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A SINGLE SHIP/MULTIPLE CRUISE MISSILE
ENGAGEMENT MODEL FOR FLEET AIR DEFENSE
PLANNING

William P. Cherry, et al

Vector Research, Incorporated

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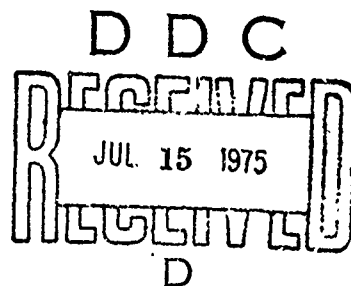Office of Naval Research

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13 ABSTRACT

A mathematical model of an engagement between a Navy ship and a number of anti-ship cruise missiles is described. The model is designed to be used in Navy planning activities involving ship/cruise missile engagements.

The engagement model described portrays the ship's defensive weapons as being rapid-fire guns and/or SAM systems, and allows for the portrayal of enemy jamming or other ECM activity. User-specified notions of partial damage to the ship's defensive weapons can also be accommodated. The outputs of the model are probability measures that the ship and each of the cruise missiles survive as functions of time. Inputs to the model are detailed performance characteristics of sensor systems, weapon systems and cruise missiles. It is intended that these inputs be supplied where necessary by engineering models, one of which is described in this report. The model is analytic in nature in that the output probability measures are obtained by numerically solving equations portraying the relationships between the processes which make up the engagement.

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Engineering Models						
Operational Models						
Ship Survival Probabilities						
Stochastic Output Descriptors						
Kill Rates for Rapid Fire Guns and SAMs						
Weapon Assignment Probabilities						
Approximation Methodology						

1a

FOREWORD

This report documents research conducted for the Office of Naval Research by Vector Research, Incorporated, (VRI) under contract N00014-72-C-0300. The research is a continuation of earlier activities conducted by VRI under the same contract and described in the report entitled *Development of Analytic Methodology for Naval Planning Areas* (VRI report number ONR-1 FR 73-1). One result of the early research was a finding that an analytic model of an engagement between a ship and a number of cruise missiles would be useful as an aid to naval planning. Such a model has been developed and is described in this report. The model is mathematical in character. This report is written for technical personnel who will program it and assist in its application.

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LIST OF FIGURES

1.0 INTRODUCTION

The research documented in this repcrt represents a continuation of work begun in 1972 which had as objectives:

(1) the study of Navy planning areas,

(2) the examination of current models, and

(3) the development of analytic structures.

Initial work in pursuit of these objectives focused on airborne anti-submarine warfare and on fleet air defense; models utilized by the Navy in conducting planning in both areas were examined. Two results of this examination were the observations that the most commonly used models were Monte Carlo simulations and that there was a strong possibility that analytic models could be developed as supplements to or substitutes for such models. Following discussions with personnel at ONR and the Office of the Chief of Naval Operations (OP 96), it was decided to concentrate activities in the area of fleet air defense, in particular fleet air defense against cruise missiles, because of the threat posed by such missiles and because of the fact that existing models were of only limited applicability in studying ship/cruise missile engagements.

The remainder of the first year's efforts were concerned with the development of an analytic model of an engagement between a single ship and a number of cruise missiles. In addition, preliminary work was carried out to determine the structural requirements of a fleet/multiple cruise missile engagement model. A hierarchical analytic structure was proposed. In this structure one set of models, referred to as engineering models, are used to map subsystem hardware characteristics into subsystem performance characteristics. A second set of models, referred to as operational

models, are used to integrate subsystem performance characteristics to provide measures of total system effectiveness (of either a ship or a fleet) in an engagement with multiple cruise missiles. The advantages of this approach which are discussed fully in [Bonder, Cherry and Miller, 1973], include not only economy of operation but also flexibility in that proposed hardware changes can be more easily evaluated and areas in which improvement is desirable or necessary can be more easily identified.

Work was carried out during the first year's contract on both operational and engineering models. A kill-rate structure was adopted for the single ship/multiple cruise missile operational model and rapid fire gun system kill rates were developed. The extension of such a structure to a fleet operational model was discussed and evaluated. Requirements for engineering models were identified and modeling was carried out in the areas of radar power management, coherent and noncoherent detection, Doppler discrimination and target tracking and prediction. The results of development activities in both operational and engineering models are described in [Bonder, Cherry and Miller, 1973].

Subsequent to the completion of the activities of the first year, initial findings and results were reviewed, and the decision made to concentrate the second year of the research activity on the development of analytic operational models of engagements between cruise missiles and single ships. A model representing engagements between a single ship and a number of cruise missiles was developed as a result of that effort and is described in this report.

The single ship/multiple cruise missile engagement model is an operational model (i.e., it maps performance descriptions for subsystems into measures of effectiveness of the engagement) and

is analytic in character. The principal mathematical structures are stochastic, and the outputs are probabilistic descriptions of measures of effectiveness (outcomes of engagements) as functions of time. The elements and processes represented in the model include:

- the ship's defensive weapon systems and weapon assignment logic,

- damage to defensive weapons,

- in-flight destruction of cruise missiles,

- jammers and other ECM systems employed against defensive weapon systems, and

- raid characteristics.

The engagement model requires as inputs the performance characteristics of systems (missiles, defensive weapons, etc.) including such items as lethality characteristics of cruise missiles with respect to the ship (and vice versa), tracking error characteristics of the ship's radar in an EW environment, etc. While it is anticipated that some of the model's inputs will be determined directly from existing data, it is believed that other inputs should be obtained as outputs of engineering submodels that map physical characteristics of subsystems into performance descriptors of those subsystems, which can then be input to the engagement model.[1] Accordingly, in addition to the engagement model itself, an engineering submodel that permits tracking error characteristics of radars and other sensors in an EW environment to be determined as a function of the physical characteristics of the radar was also developed during the second year's effort and is documented in this report.

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[1]With this arrangement the effect of variations of these subsystems on the outcome of the engagement can be examined by varying the inputs to the submodels -- a convenient and efficient approach which may facilitate the search for improved systems.

The remainder of the report consists of two chapters and three appendices. The single ship/multiple cruise missile engagement model is described in chapter 2.0. Expressions for the kill rates and weapon firing probabilities (both of which are required for the overall model) in terms of subsystems performance descriptors are also developed in chapter 2.0. Suggestions for refining the model and recommendations for extending the model to represent multiple ship/multiple cruise missile engagements are discussed in chapter 3.0. An engineering model, which may be used to generate some of the parameters involved in the expressions for kill rates, is described in appendices A and B. A system of differential equations employed in the overall model is analyzed in appendix C.

2.0 AN ANALYTIC MODEL OF A SINGLE SHIP/MULTIPLE
CRUISE MISSILE ENGAGEMENT

An analytic model of a single ship/multiple cruise missile engagement is described in detail in section 2.1, together with the principal assumptions used in the development of the model. Expressions for "kill rates" and "firing probabilities," which are fundamental to the model, are developed in sections 2.2 and 2.3 respectively.

2.1 Overall Structure of the Model

2.1.1 Principal Assumptions and Inputs to the Model

A single ship/multiple cruise missile engagement is viewed as taking place in two phases: an early phase in which the cruise missiles are engaged only by defensive aircraft at a distance from the ship, and a late phase in which the cruise missiles are engaged only by the ship's defensive weapons at relatively close ranges. Because defensive aircraft will seldom be used at the same time as the ship's defensive weapons, the view is that no such aircraft participate in the late phase of the engagement.

The engagement model described in this report portrays only the late phase of the engagement[1] of which figure 1 is an illustration.

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[1]Other models which could portray the early phase of the engagement (aircraft vs cruise missiles) are available elsewhere.
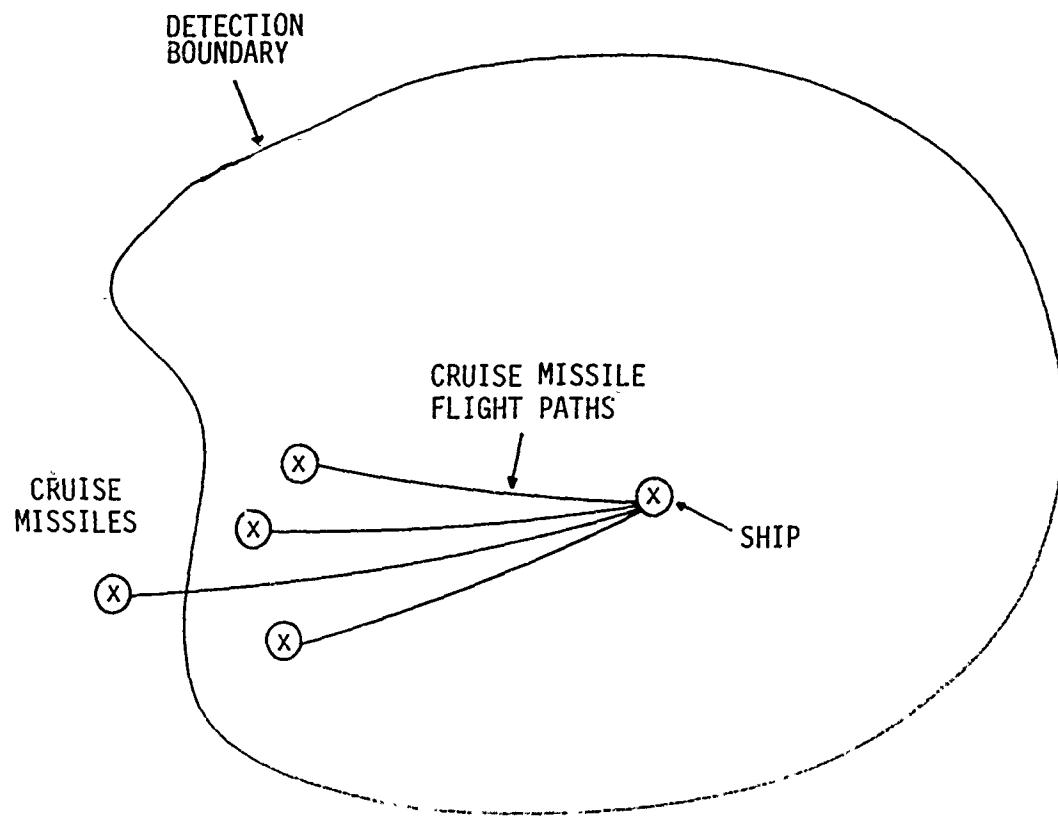
FIGURE 1: A TIME SNAPSHOT OF A SINGLE SHIP/MULTIPLE CRUISE MISSILE ENGAGEMENT

The following principal assumptions about this phase have been made:[1]

(1) The ship detects cruise missiles at deterministic ranges from the ship.[2] Detection ranges may vary with the direction of incidence of the cruise missiles. Upon detecting a cruise missile, the ship initiates tracking and continually updates its estimates of what the position of the cruise missile will be at future times.

(2) The ship's defensive weapons are limited to guns and SAM systems.

(3) The ship can redirect in-flight rounds from both guns and SAM systems at any time prior to their arrival at the cruise missile at which they have last been targeted.

(4) The ship always assigns all available defensive weapons to that surviving cruise missile which will, if not sooner destroyed, be the next to impact the ship.

The above-listed principal assumptions serve to bound (and, in the case of assumption (3), simplify) the scenario to be modeled as well as to define some of the types of input data which will be required by the model. The manner in which each assumption accomplishes these objectives is briefly discussed below.

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[1] Other assumptions, more minor in character than those listed here, will be introduced and used as needed.

[2] This assumption, and the others as well, can be relaxed. Methods for relaxing the assumptions will be discussed later.

Detection and Tracking. Since the ranges at which a ship can detect incoming cruise missiles can be predicted with reasonable confidence,[1] these ranges have been viewed as being deterministic in character. The range at which detection of each cruise missile occurs, together with the actual flight paths and velocities of the cruise missiles, are required as input to the model in order to obtain the time which the ship has to engage and destroy each missile before it impacts the ship and to fix the geometry of the engagement.

In practice, it will be convenient to represent the actual flight path of each cruise missile by a sequence of straight line segments and constant-radius turns of a specified number of degrees.[2] The speed of the cruise missile is taken to be constant on each segment.

Ship's Defensive Weapon Systems. A single ship's present defenses against cruise missiles consist of rapid-fire guns and SAM systems. The number of defensive weapon systems, along with certain performance data which affect their lethality characteristics with respect to cruise missiles, is required as input to the model.

---

[1] Perhaps the simplest method to obtain the detection range is to use the results of tests of the ship's missile detection radar against cruise-missile-like objects under conditions of the type a user desires to represent in the model (e.g., presence or absence of enemy jamming, weather conditions, the altitude of the incoming cruise missiles, etc.). Alternatively, the ship's detection range could be obtained from analytic models (see, for example, [Brennan and Hill, 1964] and [Kirkwood, 1965]). Similarly, information about flight-path estimation for incoming cruise missiles could be obtained by testing the equipment involved or by analytic modeling. An approach of the latter type is documented in [Bonder, Cherry and Miller, 1973].

[2] Procedures of this type for handling flight paths have been developed at the Systems Research Laboratory at the University of Michigan; they are documented in [SRL, 1969].

Fire Redirection. On the one hand, it is sometimes possible for a ship to redirect an in-flight SAM from the cruise missile at which it was initially targeted to another cruise missile.[1] On the other hand, redirection of in-flight gunfire by the ship does not appear to be feasible. Nevertheless it is here assumed for mathematical convenience and simplicity that redirection of both gunfire and SAMs is possible. Strictly speaking, therefore, this assumption has the effect of making the engagement model described here generate an upper bound on the effectiveness with which real ships can fight cruise missile engagements. While an analysis of such engagements is possible without this assumption, it is not clear that the additional realism thus gained would be worth the concomitant increase in mathematical and computational complexity.

In view of the assumption that the ship has the capability to redirect in-flight fire from any weapon system, it is convenient to say that a weapon system "is firing at" a cruise missile at a particular time whenever fire from that weapon is arriving at the cruise missile at that time.

Ship's Weapon Assignment Logic.[2] Although other weapon assignment logics are available, the logic described in assumption (4) above closely approximates one which has been used in practice. This logic has therefore been assumed as an initial basis for modeling.

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[1] Redirection of a SAM is usually possible in practice until the SAM enters a terminal lock-on phase.

[2] Ship's weapon assignment logics in general, and the one assumed for the engagement model in particular, are discussed at greater length in section 2.3.

Notice finally that some of the above assumptions state that certain aspects of the engagement are regarded as being deterministic instead of stochastic. For example, assumption (1) says that the ranges at which the ship detects cruise missiles is here taken to be deterministic, whereas these ranges are often treated stochastically elsewhere (i.e., in other models). These assumptions may therefore be seen as conditioning the engagement upon the indicated types of information being fixed. This conditioning could be removed, if desired, by embedding the engagement model to be described here in a richer structure in which these deterministic aspects would instead be treated stochastically. This would entail an increase in mathematical and computational complexity. Since much of the complexity of the interactions between the ship and the cruise missile can already be analyzed under the assumptions listed above, it is desirable to postpone enrichments of the types suggested above until such time as experience gained with the existing model indicates that they are needed.

2.1.2 Outputs and Overall Mathematical Structure of the Model

The outputs of the single ship/multiple cruise missile engagement model are of two types:

- the conditional probability that each cruise missile survives the engagement up to time t, given that the ship has sustained a certain degree of damage at that time (the output is provided for each time t and each possible degree of damage to the ship), and

- the probability that the ship sustains each possible degree of damage at each time t.

The overall structure of the engagement model is based on renewal theory. Fundamental to the model are "kill rates" and what are here called "firing probabilities." The kill rate associated with each defensive weapon system/cruise missile pair at time t is the conditional probability that the weapon system will kill the cruise missile in a small time interval[1] $[t, t+\delta)$, given that the cruise missile is still alive at time t and that the weapon is firing at the cruise missile at that time.[2] The firing probability p(t) associated with each defensive weapon system/ cruise missile pair is the probability that the weapon is actually firing on the missile at time t.[3]

It is convenient to number the cruise missiles in the order in which they will, if not sooner destroyed, impact the ship. Thus cruise missile 1 is that missile which, if not destroyed, will impact the ship before any of the other cruise missiles; cruise missile 2 is that missile which will, if not destroyed, be the next soonest to impact the ship, and so on. The convention here is that time is measured from zero starting with the time at which the cruise missile 1 is detected by the ship. The time at which cruise missile j will, if not sooner destroyed, impact the ship is denoted by "t_j," so that $0 \le t_1 \le t_2 \le \cdots$ follows from these conventions. The ship's defensive weapon systems may be numbered in an arbitrary order.

---

[1]Here, as elsewhere in this report, $\delta > 0$ is a small time interval.

[2]Recall that a weapon is said to be firing at a cruise missile at time t if and only if lethality from the weapon is arriving at the missile at that time.

[3]The reason why the firing of a defensive weapon upon a particular cruise missile is treated stochastically will be explained later.

Unlike the cruise missiles, the ship's defensive weapons are viewed as being capable of sustaining partial damage (as would happen, for example, when guns switch to optical tracking because of damage to the radar system). Partial damage to the ship's defensive weapons can conveniently be described by introducing and using "damage categories" for the weapons and "damage states" for the ship. The damage categories for one of the ship's defensive weapons are statements that are descriptive of that weapon's capability to continue to function in the engagement and may be chosen arbitrarily by the user of the model. For example, the damage categories for a weapon might be chosen as follows[1]:

damage category number	weapon capability
1	none (weapon destroyed)
2	marginal
3	medium
4	almost full
5	full (no damage)

The damage categories for the other defensive weapons may be chosen similarly. If there are n_i damage categories associated with defensive weapon i, and if there are W defensive weapons on the ship, then there are $\prod_{i=1}^{W} n_i$ combinations of damage categories that could characterize the status of the ship's defensive weapons at any time. It is convenient to call each such combination a damage state of the ship. Damage states are denoted

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[1]The terms which define the categories must, of course, themselves be defined.

by vectors d, where the ith component d_i of a damage state vector d is the number of the damage category of defensive weapon system i. The particular damage state corresponding to all of the ship's defensive weapons being functional at full capability is denoted by u. At time 0, the ship is assumed to be in damage state u.

The notation used is summarized below:

M = number of cruise missiles in the engagement;

W = number of defensive weapon systems on the ship;

t_j = time at which cruise missile j will, if not sooner destroyed, impact the ship ($1 \leq j \leq M$);

$m_j(t|d)$ = conditional probability that cruise missile j will be alive at time t, given that the ship is in damage state d at that time ($1 \leq j \leq M$, all d);

$\delta \cdot k_{ij}(t|d)$ = conditional probability that defensive weapon system i will kill cruise missile j in the interval [t, t+δ), given that cruise missile j is still alive at time t, that the ship is in damage state d at time t and that defensive weapon i is firing at cruise missile j at time t (a function of t for each $1 \leq i \leq W$, $1 \leq j \leq M$ and each d);

$p_{ij}(t|d)$ = conditional probability that defensive weapon system i will be firing at cruise missile j at time t, given that cruise missile j is still alive at time t and that the ship is in damage state d at that time (a function of t for each $1 \leq i \leq W$, $1 \leq j \leq M$ and each d);

$q_d(t)$ = probability that ship will be in damage state d at

time t (a function of t for each d);[1]

ℓ_{ed} = conditional probability that, given that the ship

is in damage state e, it will be in damage state d

after the impact of one more cruise missile[2] (all e,

d).

In addition, for each j = 1, $\cdots$, M, let t_j^- and t_j^+ denote times satis-
fying $t_j^- < t_j < t_j^+$ which are both very close to t_j. Put $t_{M+1}^- = t_{M+1} =$
$t_{M+1}^+ = \infty$.

The kill rates $k_{ij}(t|d)$ defined above express the lethality charac-
teristics of defensive weapon system i with respect to cruise missile j
at time t when weapon system i is firing at that cruise missile at that
time. Whether or not a weapon system is firing at cruise missile j at
some time depends (according to the assumed ship's weapon assignment
logic) on whether or not cruise missiles 1, 2, $\cdots$, j-1 have been de-
stroyed or have impacted the ship by that time. Therefore, since cruise
missile survival is here treated probabilistically, so too must be the
assignment of defensive weapons to cruise missiles at any time. The fir-
ing probabilities $p_{ij}(t)$ defined above, express the probability that

---

[1]The damage state probabilities $q_d(\cdot)$ may be interpreted as ship survival
probabilities as soon as a suitable subset of the damage states is identi-
fied with ship survival (and the complementary subset with ship non-
survival). We do not here make this association, preferring instead to
leave this to the user of the model. The $q_d(\cdot)$ are therefore made avail-
able as outputs of the model and this is regarded as being equivalent to
outputting a ship survival probability as a function of time.

[2]It is assumed that the ℓ_{ed} are known and available as input data.

weapon system i is firing at cruise missile j at time t, given that cruise missile j is still alive at that time.

Now consider a time t with $0 \leq t < t_1$. Since none of the cruise missiles can have impacted the ship by time t, we have

$$q_d(t) = \begin{cases} 1 & \text{if } d = u, \\ 0 & \text{otherwise.} \end{cases} \qquad (0 \leq t < t_1) \qquad (1)$$

In particular,

$$q_d(t_1^-) = \begin{cases} 1 & \text{if } d = u, \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

Next, choose a small $\delta > 0$ such that $t + \delta < t_1$. Since none of the cruise missiles can have impacted the ship by time $t + \delta$, and since the ship must therefore be in damage state u at that time

$$m_j(t+\delta|d) = \begin{cases} m_j(t|u) \cdot [1 - \delta \cdot \sum_{i=1}^{W} p_{ij}(t|u)k_{ij}(t|u)] & \text{if } d=u, \\ \\ \text{undefined otherwise,} \end{cases} \qquad (3)$$

for $j = 1, \cdots, M$ and $0 < t < t + \delta < t_1$.

Equation (3) can be used to determine the $m_j(t|d)$ for all d and all t in the range $0 \leq t \leq t_1$ once expressions for the firing probabilities $p_{ij}(t|d)$

in terms of the $m_j(t|d)$ are available.[1] Specifically, one could use (3) to first determine the $m_j(\delta|u)$ for $1 \le j \le M$. This information can then be used to determine $p_{ij}(\delta|u)$ for $1 \le i \le W$ and $1 \le j \le M$. This latter information could then be used in (3) to obtain the $m_j(2\delta|u)$, etc. Continuing with this "bootstrap technique" would yield values of the $m_j(t|u)$ and $p_{ij}(t|u)$ at any desired number of points in the range $0 \le t \le t_1$. The following diagram illustrates the order of the computations:

$$1 = m_j(0|u) \qquad m_j(\delta|u) \qquad m_j(2\delta|u) \qquad - - - \qquad m_j(t_1^-|u)$$

$$p_{ij}(0|u) \qquad p_{ij}(\delta|u) \qquad p_{ij}(2\delta|u) \qquad - - - \qquad p_{ij}(t_1^-|u)$$

The analysis just given, as has been indicated, suffices to determine the $m_j(t|d)$ and $q_d(t)$ only for times $t < t_1$. We now show how to compute these functions for times $t > t_1$. To this end, choose $k(1 \le k \le M)$ and suppose that the following functions and quantities have all been determined as indicated:[2]

---

[1]If the manner of dependence of the firing probabilities $p_{ij}(t|d)$ on the cruise missile survival probabilities $m_j(t|d)$ were simple, it would be possible in principle to insert these functions in (3) (which is equivalent to a system of differential equations) and integrate to obtain formulas for the $m_j(\cdot|d)$. An approach of this type is discussed in appendix C. One may, however, anticipate that the integration may be difficult if the $p_{ij}(t|d)$ are complicated functions of the $m_j(\cdot|d)$ and of the assumed weapon assignment logic for the ship. In view of this, it is better to plan to work with (3) in the form given by using suitable numerical techniques.

[2]This assumption is true when $k = 1$ as shown above (see equations (1) and (2) and recall the bootstrap technique applied to (3)).

(a) $q_d(t)$ for all d and all $t < t_k$;

(b) $q_d(t_k^-)$ for all d;

(c) $m_j(t|d)$ for $1 \leq j \leq M$, and d and all $t < t_k$;

(d) $m_j(t_k^-|d)$ for $1 \leq j \leq M$ and all d;

(e) $p_{ij}(t|d)$ for $1 \leq i \leq W$, $1 \leq j \leq M$, all d and all $t < t_k$

The probability $q_d(t_k^+)$ that the ship is in damage state d at time t_k^+ is given by

$$q_d(t_k^+) = \sum_e q_e(t_k^-) \, m_k(t_k^-|e) \, \ell_{ed} \qquad (4)$$

for all d. In fact, this relation persists for all t satisfying $t_k^+ \leq t < t_{k+1}$:

$$q_d(t) = \sum_e q_e(t_k^-) \, m_k(t_k^-|e) \, \ell_{ed} \qquad (5)$$

for all d and all t satisfying $t_k^+ \leq t < t_{k+1}$. In particular,

$$q_d(t_{k+1}^-) = \sum_e q_e(t_k^-) \, m_k(t_k^-|e) \, \ell_{ed} \qquad (6)$$

for all d. Notice that equations (4), (5), and (6) are all computable in the sense that the left-hand side of each can be determined from the corresponding right-hand side because all of the right-hand sides involve only probabilities which are, by the above assumption, known. Relation (5) therefore shows that part (a) of this assumption remains true when k is replaced by k+1. Similarly, (6) shows that part (b) of the assumption remains true when k is replaced by k+1.

Next, note that the conditional joint probability that:

•the ship is impacted at time t_k, and

•the ship thereupon goes into damage state d, and

•cruise missile $j(k+1 \leq j \leq M)$ survives until time $t_k^+ \approx t_k^-$, given that the ship is in damage state e at time t_k^-, is $m_k(t_k^-|e) \cdot \ell_{ed}$. Hence

$$m_j(t_k^+|d) \cdot q_d(t_k^+) = \sum_e m_k(t_k^-|e) \; \ell_{ed} \; q_e(t_k^-)$$

$$+ q_d(t_k^+) \cdot [m_j(t_k^-|d) - m_k(t_k^-|d)] \tag{7}$$

for $j = k+1, \cdots, M$ and all d, whereas

$$m_j(t_k^+|d) = 0 \tag{8}$$

for $j=1, \cdots, k$ and all d because cruise missiles $1, \cdots, k$ will have either been destroyed or impacted the ship by time t_k^+. Since the $q_d(t_k^+)$ which occurs on the left in (7) are given in (4) in terms of probabilities evaluated at time t_k^-, and since these latter probabilities (together with those that appear on the right in (7)) are all known, the $m_j(t_k^+|d)$ on the left in (7) and (8) are all computable.

Equation (8) persists for times t satisfying $t_k^+ \leq t < t_{k+1}$ because no cruise missiles can return to life once they have been destroyed or impacted the ship:

$$m_j(t|d) = 0 \tag{9}$$

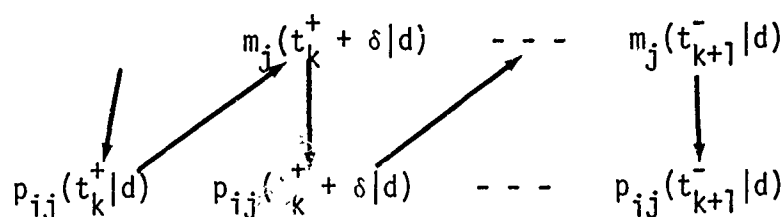for $j=1, \cdots, k$, all d and all t satisfying $t_k^+ \leq t < t_{k+1}$. In particular,

$$m_j(t_{k+1}^-|d) = 0 \tag{10}$$

for j=1, $\cdots$, k and all d. Furthermore, an argument similar in every respect to that given before shows that

$$m_j(t+\delta|d) = m_j(t|d) \cdot [1 - \delta \cdot \sum_{i=1}^{W} p_{ij}(t|d) \, k_{ij}(t|d)] \tag{11}$$

holds for all $j = k+1$, $\cdots$, M, all d and all t satisfying $t_k^+ \le t < t_{k+1}$. This equation can be numerically bootstrapped to compute the cruise missile survival probabilities $m_j(t|d)$ and the weapon system firing probabilities $p_{ij}(t|d)$ for all t in the range $t_k^+ \le t < t_{k+1}$. This bootstrapping begins with the choice $t = t_k^+$, the values of the $m_j(t_k^+|d)$ which will then be required in (11) being given in (7). The following diagram illustrates the order of the computations:

$$m_j(t_k^+ + \delta|d) \quad - - - \quad m_j(t_{k+1}^-|d)$$

$$p_{ij}(t_k^+|d) \quad p_{ij}(t_k^+ + \delta|d) \quad - - - \quad p_{ij}(t_{k+1}^-|d)$$

In particular, the $m_j(t_{k+1}^-|d)$ for $i = k+1$, $\cdots$, M and all d are produced as a result of this procedure. Thus, equations (9), (10) and (11) show that parts (c), (d) and (e) of the above assumption remain true when k is replaced by k+1.

The foregoing discussion, together with the fact that the above assumption is true for k = 1, shows by induction that this assumption is true for all k = 1, $\cdots$, M. This is to say that the output functions

$q_d(\cdot)$ and $m_j(\cdot|d)$ are computable for all damage states d and all cruise missiles $1 \le j \le M$. The formulas for performing these computations are those which have been developed above. The procedure may be recapped as follows[1]:

Step (1). Put k = 1 and compute

$$q_d(t) = \begin{cases} 1 & \text{if } d = u \\ 0 & \text{if otherwise} \end{cases} \qquad \text{all } 0 \le t \le t_1 \qquad (12)$$

$$q_d(t_1^-) \begin{cases} 1 & \text{if } d = u \\ 0 & \text{otherwise} \end{cases} \qquad\qquad (13)$$

Compute also[2] the $m_j(t|u)$ for $1 \le j \le M$ and $0 \le t < t_1$ by numerically bootstrapping the following differential equation:

$$m_j(t+\delta|u) = m_j(t|u) \cdot [1 - \delta \cdot \sum_{i=1} p_{ij}(t|u) \, k_{ij}(t|u)] \; . \qquad (14)$$

Obtain, in particular, the values $m_j(t_1^-|u)$ for $1 \le j \le M$.

Step (2). Compute

$$q_d(t_k^+) = \sum_e q_e(t_k^-) m_k(t_k^-|d) \, \ell_{ed} \qquad \text{all } d, \qquad (15)$$

$$q_d(t) = \sum_e q_e(t_k^-) m_k(t_k^-|e) \, \ell_{ed} \quad \text{all } d, \text{ all } t_k \le t < t_{k+1}, \qquad (16)$$

---

[1]Notice that the computations in step (2) of the algorithm should be performed in the order indicated because the $q_d(t_k^+)$ which occurs in (18) must first be computed as in (15) and the $m_j(t_k^+|d)$ which will be required to start the bootstrapping in (21) must first be obtained as in (18).

[2]Note that the $m_j(t|d)$ for $1 \le j \le M$, all $0 \le t < t_1$ and $d \ne u$ are undefined.

$$q_d(t_{k+1}^-) = \sum_e q_e(t_k^-) m_k(t_k^- e) \ell_{ed} \quad \text{all } d, \tag{17}$$

$$m_j(t_k^+|d) = \begin{cases} \dfrac{1}{q_d(t_k^+)} [\sum_e m_k(t_k^-|e) \ell_{ed} q_e(t_k^-) \\ \quad + q_d(t_k^-)\{m_j(t_k^-|d) - m_k(t_k^-|d)\}] \\ \qquad\qquad \text{all } k+1 \leq j \leq M, \text{ all } d, \\ 0 \text{ all } 1 \leq j \leq k, \text{ all } d, \end{cases} \tag{18}$$

$$m_j(t|d) = 0 \quad \text{all } 1 \leq j \leq k, \text{ all } d, \text{ all } t_k^+ \leq t \leq t_{k+1}, \tag{19}$$

$$m_j(t_{k+1}|d) = 0 \quad \text{all } 1 \leq j \leq k, \text{ all } d. \tag{20}$$

In addition, compute the $m_j(t|d)$ for $k+1 \leq j \leq M$, all d and all t in the range $t_k^+ \leq t \leq t_{k+1}$ by numerically bootstrapping the following differential equation:

$$m_j(t+\delta|d) = m_j(t|d) \cdot [1 - \delta \cdot \sum_{i=1} p_{ij}(t|d) k_{ij}(t|d)]. \tag{21}$$

Obtain, in particular, the values $m_j(t_{k+1}^-|d)$ for $k+1 \leq j \leq M$ and all d.

Step (3). Increment k by 1. If $k \geq M$, stop; otherwise go back to step (2).

The above algorithm constitutes the overall mathematical structure of the single ship/multiple cruise missile engagement model. In order to carry out the above procedure, however, it is necessary to first determine values for the kill rates $k_{ij}(t|d)$ and the firing probabilities $p_{ij}(t|d)$. Although this could probably be done experimentally, it would be difficult and costly to do so and would, moreover, provide only limited insight as to possible improvements in weapon design and employment tactics

for the ship. These problems can be avoided by using analytic models to predict the kill rates and firing probabilities in terms of parameters which characterize the cruise missiles and the ship's defensive weapon systems. This will facilitate using the overall model to examine promising design and tactical improvements. Models for predicting the kill rates and firing probabilities have therefore been developed and are described in the next two sections.

2.2 Determination of Kill Rates

The overall mathematical structure of the single ship/multiple cruise missile engagement model described in the previous section involved certain "kill rates" which characterize the capabilities of the ship's defensive weapon systems with respect to cruise missiles. Each kill rate $k_{ij}(t|d)$ was the instantaneous conditional probability that defensive weapon system i will kill cruise missile j in a short time interval $[t,t+\delta)$, given that cruise missile j is still alive at time t, that the ship is in damage state d at time t, and that weapon system i is firing at cruise missile j at time t. It was mentioned that kill rates for both types of the ship's defensive weapon systems -- gun emplacements and SAM systems -- may be viewed as being functions of certain parameters associated with the weapon systems and the cruise missiles. The dependence of kill rates for rapid-fire gun systems on such parameters has been described by [Bonder, Cherry and Miller, 1973]; for completeness, these ideas are discussed in section 2.2.1 below. Expressions for the kill rates for SAM systems as functions of such parameters are developed in section 2.2.2.

2.2.1 Kill Rates for Rapid Fire Gun Systems

The approach used here to determine the functional form of the kill rates for rapid-fire gun systems involves decomposing the process by which such guns kill cruise missiles into several parts.

Implicit in the notion of a kill rate[1] $k(t)$ for a particular gun-system/cruise-missile pair at time t is the assumption that the killing of the missile by rounds from the gun is probabilistic in character. If we further assume that the killing of the cruise missile at any time t by each of the rounds from the gun which arrive at the cruise missile at time t are independent events, then $k(t)$ may be written

$$k(t) = r(t) \cdot SSKP(t)$$

where:

$r(t)$ = rate at which rounds from the gun arrive at the cruise missile at time t; and

$SSKP(t)$ = single shot kill probability at time t.

Note that the rate $r(t)$ at which rounds arrive at the cruise missile at time t is not the same as the rate at which the rounds were fired at an earlier time[2] $t-t_f$ (so as to reach the missile at time t) because of Doppler effects arising from the motion of the missile relative to the gun. However, since the rounds from the gun travel so much faster than the cruise missile, these Doppler effects will be small and may therefore be neglected. The above equation for $k(t)$ then becomes

$$k(t) = F \cdot SSKP(t) \tag{22}$$

---

[1]For purposes of this and the next section, it is convenient to replace the notation $k_{ij}(t|d)$ for a kill rate by simply $k(t)$. In doing this, it is understood that we have in mind a particular defensive weapon system i, a particular cruise missile j, and a particular damage state d for the ship.

[2]Here, $t_f > 0$ represents the time of flight for rapid-fire gun rounds arriving at the cruise missile at time t.

where:

F = gun system firing rate (a constant which appropriately aver-
ages the burst and inter-burst periods for the gun); and

$SSKP(t)$ = single-shot kill probability at time t.

The firing rate F is here taken to be a basic descriptor of the gun system. The remainder of this section is, therefore, devoted to the determination of the single-shot kill probability $SSKP(t)$ at time t.

The method suggested here for computing single-shot kill probabilities for rapid-fire gun systems is based on the following assumptions:

- for purposes of the computation of the $SSKP(\cdot)$, the cruise missile is equivalent to an effective vulnerable area -- a plane area of suitable size and shape in a plane perpendicular to the line of sight between the gun system and the missile at time t;

- a round kills the cruise missile if and only if it intercepts the missile's effective vulnerable area;

- rounds are aimed (at time $t-t_f$) at the estimated center of the cruise missile's effective vulnerable area at time t, but errors in both the estimation of future position and in the delivery points of rounds may occur.

Under these conditions, the single-shot kill probability $SSKP(t)$ at time t may be modeled as

$$SSKP(t) \approx A(t) \cdot p(0,0) \tag{23}$$

where:

$A(t)$ = area of the cruise missile's effective vulnerable area at time t, and

$p(0,0)$ = value of a probability density function of the two-
dimensional delivery error evaluated at the aiming
point.

The area $A(t)$ of the effective vulnerable area of the cruise missile
will be a function of the dimensions of the missile, its aspect with
respect to the gun system at time t, and of the type of round (fuze or
impact) employed by the gun. In detailed simulations, it is common to
model targets such as cruise missiles as consisting of several vulnerable
components with vulnerable areas in six directions: front, left, right,
up, down and rear, relative to the natural coordinate system of the target.
Areas associated with each direction are projected first onto a plane
perpendicular to the closing velocity vector between projectile and target,
and then onto a plane perpendicular to the line of sight between weapon
and target. It has been found, however, that simple approximations (such
as a spherical representation) to the multiple component model produce
results which are often satisfactory. It is expected that the spherical
model will suffice to determine the effective vulnerable areas $A(t)$ of
cruise missiles, and these functions are therefore regarded as being
computable.

Consider next the intercept probability $p(0,0)$. For each round, the
delivery error at the time of predicted intercept may be modeled as a three
dimensional random vector $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ in which each component x_i ($i = 1,2,3$) is
regarded as being a sum $\sum x_{ij}$ of a number of random variables x_{ij} associ-
ated with error sources which contribute to errors in the x_i-dimension.
Under the assumptions that

- for each $i = 1,2,3$, the x_{ij} are independent,

- for any pair $1 \leq i_1 \neq i_2 \leq i_3$, the $x_{i_1 j}$ and $x_{i_2 k}$ are independent for all j and k, and

- the number of error sources which contribute to errors in each dimension is "large,"

the distribution of the miss vector $\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$ may (by the central limit theorem) be approximated as a trivariate normal distribution without correlation. To utilize the effective vulnerable area concept described above, the three dimensional miss vector must be transformed into an equivalent vector in the plane perpendicular to the line of sight between target and weapon. This transformation may be accomplished by suppressing the range dimension as described below. As will be seen, this transformation produces a bi-variate normal distribution (with correlation) for the equivalent weapon delivery error vector in the azimuth elevation plane.

The angle-like errors in elevation and azimuth are obtained by multiplying angular errors by range. The assumption is made that direct angular errors in gun pointing lead to equivalent angular errors in the Cartesian coordinate system perpendicular to the line of sight. Sources of the angular errors include the estimates of both angles, i.e., elevation and azimuth, and estimates of the first derivatives of elevation and azimuth passed from the tracking system to the fire control computer. These errors thus lead to equivalent errors in lead angle effects. By assuming that errors produced by the fire control computer are negligible, and by neglecting the rotation of the coordinate system with range, azimuth and elevation axes in the time between prediction and tracking, it may be concluded that azimuth, elevation and range errors produced by the tracking system produce equal errors in gun pointing.

Range errors in the system arise from two primary sources. The first of these is the tracking system which provides the fire control computer with estimates of range and the first derivative of range. The second source of range error is the estimation of muzzle velocity. This error is modeled as described below.

Suppose that the projectile at time of predicted intercept has velocity W in the direction $(0,0,R)$ and that the miss vector at time of predicted intercept has component Z in the range direction. It is easily seen that the time required for the projectile to reach the target plane is approximately $Z/(\dot{R}-W)$ where $\dot{R}$ is the rate of change of range. For small values of $\dot{R}$ relative to the value of W this time can be approximated by Z/W. With respect to the coordinate system with axes in the directions of range, elevation and azimuth, and by neglecting rotation the target velocity can be expressed as $(\dot{A}E, \dot{E}R, \dot{R})$ where:

  A = azimuth;

  $\dot{A}$ = azimuth rate of change;

  E = elevation;

  $\dot{E}$ = elevation rate of change;

  R = range;

  $\dot{R}$ = range rate of change.

The error component Z thus corresponds to errors in elevation and azimuth given by $E R Z/W$ and $A R Z/W$ respectively.

The magnitude of the range error is the sum of errors due to the tracking system, which are assumed to pass unaltered through the fire control system, and errors due to muzzle velocity variation. The latter range error is modeled by:

$$\text{Range error} = \left(\frac{t_f}{1 + st_f} - \frac{Vt_f^2}{(1 + st_f)^2} \left(\frac{ds}{dV} \right) \right) dV,$$

where:

t_f = projectile flight time;

V = muzzle velocity;

s = slow-down constant of projectile; and

dV = muzzle velocity error.

Under the assumptions made above, probability distributions must be obtained for the following errors:

(1) range;

(2) range rate;

(3) azimuth;

(4) azimuth rate;

(5) elevation rate; and

(6) muzzle velocity.

Since normality and independence have been assumed, mean or bias and variance are sufficient to specify the distributions required. Note, however, that for any engagement these parameters are functions of time.

The following form for the probability density function of the miss vector in a plane perpendicular to the line of sight between weapon and target incorporates the parameters listed above.

Let the total range-like and angle-like errors have variances given by:

$$\sigma_r^2 = \text{variance of total error component in range;}$$

$$\sigma_a^2 = \text{variance of total error component in azimuth; and}$$

$$\sigma_e^2 = \text{variance of total error component in elevation.}$$

Then transforming the range error into the azimuth-elevation plane results in variances

$$\sigma_A^2 = \sigma_a^2 + \sigma_r^2 \frac{\dot{A}^2}{W^2} , \quad \text{and}$$

$$\sigma_E^2 = \sigma_e^2 + \sigma_r^2 \frac{\dot{E}^2}{W^2} ,$$

where:

$$\sigma_A^2 = \text{variance of equivalent azimuth error;}$$

$$\sigma_E^2 = \text{variance of equivalent elevation error;}$$

$$\dot{A} = \text{azimuth rate;}$$

$$\dot{E} = \text{elevation rate; and}$$

$$W = \text{shell velocity at time of intercept.}$$

Assuming a two dimensional normal distribution and neglecting correlation, the value of the probability density function at the target center is

$$\frac{1}{2\pi\sigma_E\sigma_A} \ \exp \ \left(- \frac{B_E^2}{2\sigma_E^2} - \frac{B_A^2}{2\sigma_A^2} \right) ,$$

where:

$$B_E = \text{bias in elevation;}$$

$$B_A = \text{bias in azimuth (i.e., the mean vector).}$$

One final factor must be considered, namely, the correction necessary to account for the dependency between the errors associated with single

rounds fired in a burst, i.e., errors common to all rounds in the burst. The correction is obtained through consideration of the burst survival probability, BSP, given the burst error and the single shot survival probability SSP. Thus,

$$BSP = SSP^n$$

where n is number of rounds in the burst.

For any random variable X with mean $\bar{x}$, put $\varepsilon = X - \bar{x}$. Then

$$E[e^X] = E[e^{\bar{x}+\varepsilon}] \simeq E[e^{\bar{x}} + \varepsilon e^{\bar{x}} + \frac{\varepsilon^2}{2} e^{\bar{x}}] \ ,$$

$$= e^{\bar{x}} (1 + \frac{1}{2} \text{var}(X)) \ .$$

Therefore,

$$E[BSP] = E[\exp(n \ln SSP)] \simeq \exp[n\overline{\ln SSP}] (1 + \frac{n^2}{2} \text{var}(\ln SSP)).$$

Using a Taylor series expansion for $\ln(1-x)$ one obtains

$$-\ln SSP \simeq \frac{A}{2\pi\sigma_1\sigma_2} \quad \exp\left[-\left(\frac{R_1^2}{2\sigma_1^2} + \frac{R_2^2}{2\sigma_2^2}\right)\right]$$

where R_1 and R_2 are the components of the burst error vector, σ_1^2 and σ_2^2 are the intraburst variances and A is the target vulnerable area. Biases are ignored. Taking expectations,

$$-\overline{\ln SSP} = \frac{A}{2\pi(\sigma_1^2+\sigma_3^2)^{1/2} (\sigma_2^2+\sigma_4^2)^{1/2}},$$

where σ_3^2 and σ_4^2 are the variances associated with the bivariate normal distribution of R_1 and R_2. A standard calculation yields

$$\text{var} \, (\ln SSP) = (\overline{\ln SSP})^2 \left\{ \frac{(\sigma_1{}^2 + \sigma_3{}^2)(\sigma_2{}^2 + \sigma_4{}^2)}{\sigma_1 \sigma_2 (2\sigma_3{}^2 + \sigma_1{}^2)^{1/2} (2\sigma_4{}^2 + \sigma_2{}^2)^{1/2}} - 1 \right\},$$

implying that the appropriate correction to the integrated kill rate is

$$\ln \overline{BSP} - n \overline{\ln SSP}$$

$$= \ln \left\{ 1 + \frac{n^2}{2} \, \overline{\ln SSP}^2 \left(\frac{(\sigma_1{}^2 + \sigma_3{}^2)(\sigma_2{}^2 + \sigma_4{}^2)}{\sigma_1 \sigma_2 (2\sigma_3{}^2 + \sigma_1{}^2)^{1/2} (2\sigma_4{}^2 + \sigma_2{}^2)^{1/2}} - 1 \right) \right\} .$$

This correction should be applied over each burst period. It should be noted that if the higher order moments of $\ln SSP$ are large, the approximations used here are no longer valid. However, this is rarely the case with gun systems.

Assembling the methods and assumptions outlined above, the following formula is obtained for the kill rate to which the correction factor for inter- and intra-burst errors is applied:

$$k(t) = \frac{FA(t)}{2\pi \, H_A{}^{1/2} \, H_E{}^{1/2}} \, \exp \left(-\, 1/2 \left(\frac{B_A}{H_A} + \frac{B_E}{H_E} \right) \right)$$

where

$$H_A = \sigma_A{}^2 R^2 + \sigma_{\dot{A}}{}^2 t_f{}^2 R^2 + \dot{A}^2 t_f{}^2 R^2 \left(\frac{\sigma_V{}^2}{V^2} \right)$$

$$+ \dot{A}^2 t_f{}^2 \left(\sigma_R{}^2 + \sigma_{\dot{R}}{}^2 t_f{}^2 \right),$$

$$H_E = \sigma_E{}^2 R^2 + \sigma_{\dot{E}}{}^2 t_f{}^2 R^2 + \dot{E}^2 t_f{}^2 R^2 \left(\frac{\sigma_V{}^2}{V^2} \right)$$

$$+ \dot{E}^2 t_i \left(\sigma_R{}^2 + \sigma_{\dot{R}}{}^2 t_f{}^2 \right),$$

and where:

$$B_A = (b_a R + c_a)^2 \qquad (km^2)$$

$$B_E = (b_e R + c_e)^2 \qquad (km^2)$$

$$\sigma_A^2 = \sigma_{A1}^2 + \sigma_{A2}^2 \qquad (rad^2)$$

σ_{A1}^2 = **variance azimuth sensing** $\qquad (rad^2)$

σ_{A2}^2 = variance azimuth gun-pointing $\qquad (rad^2)$

$$\sigma_E^2 = \sigma_{E1}^2 + \sigma_{E2}^2 \qquad (rad^2)$$

σ_{E1}^2 = variance elevation sensing $\qquad (rad^2)$

σ_{E2}^2 = variance elevation gun-pointing $\qquad (rad^2)$

$\sigma_{\dot{A}}^2$ = variance azimuth rate input to computer $\qquad \left(\dfrac{rad^2}{sec^2}\right)$

$\sigma_{\dot{E}}^2$ = variance elevation rate input to computer $\qquad \left(\dfrac{rad^2}{sec^2}\right)$

R = range (at intercept) $\qquad (km)$

t_f = time of flight $\qquad (sec)$

σ_V^2 = variance in muzzle velocity $\qquad \left(\dfrac{km^2}{sec^2}\right)$

V = muzzle velocity $\qquad \left(\dfrac{km}{sec}\right)$

$A(t)$ = vulnerable area $\qquad (km^2)$

F = average firing rate during a burst and one interburst period $\qquad (sec^{-1})$

$\dot{A}$ = azimuth rate of target (at intercept) $\qquad (rad/sec)$

$\dot{E}$ = elevation rate of target (at intercept) $\qquad \left(\dfrac{rad}{sec}\right)$

σ_R^2 = variance range sensor (input) $\qquad (km^2)$

$\sigma_{\dot{R}}^2$ = variance range rate input to computer $\qquad \left(\dfrac{km^2}{sec^2}\right)$

$$b_a = b_{a1} + b_{a2} + b_{a3} \qquad \text{(rad)}$$

$$b_{a1} = \text{sensor azimuth bias} \qquad \text{(rad)}$$

$$b_{a2} = \text{pointing azimuth bias} \qquad \text{(rad)}$$

$$b_{a3} = \dot{A}\, R^{-1}\, b_R\, t_f \qquad \text{(rad)}$$

$$b_r = \text{range sensing bias} \qquad \text{(km)}$$

$$b_e = b_{e1} + b_{e2} = b_{e3} \qquad \text{(rad)}$$

$$b_{e1} = \text{sensor elevation bias} \qquad \text{(rad)}$$

$$b_{e2} = \text{pointing elevation bias} \qquad \text{(rad)}$$

$$b_{e3} = \dot{E}\, R^{-1}\, b_R\, t_f \qquad \text{(rad)}$$

$$c_a = c_{a1} + c_R\, R^{-1}\, \dot{A}\, t_f \qquad \text{(km)}$$

$$c_R = \begin{array}{l}\text{prediction error due to nonlinear} \\ \text{flight in range direction}[1]\end{array} \qquad \text{(km)}$$

$$c_{a1} = \text{prediction error in azimuth direction}[1] \qquad \text{(km)}$$

$$c_e = c_{e1} + c_R\, R^{-1}\, \dot{E}\, t_f \qquad \text{(km)}$$

$$c_{e1} = \begin{array}{l}\text{prediction error in elevation} \\ \text{direction}[1]\end{array} \qquad \text{(km)}$$

For the most part the parameters in the above list must be supplied by engineering models which provide values for the parameters as a function of ship systems hardware characteristics, cruise missile characteristics, and engagement geometry and time. During the initial stages of the project, work was carried out to produce engineering models which predicted biases and variances for radar systems. This work is described in appendix D to [Bonder, Cherry and Miller, 1973].

---

[1]These must be evaluated as a function of the flight path in any specific case under study.

The effects of EW, in particular enemy jamming activities, are portrayed in the model through the effects of these activities on system performance measures in the above list. The basis of engineering models capable of predicting the relationships between certain types of jamming and performance characteristics of radar systems is described in appendices A and B to this report.

2.2.2 Kill Rates for SAM Systems

Recall that the kill rate[1] $k(t)$ for a particular SAM system/cruise missile pair at time t was defined as

$\delta \cdot k(t)$ = conditional probability that the cruise missile is

killed by the SAM system in the time interval $[t, t+\delta)$,

given that the missile is still alive at time t and

that the SAM system is firing at the cruise missile

at that time.

The probability $\delta \cdot k(t)$ that the cruise missile is killed by a SAM from the SAM system in the interval $[t, t+\delta)$ can be taken in the form

$$\delta \cdot K(t) = i(t) \cdot SSKP \qquad (24)$$

where:

$i(t)$ = probability that the SAM intercepts the cruise missile

in the time interval $[t, t+\delta]$;

---

[1] As in the preceding section, the notation $\cdot k(t)$ is used instead of $\cdot kij(t \ d)$ to denote a kill rate because the damage state and the weapon system 8 and cruise missile j to which it applies is understood.

> SSKP = conditional probability that the SAM kills the cruise
>
> missile, given that it intercepts the cruise missile.

As the above notation suggests, and in contrast to the situation with rapid-fire gun systems, the single-shot kill probability SSKP for SAMs is here assumed to be a constant[1]. The value of this constant, which may depend on the type of cruise missile and the type of SAM but not on time, may be estimated using existing models. It is regarded as being a fundamental descriptor of the engagement. The determination of the kill rate $\delta \cdot k(t)$ is therefore reduced to determining $i(t)$.

Since, by assumption, the actual flight paths and speeds of the cruise missile with respect to the ship are deterministic, the flight time the SAM requires to intercept the cruise missile at any point along its path can be determined for the particular SAM type and cruise missile type involved. This flight time then determines when landing must have taken place in order for the SAM to intercept the cruise missile at that point on its flight path. One may therefore write

$$i(t) = \delta \cdot \int_0^t h(t-u)\, g(t,u)\, du \qquad (25)$$

---

[1]This assumption, which can be relatively easily relaxed, seems reasonable because SAMs are (unlike rounds from rapid-fire guns) guided onto targets.

where:

$$h(x) \cdot \delta = Pr[\text{SAM fired in } (x, x + \delta)],$$

and

$$g(t,u) = \text{probability density function for the flight time } u$$
$$\text{of a SAM fired to intercept the cruise missile at}$$
$$\text{time } t.$$

Notice that the above development of $i(t)$ exhibits the dependence of this function upon the past in that the probability of intercept at time t depends upon the status of the engagement at time $t - u$. Notice also that neither of the functions $h(\cdot)$ and $g(t,\cdot)$ will be "smooth" because the firing process is discrete and because the stochastic variation in flight times for the SAM will normally be small. However, in a limited sense, this model removes the discrete nature of the arrivals (or intercepts) of the SAMs at the cruise missile.

Under the assumption that the SAM flight time has relatively limited variation -- say a small-variance uniform random variable added to a deterministic time $\bar{u}(t)$ -- the above expression for $i(t)$ can be written

$$i(t) = \delta \cdot \int_{\bar{u}(t)-\ell}^{\bar{u}(t)+\ell} h(t - u) \frac{1}{2\omega} \, du \quad .$$

If it is further assumed, as seems reasonable, that the probability
$\delta \cdot h(t - u)$ that a SAM is fired in $[t - u, t - u + \delta]$ has the constant
value[1] $\delta \cdot h_t$ for u in the range $\bar{u}(t) - \ell \leq \bar{u}(t) + \ell$, the above expression
becomes

$$i(t) = \delta \cdot h_t \quad . \tag{26}$$

To determine the values of the h_t for use in (26), it is necessary
to consider the process by which the SAM launching system operates. The
firing process from the SAM launcher can be described by a cyclic Markov
renewal process in which firing takes place at random intervals. As in
the case of SAM flight times, the stochastic variation of these intervals
may be small, but the following analysis leads to a result identical to
the deterministic case.

---

[1] As the notation suggests, the "constant" h_t may have different values
for different times t.

Suppose that there are R rails on the SAM launcher, and consider an R state Markov renewal process with transition matrix:

$$
\begin{array}{c}
 \\
1 \\
2 \\
3 \\
\cdot \\
\cdot \\
\cdot \\
R
\end{array}
\left[
\begin{array}{cccccc}
1 & 2 & 3 & 4 & \cdots & M \\
0 & F_1(x) & 0 & 0 & \cdots & 0 \\
0 & 0 & F(x) & 0 & \cdots & 0 \\
0 & 0 & 0 & F(x) & \cdots & 0 \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
\cdot & \cdot & \cdot & \cdot & & \cdot \\
F(x) & 0 & 0 & 0 & & 0
\end{array}
\right]
$$

where

$F_1(x)$ = probability distribution of time to reload.

$F(x)$ = probability distribution of time between firings on adjacent rails.

Further, let

τ_1 = mean reload time.

τ = mean inter-firing time.

For this process, it can be shown [Cinlar, 1969], that in the limit the rate at which firings take place is given by:

$$
\frac{1}{\tau_1 + (R - 1)\tau} \quad .
$$

The case of L independent launchers can be modeled by superposing the models for the individual launchers. In this case, an extension of results in [Cherry, 1972] gives

$$\frac{L}{\tau_1 + (R - 1)\tau}$$

as the rate at which firings of all the SAM systems take place. Using a technique common to kill rate modeling, the probability $\delta \cdot h_t$ that a SAM is fired in the interval $[t, t + \delta)$ can then be approximated by the mean rate at which SAMs could be fired:

$$\delta \cdot h_t = \frac{L_t}{\tau_1 + (R - 1)\tau} \cdot \delta$$

where L_t is the number of launchers available at time $t - u$ (u being the SAM flight time yielding intercept at time t).

Finally, it should be noted that the above discussion has dealt with the rate at which SAMs can be *fired* whereas the equations introduced in the preceding section deal with the rate at which SAM's *arrive* at the cruise missile. The rate at which SAMs can be fired does not correspond to the rate at which intercepts can occur since a Doppler effect is present due to the flight times of the SAMs and the motion of the target cruise missile. Let

t_1 = time of first intercept,

t_2 = time of second intercept,

T_1 = time of flight of first SAM, and

T_2 = time of flight of second SAM.

Then the time between launchings is given by;

$$(t_2 - T_2) - (t_1 - T_1)$$

$$= (t_2 - t_1) - (T_2 - T_1)$$

The length of the inter-intercept time period in the limit as the period grows small is equal to the firing interval times $1 - D_t T_f$ where T_f is the time of flight and D_t indicates differentiation with respect to intercept time. The model firing rate thus becomes:

$$\left(\frac{L}{\tau_1 + (M-1)\,\tau} \right) \quad (1 - D_t T_f)$$

Note that the above rate corresponds to the situation in which guidance channel constraints are not operative. In the case in which guidance channel constraints are operative the rate at which SAMs can be fired corresponds to the rate at which intercepts occur. Consider the non-homogeneous Poisson process with:

$$Pr[T \geq t] = e^{-\int_0^t k(u)\,du}$$

For this process it can be shown that the mean rate of events over a time period $[0,t]$ is given by

$$\left(\int_0^t k(u)\,du \right)/t."$$

Accordingly, approximate the mean rate of kills by the expression:

$$f_1(t) = -\sum_{j=1}^{M} \frac{\ln m_j(t)}{t} \; ,$$

where

$f_1(t)$ = mean rate of kills in the period $[0,t]$,

$m_j(t)$ = Pr[cruise missile j survives to time t],

M = number of cruise missiles engaged.

This expression reflects the number of kills per time unit; the rate required is the number of intercepts. Consider a Poisson process with parameter λ and suppose that each time an event occurs in this process it is recorded with probability SSKP. It can be shown that the recorded event process is Poisson with parameter $\lambda \cdot$SSKP. Based on this analogy, the approximation $f(t) = \text{SSKP}^{-1} \cdot f_1(t)$ is used for the firing rate under guidance constraints. The firing rate used thus becomes

$$f(t) \; (1 - D_t T_f)$$

for those cases in which $f(t)$ is less than the number of guidance channels available.

2.3 *Determination of Firing Probabilities*

The firing probabilities $p_{ij}(t|d)$, which measure the likelihood that each weapon system i is actually firing at each cruise missile j at time t, may be modeled as a function of:

- whether or not cruise missile j has already been destroyed or has impacted the ship by time t; and

- the logic whereby the ship assigns defensive weapons to surviving cruise missiles.

It has been assumed (see the list of assumptions near the beginning of section 2.1) that the ship uses a weapon assignment logic in which all available defensive weapons are always assigned to the one cruise missile which will, if not sooner destroyed, be the next to impact the ship. This logic requires that the ship has the capability to evaluate the relative threats posed by the surviving cruise missiles at any time. A conceptual threat evaluation system that the ship might use for this purpose is therefore briefly discussed in section 2.3.1. The manner in which this threat evaluation system could provide data for the assumed weapon assignment logic is discussed in section 2.3.2. Finally, expressions for the firing probabilities which reflect the structure of the assumed weapon assignment logic are developed in section 2.3.3.

2.3.1 Threat Evaluation

For a single ship/multiple cruise missile engagement, the relative threat posed by each of several incoming cruise missiles may be evaluated in terms of:[1]

(1) the predicted length of time until impact;[2]

(2) the extent to which defensive weapon systems are already

committed to handling previously evaluated cruise missiles; and

---

[1]In the case of a multiple ship/multiple cruise missile engagement, other factors, such as the value of each ship, should be added to the list of items in terms of which threats should be evaluated.

[2]Depending on the configuration and capabilities of the defensive weapons on the ship, the position of a cruise missile at each time may also be important to an evaluation of the threat posed by the missile at that time. For example, a cruise missile may be observed to be about to enter a portion of its track which is not coverable by the ship's defensive weapons because of, say, elevation or azimuth limitations of these weapons so that the cruise missile, if not immediately engaged, will surely impact the ship.

(3) the extent to which defensive actions have been taken whose

outcomes have not yet occurred.

The length of time remaining until impact is clearly relevant to

evaluating the threat posed by an incoming cruise missile. Were it not

for items (2) and (3) in the above list, the threats posed by incoming

cruise missiles could perhaps reasonably be ranked solely on the basis of

remaining time to impact. Indeed, even when factors (2) and (3) are

acknowledged, one may still argue that ranking threats by remaining times

to impact regardless of the defensive actions which may have been taken

against some of the missiles and whose outcomes have not yet occurred

may be a conservative strategy. As (2) and (3) suggest, however, some

schemes for threat evaluation are structured to account for the fact that

resources already committed to a cruise missile may, after a time lag,

result in the destruction of that missile so that no further action against

the missile would be necessary.[1] However, the previously-made assumption

that the ship can redirect in-flight fire as long as the fire has not

reached the cruise missile at which it was last targeted effectively as-

sumes away items (2) and (3) in the above list. It is therefore consis-

tent to assume that the ship uses a threat evaluation system in which the

threat-rank of each cruise missile is the same as the order in which it

will, if not sooner destroyed, impact the ship and in which the threat

---

[1]Threat evaluation structures of this second type could be designed by
ranking threats by weighted remaining times to impact. The weights
assigned to cruise missiles would be functions of the probabilities
that these missiles will survive the defensive actions already taken.
Threat evaluations would then be accomplished in a computational envi-
ronment involving, in addition to the simple flight time calculations,
calculations of a more complex character.

rank of a cruise missile, once assigned, changes only when the missile is destroyed or impacts the ship[1] and does not otherwise change in time. This assumed threat ranking procedure will be used to provide input to the assumed ship's weapon assignment logic which is discussed in the next section.

2.3.2 Weapon Assignment

In a single ship/multiple cruise missile engagement, the objective of a weapon assignment logic is to assign defensive weapons to incoming cruise missiles in such a way as to minimize some measure of the damage to the ship (e.g., expected "damage", probability of impact of one or more cruise missiles, etc.). To accomplish this objective, a weapon assignment procedure should account for at least the following:

(1) the relative threat posed by each of several incoming cruise missiles;

(2) the availability of defensive weapon systems; and

(3) the characteristics of the defensive weapon systems and of the cruise missiles.

The relative threats posed by the incoming cruise missiles can be determined by a suitable threat evaluation scheme as discussed previously. The notion of defensive weapon system *availability*, as used above, includes a consideration of whether or not a given defensive weapon (together with its required support systems such as guidance links, gun

---

[1]Cruise missiles which have been destroyed or which have impacted the ship may be said to have a null threat rank.

pointing radars, etc.) is still alive as well as a consideration of whether or not the weapon is already in use and, if so, whether or not reassignment might be profitable. The characteristics of the defensive weapon systems which are relevant to the weapon assignment process clearly include such factors as range, accuracy, rate of fire, time of flight for projectiles, etc.

Two weapon assignment logics have been described by [Forsyth etal., 1973][1]. The first of these procedures has the advantage of being extremely simple. However, it appears that this procedure may, when employed against cruise missiles, result in random weapon assignments. The second procedure is considerably more complex; it involves:[2]

- the ranking of incoming cruise missiles by threat (the threat ranks may change in time and may depend on defensive actions already taken);

- the establishment of categories of threat ranks;

- the assignment of defensive weapons to cruise missiles whose threat rank is in the highest threat category until this category becomes empty (this may involve the reassignment of defensive weapons assigned to cruise missiles with threats in a lower category to cruise missiles whose threats enter a previously empty higher category).

---

[1]Both of the weapon assignment procedures described by Forsyth include some aspects of what has here been called "threat evaluation" and been viewed as being distinct from weapon assignment.

[2]The structure of this second weapon assignment procedure resembles that of a "foreground/background queue", and advantage of this correspondence could be taken in constructing a reasonably detailed model of the procedure.

The second of the above weapon assignment procedures appears to be the more promising of the two.

In view of assumptions which have previously been made, namely,

- that the ship uses a threat evaluation system in which threat ranks of a cruise missile, once assigned, change only when the missile is destroyed or impacts the ship (see the discussion in the previous section), and

- that the ship has the capability to redirect in-flight SAMs and/or gunfire to new targets as long as these projectiles have not reached the cruise missile at which they were last targeted (see the discussion at the beginning of section 2,0),

the assumed weapon assignment logic is a special case of the second weapon assignment procedure described by Forsyth in which the threat categories of the latter consist of singleton sets.

2.3.3 Determination of Firing Probabilities

The assumptions which have been made in the preceding two subsections about the ship's threat evaluation system and weapon assignment logic may be summarized as follows:

- the threat rank assigned by the ship to a cruise missile is the same as the order in which the cruise missiles will, if not sooner destroyed, impact the ship;

- the threat rank of a cruise missile, once assigned, changes only when the missile is destroyed or impacts the ship and does not otherwise change in time;

- at any time during the engagement, all available defensive
  weapons are assigned to the cruise missile having the highest
  threat rank and remain so assigned until the missile is destroyed
  or impacts the ship.

Recall that, by definition, $p_{ij}(t|d)$ is the conditional probability
that defensive weapon system i is assigned to (and firing at) cruise missile
j at time t, given that the ship is in damage state d and that cruise
missile j is still alive at that time. In view of the assumed threat
evaluation/weapon assignment logic described above, either all the ship's
available defensive weapons will be assigned to a particular cruise missile
at any time, or none will be so assigned. In particular, the available
defensive weapons will all be assigned to cruise missile j at time t if
and only if

- cruise missile j is still alive at time t, and
- either j=1 or, if $j \geq 2$, cruise missile 1, ..., j-1 have
  been destroyed or have impacted the ship by time t.

Thus

$$p_{i1}(t|d) = \begin{cases} m_1(t|u) & \text{if } d=u \text{ and } 0 \leq t < t_1, \\ \text{undefined} & \text{if } d \neq u \text{ and } 0 \leq t < t_1, \\ 0 & \text{otherwise.} \end{cases} \qquad (27)$$

Moreover, we have on the one hand $p_{ij}(t|d) = 0$ for all $1 \leq i \leq W$, all d,
all $2 \leq j \leq M$ and all $t > t_j$. On the other hand, for all $1 \leq i \leq W$, all d,
all $2 \leq j \leq M$ and all $t < t_j$, we have

$$p_{ij}(t|d) = \Pr[i \text{ firing on } j \text{ at time } t|\text{ship in } d \text{ and } j \text{ alive at}$$
$$\text{time } t]$$

$$= \frac{\text{Pr[i firing on j and j alive at t|ship in d at t]}}{\text{Pr[j alive at t|ship in d at t]}}$$

$$= \frac{\text{Pr[1, } \cdots \text{, j-1 dead at t and j alive at t|ship d at t]}}{\text{Pr[j alive at t|ship in d at t]}}$$

$$= \frac{\text{Pr[j-1 dead at t and j alive at t|ship in d at t]}}{\text{Pr[j alive at t|ship in d at t]}}$$

$$= \frac{1}{\text{Pr[j alive at t|ship in d at t]}}$$

$$\bullet \Big[\{\text{Pr[j-1 dead at t and j alive at t|ship in d at t]}$$

$$\div \text{Pr[j-1 alive at t and j alive at t|ship in d at t]}\}$$

$$-\text{Pr[j-1 alive at t and j alive at t|ship in d at t]} \Big]$$

$$= \frac{1}{\text{Pr[j alive at t|ship in d at t]}}$$

$$\bullet \Big[\text{Pr[j alive at t|ship in d at t]}$$

$$-\text{Pr[j-1 alive at t|ship in d at t]} \Big]$$

$$= \frac{1}{m_j(t|d)} \cdot [m_j(t|d) - m_{j-1}(t|d)]$$

$$= [1 - m_{j-1}(t|d)/m_j(t|d)]. \tag{28}$$

The third equality in the above string of equations follows from the fact that, according to the weapon assignment logic, defensive weapon system i will be firing at cruise missile $j \geq 2$ at time t if and only if cruise

missiles 1, $\cdots$, j-1 are dead at that time and cruise missile j is alive

at that time. The fourth equality follows from the fact that the death

of cruise missile j-1 implies that of the preceding cruise missiles, if

any. The sixth equality follows from the fact that the aliveness of cruise

missile j-1 implies that of cruise missile j (because of the structure of

the assumed weapon assignment logic). The other equations are merely

manipulative; they express facts about conditional probabilities or, in

the case of the last two equations, the definition of the $m_j(\cdot|\cdot)$.

Equations (27) and (28) may be used in (21) to compute the survival

probabilities $m_j(\cdot|d)$ for the cruise missiles and the damage state

probabilities $q_d(\cdot)$ for the ship for the case when the ship uses the threat

evaluation/weapon assignment logic which has been assumed. As indicated in

in the discussion which led to (21), the normal procedure would be to

bootstrap (21) numerically to obtain the survival probabilities $m_j(t|d)$

and $p_{ij}(t|d)$ for all time t in each time interval $[t_k^+, t_{k+1})$. However,

the simple form of (27) and (28) suggests that it may be possible to

insert these expressions in (21) and integrate the resulting system of

equations directly, thus bypassing the numerical bootstrapping. This

approach has been examined. While it appears that the use of numerical

techniques cannot be completely avoided by using the analytic approach,

it is possible to reduce the numerical problem to a form to which simple

and well-known numerical methods can be applied. Details are given in

appendix C.

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3.0 RECOMMENDATIONS FOR FUTURE RESEARCH

The single ship/multiple cruise missile engagement model described
in the previous section has been developed to a point where computer im-
plementation and initial tryout is now appropriate. VRI feels that the
model has considerable potential as an aid to analysis of ship/cruise
missile engagements and that the model may be helpful in devising counter-
measures (e.g., improved or satisfactory weapon lethality characteristics
with respect to cruise missiles) to the cruise missile threat. VRI there-
fore recommends that computer implementation and initial tryout be scheduled.

VRI also recommends that the initial runs of the model include a varia-
tional analysis to establish the sensitivity of the model outputs to the
inputs. A high degree of sensitivity of the outputs to the inputs will
suggest the need for accuracy in the input data so as to ensure reliable
outputs.

In addition to establishing the extent to which submodeling may be
necessary for data generation, the initial trial runs of the model may
suggest that modifications to the existing model structure are appropriate.
While it is unlikely that changes to the overall mathematical structure of
the model would be indicated as a result of the trial runs, it must never-
theless be recognized that the development of most useful models is an
evolutionary process and that structural changes made in the light of ex-
perience gained in using the model cannot be ruled out.

Aside from the research which may be necessary for data generation or
evolutionary structural modification as mentioned above (the need for either
of which is not yet established), there appear to be two types of research

which may be necessary in order for the model to realize its full potential; these are:

- refinements - improvement in the flexibility or degree of realism with which the model portrays engagements in the scenario as presently conceived; and

- extensions - enlargements of the scenario that the model portrays.

The most obvious example of a refinement to the existing model -- and one which should be a high-priority objective of future research -- is the portrayal of different weapon assignment logics for the ship. Indeed, the ship's weapon assignment logic is one of the primary variables of the engagement which is subject to the control of the ship, and so the degree to which the model (as presently structured or as modified in the future) will be useful in devising countermeasures to the cruise missile threat will therefore be in direct proportion to the degree of flexibility the model has in portraying different and varied weapon assignment logics.

A second refinement to the existing model is the relaxation of the assumption that the ship can redirect in-flight gunfire and SAMs; in VRI's opinion, however, this is definitely secondary in importance when compared to endowing the model with the capability to portray different and varied weapon assignment logics.

The most obvious example of an extension to the existing single ship/ multiple cruise missile engagement model -- and one that should receive serious attention in the near term -- is the enlargement of the existing model's scenario to include more than one ship. The result of research of this type would be a "multiple ship/multiple cruise missile engagement

model" which could be useful as an aid to planning for fleet air defense against cruise missiles. It may be expected that such a multiple-ship model will at least superficially resemble the single-ship model and may in fact have a closely parallel structure, so that the single-ship model may be regarded as being a prototype of the multiple-ship model -- a natural step in an evolutionary chain of models. The fact that present Navy doctrine often calls for ships (some of enormous value in terms of construction cost) to travel in company, together with the serious and recognized threat posed by cruise missiles, suggests that the multiple ship model be developed as soon as practicable.

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APPENDIX A

RADAR ANGULAR AND RANGE TRACKING ERRORS:
AMPLITUDE COMPARISON, SEQUENTIAL LOSING AND ANGLE TRACKING

A submodel for predicting values of the angular tracking errors for gun-system radars (these are the quantities denoted by $\sigma_A^{r^2}1$ and $\sigma_A^{r^2}2$ in section 2.2.1) in the presence of enemy jamming is described in this appendix. Theoretical lower bounds on the variances of angle error estimators are derived, and some of the most commonly used error estimation techniques are analyzed to determine how closely they approach these theoretical lower limits (Cramér Rao bounds).

More specifically, this appendix derives the Cramér Rao bound on the variance of error angle estimators and then discusses in some detail one implementation of such an estimator called amplitude comparison, sequential lobing. The first and second moments of the conventional sequential lober are derived and comparisons are then made with the Cramér-Rao bound. The derivation is largely based on an unpublished report[1] issued by Technology Service Corporation [Lank, Pollon, 1969].

Appendix B will discuss a second technique called amplitude comparison monopulse, which is also used to obtain angle error information.

The results concerning angle tracking errors can be directly used as an input to a target tracking analysis. See, for example, appendix D in [Bonder, Cherry, Miller, 1973].

Complex signal notation, as shown for example in [Miller, 1969] and [Reed, 1962], is employed. The notation $E(\cdot)$ stands for the expectation

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[1]The report appears in *IEEE Trans. Aerospace and Elect. Sys.*, Vol. AES-10, No. 3, May 1974, pp. 393-397.

of the quantity in the parenthesis while Re($\cdot$) and Im($\cdot$) stand for the real and imaginary parts of the indicated argument. A superscript $\ast$ denotes the conjugate of a complex number, and absolute values are denoted by $|\cdot|$. A superscript ρ denotes the transpose conjugate of a vector or matrix.[1]

A.1 *Cramer-Rao Bound: Noncoherent Processing*

It is assumed that M noncoherent received waveforms or pulses are to be processed in order to obtain the direction and the magnitude of the error angle, denoted by ε, between the target direction and the zero tracking error direction (which is usually the axis of symmetry of the antenna) in one dimension or coordinate. The purpose of this section is to derive the Cramer-Rao lower bound on the variance of an unbiased estimator of which utilizes the following complex observables when noncoherent processing is employed:

$$Z_m = AG_m(\varepsilon)e^{i\theta_m} + V_m, \quad m = 1,\ldots,M, \tag{A1}$$

---

[1]Other notation will be introduced as needed. All notation used in this appendix is independent of that used elsewhere in this report even though some overlap occurs.

where

ε is the error angle between the target direction and the zero tracking error direction,

$G(\cdot)$ are the normalized two-way antenna voltage patterns[1] measured relative to the zero tracking error direction,

A   is the observable amplitude which is normalized with respect to the antenna beam boresight direction.

θ_m is the phase of the mth observable,[1]

M   is the number of noncoherent received waveforms that are processed.

The V_m are the signals caused by system noise and are zero mean, complex Gaussian random variables which are assumed mutually independent. In addition, the in-phase and quadrature components (real and complex parts) of each V_m are assumed independent with the same variance σ^2. Under these conditions, the complex Gaussian random variables V_m have the properties [Reed, 1962]

$$E(V_m V_m^*) = 2\sigma^2 , \qquad E(V_m V_m^*)^2 = 8\sigma^4 ,$$

---

[1]The receive pattern is not necessarily the same as the transmit pattern and distinct observables may be received with the same antenna pattern.

[2]The case when the phase is the same for each target signal, i.e., monopulse processing, will be discussed in appendix B.

and all other first, second, third, and fourth order moments are zero, where

σ^2 is the variance of the in-phase and quadrature noise variates,

* is the complex conjugate operation.

The complex probability density function of the Z_m can be written as

$$p = p(Z_1,\ldots,Z_m) = C \exp\left[-\frac{1}{2}(Z - H)^\rho K^{-1}(Z - H)\right] \quad , \quad (A2)$$

where

$$Z = \begin{pmatrix} Z_1 \\ \cdot \\ \cdot \\ \cdot \\ Z_M \end{pmatrix} \quad , \quad H = A\begin{pmatrix} G_1(\varepsilon)e^{i\theta_1} \\ \cdot \\ \cdot \\ \cdot \\ G_M(\varepsilon)e^{i\theta_M} \end{pmatrix} \quad , \quad V = A\begin{pmatrix} V_1 \\ \cdot \\ \cdot \\ V_M \end{pmatrix}$$

C is a normalizing constant,

$K = \frac{1}{2}E(VV^\rho) = \sigma^2 I$,

I is the identity matrix,

ρ is the conjugate-transpose operation.

There are M + 2 unknown parameters: ε, A, θ_m, m = 1,...,M.

The likelihood function can now be written as

$$L = \ln p = \ln C - \frac{1}{2\sigma^2}\left[Z^p Z - 2\text{Re}(Z^p H) + A^2 \sum G_m^2(\varepsilon)\right] \quad , \qquad \text{(A3)}$$

where

Re is the real part,

all summations go from 1 through M.

Expanding the middle term of (A3) it is seen that the likelihood function can be written as

$$L = \ln C - \frac{1}{2\sigma^2}\left[Z^p Z - 2A \sum G_m(\varepsilon)(x_m \cos \theta_m + y_m \sin \theta_m) + A^2 \sum G_m^2(\varepsilon)\right]$$

$$\text{(A4)}$$

Making use of (A4) it is seen that the $(M + 2) \times (M + 2)$ Fisher information matrix has the form

$$M = (a_{jk}),$$

where

$$a_{11} = -E\left(\frac{\partial^2 L}{\partial \varepsilon^2}\right) = \frac{A^2}{\sigma^2} \sum \left(G_m'(\varepsilon)\right)^2 \quad ,$$

$$a_{22} = -E\left(\frac{\partial^2 L}{\partial A^2}\right) = \frac{1}{\sigma^2} \sum G^2(\varepsilon)$$

$$a_{12} = a_{21} \quad = \quad -E\left(\frac{\partial^2 L}{\partial \varepsilon \partial A}\right) = \frac{A}{\sigma^2} \sum G_m'(\varepsilon) G_m(\varepsilon) \quad ,$$

$$a_{1,k+2} \quad = \quad -E\left(\frac{\partial^2 L}{\partial \varepsilon \partial \theta_k}\right) \quad = 0, \qquad k = 1,\ldots,M \quad ,$$

$$a_{j+2,1} \quad = \quad 0, \quad j = 1,\ldots,M \quad ,$$

$$a_{2,k+2} \quad = \quad -E\left(\frac{\partial^2 L}{\partial A \partial \theta_k}\right) \quad = \quad 0, \qquad k = 1,\ldots,M \quad ,$$

$$a_{j+2,2} \quad = \quad 0, \quad j = 1,\ldots,M \quad ,$$

Thus the information matrix N can be written as

$$N \quad = \quad \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \quad , \qquad (A5)$$

where

$$P \quad = \quad \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \quad ,$$

Q   is a M X M square matrix,

all other elements of N are zero.

If $\hat{\varepsilon}$ is an unbiased estimator of ε, the error angle between the target direction and the zero tracking error direction, with variance σ_ε^2, then the Cramér-Rao theorem states that a lower bound for σ_ε^2, when noncoherent processing is utilized, has the form

$$\sigma_\varepsilon^2 \geq U_1^\rho N^{-1} U_1 \quad , \tag{A6}$$

where U_1 is a vector having a one as its first element and zeros elsewhere. Because of the form of N given by (A5) this can be written as

$$\sigma_\varepsilon^2 \geq U_1^\rho P^{-1} U_1 \quad . \tag{A7}$$

Making use of the expressions developed above it is seen that when noncoherent received waveforms are processed (A7) can be written as

$$\sigma_\varepsilon^2 \geq \frac{1}{2R} \left\{ \frac{\sum \left(G_m(\varepsilon)\right)^2}{\left[\sum \left(G_m'(\varepsilon)\right)^2\right]\left[\sum \left(G_m(\varepsilon)\right)^2\right] - \left[\sum G_m'(\varepsilon)G_m(\varepsilon)\right]^2} \right\} ,$$

$$\tag{A8}$$

where R is the per pulse signal-to-noise ratio for each beam and it is assumed that the target is located at the boresight of the beam

$$R = \frac{A^2}{2\sigma^2} = \frac{E_s}{N_o} \quad ,$$

E_s is the energy in each received waveform and N_o is the one-sided power spectral density of the system noise.

For M = 2, which is the case when two antenna patterns are used to derive the error angle ϵ, it is seen that (A8) can be written as

$$\sigma_\epsilon^2 \geq \frac{1}{2R(G_1^2 + G_2^2)} \left[\frac{\frac{G_1}{G_2} + \frac{G_2}{G_1}}{\frac{G_1'}{G_1} - \frac{G_2'}{G_2}} \right]^2 \quad , \quad \text{(A9)}$$

where

$$G_1 \equiv G_1(\epsilon) \quad , \quad G_2 \equiv G_2(\epsilon) \quad .$$

Equation (A9) is the one dimensional analog of equation (69), page 29, in [Horstetter & Delong, 1969].

A.2 Cramer-Rao Bound: Coherent Processing

If we use the same notation as developed in the previous section, then the complex radar observables when coherent processing is employed have the form

$$Z_m = AG_m(\epsilon)e^{i(\omega\tau_m - \theta)} + V_m, \quad m=1,\ldots,M \tag{A10}$$

where

ω is the doppler frequency (assumed very much smaller than the transmitted center frequency),

τ_m are known time delays relating the times of occurrence of the Z_m. For convenience it is assumed that $\tau_1 = 0$.

θ is the unknown phase corresponding to the first observable.

Thus there are four unknown parameters: ϵ, A, ω, θ.

The likelihood function corresponding to (A10) can now be written as

$$L = \ell n \, \dot{C} - \frac{1}{2\sigma^2} \left[Z^0 Z - 2A \sum G_m(\epsilon) \left\{ x_m \cos(\omega\tau_m - \theta) + y_m \sin(\omega\tau_m - \theta) \right\} \right.$$

$$\left. + A^2 \sum G_m^2(\epsilon) \right] \tag{A11}$$

The elements of the 4×4 Fisher information matrix

$$N = (a_{ij})$$

have the following values

$$a_{11} = -E\left(\frac{\partial^2 L}{\partial \epsilon^2}\right) = \frac{A^2}{\sigma^2} \sum \left(G_m'(\epsilon)\right)^2 \qquad ,$$

$$a_{22} = -E\left(\frac{\partial^2 L}{\partial A^2}\right) = \frac{1}{\sigma^2} \sum G^2(\epsilon) \qquad ,$$

$$a_{12} = a_{21} = -E\left(\frac{\partial^2 L}{\partial \epsilon \partial A}\right) = \frac{A}{\sigma^2} \sum G_m'(\epsilon) G_m(\epsilon) \quad ,$$

$$a_{13} = a_{31} = -E\left(\frac{\partial^2 L}{\partial \epsilon \partial \omega}\right) = 0 \qquad ,$$

$$a_{14} = a_{41} = -E\left(\frac{\partial^2 L}{\partial \epsilon \partial \upsilon}\right) = 0 \qquad ,$$

$$a_{23} = a_{32} = -E \left(\frac{\partial^2 L}{\partial A \partial \omega} \right) = 0 \qquad ,$$

$$a_{24} = a_{42} = -E \left(\frac{\partial^2 L}{\partial A \partial \theta} \right) = 0 \qquad .$$

Thus by using the same procedure as employed in the previous section it can be shown that the lower bound of the variance of an unbiased estimator of ε, the error angle between the target direction and the zero tracking error direction, is also given by (A8) for the case of coherent processing. Therefore, the Cramér-Rao lower bound is the same if either non-coherent and coherent processing is used to obtain an estimate of the error angle.
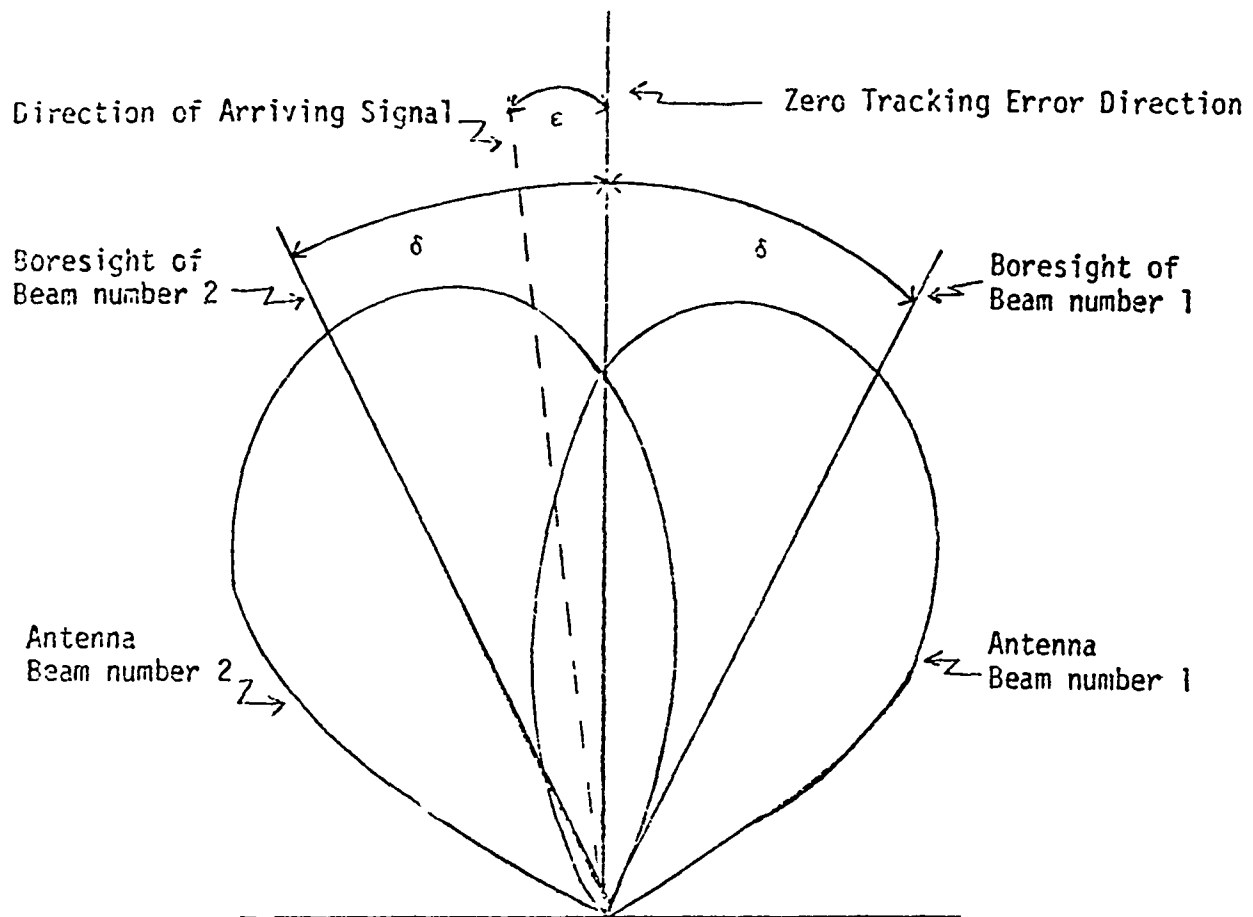
If the signal-to-noise ratio is high enough, then well-designed processors should achieve about the same performance, in terms of estimator variance, for either non-coherent or coherent radar systems. This point is noted, for example, on page 28 of the paper by [Hofstetter and DeLong, 1969]. Therefore, from this point on we shall only discuss angle estimation for noncoherent radar systems.

A.5 Discussion of the Two-Beam Estimator

Expression (A9) gives the Cramér-Rao lower bound on the variance of an unbiased estimator of the tracking error angle ε when two received waveforms are used to derive the tracking error signal. If M received waveforms are processed for each of two antenna beam positions then it is easily shown that the expression for the lower bound must be modified to read

$$\sigma_\varepsilon^2 \geq \frac{1}{2MR\left(G_1^2 + G_2^2\right)} \left[\frac{\dfrac{G_1}{G_2} + \dfrac{G_2}{G_1}}{\dfrac{G_1'}{G_1} - \dfrac{G_2'}{G_2}}\right]^2 , \qquad (A12)$$

where the notation given in the previous sections is used. Almost all operational tracking error estimators use the received signals from two antenna beam positions to derive tracking error information and the remainder of this paper will only be concerned with such two beam estimators. From this point on it will also be assumed that each antenna beam is symmetric about its boresight direction and that the beams are squinted or offset by the same amount on either side of the zero tracking error direction (which is usually the symmetry axis of the antenna). The same antenna pattern (referenced to the boresight direction) is assumed for each of the beams. The angle between the zero-tracking error direction and the antenna beam boresight directions will be denoted by δ. A picture of the geometry involved is given in figure A1.

Direction of Arriving Signal — Zero Tracking Error Direction

ϵ

δ δ

Boresight of
Beam number 2 — Boresight of
Beam number 1

Antenna
Beam number 2 — Antenna
Beam number 1

- All angles are measured in the counter-clockwise direction

- ϵ is the tracking error angle

- δ is the antenna beam squint or offset angle

- $P(\cdot)$ is the normalized two-way antenna voltage pattern measured
  with respect to the beam boresight

- $G_1(\cdot)$ and $G_2(\cdot)$ are the normalized two-way antenna voltage patterns
  of beam number 1 and beam number 2, respectively, measured
  with respect to the zero tracking error direction.

FIGURE A1:   TWO-BEAM GEOMETRY

From figure A1 it is seen that

$$G_1(\varepsilon) = P(\delta + \varepsilon) , \qquad G_2(\varepsilon) = P(-\delta + \varepsilon). \qquad \text{(A13)}$$

We now assume that ε is a small angle so that $P(\delta + \varepsilon)$ and $P(-\delta + \varepsilon)$ can be approximated by the first two terms in the Maclaurin expansion to obtain

$$G_1(\varepsilon) = P(\delta + \varepsilon) \cong P(\delta) + \varepsilon P'(\delta) ,$$

$$\text{(A14)}$$

$$G_2(\varepsilon) = P(-\delta + \varepsilon) \cong P(-\delta) + P'(-\delta).$$

From the symmetry of an antenna beam it follows that

$$P(-\delta) = P(\delta) , \qquad P'(-\delta) = -P'(\delta) , \qquad \text{(A15)}$$

so that (A14) becomes

$$G_1(\varepsilon) = P(\delta + \varepsilon) = P(\delta) + \varepsilon P'(\delta) ,$$

$$\text{(A16)}$$

$$G_2(\varepsilon) = P(-\delta + \varepsilon) = P(\delta) - \varepsilon P'(\delta).$$

If we now substitute (A16) into (A12) the following expression is obtained

$$\sigma_\epsilon^2 \geq \frac{1}{4MR} \left(\frac{1}{P'(\delta)}\right)^2 \equiv B_{CR} \qquad (A17)$$

In this equation B_{CR} is the Cramér-Rao lower bound on the variance of an unbiased estimate of the tracking error ϵ when the conditions illustrated in figure A1 are satisfied.

To illustrate the type of result that is obtained we shall assume that the normalized main lobe of the one-way antenna voltage pattern can be approximated by a Gaussian beam shape and that the normalized two-way pattern can be represented as

$$P(\psi) = \exp\left[-2K\left(\frac{\psi}{\psi_1}\right)^2\right] , \qquad (A18)$$

where ψ_1 is the half-power beamwidth[1] and $K = 2 \ln 2 \cong 1.3863$. As will be shown, the assumption of a Gaussian beamshape leads to a very simple result and is perfectly adequate for most radar system investigations.

If we now substitute (A18) into (A17) the following expression is obtained for the lower bound

$$\sigma_\epsilon^2 \geq \frac{1}{4MR} \left(\frac{\psi_1^4}{16K^2\delta^2}\right) \exp\left[4K\left(\frac{\delta}{\psi_1}\right)^2\right] . \qquad (A19)$$

---

[1] This means that $P\left(\pm \frac{\psi_1}{2}\right) = \frac{1}{2}$.

We next determine that value of δ which minimizes the right hand side of equation (A20). If we denote this optimal squint or offset angle by δ_0, then by differentiating the right hand side of (A20) with respect to δ it can be determined that

$$\delta_0 = \frac{\psi_1}{2\sqrt{K}} \cong (0.4247)\psi_1 \tag{A21}$$

Furthermore

$$P(\delta_0) = \exp\left(-\frac{1}{2}\right) \quad,$$

$$P'(\delta_0) = \left(-\frac{1}{\delta_0}\right)\exp\left(-\frac{1}{2}\right) \quad.$$

If we now substitute (A21) into (A20), the following expression is obtained for the lower bound when the optimal squint angle δ_0 is chosen

$$\sigma_\epsilon^2 \geq \frac{\psi_1^2}{32MR}\left(\frac{e}{\ln 2}\right) \quad,$$

or

$$\sigma_\epsilon^2 \geq \frac{(0.1226)\psi_1^2}{MR} \quad, \tag{A22}$$

where

ψ_1 is the half-power beamwidth of the one-way pattern,

M   is the number of pulses processed per beam position,

R   is the per pulse signal-to-noise measured relative to the nose of a beam.

Most angle processors come close to achieving the lower bound given by (A22) and for most analysis purposes it is adequate to use this lower bound as the variance of an estimate of ε, the tracking error angle. The next section will discuss one technique that is utilized for computing the tracking error angle. As will be shown, this method, called sequential lobing, asymptotically achieves the variance given by (A22) with increasing signal-to-noise ratio, R.

A.4 Amplitude Comparison, Sequential Lobing Angle Estimation

One method of obtaining the direction and the magnitude of the error angle between the target direction and the zero tracking error direction (which is usually the axis of the antenna) in one dimension or coordinate is by alternately switching the antenna beam between two positions (see figure A1). The difference in amplitude between the voltages obtained in the two switched positions is a measure of the angular displacement of the target from the zero tracking error direction or switching axis. The sign of the difference determines the direction that the switching axis must be moved in order to align the axis with the direction of the target.

Two additional beam positions are utilized to obtain the angular error in the orthogonal coordinate. Thus a two-dimensional sequential lobing radar might consist of a cluster of four feed hours or ports illuminating a single antenna, arranged so that the right-left, up-down sectors are covered by successive antenna beam positions. A cluster of five feed horns might also be employed, with the central feed used for transmission while the outer four horns or ports are used for receiving. High power RF switches are not needed in the latter arrangement since only the receiving beams, and not the transmitting beam, are switched.

In this section we shall derive the statistical properties of the usual sequential lobing angle estimator that can be implemented by the use of envelope detectors. The discussion will concern a one-dimensional, two-beam system of the type shown in figure A1. Each beam is assumed symmetric about its boresight direction and the beams are assumed to be squinted or offset by the same amount on either side of the zero tracking error direction. The analysis will first consider the case where only one received waveform is processed per beam position. The case where multiple waveforms or pulses are processed per beam position is discussed in the latter portion of the section. The analysis will employ the same notation and terminology as used in the preceeding sections.

A small tracking error will be assumed so that we can use the approximations

$$
\left.
\begin{aligned}
P(\delta + \varepsilon) &= P(\delta) + \varepsilon P'(\delta) \ , \\[2em]
P(-\delta + \varepsilon) &= P(\delta) - \varepsilon P'(\delta) \ ,
\end{aligned}
\right\} \tag{A23}
$$

where

$P(\cdot)$ is the two-way normalized antenna pattern for each beam,

δ is the beam squint or offset angle,

ϵ is the tracking error angle (assumed small).

Because of beam symmetric we have

$$P(-\delta) = P(\delta) \quad ,$$

$$P'(-\delta) = -P'(\delta) \quad . \qquad (A24)$$

A.4.1 One Waveform or Pulse Processed Per Beam Position

One frequently used error angle estimator, when one pulse is processed per beam position, has the form

$$\hat{\epsilon} = a \frac{|Z_2|^2 - |Z_1|^2}{|Z_2|^2 + |Z_1|^2} \quad , \qquad (A25)$$

where the normalizing constant a is chosen so that when there is no noise present we have

$$\hat{\epsilon} = \epsilon \quad . \qquad (A26)$$

We shall first derive an expression for a.

From the preceeding sections, it is seen that when no noise is present we have

$$|Z_1| = AP(\delta + \varepsilon) \quad , \quad \left.\begin{array}{c}\\\\\\\\\end{array}\right\}$$

$$|Z_2| = AP(-\delta + \varepsilon) \quad , \quad \left.\begin{array}{c}\\\\\\\end{array}\right\} \tag{A27}$$

so that utilizing the small angle approximations given by (A23) we have

$$\frac{|Z_2|^2 - |Z_1|^2}{|Z_2|^2 - |Z_1|^2} = -2\varepsilon \frac{P'(\delta)}{P(\delta)} \quad , \tag{A28}$$

and thus

$$a = -\frac{P(\delta)}{2P'(\delta)} \quad . \tag{A29}$$

When a Gaussian shaped main lobe is assumed of the form discussed in section A.4 and when δ is selected to obtain the smallest Cramér-Rao lower bound,[1] then the constant a has the value

$$a = \frac{\delta_0}{2} \equiv (0.2123)\psi_1 \quad , \tag{A30}$$

where ψ_1 is the half-power beamwidth.

We shall next determine the density function and the mean and variance of the estimator $\hat{\epsilon}$ given by (A25) when noise is present. It is first noted that the probability density functions of $|Z_1|$ and $|Z_2|$, which will be denoted by $f_1(r)$ and $f_2(r)$ respectively, are

$$
\left.
\begin{aligned}
f_1(r) &= \frac{r}{\sigma^2} \exp\left[-\frac{1}{2\sigma^2}\left(r^2 + A^2 P^2(\delta + \epsilon)\right)\right] I_0\left[\frac{1}{\sigma^2}\left(rAP(\delta + \epsilon)\right)\right] , \\[2em]
f_2(r) &= \frac{r}{\sigma^2} \exp\left[-\frac{1}{2\sigma^2}\left(r^2 + A^2 P^2(-\delta + \epsilon)\right)\right] I_0\left[\frac{1}{\sigma^2}\left(rAP(-\delta + \epsilon)\right)\right],
\end{aligned}
\right\} \tag{A31}
$$

---

[1] See equation (A21).

where $0 \leq r$ and $I_0(\cdot)$ is the modified Bessel function of first type and order zero. Equation (A30) expresses the classical result that $|Z_1|$ and $|Z_2|$ have a Rice distribution. If we now define the random variable s, for $0 \leq s$, with density function $g(s)$, as

$$s = \frac{|Z_2|}{|Z_1|} \quad , \tag{A32}$$

then from page 53 of [Miller, 1964] it is seen that $g(s)$ can be expressed as

$$g(s) = \frac{2s}{(1+s^2)^2} \exp\left[-(R_1 s^2 + R_2)/(1 + s^2)\right]$$

$$\times \left\{ \left[1 + \frac{(R_1 + R_2 s^2)}{(1 + s^2)} \right] \times I_0\left[\frac{2s\sqrt{R_1 R_2}}{(1 + s^2)} \right] \right.$$

$$\left. + \left[\frac{2s\sqrt{R_1 R_2}}{(1 + s^2)} \right] \times I_1\left[\frac{2s\sqrt{R_1 R_2}}{(1 + s^2)} \right] \right\} \quad , \tag{A33}$$

where

$I_1(\cdot)$ is the modified Bessel function of first type and order one,

$$R_1 = \frac{A^2 P^2 (\delta + \epsilon)}{2\sigma^2} = RP^2(\delta + \epsilon) \quad ,$$

$$R_2 = \frac{A^2 P^2 (-\delta + \epsilon)}{2\sigma^2} = RP^2(-\delta + \epsilon) \quad ,$$

$R = \dfrac{A^2}{2\sigma^2}$, is the per pulse signal-to-noise ratio for each beam when it is assumed that the target is located at the boresight of the beam.

It is seen that the random variable s is the ratio of two independent Rice variates since the additive Gaussian noise associated with each pulse or waveform that is processed is assumed independent.

We next define the modified signal-to-noise ratio R_0 as

$$R_0 = RP^2(\delta) \quad , \tag{A34}$$

where δ is the beam squint or offset angle zero tracking error direction.[1] R_0 can be considered as the effective signal-to-noise ratio due to the squinting of the antenna beams.

---

[1] When a Gaussian shaped main lobe is assumed and when δ is chosen to be the optimal squint angle defined in the previous section, then
$$R_0 = e^{-1}R \approx (0.3679)R.$$

We now see that the estimator $\hat{\epsilon}$ can be expressed as

$$\hat{\epsilon} = a \frac{s^2 - 1}{s^2 + 1} , \qquad 0 \leq s \tag{A35}$$

where the density function of the random variable s is given by (A33)
In order to find the probability density function of $\hat{\epsilon}$ the following
identities will be needed.

$$\frac{1}{s^2 + 1} = \frac{1}{2} \left[1 - \left(\frac{\hat{\epsilon}}{a} \right) \right] ,$$

$$\frac{2s}{s^2 + 1} = \left[1 - \left(\frac{\hat{\epsilon}}{a} \right)^2 \right]^{1/2}$$

$$\frac{s^2}{s^2 + 1} = \frac{1}{2} \left[1 + \left(\frac{\hat{\epsilon}}{a} \right) \right] ,$$

$$R_2 + R_1 = 2R_0 ,$$

$$R_2 - R_1 = -4\epsilon RP(\hat{\epsilon})P'(\epsilon) = 2R_0 \left(\frac{\epsilon}{a} \right) ,$$

$$\frac{R_2 - R_1}{R_2 + R_1} = \frac{\epsilon}{a} ,$$

$$\sqrt{R_1 R_2} = R_0 \left[1 - \left(\frac{\epsilon}{a} \right)^2 \right]^{1/2} ,$$

where the latter four identities utilize small angle approximations which are valid since the error angle ϵ is assumed small.

If we now make the change of variable from s to $\hat{\epsilon}$ given by (A35) and utilize the above identities it can be shown that the probability density function of $\hat{\epsilon}$, denoted by $h(\hat{\epsilon})$, can be written as

$$
h(\hat{\epsilon}) = \frac{1}{2a} \exp\left\{ \left[\left(\frac{\epsilon}{a}\right)\left(\frac{\hat{\epsilon}}{a}\right) - 1 \right] R_0 \right\}
$$

$$
\times \left\{ \left(1 + \left[\left(\frac{\epsilon}{a}\right)\left(\frac{\hat{\epsilon}}{a}\right) + 1 \right] R_0 \right) I_0 \left(R_0 \sqrt{1 - \left(\frac{\epsilon}{a}\right)^2} \sqrt{1 - \left(\frac{\hat{\epsilon}}{a}\right)^2} \right) \right.
$$

$$
\left. + \left(R_0 \sqrt{1 - \left(\frac{\epsilon}{a}\right)^2} \sqrt{1 - \left(\frac{\hat{\epsilon}}{a}\right)^2} \right) I_1 \left(R_0 \sqrt{1 - \left(\frac{\epsilon}{a}\right)^2} \sqrt{1 - \left(\frac{\hat{\epsilon}}{a}\right)^2} \right) \right\},
$$

for

$$
-a \leq \hat{\epsilon} \leq a . \tag{A36}
$$

In order to compute the moments of the estimator $\hat{\epsilon}$ we shall find its moment generating function G(s) which is defined as

$$
G(s) = \int_{-a}^{a} \exp(s\hat{\epsilon}) h(\hat{\epsilon}) d\hat{\epsilon} . \tag{A37}
$$

Making the change of variable $\hat{\epsilon} = a \cos \phi$ it is seen that the moment generating function can be expressed as

$$G(s) = \frac{1}{2} (1 + R_0) \exp(-R_0) \int_0^{\pi} \sin \phi \exp(\alpha \cos \phi) I_0(\beta \sin \phi) d\phi$$

$$+ \frac{1}{2} \left(\frac{\epsilon}{a}\right) R_0 \exp(-R_0) \int_0^{\pi} \sin \phi \cos \phi \exp(\alpha \cos \phi) I_0(\beta \sin \phi) d\phi$$

$$+ \frac{1}{2} \beta \exp(-R_0) \int_0^{\pi} \sin^2 \phi \exp(\alpha \cos \phi) I_1(\beta \sin \phi) d\phi \quad , \qquad (A38)$$

where

$$\alpha = \left(\frac{\epsilon}{a}\right) R_0 + as \quad ,$$

$$\beta = R_0 \sqrt{1 - \left(\frac{\epsilon}{a}\right)^2} \quad ,$$

so that

$$\gamma^2 = \alpha^2 + \beta^2 = R_0^2 + 2\epsilon R_0 s + a^2 s^2 \quad . \qquad (A39)$$

The three integrals given in equation (A38) are evaluated in the annex to this appendix, and the moment generating function $G(s)$ can ultimately be evaluated as

$$G(s) = \frac{\left(1 + R_0\right)}{2\gamma} e^{-R_0} (e^\gamma - e^{-\gamma})$$

$$+ \frac{\left(R_0^2 + \epsilon R_0 s\right)}{2\gamma^2} e^{-R_0} \left[(e^\gamma + e^{-\gamma}) - \frac{1}{\gamma} (e^\gamma - e^{-\gamma}) \right], \qquad \text{(A40)}$$

where γ is given by (A39). It is easily checked that $G(0) = 1$ which must hold since $h(\hat{\epsilon})$ is a density function.

Now

$$E(\hat{\epsilon}) = \left(\frac{dG}{ds} \right)_{s=0} , \qquad \text{(A41)}$$

which after some algebraic manipulation is evaluated as

$$E(\hat{\epsilon}) = \epsilon \left[1 - \frac{1}{R_0} + \frac{\left(1 - e^{-2R_0}\right)}{2R_0^2} \right]. \qquad \text{(A42)}$$

Notice that $\hat{\epsilon}$ is a biased estimator for moderate to small values of the modified signal-to-noise ratio R_0, for example,

when $\qquad R_0 = 5, \quad E(\hat{\epsilon}) = (0.82)\epsilon$,

and

when $\qquad R_0 = 20, \quad E(\hat{\epsilon}) = (0.95)\epsilon$.

As $R_0 \to \infty$, i.e., as the noise goes to zero, it is seen that the estimator $\hat{\epsilon}$ becomes unbiased.

One common way that is utilized to remove this bias is to use the estimator $\hat{\epsilon}_1$ instead of the estimator $\hat{\epsilon}$ where

$$\hat{\epsilon}_1 = a \frac{|Z_2|^2 - |Z_1|^2}{|Z_2|^2 + |Z_1|^2 + b} \quad , \qquad \text{(A43)}$$

and b is a properly chosen compensation term. The analysis of this estimator and the selection of an "optimal value" of b is quite difficult and will not be pursued further in this report [see Lank, Pollon, 1969].

Instead we shall consider the unbiased estimator $\hat{\epsilon}_2$ where

$$\hat{\epsilon}_2 = \left[1 - \frac{1}{R_0} + \frac{\left(1 - e^{-2R_0}\right)}{2R_0^2} \right]^{-1} \hat{\epsilon}$$

$$= a \left[1 - \frac{1}{R_0} + \frac{\left(1 - e^{-2R_0}\right)}{2R_0^2} \right]^{-1} \frac{|Z_2|^2 - |Z_1|^2}{|Z_2|^2 + |Z_1|^2} \qquad \text{(A44)}$$

The estimator $\hat{\epsilon}_2$ has the disadvantage[1] that an estimate of R_0, the modified signal-to-noise ratio, is needed for its implementation. It is seen, however, that the estimator is not very sensitive to errors in estimating R_0 over a large operating signal-to-noise ratio range. From (A40) it is seen that the variance of $\hat{\epsilon}_2$ depends on the value of ϵ being estimated. For convenience we shall only compute the variance of $\hat{\epsilon}$ for the value $\epsilon = 0$. However, since the estimator $\hat{\epsilon}_2$ is only suitable for estimating small error angles, the variance for non-zero values of ϵ will be closely approximated by the variance of the "$\epsilon = 0$" case.

It can now be shown that

$$\left(\frac{d^2G}{ds^2}\right)_{\substack{s=0 \\ \epsilon=0}} = \left(\frac{a^2}{R_0}\right)\left[1 - \frac{(3 + e^{-2R_0})}{2R_0} + \frac{(1 - e^{-2R_0})}{R_0^2}\right]. \quad (A45)$$

Therefore, since

$$a = -\frac{P(\delta)}{2P'(\delta)} \quad , \quad R_0 = RP^2(\delta) \quad ,$$

it follows that

$$\left(\frac{a^2}{R_0}\right) = \frac{1}{4R[P'(\delta)]^2} \equiv P_{CR} \quad . \quad (A46)$$

---

[1] Which is not present in the estimator $\hat{\epsilon}$.

Now from equation (A17) of the previous section it is seen that B_{CR} is the Cramér-Rao lower bound on the variance of an unbiased estimator of ε when one pulse or received waveform is processed per beam position (i.e., when $M = 1$).

Finally, we obtain from (A45) and (A46)

$$\text{Var}(\hat{\varepsilon}_2 | \varepsilon = 0) = B_{CR} \frac{\left[1 - \frac{(3 + e^{-2R_0})}{2R_0} + \frac{(1 - e^{-2R_0})}{R_0^2} \right]}{\left[1 - \frac{1}{R_0} + \frac{(1 - e^{-2R_0})}{2R_0^2} \right]^2} \tag{A47}$$

From (A47) it is seen that $\text{Var}(\hat{\varepsilon}_2 | \varepsilon = 0)$ approaches the Cramér-Rao lower bound with increasing signal-to-noise ratio R_0. However, even for small values of R_0 the estimator $\hat{\varepsilon}_2$ is remarkably efficient. For example,

$$\text{when } R_0 = 5, \quad \text{Var}(\hat{\varepsilon}_2 | \varepsilon = 0) \cong (1.13) B_{CR} .$$

It is of interest to note that if the bias present in the biased estimator $\hat{\varepsilon}$ given by expression (A42) can be "lived with" then the variance of this estimator is less than the Cramér-Rao lower bound on the variance of an unbiased estimator.

A.4.2 M Waveforms or Pulses Processed Per Beam Position

The preceding analysis has assumed that one pulse or received waveform is processed per beam position. If M pulses are processed per beam position,

then expression (A12) indicates that the estimator $\hat{\epsilon}_2$ should be modified to read

$$\hat{\epsilon}_2 = \frac{a}{M} \left[1 - \frac{1}{R_0} + \frac{(1 - e^{-2R_0})}{2R_0^2} \right]^{-1} \sum_{j=1}^{M} \frac{|Z_{2j}|^2 - |Z_{1j}|^2}{|Z_{2j}|^2 + |Z_{1j}|^2} \, ,$$

$$(A48)$$

where $|Z_{1j}|$ and $|Z_{2j}|$, for $j = 1,\ldots,M$, are the envelope detected outputs from beam 1 and beam 2, respectively. This expression, which gives an un-biased estimate of ϵ, has a variance reduced by a factor of M with respect to the estimator when only one pulse or pattern is processed per beam position.

Frequently the estimator $\hat{\epsilon}_3$ where

$$\hat{\epsilon}_3 = k \; \frac{\sum \left[|Z_{2j}|^2 - |Z_{1j}| \right]^2}{\sum \left[|Z_{2j}|^2 + |Z_{1j}|^2 \right] + b} \, , \qquad (A49)$$

and k is a normalization constant and b is a bias compensation term is

used instead of $\hat{\epsilon}_2$ given by (A48). The reason for using the estimator $\hat{\epsilon}_3$ is to avoid the possibility of dividing by a very small number if the amplitude of the target return signal is small. No analysis of the estimator $\hat{\epsilon}_3$ will be given in this report. A preliminary analysis is given in [Lank, Pollon, 1969] under the assumption that M is large enough so that

$$\sum |z_{1j}|^2 \quad \text{and} \quad \sum |z_{2j}|^2$$

can be approximated by Gaussian random variables (utilizing the Central Limit Theorem) for the purpose of computing moments. Also, no analysis will be given in the report on how well the above estimators operate in the presence of a fluctuating target signal. The analysis given above has assumed that the amplitude of the received target signal remains essentially constant over all the pulses that are processed.

A.5 DISCUSSION

Section A.4 of this report derived the Cramér-Rao lower bound on the variance of estimating the error angle between the target direction and the zero tracking error direction. A very simple expression was obtained for the case of a two-beam system where each beam is squinted at the same angle on either side of the zero tracking error direction. The expression obtained assumed that the main lobe of each of the two beams has the same shape and that the normalized two-way power pattern can be approximated by

$$P(\psi) = \exp\left[-2K\left(\frac{\psi}{\psi_1}\right)^2\right] , \tag{A50}$$

where ψ_1 is the half-power beamwidth and $K = 2\ell n2 \cong 1.3863$. The expression obtained also assumed that the two beams were squinted at an optimal angle from the zero tracking error direction so as to achieve the best possible lower bound. It was shown in section A.4 that this optimal squint or off-set angle δ_0 could be expressed as

$$\delta_0 = \frac{\psi_1}{2\sqrt{K}} \cong (0.4247)\psi_1 . \tag{A51}$$

Under the above conditions, the following inequality was obtained

$$\sigma_{\hat{\epsilon}}^2 \geq \frac{(0.1226)\psi_1^2}{MR} \quad , \tag{A52}$$

where

ψ_1 is the half-power beamwidth (radians) of the one-way voltage pattern,

M   is the number of pulses or received waveforms processed for each of the two beam positions,

R   is the per-pulse signal-to-noise ratio measured relative to the nose of the beam,

$\sigma_{\hat{\epsilon}}^2$ is the variance of any unbiased estimator $\hat{\epsilon}$ of the tracking error angle ϵ.

The derivation of equation (A52) assumed that the noise variates added to the output observables are independent, zero mean, complex Gaussian random variables whose moments are given by (A3).

Most error angle processors come close to achieving the lower bound given by equation (A52) and for most analysis purposes it is adequate to use this lower bound as the variance of an estimate of ϵ, the tracking error angle. Section A.5 discussed amplitude comparison, sequential lobing which is one common technique used to accomplish error angle estimation. Under conditions of moderate signal-to-noise ratio,

this estimation procedure approaches the variance given by the Cramer-Rao bound. The procedure can be implemented by the use of envelope detectors and is sequential in nature, that is, the signal pulses occur sequentially in time. The two antenna beams are produced by using two separate horns or ports to feed the antenna.

Another technique that is commonly used to accomplish error angle estimation is called amplitude comparison monopulse. This procedure is not discussed here and the reader is referred to the papers referenced in section A.8 and to appendix B. The method employs a simultaneous-lobing technique in which the RF signals received from two offset or squinted antenna beams are combined so that the sum and difference signals are obtained simultaneously. The sum and difference signals are then processed to obtain both the magnitude and direction of the error signal. All the information necessary to determine the angular error is obtained on the basis of a single pulse, hence the name monopulse.

An amplitude comparison monopulse system is less susceptible to errors caused by target cross section fluctuation since the returns from both antenna beams are received at the same time. This type of estimator also achieves a variance which approaches the Cramer-Rao bound with increasing signal-to-noise ratio. A derivation of the statistical properties of this estimator will be given in appendix B.

For most radar systems the output error signal $\hat{\varepsilon}$ is used to control a tracking servo which positions the two squinted or offset antenna beams so that the new zero tracking error direction corresponds to the predicted target position when the next sequence of tracking pulses are emitted.

The direction of zero tracking error is changed by either physically moving the antenna, as is done for the conventional tracking radar, or by electronically moving the beams, as is done when a phased array radar is employed. The tracking servo is usually either of the constant velocity or constant acceleration type. The analysis of such a combination is given in [Swerling, 1954]. Of course, the new zero tracking error direction will not exactly correspond to the true target direction because of noise effects in estimating the previous error angle (see equation (A52) and because of servo noise. If the noise power is large enough it may happen that the variance of the estimate of the tracking error $\hat{\varepsilon}$ may be so large that there is a "significant" probability of the new zero tracking error direction being widely separated from the actual target direction. In this case the squinted antenna beams will not be positioned properly to detect the next set of target returns and the tracking process will be interrupted. This is one effect that intense noise jamming attempts to accomplish in addition to increasing the tracking angle variances.

Equation (A52) can be used to evaluate the effect of CW noise jamming on an amplitude comparison, sequential lobing, angle tracker. The value of R in (A52) must be interpreted as the signal-to-jamming power ratio. The jamming power is computed after taking account of the beam gain in the direction of the jamming emitter. Usually the jamming enters through the antenna sidelobes where the antenna gain is greatly reduced as compared to the main lobe gain. The signal-to-jamming ratio must take account of any ECCM capabilities present in the radar, for example, adoptive sidelobe cancellation which places nulls in the receive antenna pattern in the

receive antenna pattern in the direction of the noise sources. An additional study would be required to analyze the effect of CW noise jamming on an amplitude comparison monopulse system since the derivation of (A52) assumed that the observables contained independent white, additive Gaussian noise. Such would not be the case in a monopulse system where the returns from both offset beam positions are obtained simultaneously. Another type of jamming that may be present is large amplitude, short duration, pulse jamming. The effect of such jamming is not considered in this report.

When two or more targets are present, the radar tracking system must be capable of distinguishing between them if either one is to be tracked accurately. Without range or velocity differences, the conventional angular tracking methods cannot separate targets when they are separated by much less than one beamwidth. The theory and design of such multiple angle tracking radars is given in [Lank, Pollon, 1968] and [Pollon, 1968] and is not considered in this report. These references derive the form of the data processor and analyze the variance of the multiple angle estimates and their relationship to the Cramér-Rao lower bound.

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ANNEX

The following expression is derived on pages 58-59 of [Miller, 1964]

$$\int_0^\pi (\sin \phi)^{n/2} \exp(\alpha \cos \phi) I_{(n-2)/2}(\beta \sin \phi) \, d\phi$$

$$= \sqrt{\frac{2\pi}{\beta}} \left[\frac{\beta^2}{\alpha^2 + \beta^2} \right]^{(n-1)/4} I_{(n-1)/2}\left(\sqrt{\alpha^2 + \beta^2} \right) \qquad [A1]$$

where $I_k(\cdot)$ is the modified Bessel function of first type and order k. If we let n = 2 in this expression and use the identity

$$\sqrt{\frac{\pi}{2Z}} \; I_{1/2}(Z) = \frac{\sinh(Z)}{Z} = \frac{1}{2Z}(e^Z - e^{-Z}) \; ,$$

then we obtain the expression

$$\int_0^\pi \sin \phi \; \exp(\alpha \cos \phi) \, I_0(\beta \sin \phi) \, d\phi$$

$$= \frac{\exp\left(\sqrt{\alpha^2 + \beta^2} \right) - \exp\left(-\sqrt{\alpha^2 + \beta^2} \right)}{\sqrt{\alpha^2 + \beta^2}} \qquad [A2]$$

Similarly, if we let n = 4 in expression [A1] and use the identity

$$\sqrt{\frac{\pi}{2Z}} \; I_{3/2}(Z) = \frac{\cosh(Z)}{Z} - \frac{\sinh(Z)}{Z^2}$$

$$= \frac{1}{2Z} \left[e^Z + e^{-Z} - \frac{1}{Z} \left(e^Z - e^{-Z} \right) \right] ,$$

then we obtain the expression

$$\int_0^{\pi} \sin^2\phi \; \exp(\alpha \cos \phi) \; I_1(\beta \sin \phi) d\phi$$

$$= \frac{\beta}{(\alpha^2 + \beta^2)} \left[\exp\left(\sqrt{\alpha^2 + \beta^2} \right) + \exp\left(-\sqrt{\alpha^2 + \beta^2} \right) \right.$$

$$\left. - \frac{1}{\sqrt{\alpha^2 + \beta^2}} \left\{ \exp\left(\sqrt{\alpha^2 + \beta^2} \right) - \exp\left(-\sqrt{\alpha^2 + \beta^2} \right) \right\} \right]. \quad [A3]$$

Finally, if we differentiate equation [A2] with respect to α we obtain the expression

$$\int_0^\pi \sin \phi \cos \phi \exp(\alpha \cos \phi) I_0(\beta \sin \phi) \, d\phi$$

$$= \frac{\alpha}{(\alpha^2 + \beta^2)} \left[\exp\left(\sqrt{\alpha^2 + \beta^2}\right) + \exp\left(-\sqrt{\alpha^2 + \beta^2}\right) \right.$$

$$\frac{1}{\sqrt{\alpha^2 + \beta^2}} \left\{ \exp\left(\sqrt{\alpha^2 + \beta^2}\right) - \exp\left(-\sqrt{\alpha^2 + \beta^2}\right) \right\} \left. \vphantom{\frac{1}{1}} \right].$$

[A4]

APPENDIX B

RADAR ANGULAR TRACKING ERRORS: AMPLITUDE
COMPARISON, MONOPULSE, ANGLE TRACKING

B.1 Introduction

A variation of the model described in appendix A for predicting angular tracking errors for the radar tracking the cruise missile in the presence of enemy noise jamming is described in this appendix. The variation described in this appendix differs from that of appendix A in what is assumed about the radar's method of angular tracking; in appendix A it was assumed that the radar uses a method called "amplitude comparison, sequential lobing," whereas in this appendix it is assumed that the radar uses another method called "amplitude comparison, monopulse." This second method employs a simultaneous-lobing technique in which RF signals received from two offset or squinted antenna beams are obtained simultaneously so that they have the same phase. An advantage of this method is a smaller bias with the same signal-to-noise ratio.

The form of the maximum likelihood angle error estimator is derived for the case where the radar uses the amplitude comparison, monopulse method. It is shown that this estimator can be implemented with a phase detector. In addition, the first two moments of the monopulse maximum likelihood estimator are derived and a comparison made with the Cramer-Rao lower bound. The probability density function of the estimator is not obtained. The bias of the monopulse estimator is less than the bias of the sequential lober as derived in appendix A.

The same complex signal notation used in appendix A is used in this appendix. The notation E(·) stands for the expectation of the quantity in the parenthesis while Re(·) and Im(·) stand for the real and imaginary parts, and |·| stands for the absolute value. A superscript ρ denotes the transpose conjugate of a vector or matrix and a superscript * denotes the conjugate of a complex number.

The short bibliography started in appendix A is continued in Section B.5.

B.2 Likelihood Equations and Maximum Likelihood Estimator

This section will derive the likelihood equations and the form of the maximum likelihood estimator when a simultaneous-lobing technique is used to derive error angle information. In this method the RF signals received from two offset or squinted antenna beams are combined so that the sum and difference signals are obtained simultaneously. All the information necessary to determine the angular error is obtained on the basis of a single pulse received through the two antenna beams, hence the name amplitude comparison, monopulse is employed.

It will be assumed that each antenna beam is symmetric about its boresight direction and that the beams are squinted or offset by the same amount on either side of the zero tracking error direction (which is usually the axis of symmetry of the antenna). The same antenna pattern (referenced to the boresight direction) is assumed for each of the beams. The angle between the zero-tracking error direction and the antenna beam boresight direction will be denoted by δ. A picture of the geometry involved is given in figure A1 of appendix A.

We shall first discuss the case when one received waveform or pulse is simultaneously received by the two squinted antenna beams. The following two complex observables are utilized in the derivation of the maximum likelihood estimator:

$$Z_1 = A[G_1\ (\varepsilon) + G_2\ (\varepsilon)]\ e^{i\theta} + V_1, \tag{B1}$$

$$Z_2 = A[G_1\ (\varepsilon) - G_2\ (\varepsilon)]\ e^{i\theta} + V_2,$$

where

$G_1(\cdot)$ and $G_2(\cdot)$ are the normalized two-way voltage patterns of beam number 1 and beam number 2, respectively, measured with respect to the zero tracking error direction,

A   is the amplitude of the received pulse normalized to the antenna beam boresight direction,

ε is the angle between the target direction and the zero tracking error direction,

θ is the phase of the received signal.

The V_m are the signals caused by system noise and are assumed to be zero mean, complex Gaussian random variables which are mutually independent. In addition, the in-phase and quadrature (real and complex parts) of each V_m are assumed independent with the same variance σ^2. Under these conditions, the complex Gaussian random variables V_m have the properties [Reed, 1962]

$$E(V_m V_m^*) = 2\sigma^2 \quad , \quad E(V_m V_m^*)^2 = 8\sigma^4 \quad , \tag{B2}$$

and all other first, second, third, and fourth order moments are zero, where

σ^2 is the variance of the in-phase and quadrature noise variates,

*   is the complex conjugate operation.

An amplitude comparison monopulse system is less susceptible to errors caused by target cross section fluctuation than a sequential lober since the returns from both antenna beams are received at the same time. This technique is particularly useful where pulse-to-pulse amplitude fluctuations due to target variations or interference signals can degrade conical or sequential scanning tracking techniques. The phase shifts through the RF and IF portions of the system must be carefully equalized to maintain the equal phase relationship in both channels that is indicated by equation (B1). In addition, the tolerances between the receiving horns and the comparator section of the feed assembly must be very closely controlled. This appendix will not discuss the degradation in tracking performance caused by a phase unbalance in the two channels.

As illustrated in sections A.2 and A.3, the likelihood function L associated with (B1) can be written as

$$L = \ln C - \frac{1}{2\sigma^2} |W-K|^2 \qquad (B3)$$

where

C   is a normalizing constant,

$$W = \begin{bmatrix} Z_1 \\ \\ Z_2 \end{bmatrix}, \qquad K = A\begin{bmatrix} G_1(\epsilon) + G_2(\epsilon) \\ \\ G_1(\epsilon) - G_2(\epsilon) \end{bmatrix} e^{i\theta}.$$

The unknowns in equation (B3) are ϵ, A and θ.

Now, if $P(\cdot)$ is the normalized two-way antenna pattern measured with respect to beam boresight and δ is the offset or squint angle associated with each beam, then as shown in appendix A the following relationships are approximately true when the tracking error angle ϵ is small,

$$G_1(\epsilon) = P(\delta + \epsilon) = P(\delta) + \epsilon P'(\delta),$$

$$G_2(\epsilon) = P(-\delta + \epsilon) = P(\delta) - \epsilon P'(\delta) \quad .$$

It will also be assumed that the slope of the antenna pattern is almost constant in the vicinity of the offset angle δ and that we can use the following relations for a symmetric pattern:

$$P(S) = P(-S), \quad P'(S) = -P'(-S) \tag{B4}$$

Using the above relationship it is seen that equation (B3) now becomes:

$$L = \ln C - \frac{1}{2\sigma^2} |W-K|^2,$$

where

    C is a normalizing constant,

$$W = \begin{bmatrix} Z_1 \\ \\ Z_2 \end{bmatrix}, \quad K = 2A \begin{bmatrix} P(\delta) \\ \\ \epsilon P'(\delta) \end{bmatrix} e^{i\theta} \quad .$$

The three likelihood equations have the form:

$$\frac{\partial L}{\partial \epsilon} = 0, \quad \frac{\partial L}{\partial A} = 0, \quad \frac{\partial L}{\partial \theta} = 0 \quad . \tag{B5}$$

The first equation of (B5) leads to the expression

$$\varepsilon = \frac{1}{2AP'(\delta)} \, Re(Z_2 e^{-i\theta}) \, , \tag{B6}$$

where $Re(\cdot)$ stands for the "real part of." The second equation of (B5) leads to the expression

$$A = \frac{1}{2AP(\delta)} \, Re(Z_1 e^{-i\theta}) \, , \tag{B7}$$

where it is assumed that terms involving ε^2 and ε times the voltage in the difference channel can be neglected compared to the other terms.

Combining (B6) and (B7) we obtain

$$\varepsilon = \frac{b \, Re \, (Z_2 e^{-i\theta})}{Re \, (Z_1 e^{-i\theta})} \, , \tag{B8}$$

where b is defined as:

$$b = \frac{P(\delta)}{P'(\delta)} \quad .$$

The last equation in (B5) leads to

$$Im(Z_1 e^{-i\theta}) = 0, \tag{B9}$$

where $Im(\cdot)$ denotes the "imaginary part of" and again terms involving ε times the voltage in the difference channel are neglected. It is easily seen that (B9) leads to

$$e^{-i\theta} = \frac{Z_1^*}{|Z_1|} \quad ,$$

so that from (B8) the maximum likelihood estimates for ε, which will be denoted by $\hat{\varepsilon}$, can be expressed as

$$\hat{\varepsilon} = \frac{b\,Re(Z_2 Z_1^*)}{|Z_1|^2} \quad . \tag{B10}$$

It is easily seen that $\hat{\varepsilon}$ is consistant, i.e., if no noise is present then

$$E(\hat{\varepsilon}) = \varepsilon \quad . \tag{B11}$$

If we note that

$$Re[(Z_2 + iZ_1)Z_1^*] = Re(Z_2 Z_1^*) \quad , \tag{B12}$$

then it follows that (B12) can be written as

$$\hat{\varepsilon} = \frac{b|Z_2 + iZ_1|}{|Z_1|} \; Re\left[\frac{(Z_2 + iZ_1)}{|Z_2 + iZ_1|} \times \frac{Z_1^*}{|Z_1|}\right] \quad . \tag{B13}$$

Under conditions of small error angle ε and moderate to large signal-to-noise ratio in the sum channel it follows that

$$\frac{|Z_2 + Z_1|}{|Z_1|} \approx 1 \quad , \tag{B14}$$

so that an estimator frequently used as an approximation to (B13) is

$$\varepsilon = bRe \left[\frac{(Z_2 + iZ_1)}{|Z_2 + iZ_1|} \times \frac{Z_1^*}{|Z_1|} \right] \quad . \tag{B15}$$

The estimation given by (B15) is very easy to implement with a phase detector used to obtain the term of the form $Re(ab^*)$.

The estimator given by (B10) was first discussed in this form by [Mosca, 1969]. The estimator computes an angle estimate once per pulse. A simple stretching or "boxcar" operation followed by low pass filtering supplies d-c inputs to the antenna servo amplifiers. Additional references related to this estimator are given by [McGinn, 1966], and [Hofstetter and Delong, 1969].

The likelihood function (B3) can also be used to derive the Cramér-Rao lower bound on the variance of an unbiased estimator of ε, the tracking error angle. Following the procedure given in appendix A and using the small error angle simplifications given by (B4) it follows that the lower bound can be written as

$$\sigma_\varepsilon^2 \geq \frac{1}{4R} \left(\frac{1}{P'(\delta)} \right)^2 \equiv B_{CR} \quad , \tag{B16}$$

where σ_ε^2 is the variance of any unbiased estimator of ε and B_{CR} is the Cramér-Rao lower bound and the signal-to-noise ratio R is defined as

$$R = \frac{A^2}{2\sigma^2} \quad . \tag{B17}$$

Equation (B16) can also be written as

$$\sigma_\epsilon^2 \geq \frac{b^2}{4R_s} \quad , \tag{B18}$$

where R_s is the signal-to-noise ratio in the sum channel, i.e.,

$$R_s = \frac{A^2 p^2(\delta)}{2\sigma^2} \quad . \tag{B19}$$

Equation (B18) indicates the bound on the performance of an unbiased angle estimator as a function of the sum channel signal-to-voise ratio, R_s, and a factor b which depends on the slope of the antenna pattern and the beam offset or squint angle δ. References in addition to those listed above which discuss the variance of monopulse estimators are [Blackman, 1971], [Lipman, 1971], [Kerr, 1968], [Sharenson, 1962], [Manasse, 1960], [Nester, 1962] and [Urkowitz, 1964].

B.3 Mean and Variance of the Angle Estimator

We shall next determine the mean and variance of the estimator given by (B10), namely

$$\hat{\wedge} = \frac{b}{|Z_1|^2} \operatorname{Re}(Z_2 Z_1^*) \quad . \tag{B20}$$

Since Z_1 and Z_2 have the joint density function indicated by (B3), the simplist way to determine the mean of (B20) is to convert to polar coordinate

$$Z_1 = r_1 e^{i\Theta_1} \quad , \qquad Z_2 = r_2 e^{i\Theta_2} \quad , \tag{B21}$$

with Jacobian r_1r_2. In these new coordinates, the estimator becomes

$$\hat{\epsilon} = b \frac{r_2}{r_1} \cos(\Theta_2 - \Theta_1), \qquad (B22)$$

and the expression for the mean of $\hat{\epsilon}$ becomes

$$E(\hat{\epsilon}) = \int_0^{\infty}\int_0^{\infty}\int_0^{2\pi}\int_0^{2\pi} r_1 r_2 \hat{\epsilon} f(r_1, r_2, \Theta_1, \Theta_2) \, d\Theta_1 d\Theta_2 dr_1 dr_2 ,$$

where $\hat{\epsilon}$ is given by (B22) and $f(r_1, r_2, \Theta_1, \Theta_2,)$ is the joint density function in polar coordinates. Rather than provide all details of this integration we shall instead indicate the key steps.

The integration with respect to Θ_1 and Θ_2 can be performed using the identities

$$\int_0^{2\pi} d\psi \cos(\psi-\Theta)\exp[\alpha\cos(\psi-\Theta)] = 2\pi I_1(\alpha), \qquad (B23)$$

$$\int_0^{2\pi} d\psi \sin(\psi-\Theta)\exp[\alpha\cos(\psi-\Theta)] = 0, \qquad (B24)$$

where $I_1(\cdot)$ indicates the modified Bessel function of first type and order 1. Equation (B23) is the standard definition of the modified Bessel function and (B24) can be directly integrated after making the change of variable $\gamma = \alpha \cos(\psi-\theta)$.

The integrations with respect to r_1 and r_2 are performed using the identities

$$\int_0^\infty dx\, x^2\, e^{-\alpha x^2} I_1(\gamma x) = \frac{\gamma}{4\alpha^2}{}_1F_1(2;2;\frac{\gamma^2}{4\alpha}) , \qquad (B25)$$

$$\int_0^\infty dx\, e^{-\alpha x^2} I_1(\gamma x) = \frac{\gamma}{4\alpha}\, {}_1F_1(1;2;\frac{\gamma^2}{4\alpha}) , \qquad (B26)$$

where ${}_1F_1(\cdot,\cdot,\cdot)$ is the confluent hypergeometric function. A listing of such integrals can be found in [Miller, 1964 appendix 1].

We can convert the hypergeometric functions to more recognizable forms by using the identities

$$_1F_1(2;2;x) = e^x, \tag{B27}$$

$$_1F_1(1;2;x) = \frac{1}{x}[e^x-1], \tag{B28}$$

which can be found in [GPO, 1964, page 509].

The final result after performing all the integrations is

$$E(\hat{\epsilon}) = \epsilon[1-\exp(1R_s)] \tag{B29}$$

where R_s is the signal-to-noise ratio in the sum channel given by (B19). This equation corresponds to equation (20) of the paper by Mosca, which was not derived, and is also the same as equation (30) in a report by McAulay. The McAuley report derives this equation in an entirely different fashion.

Equation (B29) shows that $\hat{\epsilon}$ is a biased estimator, however, the bias is negligible when the signal-to-noise ratio R_s is moderate to large. Comparing equation (B29) above with equation (25) of the paper by Lank and Pollon, which gives the analogous expression for an angle tracking radar employing sequential lobing,[1] one sees that the bias for the amplitude comparison monopulse is much reduced over that of a sequential lober, for small to moderate . signal-to-noise ratios; the processing of phase information in an amplitude comparison monopulse rather than only envelope information is the cause of this improvement.

$$E(\hat{\epsilon}^2) = \int_0^\infty \int_0^\infty \int_0^{2\pi} \int_0^{2\pi} r_1 r_2(\hat{\epsilon}^2) f(r_1,r_2,\theta_1,\theta_2) d\theta_1 d\theta_2 dr_1 dr_2, \tag{B30}$$

_____

[1] Also derived in appendix A.

and is easily seen to be infinite. The quickest way to show this is to note that the existance or nonexistance of the second moment is not affected by the fact that a signal is present. Thus, when (B30) is integrated with only noise present, the result is that the second moment is infinite. The basic reason is that the denominator of the integrand has a singularity at $r_1 = 0$. This result was noted in the paper by Mosca.

The effect of the singularity is to make the "tails" of the density function of $\hat{\epsilon}$ slightly too high so that the second moment does not exist. A similar phenomenon happens, for example, for the quotient of two normal distributions which results in a Rayleigh distribution for which no moments exist.

However, the behavior of the estimator $\hat{\epsilon}$ can be very closely approximated near the value $\epsilon=0$ (which is of most interest) by a random variable with a finite second moment.

It can be shown that, under most conditions, $|Z_1|^2$ can be approximated by

$$|Z_1|^2 \doteq 4A^2P^2(\delta) = |E(Z_1)|^2 . \qquad (B31)$$

Thus, as an approximation to $\hat{\epsilon}$ we can utilize the computationally simpler estimator $\hat{\tau}$ given by

$$\hat{\tau} = b \ \frac{Re(Z_2 Z_1^*)}{|E(Z_1)|^2} , \qquad (B32)$$

to evaluate the operation of $\hat{\epsilon}$. It is seen that all moments of $\hat{t}$ exist. The computation of the first moment of $\hat{t}$ results in

$$E(\hat{t}) = \epsilon, \tag{B33}$$

where we utilize the fact that Z_1 and Z_2 are independent and that $E(Re(Z_2 Z_1^*)) = Re(E(Z_2 Z_1^*))$. The variance of $\hat{t}$ is

$$Var(\hat{t}) = \frac{b^2}{4R_s} \left[1 + \frac{\epsilon^2}{b^2} + \frac{1}{R_s} \right], \tag{B34}$$

so that for small error angles, ϵ, and moderate to large signal-to-noise ration R_s, we can approximate the variance by

$$Var(\hat{t}) = \frac{b^2}{4R_s}, \tag{B35}$$

which is the Cramér-Rao bound given by (B18). The computation of (B34) uses the fact that

$$E\left[Re(Z_2 Z_1^*)\right]^2 = \frac{1}{2} Re\, E(Z_2 Z_2^* Z_1 Z_1^*) + \frac{1}{2} Re\, E(Z_2 Z_2 Z_1^* Z_1^*), \tag{B37}$$

and the moment theorem of Reed.

The key assumption for the approximation of $\hat{\epsilon}$ given by (B32) is that the angle estimator is operating in a situation where the total power in the sun channel can be closely approximated by the signal power only. Of course (B32) cannot be used in an operational estimator since the signal power is actually unknown to the angle estimator and only the total power

in the sum channel can be measured. Operationally, the way to produce
an angle estimator for which all moments exist (similiar, for example,
the angle estimator when sequential lobing is employed) is to add a
small positive bias, α, to the denominator of (B20) so that it becomes

$$\hat{\epsilon}_\alpha = \frac{\text{Re } (Z_2 Z_1)}{(|Z_1| + \alpha)^2} \qquad (B38)$$

All moments for $\hat{\epsilon}_\alpha$ exist since the denominator is always greater than
zero. The analysis of $\hat{\epsilon}_\alpha$ is quite involved and has not been undertaken.
Depending on the operating range of R_s expected, the normalizing constant
b can be chosen so that $\hat{\epsilon}_\alpha$ is unbiased. The use of such a bias term might
be desirable in order to limit the estimated value of ϵ when several signals
interfere in the sum channel.

It can be shown (see [Cramér, 1946], pp 500-504) that as R_s increases
the maximum likelihood estimator $\hat{\epsilon}$ given by (B20) becomes asymptotically
unbiased and is an asymptotically normal and asymptotically efficient
estimator of ϵ, i.e., the variance approaches the Cramér-Rao lower bound.
T is means that $\hat{\epsilon}$ and also $\hat{\tau}$ can be approximated by a normal distribution
with mean ϵ and variance $B_{CR} = \frac{b^2}{4R_s}$ in the vicinity of the mean. Precisely
how far out on the tails of the normal density function this approximation
holds has not been investigated, but the approximation should be very
adequate in the vicinity of ϵ. This is the region of interest in most
applications which involve the correction of small tracking errors.

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APPENDIX C

THE DIFFERENTIAL EQUATIONS: AN ANALYTIC APPROACH

The overall mathematical structure of the single ship/multiple cruise missile engagement model involves the solution of a sequence of systems of differential equations

$$m_j(t+\delta|d) = m_j(t|d) \cdot \left[1-\delta\cdot\sum_{i=1}^{W} p_{ij}(t|d) \, k_{ij}(t|d)\right] \qquad (C1)$$

for $t_k^+ \le t < t_{k+1}$, $k+1 \le j \le M$ and all d. There is one such system for each of M+1 time intervals $[t_k^+, t_{k+1})$. The assumed ship's weapon assignment logic led to expressions for the firing probabilities of the form

$$p_{11}(t|d) = \begin{cases} m_1(t|u) & \text{if } d=u \text{ and } 0 \le t < t_1, \\ \text{undefined} & \text{if } d \ne u \text{ and } 0 \le t < t_1, \\ 0 & \text{otherwise} \end{cases}$$

and, for $2 \le j \le M$, (C2)

$$p_{ij}(t|d) = \begin{cases} [1-m_{j-1}(t|d)/m_j(t|d)] & 1 \le i \le W, \\ & \text{all d and } 0 \le t < t_j \\ 0 & 1 \le i \le W, \text{ all d and } t \ge t_j \end{cases}$$

As has been explained, equations (C1) and (C2) can be used in tandem (i.e., by what has here been called the "bootstrap technique") to compute the cruise missile survival probabilities $m_j(\cdot|d)$.

It is interesting to note that it is possible to analyze the systems (C1) and (C2) to a certain extent using only _analytical_ methods. For some purposes - - such as obtaining information about the structure of the solution to (C1) and (C2) but not necessarily the solution itself -- these analytic methods may suffice and be more convenient than the usual solution procedure involving numerical bootstrapping. The purpose of this appendix is to briefly describe the analytic approach to (C1) and (C2).

To describe the analytical approach to working with (C1) and (C2), it suffices to discuss the case of one of the time intervals -- say $t_k^+ \leq t \leq t_{k+1}^-$ for some fixed k with $1 \leq k < M$ -- and one of the damage states d; the analysis for the other time intervals and damage states is the same.

Writing the differential equations (C1) in the more customary form

$$\frac{dm_j(t|d)}{dt} = -m_j(t|d) \cdot \sum_{i=1}^{W} p_{ij}(t|d) \cdot k_{ij}(t|d)$$

and inserting the expressions (C2) for the firing probabilities $p_{ij}(t|d)$ gives

$$\frac{dm_j(t|d)}{dt} = [m_{j-1}(t|d) - m_j(t|d)] \cdot \sum_{i=1}^{W} k_{ij}(t|d)$$

$$= a_j(t|d) \cdot [m_{j-1}(t|d) - m_j(t|d)] \tag{C3}$$

for j=k+1, $\cdots$, M and $t_k^+ \leq t < t_{k+1}$, where

$$a_j(t|d) = \sum_{i=1}^{W} k_{ij}(t|d)$$

for $j=k+1, \cdots, M$ and $t_k^+ \leq t < t_{k+1}$. The initial condition for the system is the one which applies to the time interval $[t_k^+, t_{k+1}]$ namely[1]

$$m_j(t|d) = m_j(t_k^+ |d) \text{ when } t=t_k^+, \; k+1 \leq j \leq M.$$

The equations in the system (C3) have a special form; they are often called "first-order linear" equations.

It is convenient to write (C3) and (C4) in a more compact form by using vector notation. Put

$$\vec{m}(t|d) \quad \begin{pmatrix} m_{k+1}(t|d) \\ \vdots \\ m_M(t|d) \end{pmatrix}$$

$$\vec{m}(t_k^+|d) = \begin{pmatrix} m_{k+1}(t_k^+|d) \\ \vdots \\ m_M(t_k^+|d) \end{pmatrix}$$

$$A(t) = \begin{pmatrix} a_{k+1}(t|d) & -a_{k+1}(t|d) & 0 & \cdots 0 \\ 0 & a_{k+2}(t|d) & -a_{k+2}(t|d) & \cdots 0 \\ 0 & 0 & 0 & a_M(t|d) \end{pmatrix}$$

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[1] The values $m_j(t_k^+|d)$ may be computed from (18) in the usual way.

Then (C3) and (C4) may be written

$$\frac{d}{dt}\overrightarrow{m}(t) = A(t)\,\overrightarrow{m}(t), \tag{C5}$$

$$\overrightarrow{m}(t|d) = m(t_k^+|d) \text{ when } t=t_k^+.$$

If $A(t)$ were a constant matrix -- say $A(t)=A$ for all t in the range $t_k^+ \leq t < t_{k+1}$ -- then the unique solution to the system (C5) would be[1]

$$\overrightarrow{m}(t|d) = \exp[A(t-t_k)]\cdot\overrightarrow{m}(t_k^+|d) \qquad (t_k^+ \leq t < t_{k+1}).$$

Since A is a band-diagonal matrix, the computation of the exponential matrix would be straightforward.

More generally, when $A(t)$ is not a constant matrix (this is the case most likely to arise in practice), it may be shown (see, e.g., [Bellman, 1970]) that the unique solution to (C5) is

$$\overrightarrow{m}(t|d) = X(t)\cdot\overrightarrow{m}(t_k|d) \qquad (t_k^+ \leq t \leq t_{k+1}^-)$$

where $X(t)$ is the unique matrix satisfying

$$\frac{dX(t)}{dt} = A(t)\,X(t),$$

$$X(t_k^+) = I.$$

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[1]In particular, $\overrightarrow{m}(t_{k+1}^-|d)$ would be given by $\overrightarrow{m}(t_{k+1}^-|d) = \exp[A(t_{k+1}-t_k)]\cdot\overrightarrow{m}(t_k^+|d)$ and these values of the $m_j(t_{k+1}^-|d)$ would be used in computing the $m_j(t_{k+1}^+|d)$ in the usual way for the next time interval.

The matrix $X(t)$ may be computed using the so-called "method of successive approximations":

$$X(t) = \lim_{n \to \infty} X_n(t)$$

where

$$X_0(t) = I$$

$$X_n(t) = I + \int_{t_k}^{t} A(s) X_n(s) ds \qquad n=0,1,2 \cdots.$$

Notice finally that the above described "analytical approach" does not completely avoid the use of numerical techniques because the method of excessive approximations is inherently numerical in character and because the computation of the kernel matrix $A(t)$ (for use in C6) will normally be accomplished numerically.