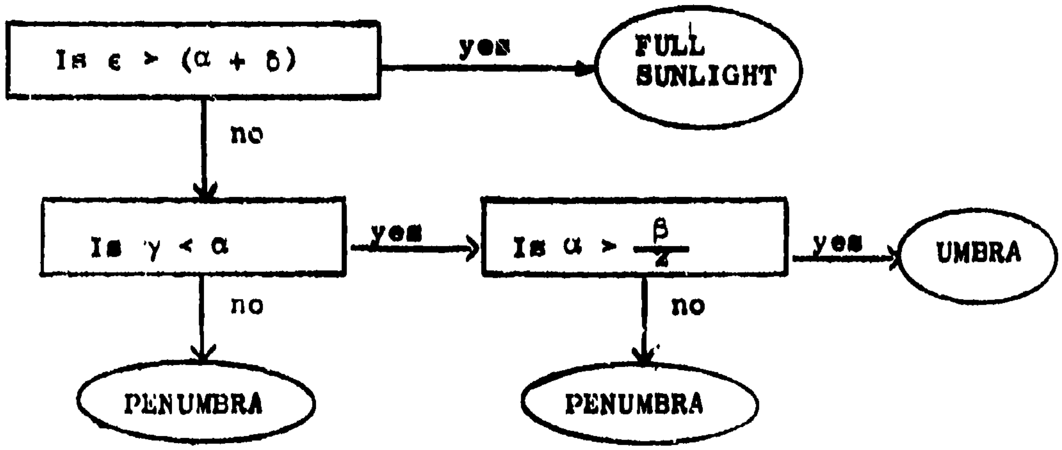
$$\delta = \sin^{-1} \frac{a_s}{R_s}$$

$$\epsilon = \cos^{-1} \frac{\underline{R}_s \cdot \underline{R}_e}{R_s R_e}$$

$$\gamma = \delta + \epsilon$$

$$\alpha = \sin^{-1} \frac{a_e}{R_e}$$

The logic to determine the location of P is as follows:



The same logic, of course, can be used to determine whether the tracking station is in sunlight or shadow.

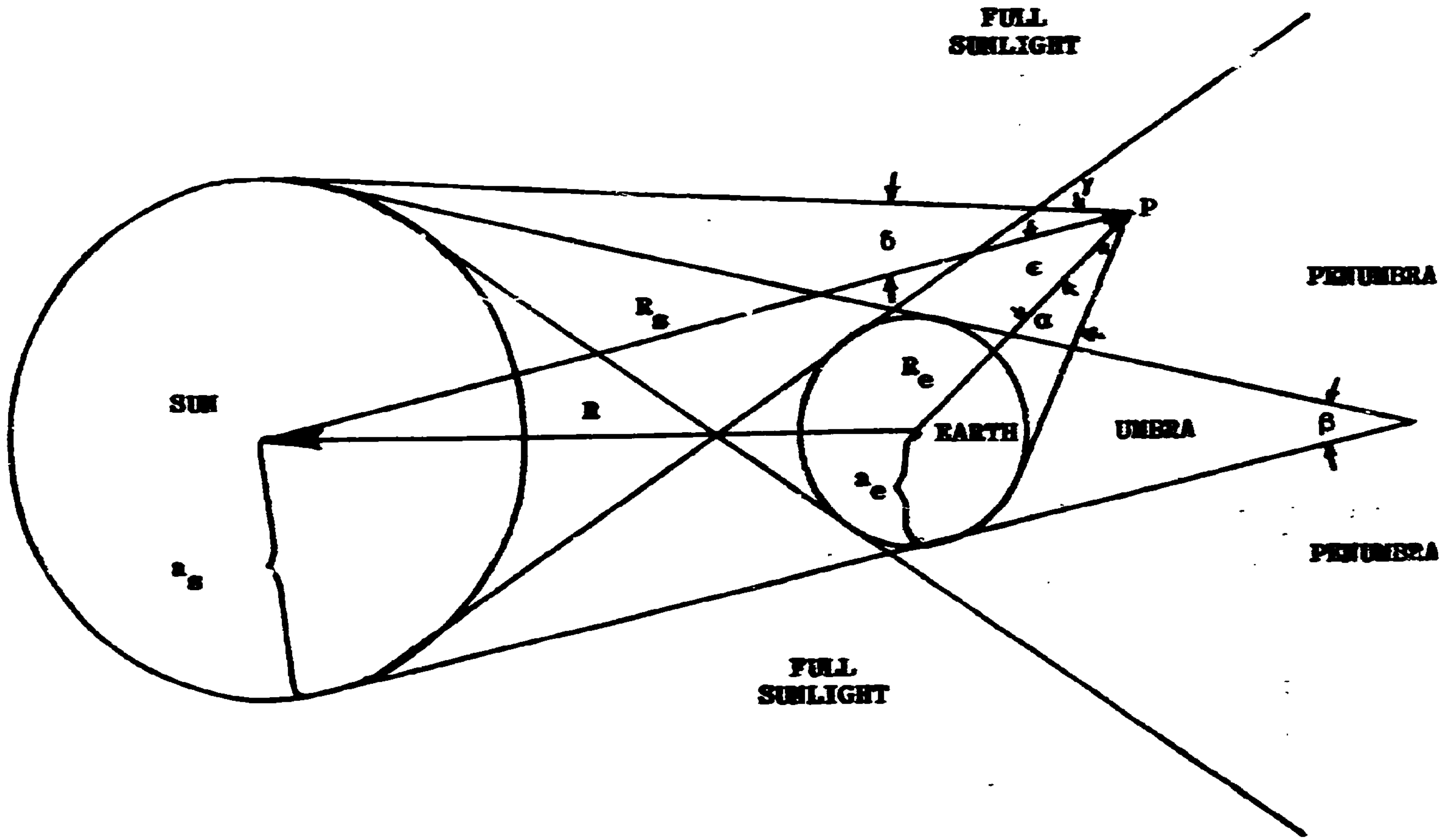


FIGURE 1. EARTH SHADOW GEOMETRY

This algorithm requires the computation of the vectors $\underline{R}_e$ and $\underline{R}_S$. The vector $\underline{R}_e$ is computed in the usual way using the orbital elements of the satellite. $\underline{R}_S$ is seen from the diagram to be $\underline{R}_S = \underline{R}_e - \underline{R}$. The vector $\underline{R}$ is available from an ephemeris tape if such a tape is convenient. For most purposes the geocentric inertial coordinates (X,Y,Z) of the vector $\underline{R}$ can be computed satisfactorily from elliptic expansions using Bessel functions of the first kind. No significant error is introduced by truncating terms with powers of eccentricity greater than three. We have therefore

$$f = M + (2e - \frac{1}{4} e^3) \sin M + (\frac{5}{4} e^2) \sin 2 M + (\frac{13}{12} e^3) \sin 3 M + \ldots,$$

$$\frac{R}{a} = 1 + \frac{1}{2} e^2 + (- e + \frac{3}{8} e^3) \cos M + (- \frac{1}{2} e^2) \cos 2 M + (- \frac{3}{8} e^3) \cos 3 M + \ldots ,$$

$$M = n (t - t_p) ,$$

$$u = \omega + f ,$$

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \epsilon & -\sin \epsilon \\ 0 & \sin \epsilon & \cos \epsilon \end{bmatrix} \begin{bmatrix} \cos u \\ \sin u \\ 0 \end{bmatrix} R \quad \text{, where}$$

f is true anomaly,

e is eccentricity,

M is mean anomaly,

a is semi-major axis of earth orbit about the sun,

n is mean motion = 0.01720279 rad/day,

t is time,

t_p is time of perigee passage, an input constant obtained from American Ephemeris,

ω is argument of perigee, an input constant obtained from American Ephemeris,

u is argument of the latitude,

ϵ is mean obliquity = 23.443 deg.

APPENDIX Y

SOME SPHERICAL-EARTH GEOMETRICAL FORMULAS

The following simple formulas – exact for spherical earth – are presented primarily for planning purposes and simple error analyses. Exact and necessarily more complex computational methods for the oblate earth are presented elsewhere in this report.

1.0 PLANE TRIANGLE FORMULAS

The triangle is defined by one side extending from the center of the earth to the observer on the surface of the earth, one side extending from the center of the earth to the tracked object above the earth, and one side extending from the observer to the tracked object. See Figure 1.

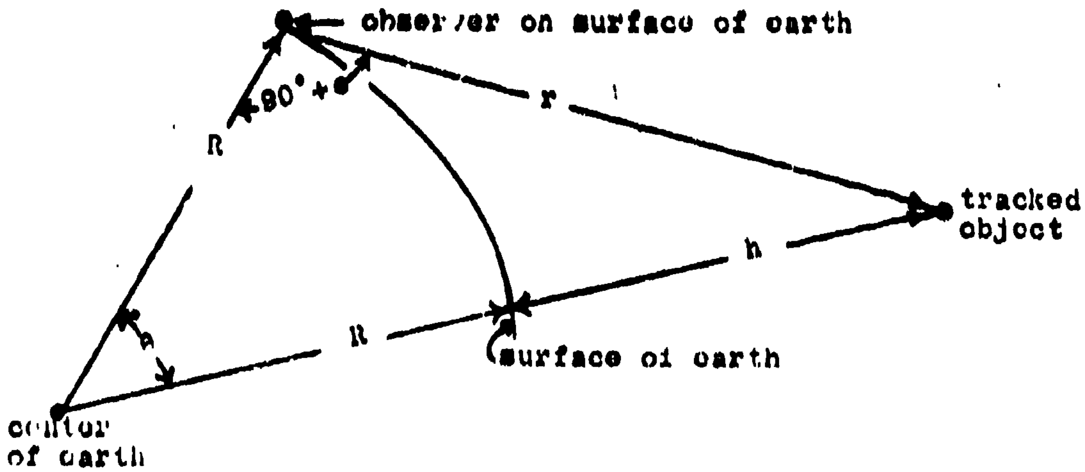


FIGURE 1

Symbols in Figure 1 are defined as follows:

r is slant range from observer to tracked object.

e is elevation angle of tracked object above local horizontal, positive above and negative below.

h is altitude of tracked object above surface of earth.

R is earth radius, approximately 3440 nm.

θ is central angle between vector from center of earth to observer and the vector from center of earth to tracked object, where $0 \leq \theta \leq 180°$.

The following formulas apply:

(1) Ground range $=$ $R\theta$, where θ is in radians.

(2) $$r = +\sqrt{2\,R\,(R+h)(1-\cos\theta) + h^2}$$

(3) $$e = \tan^{-1}\left[\frac{1}{\tan\theta} - \frac{R}{(R+h)\sin\theta}\right], \text{ where}$$

$$(-90) \leq e \leq (+90^\circ).$$

(4) $$h = R\left[\frac{\cos e}{\cos(e+\theta)} - 1\right].$$

(5) $$h = +\sqrt{R^2 + r^2 + 2\,r\,R\sin e} - R.$$

(6) $$\theta = \cos^{-1}\left[\frac{R\cos e}{R+h}\right] - e, \quad \text{where } 0^\circ \leq \theta \leq 180^\circ.$$

Simple considerations permit these formulas to be used with an observer above the surface of the earth. Note that these formulas are strictly geometrical and do not consider refraction effects.

2.0 SPHERICAL TRIANGLE FORMULAS

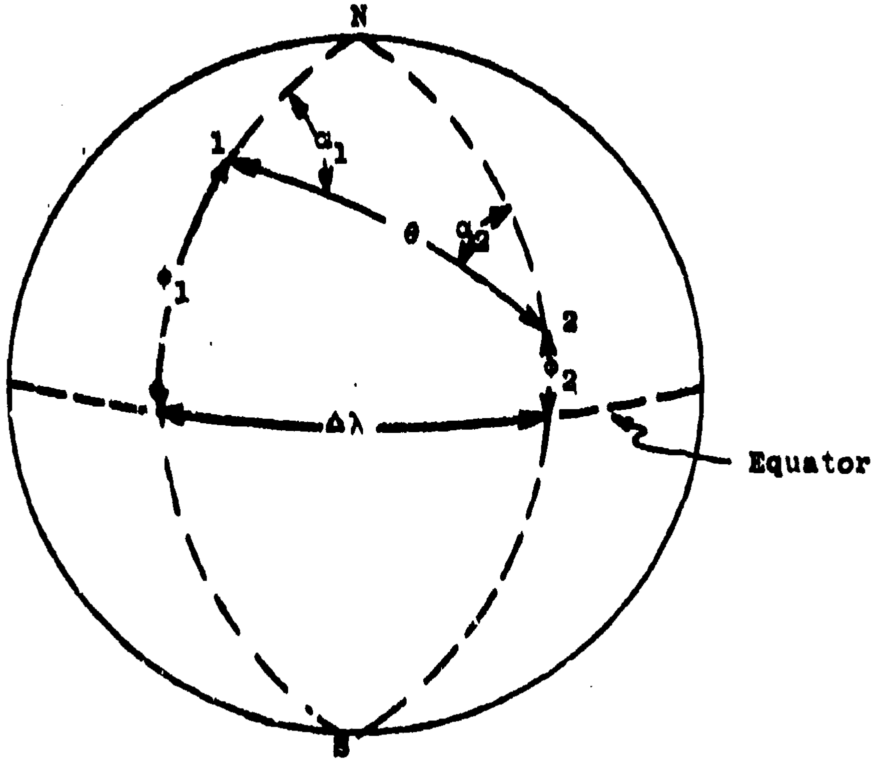


FIGURE 2

Symbols in Figure 2 are defined as follows:

N, S	represent respectively the north and south poles of the earth.
1, 2	designate Sites 1 and 2, respectively.
ϕ	is latitude, positive north of equator and negative south of equator.
$\Delta\lambda$	is longitude difference between sites.
θ	is central angle between vectors extending from center of earth to two sites.
α's	are azimuth angles as indicated in diagram.

The following formulas apply:

$$(7)\quad \theta = \cos^{-1}\left[\sin\phi_1 \sin\phi_2 + \cos\phi_1 \cos\phi_2 \cos(\Delta\lambda)\right],$$

where $0^\circ \leq \theta \leq 180^\circ$.

$$(8)\quad \alpha_1 = 2\sin^{-1}\sqrt{\frac{\sin\left(\frac{\theta+\phi_1-\phi_2}{2}\right)\cos\left(\frac{\theta+\phi_1+\phi_2}{2}\right)}{\sin\theta\cos\phi_1}},$$

where $0^\circ \leq \alpha_1 \leq 180^\circ$.

$$(9)\quad \phi_2 = \sin^{-1}\left[\sin\phi_1 \cos\theta + \cos\phi_1 \sin\theta \cos\alpha_1\right],$$

where $(-90^\circ) \leq \phi_2 \leq (+90^\circ)$.

$$(10)\quad \Delta\lambda = \tan^{-1}\left[\frac{\tan\alpha_1 \sin D_1}{\sin(\phi_1+D_1)}\right], \text{ where } 0^\circ \leq \Delta\lambda \leq 180^\circ$$

and where

$$(11)\quad D_1 = \tan^{-1}\left[\tan\theta\cos\alpha_1\right], \text{ where } 0^\circ \leq D_1 \leq 180^\circ .$$

An alternative form for each of equations 8, 9, 10 and 11 can be obtained by substituting subscript 1 for subscript 2 and substituting subscript 2 for subscript 1 throughout the particular equation.

APPENDIX Z

NUMERICAL METHODS - INTERPOLATION, DIFFERENTIATION, QUADRATURE AND INTEGRATION

INTERPOLATION

The problem of interpolation consists in constructing a continuous function which fulfills certain conditions at a finite number of discrete points. The condition usually imposed is the coincidence in numerical values of the interpolating function with the numerical values of a finite number of tabulated points. The type of function includes power polynomials, rational functions and trigonometric polynomials. There are special applications in which each type is preferable. However, considerations of speed, simplicity, versatility - as well as possible requirements for differentiation or integration - generally result in the selection of power polynomials of the type

$$f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \quad .$$

The legitimacy of the power polynomial representation is based upon a theorem of Weierstrass, which can be stated as follows: Let $f(x)$ be an arbitrary continuous function defined in a finite interval $a \leq x \leq b$. It is always possible to approximate $f(x)$ over the whole interval (a,b) as closely as we please by a power polynomial of sufficiently high degree. This obviously does not mean that we can interpolate in a table as accurately as we please by choosing a power polynomial of sufficiently high degree.

Most of the discussion in this appendix will be focussed on power polynomials. All of the interpolation formulas (of degree n) in this appendix will precisely fit or go through (n+1) points in the table. An individual formula is selected for a particular application primarily on the basis of computational convenience and upon special limitations and advantages associated with distribution of the table entries.

Selection of the proper degree of the interpolating polynomial is done on the basis of an error analysis, which should be performed by the mathematician who prepares the table. First degree corresponds to linear interpolation. Interpolation beyond sixth degree is virtually unknown, and a good practical limit is fourth degree.

NOTATION FOR NEWTON'S, BESSEL'S AND EVERETT'S FORMULAS

The following table showing differences up to the fifth will serve to explain the notation:

ENTRY	TABLE VALUE	1ST DIFF.	2ND DIFF.	3RD DIFF.	4TH DIFF.	5TH DIFF.
x_{-2}	y_{-2}					
		$\Delta_{-1.5}$				
x_{-1}	y_{-1}		Δ^2_{-1}			
		$\Delta_{-0.5}$		$\Delta^3_{-0.5}$		
x_0	y_0		Δ^2_0		Δ^4_0	
		$\Delta_{0.5}$		$\Delta^3_{0.5}$		$\Delta^5_{0.5}$
x_1	y_1		Δ^2_1		Δ^4_1	
		$\Delta_{1.5}$		$\Delta^3_{1.5}$		
x_2	y_2		Δ^2_2			
		$\Delta_{2.5}$				
x_3	y_3					

<u>It is necessary that the entries in the table be listed at equal intervals.</u> First differences are defined as follows:

$$\Delta_{-1.5} = y_{-1} - y_{-2}\ ;\quad \Delta_{0.5} = y_1 - y_0\ ;\quad \text{etc.}$$

Second differences are defined as follows:

$$\Delta^2_{-1} = \Delta_{-0.5} - \Delta_{-1.5}\ ;\quad \Delta^2_1 = \Delta_{1.5} - \Delta_{0.5}\ ;\quad \text{etc.}$$

Similarly for third, fourth, fifth, sixth, etc. differences.
Let $\delta_x = x_3 - x_2 = x_2 - x_1 = x_0 - x_{-1}$, etc.

NEWTON'S FORWARD AND BACKWARD DIFFERENCE INTERPOLATION FORMULAS

These formulas are normally used only near the beginning or end of a table where other and more rapidly converging formulas such as Bessel's and Everett's are not applicable. Let us assume that x_0 is the first entry in the table. The entry, x, for which we want to compute a table value, y, is located at $(x_0+p\delta x)$, where p is some positive fraction less than unity. Newton's Forward Difference Formula is applicable:

$$y = y_0 + p\,\Delta_{0.5} + \frac{1}{2}\,p\,(p-1)\,\Delta_1^2 + \frac{1}{6}\,p\,(p-1)(p-2)\,\Delta_{1.5}^3 + \ldots$$

Next let us assume that x_0 is the last entry in the table. The entry, x, for which we want to compute a table value, y, is located at $(x_0-p\delta x)$, where p is some positive fraction less than unity. Newton's Backward Difference Formula is applicable:

$$y = y_0 - p\,\Delta_{-0.5} + \frac{1}{2}\,p\,(p-1)\,\Delta_{-1}^2 - \frac{1}{6}\,p\,(p-1)(p-2)\,\Delta_{-1.5}^3 + \ldots$$

Clearly these formulas may be used where x does not fall within the first or last interval of the table. In fact, for reasons which will be apparent later, it is usually necessary to apply these formulas when x is less than two intervals from the beginning or end of the table. Newton's Forward and Backward Difference Formulas have been shown extended out to third differences, but there is little if any advantage in using differences beyond the second, because higher differences are too far removed from the entry of interest.

BESSEL'S INTERPOLATION FORMULA

This is probably the most widely used of all interpolation formulas. It is a central difference formula, sometimes extended to fourth differences, but if differences higher than the third are desired, Everett's formula is generally preferred. The entry, x, for which we want to compute a table value, y, is located at $(x_0 + p\,\delta\,x)$, where p is some positive fraction less than unity. Bessel's formula is as follows:

$$y = 0.5\,(y_0+y_1) + (p-0.5)\,\Delta_{0.5} + \frac{p(p-1)}{4}\,(\Delta_0{}^2 + \Delta_1{}^2)$$

$$+ \frac{p(p-0.5)(p-1)}{6}\,\Delta^3_{0.5} + \frac{(p+1)p(p-1)(p-2)}{48}\,(\Delta_0{}^4 + \Delta_1{}^4) + \ldots$$

In cases where fourth differences are significant, it is frequently satisfactory and somewhat simpler to "throw back" the fourth differences into the second as follows:

$$y \cong 0.5\,(y_0+y_1) + (p-0.5)\,\Delta_{0.5}$$

$$+ \frac{p(p-1)}{4}\left[\Delta_0{}^2 + \Delta_1{}^2 - 0.184\,(\Delta^4{}_0 + \Delta^4{}_1)\right]$$

$$+ \frac{p(p-0.5)(p-1)}{6}\,\Delta^3{}_{0.5}$$

EVERETT'S INTERPOLATION FORMULA

Everett's formula is generally preferred to Bessel's when higher differences are necessary. Everett's formula shown extended out to sixth differences – usually the ultimate limit of usefulness – is as follows:

$$y = q\left[y_0 + \frac{1}{6}\,(q^2-1)\,\Delta_0{}^2 + \frac{1}{120}\,(q^2-1)(q^2-4)\,\Delta_0{}^4\right.$$

$$\left. + \frac{1}{5040}\,(q^2-1)(q^2-4)(q^2-9)\,\Delta_0{}^6 + \ldots\right]$$

$$+ p\left[y_1 + \frac{1}{6}\,(p^2-1)\,\Delta_1{}^2 + \frac{1}{120}\,(p^2-1)(p^2-4)\,\Delta_1{}^4\right.$$

$$\left. + \frac{1}{5040}\,(p^2-1)(p^2-4)(p^2-9)\,\Delta_1{}^6 + \ldots\right] ,$$

where $q = (1-p)$ and where the entry, x, for which we want to compute a table value, y, is located at $(x_0+p\,\delta_x)$, where p is some positive fraction less than unity.

LAGRANGE'S FORMULA

Lagrange's formula is more general than the formulas previously treated. Equal intervals in the table entries are not required. Interpolation can be accomplished anywhere within the table including the first and last interval, and also extrapolation can be performed (with caution). Most analysts prefer formulas given previously, however, for tables with entries at equal intervals. Let the table be of the form:

ENTRY	TABLE VALUE
x_0	y_0
x_1	y_1
x_2	y_2
x_3	y_3
.	.
.	.
.	.
x_n	y_n

Given some entry, x, we wish to estimate the corresponding table value, y. Lagrange's formula is as follows:

$$
\begin{aligned}
y = {} & \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\, y_0 \\
& + \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\, y_1 \\
& + \frac{(x-x_0)(x-x_1)(x-x_3)\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)}\, y_2 + \ldots \\
& + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})}\, y_n
\end{aligned}
$$

HERMITE'S FORMULA

This is a trigonometric analog to Lagrange's formula. Equal intervals in the table entries are not required. Interpolation can be accomplished anywhere within the table including the first and last interval, and also extrapolation can be performed (with caution). When a function represented by a table is known to be periodic, Hermite's formula is frequently preferred over Lagrange's in spite of the much larger computation time. Hermite's formula is as follows:

$$
\begin{aligned}
y = {} & \frac{\sin(x-x_1)\sin(x-x_2)\ldots\sin(x-x_n)}{\sin(x_0-x_1)\sin(x_0-x_2)\ldots\sin(x_0-x_n)}\,y_0 \\
& + \frac{\sin(x-x_0)\sin(x-x_2)\ldots\sin(x-x_n)}{\sin(x_1-x_0)\sin(x_1-x_2)\ldots\sin(x_1-x_n)}\,y_1 \\
& + \frac{\sin(x-x_0)\sin(x-x_1)\sin(x-x_3)\ldots\sin(x-x_n)}{\sin(x_2-x_0)\sin(x_2-x_1)\sin(x_2-x_3)\ldots\sin(x_2-x_n)}\,y_2 + \ldots \\
& + \frac{\sin(x-x_0)\sin(x-x_1)\ldots\sin(x-x_{n-1})}{\sin(x_n-x_0)\sin(x_n-x_1)\ldots\sin(x_n-x_{n-1})}\,y_n
\end{aligned}
$$

NOTE

The mathematics literature contains many interpolation formulas not included here. Among these are Gregory's, Newton's Divided Differences, Aitkins, Neville's, Gauss', Stirling's, Thiele's and Steffensen's. After careful consideration and evaluation these were omitted, because in our field of interest they appear to contribute little to the capability represented by the formulas included.

INVERSE INTERPOLATION

A table supplied by a competent mathematician is amenable to interpolation. If such a supplier does not specify any particular interpolation procedure, then it is safe to assume that one of the direct interpolation procedures previously discussed is applicable. Occasions arise, however, when the problem is not the standard one of determining the table value from a given entry but rather determining the entry from a given table value. The process by which the entry is determined from a given table value is called inverse interpolation.

In inverse interpolation the analyst is using a table for a purpose for which it was not designed, and he is consequently responsible for checking out the accuracy of any procedure he uses. The most efficient method of inverse interpolation involves a reversal of the roles of entry and table value in interpolation formulas already discussed. In other words, instead of expressing the table value in terms of a polynomial in the entry, we express the entry in terms of a polynomial in the table value. With the original table, however, there is no assurance that the entry can be adequately expressed in terms of a simple polynomial in the table value. In fact, reversal of the roles of entry and table value can easily result in an interpolated value in the wrong interval of the table. The accuracy of this reversal procedure is dependent upon the particular table being used and may frequently be improved by using prior direct interpolation to reduce the table interval size. If reversal of the role of entry and table value gives unsatisfactory results with a given table and if it is impractical to reduce the table interval size, then one can resort to successive approximations. In successive approximations one applies direct interpolation procedures using closer and closer approximations to the true entry corresponding to a given table value. These two general approaches will be discussed in further detail.

Reversal of Role of Entry and Table Value

Because Lagrange's and Hermite's formulas do not require equal intervals between successive values of the entry, the roles of entry and table value may be readily interchanged. This is a perfectly general method and requires no further discussion.

Interpolation formulas developed in the form of convergent power series can always be reverted. Without developing the reverted series for any specific interpolation formula, we will show the general case. The power series

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_n x^n$$

when reverted becomes

$$x = \left(\frac{y-a_0}{a_1}\right) + c_1\left(\frac{y-a_0}{a_1}\right)^2 + c_2\left(\frac{y-a_0}{a_2}\right)^3 + c_3\left(\frac{y-a_0}{a_3}\right)^4 + \dots + c_{n-1}\left(\frac{y-a_0}{a_1}\right)^n ,$$

where

$$c_1 = -\frac{a_2}{a_1} ,$$

$$c_2 = -\frac{a_3}{a_1} + 2\left(\frac{a_2}{a_1}\right)^2 ,$$

$$c_3 = -\frac{a_4}{a_1} + 5\left(\frac{a_2 a_3}{a_1^2}\right) - 5\left(\frac{a_2}{a_1}\right)^3 ,$$

$$c_4 = -\frac{a_5}{a_1} + 6\left(\frac{a_2 a_4}{a_1^2}\right) + 3\left(\frac{a_3}{a_1}\right)^2 - 21\left(\frac{a_2^2 a_3}{a_1^3}\right) + 14\left(\frac{a_2}{a_1}\right)^4 ,$$

$$c_5 = -\frac{a_6}{a_1} + 7\left(\frac{a_2 a_5 + a_3 a_4}{a_1^2}\right) - 28\left(\frac{a_2^2 a_4 + a_2 a_3^2}{a_1^3}\right) + 84\left(\frac{a_2^3 a_3}{a_1^4}\right) - 42\left(\frac{a_2}{a_1}\right)^5 ,$$

$$\vdots$$

Successive Approximations

Bessel's formula will be used to illustrate a general procedure. If we transpose and divide through by $\Delta_{0.5}$, we obtain

$$p = 0.5 + \frac{2y-y_0-y_1}{2\Delta_{0.5}} - \frac{p(p-1)}{4}\left(\frac{\Delta_0^{2}+\Delta_1^{2}}{\Delta_{0.5}}\right)$$

$$- \frac{p(p-0.5)(p-1)}{6}\left(\frac{\Delta^3_{0.5}}{\Delta_{0.5}}\right) - \frac{(p+1)p(p-1)(p-2)}{48}\left(\frac{\Delta_0^{4}+\Delta_1^{4}}{\Delta_{0.5}}\right) .$$

To a first approximation for p, we neglect all differences higher than the first and therefore have

$$p^{(1)} = 0.5 + \frac{2y-y_0-y_1}{2\Delta_{0.5}} .$$

The second approximation is obtained by substituting $p^{(1)}$ in the right-hand side of the series expression for p. The resulting second approximation $p^{(2)}$ is then substituted in the right-hand side of the series expression for p, and a third approximation $p^{(3)}$ is obtained — and so on for higher approximations until the difference in successive approximations for p become negligible.

A more general but usually more slowly converging method of successive approximations makes use of Lagrange's formula. To illustrate, let y lie between y_0 and y_1. Then we know that x lies between x_0 and x_1. As a first approximation for x, we have

$$x^{(1)} = x_0 + \left(\frac{y-y_0}{y_1-y_0}\right)\left(x_1-x_0\right) .$$

Supplying $x^{(1)}$ to Lagrange's formula will result in a corresponding value for $y^{(1)}$, in general not equal to y. If y lies between $y^{(1)}$ and y_1, we perform another linear interpolation as follows:

$$x^{(2)} = x^{(1)} + \frac{y - y^{(1)}}{y_1 - y^{(1)}} \left(x_1 - x^{(1)}\right) \quad .$$

If on the other hand y lies between $y^{(1)}$ and y_0, we compute

$$x^{(2)} = x_0 + \frac{y - y_0}{y^{(1)} - y_0} \left(x^{(1)} - x_0\right) \quad .$$

Lagrange's formula will provide a corresponding $y^{(2)}$ with which to estimate $x^{(3)}$, etc.

Double Interpolation

Suppose we are given a table of the form

	x_0	x_1	x_2	x_3	.	.	.
w_0	y_{00}	y_{01}	y_{02}	y_{03}	.	.	.
w_1	y_{10}	y_{11}	y_{12}	y_{13}	.	.	.
w_2	y_{20}	y_{21}	y_{22}	y_{23}	.	.	.
w_3	y_{30}	y_{31}	y_{32}	y_{33}	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

In this table we have double entries, w and x, associated with each table value y. The problem is to determine the table value y_{gh}, given (w_g, x_h). Special formulas have been derived for this problem based upon assumption of simple interpolation formulas. The more general procedure is to reduce the double entry table to a single entry table and then proceed with one of the formulas previously discussed. For example, by standard interpolation procedures already described, we first compute

$$y_{oh},\ y_{1h},\ y_{2h},\ y_{3h},\ \ldots \quad .$$

We then interpolate within this set, again by standard interpolation procedures already described, to obtain y_{gh}.

Differentiation

The procedure is to differentiate the appropriate interpolation formula. Bessel's formula will be used to illustrate the general method. We recognize that

$$\frac{dy}{dx} = \frac{dy}{dp}\,\frac{dp}{dx} = \frac{1}{\delta x}\,\frac{dy}{dp}$$

Hence,

$$\frac{dy}{dx} = \frac{1}{\delta x}\left[\Delta_{0.5} + \frac{(2p-1)}{4}\left(\Delta_0^{\,2} + \Delta_1^{\,2}\right) + \frac{3p^2-3p+0.5}{6}\,\Delta_{0.5}^3 + \frac{4p^3-6p^2-2p+2}{48}\left(\Delta_0^{\,4} + \Delta_1^{\,4}\right) + \ldots\right] \quad .$$

Repeated differentiation gives higher order derivatives, where

$$\frac{d^2y}{dx^2} = \frac{1}{(\delta x)^2}\,\frac{d^2y}{dp^2}\ ; \quad \frac{d^3y}{dx^3} = \frac{1}{(\delta x)^3}\,\frac{d^3y}{dx^3}\ ; \quad \text{etc.}$$

If the table values are not given for equidistant values of the entry, we may represent the table values by Lagrange's or Hermite's formula prior to differentiation. For example, in the case of a second degree polynomial, Lagrange's formula gives

$$\frac{dy}{dx} = \frac{2x-x_1-x_2}{(x_0-x_1)(x_0-x_2)}\, y_0 + \frac{2x-x_0-x_2}{(x_1-x_0)(x_1-x_2)}\, y_1 + \frac{2x-x_0-x_1}{(x_2-x_0)(x_2-x_1)}\, y_2 \quad .$$

Quadrature

Newton-Cotes Quadratures

These formulas require equal intervals between the table entries or independent variable. There is no limit to the number of these formulas, but there are only four that are used commonly. These are the trapezoid rule, Simpson's rule, three-eighths rule, and Weddle's rule. The total number of intervals covered in the integration is not restricted when the trapezoid rule is used. An even number of intervals must be covered in the integration when Simpson's rule is used. The number of intervals covered in the integration must be a multiple of three when the three-eighths rule is used. The number of intervals covered in the integration must be a multiple of six when Weddle's rule is used. In terms of accuracy these rules rank in descending order: Weddle's, three-eighths, Simpson's and trapezoid. Simpson's rule is probably the most useful of all formulas for quadrature.

The notation followed in this discussion of Newton-Cotes quadrature is explained by the following table:

value of independent variable x	value of corresponding function y=f(x)
x_0	y_0
$x_0+\delta_x$	y_1
$x_0+2\delta_x$	y_2
$x_0+3\delta_x$	y_3
.	.
.	.
.	.
$x_0+n\delta_x$	y_n

Trapezoid Rule

$$\int_{x_0}^{x_0+n\delta_x} f(x)dx = \frac{\delta_x}{2}\left[y_0+2y_1+2y_2+\ldots+2y_{n-1}+y_n\right]$$

where, excluding y_0 and y_n each of which has a coefficient of 1, every value of y has a coefficient of 2 inside the brackets.

Simpson's Rule

$$\int_{x_0}^{x_0+n\delta_x} f(x)dx = \frac{\delta_x}{3}\left[y_0+4y_1+2y_2+4y_3+2y_4+\ldots+2y_{n-2}+4y_{n-1}+y_n\right]$$

where, excluding y_0 and y_n each of which has a coefficient of 1, the repeating pattern of coefficients beginning with $y=y_1$ is 4,2 inside the brackets.

Three-Eighths Rule

$$\int_{x_0}^{x_0+n\delta} f(x)dx = \frac{3\delta_x}{8}\Big[y_0+3y_1+3y_2+2y_3+3y_4+3y_5$$
$$+2y_6+\ldots+2y_{n-3}+3y_{n-2}+3y_{n-1}+y_n\Big]$$

where, excluding y_0 and y_n each of which has a coefficient of 1, the repeating pattern of coefficients beginning with $y=y_1$ is 3,3,2 inside the brackets.

Weddle's Rule

$$\int_{x_0}^{x_0+n\delta_x} f(x)\,dx = \frac{3\delta_x}{10}\Big[y_0+5y_1+y_2+6y_3+y_4+5y_5+2y_6+5y_7+y_8+6y_9+y_{10}$$
$$+5y_{11}+2y_{12}+\ldots+2y_{n-6}+5y_{n-5}$$
$$+y_{n-4}+6y_{n-3}+y_{n-2}+5y_{n-1}+y_n\Big]$$

where, excluding y_0 and y_n each of which has a coefficient of 1, the repeating pattern of coefficients beginning with $y=y_1$ is 5,1,6,1,5,2 inside the brackets.

Gauss Quadrature

We set

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n} H_j\, f(a_j) \quad ,$$

an equation which holds exactly if f(x) is a power polynomial of degree (2n-1) or less. The discrete abscissae a_j and the corresponding weight coefficients H_j must be determined. Based upon Weierstrass' theorem, we conclude that the resulting quadrature formula should be capable of approximating definite integrals of any continuous function within a margin of error which should diminish, in general, with increasing value of n.

If a and b are finite, it is convenient and entails no loss in generality to assume that (b=+1) and that (a=-1). If this assumption is not true in an actual problem, it can be realized by an appropriate linear transformation of variables:

$$\int_{a}^{b} f(y)\,dy = \frac{b-a}{2} \int_{-1}^{+1} g(z)\,dz$$

where $g(z)\,dz$ is obtained by substituting $y = 1/2\left[(b-a)z + a+b\right]$ into $f(y)\,dy$. Consequently the problem is to evaluate the H_j and the a_j for the following equation:

$$\int_{-1}^{+1} f(x)\,dx = \sum_{j=1}^{n} H_j\, f(a_j) \quad .$$

Gauss has shown that if (f(x) is a power polynomial of (2n-1) degree or lower, then

$$\int_{-1}^{+1} f(x)\,dx = 2 \sum_{j=1}^{n} \frac{f(a_j)}{(1-a_j^2)\left[P_n'(a_j)\right]^2} \quad ,$$

where the a_j are roots of the Legendre polynomials $P_n(x)$ and where the $P_n'(a_j)$ are derivatives of $P_n(x)$ evaluated at a_j. Finding the roots of a Legendre polynomial $P_n(x)$ is equivalent to solving an algebraic equation with terms up to degree n. Having these roots, one can calculate H_j with little difficulty. Because of the difficulty in obtaining the roots, a_j, it is customary to use tabulated values for both a_j and H_j even in computer applications. Table Z-1 (from Bull. Amer. Math. Soc., 48, 739, 1942) in this appendix shows H and a to fifteen decimal places for n up to 16. Note that all weights H are positive. Because of the symmetry of the Legendre polynomials about their origin, their non-zero roots occur in pairs, plus and minus a_j, both members of a pair having the same weight.

TABLE Z-1

H_j			a_j
	n=2		
1.00000 00000 00000		± (and)	0.57735 02691 89626
	n=3		
0.55555 55555 55556		± (and)	0.77459 66692 41483
0.88888 88888 88889			0.00000 00000 00000
	n=4		
0.34785 48451 37454		± (and)	0.86113 63115 94053
0.65214 51548 62546		± (and)	0.33998 10435 84856
	n=5		
0.23692 68850 56189		± (and)	0.90617 98459 38664
0.47862 86704 99366		± (and)	0.53846 93101 05683
0.56888 88888 88889			0.00000 00000 00000
	n=6		
0.17132 44923 79170		± (and)	0.93246 95143 03152
0.36076 15730 48139		± (and)	0.66120 93864 66265
0.46791 39345 72691		± (and)	0.23861 91860 83197
	n=7		
0.12948 49661 68870		± (and)	0.94910 79123 42759
0.27970 53914 89277		± (and)	0.74153 11855 99394
0.38183 00505 05119		± (and)	0.40584 51513 77397
0.41795 91836 73469			0.00000 00000 00000

TABLE Z-1 (Cont'd)

H_j		a_j
	n=8	
0.10122 85362 90376	($\pm$ and)	0.96028 98564 97536
0.22238 10344 53374	($\pm$ and)	0.79666 64774 13627
0.31370 66458 77887	($\pm$ and)	0.52553 24099 16329
0.36268 37833 78362	($\pm$ and)	0.18343 46424 95650
	n=9	
0.08127 43883 61574	($\pm$ and)	0.96816 02395 07626
0.18064 81606 94857	($\pm$ and)	0.83603 11073 26636
0.26061 06964 02935	($\pm$ and)	0.61337 14327 00590
0.31234 70770 40003	($\pm$ and)	0.32425 34234 03809
0.33023 93550 01260		0.00000 00000 00000
	n=10	
0.06667 13443 08688	($\pm$ and)	0.97390 65285 17172
0.14945 13491 50581	($\pm$ and)	0.86506 33666 88985
0.21908 63625 15982	($\pm$ and)	0.67940 95682 99024
0.26926 67193 09996	($\pm$ and)	0.43339 53941 29247
0.29552 42247 14753	($\pm$ and)	0.14887 43389 81631
	n=11	
0.05566 85671 16174	($\pm$ and)	0.97822 86581 46057
0.12558 03694 64905	($\pm$ and)	0.88706 25997 68095
0.18629 02109 27734	($\pm$ and)	0.73015 20055 74049
0.23319 37645 91990	($\pm$ and)	0.51909 61292 06812
0.26280 45445 10247	($\pm$ and)	0.26954 31559 52345
0.27292 50867 77901		0.00000 00000 00000

TABLE Z-1 (Cont'd)

H_j		a_j
	n=12	
0.04717 53363 86512	($\pm$ and)	0.98156 06342 46719
0.10693 93259 95318	($\pm$ and)	0.90411 72563 70475
0.16007 83285 43346	($\pm$ and)	0.76990 26741 94305
0.20316 74267 23066	($\pm$ and)	0.58731 79542 86617
0.23349 25365 38355	($\pm$ and)	0.36783 14989 98180
0.24914 70458 13403	($\pm$ and)	0.12523 34085 11469
	n=13	
0.04048 40047 65316	($\pm$ and)	0.98418 30547 18588
0.09212 14998 37728	($\pm$ and)	0.91759 83992 22978
0.13887 35102 19787	($\pm$ and)	0.80157 80907 33310
0.17814 59807 61946	($\pm$ and)	0.64234 93394 40340
0.20781 60475 36889	($\pm$ and)	0.44849 27510 36447
0.22628 31802 62897	($\pm$ and)	0.23045 83159 55135
0.23255 15532 30874		0.00000 00000 00000
	n=14	
0.03511 94603 31752	($\pm$ and)	0.98628 38086 96812
0.08015 80871 59760	($\pm$ and)	0.92843 48836 63574
0.12151 85706 87903	($\pm$ and)	0.82720 13150 69765
0.15720 31671 58194	($\pm$ and)	0.68729 29048 11685
0.18553 83974 77938	($\pm$ and)	0.51524 86363 58154
0.20519 84637 21296	($\pm$ and)	0.31911 23689 27890
0.21526 38534 63158	($\pm$ and)	0.10805 49487 07344

TABLE Z-1 (Cont'd)

H_j			a_j
	n=15		
0.03075 32419 96117		(and $\pm$)	0.98799 25180 20485
0.07036 60474 88108		(and $\pm$)	0.93727 33924 00706
0.10715 92204 67172		(and $\pm$)	0.84820 65834 10427
0.13957 06779 26154		(and $\pm$)	0.72441 77313 60170
0.16626 92058 16994		(and $\pm$)	0.57097 21726 08539
0.18616 10000 15562		(and $\pm$)	0.39415 13470 77563
0.19843 14853 27112		(and $\pm$)	0.20119 40939 97435
0.20257 82419 25561			0.00000 00000 00000
	n=16		
0.02715 24594 11754		(and $\pm$)	0.98940 09349 91650
0.06225 35239 38648		(and $\pm$)	0.94457 50230 73233
0.09515 85116 82493		(and $\pm$)	0.86563 12023 87832
0.12462 89712 55534		(and $\pm$)	0.75540 44083 55003
0.14959 59888 16577		(and $\pm$)	0.61787 62444 02644
0.16915 65193 95003		(and $\pm$)	0.45801 67776 57227
0.18260 34150 44924		(and $\pm$)	0.28160 35507 79259
0.18945 06104 55068		(and $\pm$)	0.09501 25098 37637

Numerical Integration

A general treatment of this topic is far outside the scope of this report. The particular problem to be discussed here is the type associated with the solution of the equations of motion of a missile or satellite. We have a system of equations defined by

$$y_i'' = f_i(t, y_1, \ldots, y_N, y_1', \ldots, y_N')\ (i=1,2,\ldots,N),$$

where we are given as initial conditions

$$y_i(t_o) = y_{io},\ y_i'(t_o) = y_{io}'\ (i=1,2,\ldots,N) \quad .$$

For a rectangular coordinate system and no variational equations N would be 3, but with variational equations N may be much larger. The fourth order Runge-Kutta process is almost universally used to start the integration.

The Fourth Order Runge-Kutta Process

If we denote the values of y_i and y_i' at $t=t_n$ by y_{in} and y_{in}', respectively, and if we let h be the step size of the independent variable t, then the following Runge-Kutta algorithm is applied to each of the N simultaneous equations in order to complete one integration step:

$$k_{i1} = h\ f_i(t_n, y_{in}, y_{in}') \quad ,$$

$$k_{i2} = h\ f_i(t_n + \frac{h}{2}, y_{in} + \frac{h}{2} y_{in}' + \frac{h}{8} k_{i1}, y_{in}' + \frac{k_{i1}}{2}) \quad ,$$

$$k_{i3} = h\ f_i(t_n + \frac{h}{2}, y_{in} + \frac{h}{2} y_{in}' + \frac{h}{8} k_{i1}, y_{in}' + \frac{k_{i2}}{2}) \quad ,$$

$$k_{i4} = h\ f_i(t_n + h, y_{in} + h\ y_{in}' + \frac{h}{2} k_{i3}, y_{in}' + k_{i3}) \quad ,$$

$$\Delta y_{in} = h \left[y_{in}' + \frac{1}{6} (k_{i1} + k_{i2} + k_{i3}) \right] \quad ,$$

$$\Delta y_{in}' = \frac{1}{6} \left[k_{i1} + 2\ k_{i2} + 2\ k_{i3} + k_{i4} \right] \quad ,$$

$$y_{i,n+1} = y_{in} + \Delta y_{in} \quad ,$$

$$y_{i,n+1}' = y_{in}' + \Delta y_{in}' \quad .$$

The output of the first integration step serves as the input for the next integration step and so on. We require n integration steps through Runge-Kutta to start an n^{th} order Cowell process. The Cowell process is almost universally used for orbital and long range missile trajectories.

Cowell Integration Process

The Cowell integration process is widely known as the Gauss-Jackson integration process and also simply as the second-sum integration process. The procedure is usually characterized by a table of differences. In some research investigations — particularly in the field of celestial mechanics — differences up to the tenth are used. At the Eastern Test Range differences up to the sixth are adequate for over 99% of the applications, and differences higher than the eighth are never used. An eighth-difference algorithm presented here follows closely that used by Aerospace in the TRACE-66 computer program. A slight departure has been incorporated in the Aerospace algorithm in that all major equations are given in terms of differences. This departure facilitates scaling the algorithm down or up from eighth order. For lower order applications one may simply drop off higher differences. For higher order applications one must establish the necessary constants by integrating the appropriate interpolation formulas.

Consider the difference table, Table Z-2. In order to start the eighth-order Cowell process, we need the eleven values immediately above the diagonal line. All the second derivatives of y_i except y''_{19} are obtained in the preceding Runge-Kutta integration. Each Δ in the triangular array above the diagonal line is then computed by differencing the value immediately above it from the value immediately below it in the preceding column. For example, $\Delta'_{14} = y''_{15} - y''_{14}$; $\Delta'_{15} = y''_{16} - y''_{15}$; $\Delta''_{14} = \Delta'_{15} - \Delta'_{14}$; etc. Initially, therefore, all the values in the triangular array above the diagonal line and to the right of the vertical line come from Runge-Kutta. We now proceed to fill in the table values that lie above the diagonal line and to the left of the vertical line. With an integration step size of h, compute

$$'F_{15} = y'_{14}/h - D_0 y''_{14} - D_1 \Delta^{I}_{14} - D_2 \Delta^{II}_{13} - D_3 \Delta^{III}_{13}$$

$$- D_4 \Delta^{IV}_{12} - D_5 \Delta^{V}_{12} - D_6 \Delta^{VI}_{11} - D_7 \Delta^{VII}_{11} - D_8 \Delta^{VIII}_{10}$$

and

$$''F_{15} = y_{14}/h^2 - C_0 Y''_{14} - C_2 \Delta^{II}_{13} - C_4 \Delta^{IV}_{12} - C_6 \Delta^{VI}_{11} - C_8 \Delta^{VIII}_{10} \quad ,$$

where the C and D cc fficients are given (along with some others needed later) in Table Z-3. The remaining F values above the diagonal line in Table Z-2 are computed recursively in the following sequence:

$$'F_{1K+1} = 'F_{1K} + y''_{1K} \qquad (K = 5, 6, \ldots, 8)$$

$$''F_{1K+1} = ''F_{1K} + 'F_{1K} \qquad (K = 5, 6, \ldots, 9)$$

It may be remarked here that all the values in the difference table above the diagonal line are now complete and final for this integration step. The Cowell prediction-correction process will now make three successive estimates for y_{19} and also y'_{19}, two successive estimates for each value that lies beneath the diagonal line and to the right of the vertical line, and one estimate for each value that lies beneath the diagonal line and to the left of the vertical line.

The prediction process proceeds as follows:

$$y_{19} = h^2 (''F_{110} + A_0 y''_{18} + A_1 \Delta^{I}_{17} + A_2 \Delta^{II}_{16} + A_3 \Delta^{III}_{15}$$

$$+ A_4 \Delta^{IV}_{14} + A_5 \Delta^{V}_{13} + A_6 \Delta^{VI}_{12} + A_7 \Delta^{VII}_{11} + A_8 \Delta^{VIII}_{10})$$

$$y'_{19} = h ('F_{19} + \dot{A}_0 y''_{18} + \dot{A}_1 \Delta^{I}_{17} + \dot{A}_2 \Delta^{II}_{16} + \dot{A}_3 \Delta^{III}_{15}$$

$$+ \dot{A}_4 \Delta^{IV}_{14} + \dot{A}_5 \Delta^{V}_{13} + \dot{A}_6 \Delta^{VI}_{12} + \dot{A}_7 \Delta^{VII}_{11} + \dot{A}_8 \Delta^{VIII}_{10})$$

where the values for A and $\dot{A}$ are given in Table Z-3.

Note that this prediction process uses only the row of table values immediately above the diagonal line. Substitution of these estimates of y_{i9} and y'_{i9} into the original set of simultaneous equations provides a set of estimates for y''_{i9}. We are now in a position to calculate the remaining values in the difference table that lie in the row beneath the diagonal line and to the right of the vertical line. For example: $\Delta^{I}_{i8} = y''_{i9} - y''_{i8}$; $\Delta^{II}_{i7} = \Delta^{I}_{i8} - \Delta^{I}_{i7}$; etc.

We now apply this row of values directly beneath the diagonal line to correct our estimates for y_{i9} and y'_{i9}:

$$y_{i9} = h^2({}''F_{i10} + B_0 y''_{i9} + B_1 \Delta^{I}_{i8} + B_2 \Delta^{II}_{i7} + B_3 \Delta^{III}_{i6}$$

$$+ B_4 \Delta^{IV}_{i5} + B_5 \Delta^{V}_{i4} + B_6 \Delta^{VI}_{i3} + B_7 \Delta^{VII}_{i2} + B_8 \Delta^{VIII}_{i1})$$

$$y'_{i9} = h({}'F_{i9} + \dot{B}_0 y''_{i9} + \dot{B}_1 \Delta^{I}_{i8} + \dot{B}_2 \Delta^{II}_{i7} + \dot{B}_3 \Delta^{III}_{i6}$$

$$+ \dot{B}_4 \Delta^{IV}_{i5} + \dot{B}_5 \Delta^{V}_{i4} + \dot{B}_6 \Delta^{VI}_{i3} + \dot{B}_7 \Delta^{VII}_{i2} + \dot{B}_8 \Delta^{VIII}_{i1})$$

where the values for B and $\dot{B}$ are given in Table Z-3. Substitution of these new estimates of y_{i9} and y'_{i9} into the original set of simultaneous equations provides a new and final set of estimates for y''_{i9}. We now calculate new and final estimates for the remaining values in the difference table that lie in the row beneath the diagonal line and to the right of the vertical line: $\Delta^{I}_{i8} = y''_{i9} - y''_{i8}$; etc.

We now apply this row of values directly beneath the diagonal line to recorrect our corrected estimates of y_{i9} and y'_{i9}, using the correction equations just presented. These recorrected estimates constitute the final estimates for y_{i9} and y'_{i9}. This integration step is completed by computing ${}'F_{i10} = {}'F_{i9} + y''_{i9}$ and ${}''F_{i11} = {}''F_{i10} + {}'F_{i10}$.

Note that one must carry out each stage of the integration step with each of the N simultaneous equations before proceeding with the next stage.

Before beginning the next integration step we must effectively shift every element in the difference table upward so that the row immediately below the diagonal line becomes the row immediately above the diagonal line.

Only the basic essentials of the Cowell process have been presented. A step-size selector is always incorporated in the algorithm. It is possible to estimate the error in the integration by comparing predicted and corrected estimates for the position and velocity vectors. If the error is excessive, the step size is reduced. If conditions permit, the step size is increased. Reduction in step size necessitates a return to the Runge-Kutta process for a new start.

TABLE Z-2

DIFFERENCE TABLE

		y''_{i0}								
			Δ^{I}_{i0}							
		y'_{i1}		Δ^{II}_{i0}						
			Δ^{I}_{i1}		Δ^{III}_{i0}					
		y''_{i2}		Δ^{II}_{i1}		Δ^{IV}_{i0}				
			Δ^{I}_{i2}		Δ^{III}_{i1}		Δ^{V}_{i0}			
		y''_{i3}		Δ^{II}_{i2}		Δ^{IV}_{i1}		Δ^{VI}_{i0}		
			Δ^{I}_{i3}		Δ^{III}_{i2}		Δ^{V}_{i1}		Δ^{VII}_{i0}	
${}''F_{i5}$		y''_{i4}		Δ^{II}_{i3}		Δ^{IV}_{i2}		Δ^{VI}_{i1}		Δ^{VIII}_{i0}
	${}'F_{i5}$		Δ^{I}_{i4}		Δ^{III}_{i3}		Δ^{V}_{i2}		Δ^{VII}_{i1}	
${}''F_{i6}$		y''_{i5}		Δ^{II}_{i4}		Δ^{IV}_{i3}		Δ^{VI}_{i2}		Δ^{VIII}_{i1}
	${}'F_{i6}$		Δ^{I}_{i5}		Δ^{III}_{i4}		Δ^{V}_{i3}		Δ^{VII}_{i2}	
${}''F_{i7}$		y''_{i6}		Δ^{II}_{i5}		Δ^{IV}_{i4}		Δ^{VI}_{i3}		
	${}'F_{i7}$		Δ^{I}_{i6}		Δ^{III}_{i5}		Δ^{V}_{i4}			
${}''F_{i8}$		y''_{i7}		Δ^{II}_{i6}		Δ^{IV}_{i5}				
	${}'F_{i8}$		Δ^{I}_{i7}		Δ^{III}_{i6}					
${}''F_{i9}$		y''_{i8}		Δ^{II}_{i7}						
	${}'F_{i9}$		Δ^{I}_{i8}							
${}''F_{i10}$		y''_{i9}								
	${}'F_{i10}$									
${}''F_{i11}$										

TABLE Z-3

CONSTANT COEFFICIENTS

$i =$	0	1	2	3	4	5	6	7	8
D_i	$-\frac{1}{2}$	$\frac{-1}{12}$	$\frac{1}{24}$	$\frac{11}{720}$	$\frac{-11}{1440}$	$\frac{-191}{60480}$	$\frac{191}{120960}$	$\frac{2497}{3628800}$	$-\frac{2497}{7257600}$
C_i	$\frac{1}{12}$	0	$\frac{-1}{240}$	0	$\frac{31}{60480}$	0	$\frac{-289}{3628800}$	0	$\frac{317}{22809600}$
A_i	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{19}{240}$	$\frac{3}{40}$	$\frac{863}{12096}$	$\frac{275}{4032}$	$\frac{33953}{518400}$	$\frac{8183}{129600}$	$\frac{3250433}{53222400}$
$\dot{A}_i$	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$	$\frac{19087}{60480}$	$\frac{5257}{17280}$	$\frac{1070017}{3628800}$	$\frac{25713}{89600}$
B_i	$\frac{1}{12}$	0	$\frac{-1}{240}$	$\frac{-1}{240}$	$\frac{-221}{60480}$	$\frac{-19}{60480}$	$\frac{-9829}{3628800}$	$-\frac{407}{172800}$	$\frac{-330157}{159667200}$
$\dot{B}_i$	$\frac{1}{2}$	$\frac{-1}{12}$	$\frac{-1}{24}$	$\frac{-19}{720}$	$\frac{-3}{160}$	$\frac{-863}{60480}$	$\frac{-275}{24192}$	$\frac{-33953}{3628800}$	$-\frac{8183}{1036800}$

APPENDIX AA

BASIC REGRESSION RELATIONS - RIDGE REGRESSION

1.0 INTRODUCTION

Well-known to analysts associated with regression studies are the problems related to ill-conditioned matrices - problems leading to loss of precision, grossly inaccurate (inflated) estimates of the parameters and gross underestimates of the errors in the estimates. These problems have been the subject of a great many investigations. One of the most fruitful approaches was developed by Hoerl and Kennard and labeled by them "ridge regression". The Hoerl-Kennard (HK) estimation process is inherently Bayesian in nature. It assumes expected values of zero for the adjustable parameters and tends to constrain the adjusted values as close as possible to zero without unduly enlarging the residuals.

We extend ridge regression to cases where a priori expected values and variances of the adjusted parameters may be used. The resulting ridge estimator is general in nature and specifically adaptable to ordinary orbit determination computer programs. Some appropriate background material is furnished in Section 2. This is followed in Section 3 by the mathematical development of the ridge estimator using prior information. In Section 4, an application is shown based on a satellite orbit estimated from radar tracking data, and this is followed in Section 5 by a brief summary of other ridge estimators and in Section 6 by some concluding remarks.

2.0 STANDARD REGRESSION USING PRIOR INFORMATION

We assume the standard linear approximation to the general nonlinear multiple regression problem:

$$Y = X\beta + \epsilon_m \qquad (2.1)$$

where Y is (nx1) and denotes the measurement vector; X is (nxj) and of rank j and represents the partial derivative matrix of nonstochastic elements relating the mean values of the measurements to the adjusted parameters; β is $(j\times1)$ and designates the true fixed but unknown parameter vector; ε_m is $(n\times1)$ and constitutes the measurement error vector. We assume $E(\varepsilon_m)$ is zero and that $E(\varepsilon_m\varepsilon_m') = VAR(\varepsilon_m) = \Sigma_m$, where Σ_m is $(n\times n)$ and known.

In addition we have prior information consisting of a k element parameter vector β_o, which estimates $R\beta$ and a k element parameter error vector ε_p. β_o is known from introspection or from previous independent measurements. Therefore

$$\beta_o = R\beta + \varepsilon_p \tag{2.2}$$

where R is $(k\times j)$ of rank k and consists of known nonstochastic elements. If, for example, $R = [I0]$, where I is a $(k\times k)$ unit matrix and 0 is a $[k \times (j-k)]$ zero matrix, then β_o represents estimates of the first k elements of β. Equation (2.2) assumes β_o is random and hence represents a departure from the Bayesian approach, which assumes a prior distribution on β, here considered fixed. In addition $E(\varepsilon_p)$ is zero and $E(\varepsilon_p\varepsilon_p') = VAR(\varepsilon_p) = \Sigma_p$, where Σ_p is $(k\times k)$ and assumed to be known. Furthermore we assume $COV(\varepsilon_m, \varepsilon_p)$ is zero.

In order to include the prior information in the estimation of β, we combine equations (2.1) and (2.2) as follows:

$$\begin{bmatrix} Y \\ \beta_o \end{bmatrix} = \begin{bmatrix} X \\ R \end{bmatrix} \beta + \begin{bmatrix} \varepsilon_m \\ \varepsilon_p \end{bmatrix} \tag{2.3}$$

or in an obvious change of notation

$$\tilde{Y} = \tilde{X}\beta + \tilde{\varepsilon} \tag{2.4}$$

where

$$VAR(\tilde{\varepsilon}) = \begin{bmatrix} \Sigma_m & 0 \\ 0 & \Sigma_p \end{bmatrix} = \tilde{\Sigma} \quad .$$

The prior information has thus assumed the role of measurements. Applying generalized least squares to (2.4), we obtain the following relation for the estimator $\hat{\beta}$ of the parameter vector β:

$$\hat{\beta} = (\tilde{X}' \tilde{\Sigma}^{-1} \tilde{X})^{-1} \quad (\tilde{X}' \tilde{\Sigma}^{-1} \tilde{Y}) \quad . \tag{2.5}$$

This converts by simple substitution to

$$\hat{\beta} = (X' \Sigma_m^{-1} X + R' \Sigma_p^{-1} R)^{-1} \quad (X' \Sigma_m^{-1} Y + R' \Sigma_p^{-1} \beta_o) \tag{2.6}$$

or in an obvious change in notation

$$\hat{\beta} = P(X' \Sigma_m^{-1} Y + R' \Sigma_p^{-1} \beta_o) \quad . \tag{2.7}$$

It can be shown that $\hat{\beta}$ is an unbiased estimator of β and that $VAR(\hat{\beta}) = P$.

The fractional part of posterior precision due to sample information is

$$(1/j) \ tr \left[X' \Sigma_m^{-1} X \ (X' \Sigma_m^{-1} X + R' \Sigma_p^{-1} R)^{-1} \right] \quad .$$

3.0 RIDGE ESTIMATOR USING PRIOR INFORMATION

Highly correlated errors in the parameter estimates result in poor conditioning of the $(X' \Sigma_m^{-1} X + R' \Sigma_p^{-1} R)$ matrix in Equation (2.6). The poorer the condition of this matrix, the more $(\hat{\beta}-\beta_o)$ can be expected to be too long, thereby resulting in considerable disagreement between $\hat{\beta}$ and the true vector β. On the other hand, the worse the conditioning, the less is the sensitivity of the residual sum of squares to small departures from $\hat{\beta}$. Following the concept of Hoerl and Kennard, we impose an accessory condition upon the least squares criterion, thereby restraining the length of the vector $(\hat{\beta}-\beta_o)$ without greatly influencing the residual sum of squares.

Let B be any estimate of the vector β. Then the sum of squares of the weighted "measurement" residuals is given by $(\tilde{Y}-\tilde{X}B)' \tilde{\Sigma}^{-1} (\tilde{Y}-\tilde{X}B)$.

Using Lagrangian constraints, we minimize

$$F = (B-\beta_o)'\Sigma_p^{-1}(B-\beta_o) + \frac{1}{h} \left[(\tilde{Y}-\tilde{X}B)'\tilde{\Sigma}^{-1}(\tilde{Y}-\tilde{X}B) - \phi\right] \quad (3.1)$$

where (1/h) is the multiplier and ϕ is the total sum of squares. We obtain therefore

$$\frac{\partial F}{\partial B} = 0 = \Sigma_p^{-1}(B-\beta_o) - \left(\frac{1}{h}\right) \tilde{X}'\tilde{\Sigma}^{-1}(\tilde{Y}-\tilde{X}B) \quad . \quad (3.2)$$

This reduces to

$$B = \hat{\beta}^* = \left[X'\Sigma_m^{-1} X+(h+1)R'\Sigma_p^{-1} R\right]^{-1}\left[X'\Sigma_m^{-1} Y+(h+1)R'\Sigma_p^{-1} \beta_o\right]; \ h \geq 0$$

or in an obvious change in notation (3.3)

$$\hat{\beta}^* = Q\left[X'\Sigma_m^{-1} Y + (h+1) R'\Sigma_p^{-1} \beta_o\right] \ ; \ h \geq 0 \quad . \quad (3.4)$$

Equation (3.3) constitutes the ridge estimator with prior information and has the same form as Equation (2.6). The input constant $[(h+1)\Sigma_p^{-1}]$ has replaced the input constant (Σ_p^{-1}). Computation and evaluation of the ridge trace is handled in a way similar to that used with the (HK) estimator and will be demonstrated in Section 4.

$\hat{\beta}^*$ is related to $\hat{\beta}$ as follows:

$$\hat{\beta}^* = Q P^{-1} \hat{\beta} + Q h R' \Sigma_p^{-1} \beta_o \quad . \quad (3.5)$$

Application of the law of covariance propagation gives

$$VAR(\hat{\beta}^*) = Q\left[X'\Sigma_m^{-1} X + (h+1)^2 R'\Sigma_p^{-1} R\right]Q' \quad . \quad (3.6)$$

The sum of the squared biases of all the adjusted parameters is given by

$$\beta' \left[Q\, P^{-1} + Q\, h\, R' \Sigma_p^{-1} R - I_j\right]' \left[Q\, P^{-1} + Q\, h\, R' \Sigma_p^{-1} R - I_j\right] \beta = 0\ . \tag{3.7}$$

For computational convenience, Equation (3.3) may be linearized to obtain the following iterative form:

$$\Delta\hat{\beta}^* = \left[X' \Sigma_m^{-1} X + (h+1) R' \Sigma_p^{-1} R\right]^{-1} \left[X' \Sigma_m (\Delta Y) + (h+1) R' \Sigma_p^{-1} (\Delta\beta_o)\right] \quad ;\ h \geq 0 \tag{3.8}$$

where $\Delta\hat{\beta}^*$ is the vector of corrections to the current estimates of the adjusted parameters; ΔY is the vector of measurement residuals; $\Delta\beta_o$ is the vector of differences between current and a priori estimates of the parameters. When $h = 0$, Equation (3.8) reduces to a linearized form of Equation (2.6).

4.0 APPLICATION IN ORBIT DETERMINATION

The readers with a knowledge of regression analysis already have an explicit concept of the process of orbit determination. In standard orbit determination the parameters regressed upon are the orbital elements plus usually a dozen or more other parameters relating to the environment and the tracker characteristics.

The present application involves a standard satellite orbit determination (Cowell, special perturbations, batch processing) in which the adjustable parameters include eight radar coefficients and six orbital elements. The measurements are radar track data. The radar track data are characterized by certain errors which may be expressed as linear terms in the so-called radar measurement equations. The radar measurement equations, abbreviated so as to contain only terms of present interest, are as follows:

$$\underbrace{R}_{\text{measurement}} = \underbrace{R_t}_{\text{true}} + \underbrace{r_1}_{\text{zero set}} + \underbrace{\varepsilon_R}_{\text{random error}} \tag{4.1}$$

$$\begin{aligned}\underbrace{A}_{\text{measurement}} &= \underbrace{A_t}_{\text{true}} + \underbrace{a_1}_{\text{zero set}} + \underbrace{a_2 \sec E_t}_{\text{collimation}} \\ &+ \underbrace{a_3 \tan E_t}_{\text{nonorthogonality}} + \underbrace{u \sin A_t \tan E_t - v \cos A_t \tan E_t}_{\text{mislevel}} \\ &+ \underbrace{\varepsilon_A}_{\text{random error}}\end{aligned} \tag{4.2}$$

$$\begin{aligned}\underbrace{E}_{\text{measurement}} &= \underbrace{E_t}_{\text{true}} + \underbrace{e_1}_{\text{zero set}} + \underbrace{e_2 \cos E_t}_{\text{droop}} \\ &+ \underbrace{u \cos A_t + v \sin A_t}_{\text{mislevel}} + \underbrace{\varepsilon_E}_{\text{random error}}\end{aligned} \tag{4.3}$$

In these equations R represents range; A, azimuth; E, elevation. The zero-set errors are constant bias or off-set values. Collimation represents the lack of perpendicularity between the radar beam and the elevation axis. Mislevel represents the tilt of the azimuth plane - u being the northward component and v being the eastward component. This tilt is defined with respect to the local horizontal to the geodetic spheroid. Droop represents the sag of the radar beam axis. The random errors represent noise in the data and have zero means. The adjustment exercise provides estimates for r_1, a_1, a_2, a_3, e_1, e_2, u and v. Along with these coefficients, estimates are also made of six orbital elements.

The ($\alpha\delta\gamma ARV$) orbital element set is used in this illustration. The individual elements of this set are respectively right ascension and declination of the position vector, flight path angle, azimuth of the velocity vector, length of the position vector and magnitude of the velocity vector. In contrast to the classical elements, the ($\alpha\delta\gamma ARV$) set varies rapidly with time, a characteristic which is sometimes a disadvantage although not in the present application.

The mathematical adjustment procedure is iterative and is based upon Eq. (3.8), the equations of motion of the satellite, and the radar measurement equations. Initially, with $h = 0$, the procedure is the standard one in orbit determination. After a converged solution with $h = 0$ has been obtained, then ten or so additional solutions with different values of h are computed in order to define the curves comprising the ridge trace. The correlation matrix with $h = 0$ is shown in Table I with elements rounded to three digits.

In the ridge trace, Figure 1, we plot $(\hat{\beta}_i^* - \beta_{oi})/\sigma_{oi}$ vs h and also show the root-mean-square of the weighted measurement residuals vs h. The symbol σ_{oi} is used for the a priori standard deviation in β_{oi}, where the subscript i designates the i^{th} element in the parameter vector. Note the typically inflated values for the estimates of e_1 and e_2 at $h = 0$. We see that the estimate of e_1 is 4.4 standard deviations larger algebraically than its expected value, and the estimate of e_2 is 3.7 standard deviations smaller algebraically than its expected value. This inflation is associated with a correlation coefficient of -0.985 between the errors in the estimates of e_1 and e_2. At $h = 40$, the estimated e_1 is only 1.7 standard deviations from its expected value and the estimated e_2 is only 0.1 standard deviations from its expected value. Stability is achieved at around $h = 40$, at which time $\hat{\beta}^*$ behaves more like an orthogonal vector. Also note that the root-mean-square of the weighted measurement residuals has increased only from 0.991 to 0.994 as h moves from zero to forty. Inflation to some extent is also shown at h = 0 by the estimates of a_1, a_2 and a_3. Their absolute values contract as h moves from zero to forty. The remaining coefficients are only slightly affected by changes in h. Without significantly increasing the residual sum of squares, this numerical example provides a set of radar coefficients more consistent with the a priori values and variances than would have been obtained from standard regression procedures.

5.0 OTHER RIDGE ESTIMATORS

In the previous discussion we have chosen to minimize the sum of the weighted squared differences between the regression estimates and a priori estimates subject to a side condition which places a limit on the amount by which the residual weighted sum of squares may exceed the minimum value. The resulting estimator is more flexible than the other two to be mentioned in this section and is also directly applicable to standard orbit determination programs. It provides for the explicit inclusion of both a priori estimates of the parameters and the covariance matrix of the a priori estimates. The estimator is given in Equation (3.3); the variance in the estimate is given in Equation (3.6); and the estimator is unbiased.

Utilizing the same mathematical procedure, one may minimize the sum of the squared differences between the regression coefficients and a priori estimates subject to a side condition which places a limit on the amount by which the residual sum of squares may exceed the minimum value. The resulting estimator is

$$\hat{\beta}^* = \left[X'X + k\,I\right]^{-1}\left[X'Y + k\,\beta_o\right] \quad : \quad k \geq o \quad , \qquad (5.1)$$

where $X'X$ is in the form of a correlation matrix; measurements are uncorrelated and units are chosen to achieve common variance σ^2 in the measurements; I is identity matrix. The variance in the estimates is given by

$$VAR\ (\hat{\beta}^*) = \sigma^2\left[I + k\,(X'X)^{-1}\right]^{-1}(X'X)^{-1}\left[I + k\,(X'X)^{-1}\right]^{-1} . \qquad (5.2)$$

The bias in the estimates is given by

$$(E\ \hat{\hat{\beta}}* - \beta)'\ (E\ \hat{\hat{\beta}}* - \beta) - k^2\ (\beta-\beta_o)'\ (X'\ X + k\ I)^{-2}\ (\beta-\beta_o) \quad . \qquad (5.3)$$

Hoerl and Kennard assume the a priori estimates of the parameters to be zero and hence they minimize the sum of squares of the regression estimates subject to a side condition which places a limit on the amount by which the residual sum of squares may exceed the minimum value. The resulting estimator, variance and bias are given by Equations (5.1),(5.2) and (5.3) respectively with β_o set to zero.

6.0 CONCLUDING REMARKS

This discussion illustrates an important deficiency in the standard least-squares, point-estimation procedure: In the case of high correlation among the errors in the parameter estimates, there may be a gross inflation of the adjustment vector in order to achieve a final minuscular reduction in the sum of squares of the residuals.

All three estimators discussed in the previous section fulfill the Bayesian desideratum that estimates be held as closely as possible to a priori expected values so long as the residuals are not significantly larger than in standard least-squares regression. In mean-square-error characteristics, however, these estimators differ somewhat. The estimator given in Equation (5.1), with or without non-zero β_o, is biased. The bias tends to increase the mean square error relative to standard least squares, which is unbiased, but the variance associated with the (5.1) estimator is so much less than that from standard least squares that over some (unknown) range in k values the mean square error is less than that from standard least squares. The use of the ridge trace with the (5.1) estimator does not assure reduced mean square error. The estimator given in Equation (3.3) has the generally desirable property of being unbiased, but the variance associated with it is generally larger than that from standard least squares. (If one subtracts the variance given in Equation 3.6 with h = 0 from the corresponding variance with $h > 0$, one obtains a positive definite matrix.) Hence with exact models the estimator given in Equation (3.3) will generally produce larger mean square error than does standard least squares.

In real life one is likely to encounter under-specified models whose use leads to inflated estimates in cases of high correlation. In such circumstances all three estimators discussed in the previous section usually give smaller mean square error than does standard least squares. Inflation of the type shown in Figure 1 is usually characteristic of a misfit between the data and the model — a circumstance which is frequently unavoidable. A typical situation associated with inflation is high correlation between the errors in two or more parameters in the specified model, accompanied by high correlation between these errors and the error in an unmodeled term. Application of the ridge estimators does not of course eliminate the effect from the error in the unmodeled term, but it reduces the effect to a value comparable to the error — hopefully small — in the unmodeled term.

At first glance the reader might be alarmed at the rather large value of 40 arrived at for h in the numerical example. It appears that the prior information has been given (nearly) full weight. Actually this is not the case. If the prior information had been given full weight, then the curves in Figure 1 would show a general tendency to be tangent to the zero line at $h = 40$, whereas most of them show a strong disinclination to approach zero even at $h = 70$. Furthermore, if h had been increased to the point where prior information was given full weight the residual sum of squares would — except in a prohibitively unlikely coincidence — have shown a marked increase. In reality the demonstrated ridge estimation procedure has a significant effect only upon the parameter estimates whose errors are mutually correlated, and with these the adjustments are minimized and portioned out inversely according to their a priori variances so far as possible, without unduly enlarging the residuals.

An interesting characteristic of a typical ridge trace is that for all values of the multiplier the following simple equation holds quite closely:

$$a_1 x_1 + a_2 x_2 + \ldots a_n x_n = 1 \quad (6.1)$$

In this equation the x's represent the parameter estimates*, and the a's are empirical weight constants calculated from the ridge trace. Each "a" is a measure of the "effect" of unit value of its respective parameter. The total "effect" is constant and arbitrarily designated as unity.

The R matrix is included in Equation (3.3) for mathematical generality, but there is a possibility of confusion in using $R \neq I$ in ridge regression. For example, if $R \neq I$, then one or more of the adjusted parameters are effectively assigned infinite a priori variances and cannot be represented in the ridge trace. In the event that one of the parameters with infinite variance has error highly correlated with that of a parameter with pre-assigned small finite variance, then the ridge analysis may result in practically all of the adjustment being thrown into the parameter with infinite variance. (In this circumstance, incidentally, Equation (6.1) will not hold.) The final result may nevertheless be acceptable, but the analyst should realize that he has effectively discarded that particular parameter with small finite variance from the ridge regression and assigned it its a priori value.

In many applications — for example, orbit determination — the number of observations may run into the thousands, and the resulting $X'X$ matrix (or equivalent) becomes too large to invert on a practical basis. In such a circumstance the analyst may choose to reduce the data density to a rate where serial correlation is negligible. If serial correlation can be ignored, then the size

*that is, the values plotted in the ridge trace as a function of the multiplier.

of the matrix to be inverted is only j by j, where j is the number of adjusted parameters. For example, Equation (3.8) with $h = 0$ and $R = I$ simplifies to

$$\Delta\hat{\beta} = \left[\sum_{i=1}^{n} (X_i' \Sigma_{m_i}^{-1} X_i) + \Sigma_p^{-1}\right]^{-1} \left\{\sum_{i=1}^{n}\left[X_i' \Sigma_{m_i}^{-1} (\Delta Y)_i\right] + \Sigma_p^{-1} (\Delta\beta_o)\right\}, \tag{6.2}$$

where n is the number of observations.

Equation (6.2) with some changes in notation is used and discussed in Appendix S.

TABLE I

CORRELATION COEFFICIENTS FOR 14 FACTOR EXAMPLE

r_1	a_1	a_2	a_3	e_1	e_2	u	v	α	δ	γ	A	R	V
1													
-0.008	1												
-0.003	-0.965	1											
0.002	0.313	-0.516	1										
-0.005	0.000	0.001	0.000	1									
0.011	0.001	-0.001	-0.001	-0.985	1								
0.021	-0.018	0.011	0.008	-0.088	0.095	1							
0.004	-0.061	0.047	0.037	0.045	-0.060	-0.133	1						
-0.557	0.011	0.005	-0.001	0.028	-0.018	0.109	0.009	1					
0.431	-0.011	-0.005	0.002	-0.041	0.021	0.083	0.012	-0.336	1				
0.210	0.002	-0.000	-0.002	0.014	-0.003	0.194	0.008	0.498	0.367	1			
0.512	-0.014	-0.066	0.003	-0.047	0.025	-0.044	0.013	-0.764	0.796	-0.158	1		
-0.007	-0.001	-0.002	-0.002	-0.012	0.005	0.174	0.011	0.456	0.625	0.833	0.096	1	
0.004	0.001	0.003	0.003	0.014	-0.005	-0.171	-0.012	-0.433	-0.642	-0.817	-0.123	-0.999	1

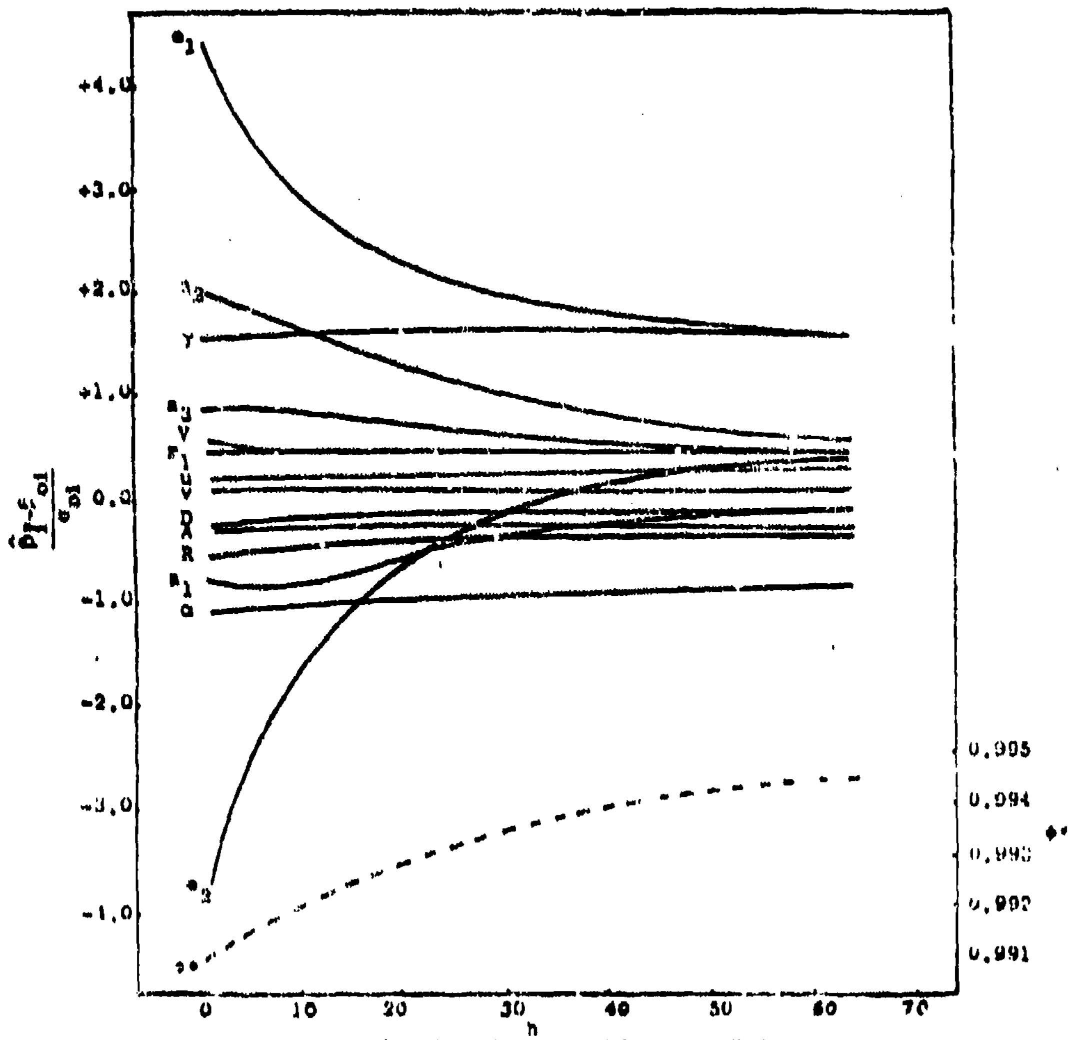


FIGURE 1. RIDGE TRACE 14-FACTOR EXAMPLE

APPENDIX AB

DERIVATIVES OF MATRICES

Because of the necessity for limiting the size of this report, it has been necessary to omit many derivations or at least many steps in the derivations. The material of this appendix - perhaps not immediately available elsewhere - is offered as background for the reader who may wish to derive equations given earlier. Applications are primarily in the realm of statistical transformations.

There is no firm convention for the forms of many of these derivatives, and consequently the mathematician is free to adopt his own forms so long as he defines them. The rather loose conventions - so far as they exist - are given at this point. For purposes of illustration, we first define the following five quantities:

S is (1×1)

V is $(p \times 1)$ with elements v_i

T is $(q \times 1)$

M is $(g \times h)$ with elements m_{ij}

N is $(r \times t)$

The corresponding derivative forms are as follows:

$$\underset{(p \times 1)}{\frac{\partial V}{\partial S}} = \begin{bmatrix} \frac{\partial v_1}{\partial S} \\ \frac{\partial v_2}{\partial S} \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial v_p}{\partial S} \end{bmatrix}$$

$$\underset{(g\times h)}{\frac{\partial M}{\partial s}} = \begin{bmatrix} \frac{\partial m_{11}}{\partial s} & \frac{\partial m_{12}}{\partial s} & \cdots & \frac{\partial m_{1h}}{\partial s} \\ \frac{\partial m_{21}}{\partial s} & \frac{\partial m_{22}}{\partial s} & \cdots & \frac{\partial m_{2h}}{\partial s} \\ \vdots & \vdots & & \vdots \\ \frac{\partial m_{g1}}{\partial s} & \frac{\partial m_{g2}}{\partial s} & \cdots & \frac{\partial m_{gh}}{\partial s} \end{bmatrix}$$

$$\underset{(p\times 1)}{\frac{\partial s}{\partial V}} = \begin{bmatrix} \frac{\partial s}{\partial v_1} \\ \frac{\partial s}{\partial v_2} \\ \vdots \\ \frac{\partial s}{\partial v_p} \end{bmatrix}$$

$$\underset{(g\times h)}{\frac{\partial M}{\partial s}} = \begin{bmatrix} \frac{\partial m_{11}}{\partial s} & \frac{\partial m_{12}}{\partial s} & \cdots & \frac{\partial m_{1h}}{\partial s} \\ \frac{\partial m_{21}}{\partial s} & \frac{\partial m_{22}}{\partial s} & \cdots & \frac{\partial m_{2h}}{\partial s} \\ \vdots & \vdots & & \vdots \\ \frac{\partial m_{g1}}{\partial s} & \frac{\partial m_{g2}}{\partial s} & \cdots & \frac{\partial m_{gh}}{\partial s} \end{bmatrix}$$

$$\underset{(p\times 1)}{\frac{\partial s}{\partial V}} = \begin{bmatrix} \frac{\partial s}{\partial v_1} \\ \frac{\partial s}{\partial v_2} \\ \vdots \\ \frac{\partial s}{\partial v_p} \end{bmatrix}$$

$$\underset{(g \times h)}{\frac{\partial s}{\partial M}} = \begin{bmatrix} \frac{\partial s}{\partial m_{11}} & \frac{\partial s}{\partial m_{12}} & \cdots & \frac{\partial s}{\partial m_{1h}} \\ \frac{\partial s}{\partial m_{21}} & \frac{\partial s}{\partial m_{22}} & \cdots & \frac{\partial s}{\partial m_{2h}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial s}{\partial m_{g1}} & \frac{\partial s}{\partial m_{g2}} & \cdots & \frac{\partial s}{\partial m_{gh}} \end{bmatrix}$$

$$\underset{(q \times p)}{\frac{\partial V}{\partial T}} = \begin{bmatrix} \frac{\partial v_1}{\partial T} & \frac{\partial v_2}{\partial T} & \cdots & \frac{\partial v_p}{\partial T} \end{bmatrix}$$

$$\underset{(g \times hp)}{\frac{\partial V}{\partial M}} = \begin{bmatrix} \frac{\partial v_1}{\partial M} & \frac{\partial v_2}{\partial M} & \cdots & \frac{\partial v_p}{\partial M} \end{bmatrix}$$

$$\underset{(hp \times g)}{\frac{\partial M}{\partial V}} = \begin{bmatrix} \frac{\partial m_{11}}{\partial V} & \frac{\partial m_{21}}{\partial V} & \cdots & \frac{\partial m_{g1}}{\partial V} \\ \frac{\partial m_{12}}{\partial V} & \frac{\partial m_{22}}{\partial V} & \cdots & \frac{\partial m_{g2}}{\partial V} \\ \vdots & \vdots & & \vdots \\ \frac{\partial m_{1h}}{\partial V} & \frac{\partial m_{2h}}{\partial V} & \cdots & \frac{\partial m_{gh}}{\partial V} \end{bmatrix}$$

$$
\underset{(hr\times gt)}{\frac{\partial M}{\partial N}} = \begin{bmatrix} \frac{\partial m_{11}}{\partial N} & \frac{\partial m_{21}}{\partial N} & \cdots & \frac{\partial m_{g1}}{\partial N} \\ \frac{\partial m_{12}}{\partial N} & \frac{\partial m_{22}}{\partial N} & \cdots & \frac{\partial m_{g2}}{\partial N} \\ \vdots & \vdots & & \vdots \\ \frac{\partial m_{1h}}{\partial N} & \frac{\partial m_{2h}}{\partial N} & \cdots & \frac{\partial m_{gh}}{\partial N} \end{bmatrix}
$$

There are four particularly useful theorems whose proofs are elementary. In order to present these, we define the following vectors and matrices (subject to restrictions given later):

A is (px1); B is (px1); C is (qx1); E is (pxq); F is (pxp). Combinations of these give the scalars X, Y, Z, where

$$X = A'B = B'A ,$$

$$Y = A'E\,C ,$$

$$Z = A'F\,A .$$

Then we have the theorems

I $\qquad \frac{\partial X}{\partial A} = B$,

II $\qquad \frac{\partial Y}{\partial E} = AC'$,

III $\qquad \frac{\partial Z}{\partial A} = 2FA$,

IV $\qquad \frac{\partial Z}{\partial F} = 2AA' - d(AA')$, where $d(AA')$ denotes a diagonal matrix with elements the same as those in AA' .

Restrictions applying to theorems I through IV respectively are as follows:

I. A has independent elements. The elements of A are independent of those in B.

II. E has independent elements. The elements of E are independent of those in A and C.

III. A has independent elements. The elements in A are independent of those in F. F is symmetric.

IV. F is symmetric. Except for the dependence associated with the definition of symmetry, the elements of F are independent. The elements of F are independent of those in A.

APPENDIX AC

IMPACT PREDICTION

The following algorithm may be used in real time to compute the longitude and latitude of impact under vacuum assumptions. Gravity is represented only by the central gravitational parameter. The input is a position vector and a velocity vector. Because of conflicts in notation and some new interpretations, all symbols are defined at the end of this appendix.

(1)	r	=	$(x^2 + y^2 + z^2)^{\frac{1}{2}}$
(2)	v^2	=	$\dot{x}^2 + \dot{y}^2 + \dot{z}^2$
(3)	$e \cos E$	=	$(rv^2/K) - 1$
(4)	a	=	$r/(1 - e \cos E)$
(5)	$e \sin E$	=	$(x\dot{x} + y\dot{y} + z\dot{z})/(Ka)^{\frac{1}{2}}$
(6)	e^2	=	$(e \sin E)^2 + (e \cos E)^2$
(7)	$e \cos E_I$	=	$(a - r_{I1})/a$
(8)	$e \sin E_I$	=	$-[e^2 - (e \cos E_I)^2]^{\frac{1}{2}}$
(9)	$\cos (E_I - E)$	=	$[(e \cos E_I)(e \cos E) + (e \sin E_I)(e \sin E)]/e^2$
(10)	$\sin (E_I - E)$	=	$[(e \sin E_I)(e \cos E) - (e \cos E_I)(e \sin E)]/e^2$
(11)	f	=	$[\cos (E_I - E) - e \cos E_I]/[1 - e \cos E]$
(12)	g	=	$[\sin (E_I - E) + e \sin E - e \sin E_I][a^3/K]^{\frac{1}{2}}$
(13)	x_I	=	$fx + g\dot{x}$
(14)	y_I	=	$fy + g\dot{y}$
(15)	z_I	=	$fz + g\dot{z}$
(16)	$r_{I2}{}^2$	=	$x_I{}^2 + y_I{}^2 + z_I{}^2$
(17)	r_{I3}	=	$[A^2(x_I{}^2 + y_I{}^2) + B^2 z_I{}^2]^{\frac{1}{2}}/r_{I2}$

If r_{I3} differs excessively from r_{I2}, then introduce r_{I3} in place of r_{I1} in Eq. 7 and loop through Equations 7 - 17 up to a maximum of three times to improve the agreement. Retain the final r_{I3} to use in place of r_{I1} for the next cycle beginning at Eq. 1.

(18) $(E_I - E) = \arctan [\sin(E_I - E)/\cos(E_I - E)]$

(19) $t = [(E_I - E) + e \sin E - e \sin E_I][a^3/K]^{\frac{1}{2}}$

(20) $\phi = \arcsin (z_I/r_{I3})$

(21) $\gamma = \arctan (y_I/x_I) - \omega t$

NOTE: A suitable approximation to Eq. 17 may be used in place of Eq. 17 in order to reduce computer time if necessary. It is customary to use canonical units (see Appendix O) and thus reduce K to unity.

It is possible to incorporate nominal corrections for atmosphere, second gravity harmonic, and geocentric to geodetic latitude by computing these corrections before launch in terms of (Δ lat.) and (Δ long.) at impact, algebraically summing the corrections, and fitting to a pair of low order polynomials. The values of ϕ and γ determined from Equations 20 and 21, respectively can then be improved in real time by adding the corrections indicated from evaluation of the two polynomials. The independent variable in the polynomials may, for example, be γ if the missile is launched generally eastward or westward.

GLOSSARY

E	eccentric anomaly at epoch.
E_I	eccentric anomaly at impact.
μ	central gravitational parameter.
A	semi-major axis of earth spheroid.
B	semi-minor axis of earth spheroid.
a	semi-major axis of trajectory ellipse.
e	eccentricity of trajectory ellipse.
f	defined by Eq. 11.
g	defined by Eq. 12.
r	distance from geocenter to missile position at epoch.
r_{I1}, r_{I2}, r_{I3}	various approximations for distance from geocenter to missile position at impact.*
t	time of flight from epoch to impact.
v	magnitude of inertial velocity vector at epoch.
$\begin{pmatrix} x, y, z \\ \dot{x}, \dot{y}, \dot{z} \end{pmatrix}$	epochal inertial position and velocity components similar to Figure 6, Page 15 with one important exception: The x axis goes through the meridian of Greenwich at epoch.
$\begin{pmatrix} x_I, y_I, z_I \\ \dot{x}_I, \dot{y}_I, \dot{z}_I \end{pmatrix}$	inertial position and velocity components at impact in same coordinate system as above.
Φ	geocentric latitude.
λ	geodetic longitude (East)
ω	earth rotation rate.

* The equatorial radius of the earth can be used for r_{I1}, in the first cycle through the algorithm if no better approximation is available.

APPENDIX AD

SELENOGRAPHIC COORDINATE TRANSFORMATIONS

1.0 DEFINITION OF SELENOGRAPHIC COORDINATE SYSTEM

This system is Moon-centered, Moon-fixed and either spherical or rectangular. Selenographic latitude $\phi_☾$ is analogous to geocentric latitude and is measured from the lunar equator, positive to the north; that is, in the hemisphere containing Mare Serenitatis. The Moon's equator is a great circle containing the center of mass of the Moon and lying in a plane perpendicular to the Moon's axis of rotation. Selenographic east longitude $\gamma_☾$ is measured from the lunar prime meridian positively along the equator toward Mare Crisium. The lunar prime meridian is defined as being the meridian that passes through the mean center of the Moon's disk, where the mean center is taken to be the point on the lunar face intersected by the lunar radius that is directed toward the Earth's center when the Moon is at the mean ascending node and when the node coincides with the mean perigee or mean apogee. The third coordinate is distance from the Moon's center of mass, $r_☾$. There is as yet no selenoid analogous to the geoid and no ellipsoid corresponding to the Earth ellipsoid.

Selenographic coordinates of an object may also be expressed in a rectangular, right-handed, Moon-centered system ($E_☾$, $F_☾$, $G_☾$) analogous to the geocentric system (E, F, G) (p. 14) defined by

$$E_☾ = r_☾ \cos \phi_☾ \cos \gamma_☾$$

$$F_☾ = r_☾ \cos \phi_☾ \sin \gamma_☾$$

$$G_☾ = r_☾ \sin \phi_☾$$

The ($E_☾$ $F_☾$ $G_☾$) coordinate axes are not in general parallel to the EFG coordinate axes.

2.0 DEFINITION OF SELENOCENTRIC EQUATORIAL COORDINATE SYSTEM

This system is Moon-centered inertial and has coordinate axes ($x_{☾}$, $y_{☾}$, $z_{☾}$) parallel respectively to the axes of the geocentric inertial equatorial system (x,y,z) (p.14) and has origin at the center of mass of the Moon.

3.0 CASSINI'S LAWS

The statement of these laws will facilitate a clearer understanding of the relative motion of the coordinate systems of the Moon and the Earth. These laws are as follows:

I. The Moon rotates uniformly direct about on axis which is fixed with respect to the Moon itself. The period of rotation is identical with the sidereal period of the Moon in its direct orbit about the Earth, namely 27.321661 days, and as a consequence the Moon presents almost the same face toward the Earth.

II. The plane of the lunar equator intersects the ecliptic plane at a constant angle of about $1° 35'$.

III. The plane of the Moon's orbit intersects the ecliptic plane at a nearly constant angle of $5° 9'$, while the node of the Moon's orbit regresses with a period of about 18 2/3 years. Moreover, the following three planes intersect in a common line: (1) the plane determined by the Earth orbiting about the Moon, (2) the plane through the Moon parallel to the ecliptic plane, (3) the Moon's equatorial plane. The second mentioned plane lies between the first and third. (The ecliptic is the plane of the Earth's orbit about the Sun.)

These "laws" are obviously only approximate, but they are very close approximations. There are various months associated with the Moon's motion. The sidereal month of 27.321661 days is the true period of revolution of the Moon about the Earth with respect to inertial space. The synodic month of 29.530589 days is the period between two conjunctions (new Moons) or oppositions (full Moons) with the Sun. The tropical month of 27.321582 days is the period between two passages of the Moon over the vernal equinox. The anomalistic month of 27.554551 days is the period between two successive perigee passages of the Moon. The nodical or Draconitic month of 27.212220 days is the period between two successive nodal passages of the Moon as its orbit intersects the plane of the ecliptic. The "age" of the Moon is the elapsed time since the previous new Moon, full Moon thus occurring at an age of about 14 3/4 days, at which time the fraction illuminated is 1.00.

4.0 LIBRATION

Libration refers to the oscillations or nodding of the Moon's face as seen by an observer on the Earth. There are three general types of libration: (1) Optical or geometric, (2) Dynamic or physical, (3) Diurnal or parallactic. The resultant of the first two at some instant is described by the departure of the selenographic coordinates of the Earth's center from zero. These coordinates are tabulated for each day of the year in the American Ephemeris.

4.1 Optical or Geometric

This type represents by far the major part of the libration. It is described by Cassini's laws and calculated mathematically from Encke's (1843) formulas. Because the Moon's equator does not lie in the orbital plane of the Moon about the Earth, an observer on the Earth sees more than half the northern hemisphere of the Moon at one period during the month and more than half the southern hemisphere of the Moon at another period during each month. This dispersion in latitude is about ± 6.5 degrees. That is, the selenographic latitude of the Earth varies from about +6.5 degrees to -6.5 degrees. Because of the eccentricity of the Moon's orbit about the Earth, the radius vector from the Earth to the Moon does not maintain a constant angular rate (Kepler's Laws). Therefore, since the Moon's rotation about its own axis is nearly uniform, an observer on the Earth sees relatively farther about the eastern limb and farther about the western limb at different periods during the month. As a consequence, there is a longitude dispersion of about $\pm$ 7.5 degrees. That is, the selenographic longitude of the Earth varies from about +7.5 degrees to -7.5 degrees. Because of the geometrical librations, which constitute practically all of the librations, an observer on the Earth is able to see approximately 59% of the Moon's surface over a period of a month.

4.2 Dynamic or Physical

These librations result from differences between the principal moments of inertia of the Moon in association with the irregularities in the Moon's orbital motion. The longest diameter of the Moon is directed generally toward the Earth and the shortest along the axis of rotation. Because of the geometrical libration, the longest diameter does not point directly toward the Earth, and therefore the attraction of the Earth on this bulge exerts a torque causing the long axis to precess with a small apical angle. Standard mathematical theory develops the forced vibrations and free vibrations associated with the constants of integration. The free vibrations are presently considered negligible. Physical librations are calculated from Hayn's (1907) formulas.

4.3 Diurnal or Parallactic

Because the Moon is approximately only 60 Earth radii from the Earth, at any particular instant observers on different parts of the Earth will have significantly different selenographic coordinates and hence topocentric corrections must be considered.

5.0 MOON'S ORBITAL MOTION

The geocentric coordinates of the Moon can be described by Brown's Lunar Theory originally published in 1905 and containing over 1650 terms in the equations of motion. With improvements by Eckert and others, this theory still forms the basis for most lunar ephemerides. Some ephemerides are computed using special perturbations and numerical integration. In this country the JPL Lunar Ephemeris Tapes are used almost exclusively. They are constantly being improved. Although the Moon's position can be calculated approximately from orbital elements and a relatively simple computer program, the JPL tapes are used for serious and accurate work on a computer. For hand computations the ephemeris given in the American Ephemeris is more convenient and is satisfactory.

6.0 SEQUENCE OF TRANSFORMATIONS

There are two general types of problems: (1) Given the selenographic coordinates of a point, what are the apparent coordinates for an observer on the surface of the Earth? (2) Given the apparent coordinates of a point for an observer on the surface of the Earth, what are the selenographic coordinates? For problem 1, the following sequence applies:

$V_3 = T_8 T_7 T_6 T_5 T_4 [T_3 T_2 V_2 + T_1 V_1]$, where

V_1 is input selenographic vector.

T_1 transforms to the selenocentric equatorial system expressed in the true equator and equinox of date.

V_2 is the input geocentric equatorial coordinates of the Moon's center expressed in the mean equator and equinox of 1950.0.

T_2 transforms to the mean equator and equinox of date.

T_3 transforms to the true equator and equinox of date.

T_4 transforms to the Earth-fixed, Earth-centered, rectangular coordinates based on Conventional International Origin.

T_5 transforms to the true pole of date.

T_6 transforms to radar coordinates as affected by deflection of the vertical.

T_7 introduces aberration effects.

T_8 introduces refraction effects.

V_3 is vector of final apparent radar coordinates.

For problem 2, the following sequence applies in which all terms have the same meanings:

$$V_1 = T_1^{-1} \left[T_4^{-1} T_5^{-1} T_6^{-1} T_7^{-1} T_8^{-1} V_3 - T_3 T_2 V_2 \right] .$$

7.0 MATHEMATICAL DESCRIPTION

The only transformation peculiar to the problem being discussed in this appendix is that associated with $T_1 V_1$. All other transformations are discussed elsewhere in this report. V_1 is defined by

$$V_1 = \begin{bmatrix} E_{☾} \\ F_{☾} \\ G_{☾} \end{bmatrix} .$$

T_1 is defined by

$$T_1 = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} ,$$

where the elements of T_1 are functions of the Euler angles (Λ, i, Ω) defining the Moon's orientation:

$$C_{11} = \cos\Lambda \cos\Omega' - \cos i \sin\Omega' \sin\Lambda$$

$$C_{12} = -\sin\Lambda \cos\Omega' - \cos i \sin\Omega' \cos\Lambda$$

$$C_{13} = \sin i \sin\Omega'$$

$$C_{21} = \cos\Lambda \sin\Omega' + \cos i \cos\Omega' \sin\Lambda$$

$$C_{22} = -\sin\Lambda \sin\Omega' + \cos i \cos\Omega' \cos\Lambda$$

$$C_{23} = -\sin i \cos\Omega'$$

$$C_{31} = \sin i \sin\Lambda$$

$$C_{32} = \sin i \cos\Lambda$$

$$C_{33} = \cos i$$

Prerequisite to the computation of the Moon's orientation angles are the physical libration constants (σ, τ, ρ) which may be computed as follows:

g = $☾ - \Gamma'$, the mean anomaly of the Moon.

g' = $L - \Gamma$, the mean anomaly of the Sun.

ω = $\Gamma' - \Omega$, the argument of perigee of the Moon.

Expressions for ☾, Γ' , L , Γ and Ω are given on Page 45. Then

$$\begin{aligned} \sigma \sin I = & -0^{\circ}.0302777 \sin g \\ & +0^{\circ}.0102777 \sin (g + 2\omega) \\ & -0^{\circ}.00305555 \sin (2g + 2\omega) \quad , \end{aligned}$$

$$\begin{aligned} \tau = & -0^{\circ}.003333 \sin g + 0^{\circ}.0163888 \sin g' \\ & +0^{\circ}.005 \sin 2\omega \quad , \end{aligned}$$

$$\begin{aligned} \rho = & -0^{\circ}.0297222 \cos g + 0^{\circ}.0102777 \cos (g + 2\omega) \\ & -0^{\circ}.00305555 \cos (2g + 2\omega) \quad , \end{aligned}$$

where $I = 1^{\circ}.535$.

In addition, we require ϵ from Eq. (75a) and Δ, where

$$\Delta = \tan^{-1}\left[\frac{-\sin\epsilon\sin(\Omega+\sigma+\Delta\lambda)}{\sin(I+\rho)\cos\epsilon-\cos(I+\rho)\sin\epsilon\cos(\Omega+\sigma+\Delta\lambda)}\right],$$

$$0 \leq \Delta < 360^\circ \quad ,$$

where $\Delta\lambda$ is given on Page 44.

The final orientation angles are obtained from the following relations:

$$\cos i = \cos(\Omega+\sigma+\Delta\lambda)\sin\epsilon\,(I+\rho) + \cos\epsilon\cos(I+\rho)\ ,$$

$$0 < i < 90^\circ \quad ,$$

$$\sin\Omega' = -\sin(\Omega+\sigma+\Delta\lambda)\sin(I+\rho)\csc i\ ,$$

$$-90^\circ < \Omega' < 90^\circ \quad ,$$

$$\Lambda = \Delta + ☾ + \tau - \Omega - \sigma \quad , \quad 0 \leq \Lambda < 360^\circ \quad .$$

This completes the algorithm for T_1. The other transformations are given elsewhere in this report and need not be repeated here. A note regarding aberration:

> Most tables of the Moon are constructed so as to give the apparent position of the Moon directly, by slightly modifying the geometric elements of the Moon's orbit, and hence with such tables only the small diurnal correction is missing. When a geometric lunar ephemeris is used, then the full planetary aberration corrections must be made. The JPL tapes are geometric.

APPENDIX AE

ABERRATION

1.0 INTRODUCTION

Because of the finite velocity of light, the apparent position of any celestial body depends upon the motions both of this body and of the observer during the interval of time required for light to travel from the body to the observer.

The displacement of the apparent position of the celestial body from its actual geometric position at the instant when the light left it is called stellar aberration. Stellar aberration is due to the instantaneous velocity of the observer in an inertial system with origin at the center of mass of the solar system. Stellar aberration is independent of the motion or distance of the observed body, but the computations are somewhat simpler if the distance is known. It is convenient to separate stellar aberration into two parts: diurnal (due to rotational velocity of the observer about the Earth's axis) and annual (due to Earth's orbital velocity). There is a third part associated with the motion of the solar system in space which is always ignored.

The displacement of the apparent position of the celestial body from its actual geometric position at the instant of observation is known as planetary aberration. Planetary aberration is thus the resultant of stellar aberration and the geometric displacement of the body in space due to its motion while the light was traveling to the observer, the latter component being called "light time" correction. To the order of accuracy that the motion of the object during the light time is rectilinear and uniform, the planetary aberration depends upon the instantaneous velocity of the observer <u>relative</u> to the object at the time of observation in exactly the same way as stellar aberration depends upon the instantaneous total velocity of the observer.

AFETR is concerned with missiles and satellites at distances no greater than the Moon. Since these objects partake of the same orbital motion as the Earth about the Sun, the annual component of aberration is negligible. It is satisfactory therefore to work in a geocentric inertial coordinate system. Expressed in this system, the corrections to be added to the apparent observations are

$$\begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \tau \begin{bmatrix} \Delta \dot{X} \\ \Delta \dot{Y} \\ \Delta \dot{Z} \end{bmatrix} \qquad (1)$$

Where (X,Y,Z) form a geocentric inertial right-handed rectangular system with X and Y in the equatorial plane, Z extending northward along the spin axis and X extending through the meridian of the site at the instant of observation. τ is light travel time from object to radar. ($\Delta\dot{X}$, $\Delta\dot{Y}$, $\Delta\dot{Z}$) designate the velocity components of the observed object relative to the corresponding velocity components of the site.

Equation (1) is similar in form to the corrections currently used at AFETR for aberration or transit time with radar:

$$\begin{bmatrix} \Delta R \\ \Delta A \\ \Delta E \end{bmatrix} = \tau \begin{bmatrix} \dot{R} \\ \dot{A} \\ \dot{E} \end{bmatrix} \qquad (2)$$

where (R,A,E) form an Earth-fixed polar topocentric system with R designating the slant range, A the azimuth measured in the horizontal plane positive eastward from north, E the elevation measured positive above the horizontal plane, origin at the radar gimbals. Since the (R,A,E) system is not inertial, Equation (2) is accurate only for ΔR. However, since for all satellites within a few thousand miles of the Earth, the corrections ΔA and ΔE are extremely small, the errors in ΔA and ΔE

computed from Equation (2) are not important. At distances equal to that of the Moon, however, Equation (2) gives significant errors in ΔA and ΔE. (Aberration corrections to be used in star observations are discussed in Appendix C.)

The problem here is to transform Equation (1) from the (X,Y,Z) system to the (R,A,E) system for practical applications.

ALGORITHM

An intermediate (u,v,w) coordinate system is useful. It has origin at the radar gimbals located at geodetic latitude ϕ_o. The u and v axes lie in the local horizontal plane, u pointing eastward and v northward. The w axis extends upward.

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} R \cos E \sin A \\ R \cos E \cos A \\ R \sin E \end{bmatrix} , \qquad (3)$$

$$\begin{bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{bmatrix} = \begin{bmatrix} \dot{R} \cos E \sin A - \dot{E} R \sin E \sin A + \dot{A} R \cos E \cos A \\ \dot{R} \cos E \cos A - \dot{E} R \sin E \cos A - \dot{A} R \cos E \sin A \\ \dot{R} \sin E + \dot{E} R \cos E \end{bmatrix} . \qquad (4)$$

The light travel time from object to radar is given by

$$\tau = R/C , \qquad (5)$$

where C is velocity of light.

$$\begin{bmatrix} \Delta u \\ \Delta v \\ \Delta w \end{bmatrix} = \tau \begin{bmatrix} \dot{u} - \omega v \sin \phi_o + \omega w \cos \phi_o \\ \dot{v} + \omega u \sin \phi_o \\ \dot{w} - \omega u \cos \phi_o \end{bmatrix} , \qquad (6)$$

where ω is Earth rotation rate.

$$\begin{bmatrix} \Delta R \\ \Delta A \\ \Delta E \end{bmatrix} = \begin{bmatrix} \dfrac{u\,\Delta u + v\,\Delta v + w\,\Delta w}{R} \\ \dfrac{v\,\Delta u - u\,\Delta v}{u^2 + v^2} \\ \dfrac{R\,\Delta w - w\,\Delta R}{R(R^2 - w^2)^{1/2}} \end{bmatrix} \quad . \tag{7}$$

It is evident in the case of ΔR the correction is identical to that in Equation (2). In radar traverse and elevation angles, a typical disagreement between this algorithm and Equation (2) is of the order of 0.1 milliradian at lunar distances.

Instead of correcting the observation and keeping the time fixed as in Equations (1) and (2), one may correct the time of the observation and leave the observation fixed. Analogous to Equation (1), we have

$$\underset{\text{true or geometric}}{\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{t-\tau}} = \underset{\text{apparent}}{\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}_{t}} \tag{1a}$$

where t is the time of observation.

Analogous to Equation (2), we have

$$\underset{\text{"true or geometric"}}{\begin{bmatrix} R \\ A \\ E \end{bmatrix}_{t-\tau}} = \underset{\text{apparent}}{\begin{bmatrix} R \\ A \\ E \end{bmatrix}_{t}} \tag{2a}$$

Equation (2a) like Equation (2) has limited accuracy.

BIBLIOGRAPHY

1. Frazer, J. B., "On the Motion of an Artificial Earth Satellite." The Mitre Corporation, Air Force Systems Command ESD-TR-66-293, September 1966.

2. Hoerl, A. E., and Kennard, R. W., "Ridge Regression: Biased Estimation for Nonorthogonal Problems," Technometrics, Vol. 12, No. 1, February 1970.

3. Kalman, R. E., "A New Approach to Linear Filtering and Prediction Problems," Journal of Basic Engineering, Vol. 82D, No. 1, March 1960.

4. Kozai, Y., "The Motion of a Close Earth Satellite," Astronomical Journal, Vol. 64, No. 9, November 1959.

5. O'Connor, J. J., "Geopotential Model Errors," RCA Technical Report ETR-TR-69-5, March 1969.

6. Scott, F. P., and Hughes, J. A., "Computation of Apparent Places for the Southern Star Program," Astronomical Journal, Vol. 69, No. 5, June 1964.

7. Trimble, G. D., "Ballistic Camera Refraction Routine," RCA Computer Program No. 30, February 1969.

8. Ward, J. A., "Automatic Multi-Satellite Ephemeris Maintenance," RCA Technical Report ETR-TR-71-1, August 1971.