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INEAR ANALYSIS OF THE DEFORMATION OF PRESSURE STABILIZED BEAMS

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A linear theory for the behavior of pressure stabilized beams is derived and solutions to the resulting equations are obtained for uniform and concentrated loading for both simply supported and fixed end restraints. The concentrated load solution is in the form of a Green's function for application to more general loading situations.

Results from these solutions are presented which illustrate the deformation behavior and load-carrying capacity of pressure stabilized beams as a function of the pressure level, beam geometry and material properties.

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FOREWORD

This work was carried out in response to the results of the systems analysis conducted in connection with the preparation of the QMDO for Functional Field Shelters. This analysis identified the pressurized rib concept as being the most promising structural alternative with regard to meeting the Army requirement for lightweight highly mobile tentage. A project was initiated in FY 71 to investigate the feasibility of the pressurized rib concept and to develop the design data on pressure stabilized structural elements necessary for Army tentage applications. Support for this project has been provided through the In-House Laboratory Independent Research program.

This report presents a mathematical analysis of the deformation behavior of these structural elements under load and the development of the equations which will be used in the design of structures. Subsequent reports will present experimental studies designed to test this theoretical analysis and will describe extensions of this study to the performance of arches.

TABLE OF CONTENTS

Page

LIST OF FIGURES

iii

LIST OF SYMBOLS

A_i, B_i, C_i	Constants of integration
A	Area
a	Tube radius
C_{ij}	Elastic constants
$\bar{C}_{11}$	Reference value of C_{11}
c	Ratio $C_{33}/\bar{C}_{11}$
d	Ratio $C_{11}/\bar{C}_{11}$
e_1	One-dimensional axial strain measure
e_{12}	One-dimensional shear strain measure
$\bar{F}$	Surface load acting on the beam
f	Generalized force associated with $\bar{F}$
f_w	Magnitude of f at wrinkling
G	Circumferential line load acting on beam
g	Generalized force associated with G
g_w	Magnitude of g at wrinkling
l	Half length of the beam
N_{ij}	Stress resultants
n_{ij}	Nondimensional stress resultants
n	Axial stress resultant due to pressurization
p	Pressure
r	Radial coordinate

LIST OF SYMBOLS (cont'd)

T,Q,M   One-dimensional stress resultants

u_i Displacement components resulting from external load

$\overline{U}$ Axial displacement of beam cross-section

U   Nondimensional form of $\overline{U}$

v_i General displacement components

w_i Displacement components resulting from pressurization

$\overline{W}$ Transverse displacement of beam cross-section

W   Nondimensional form of $\overline{W}$

x_i Independent coordinates

β Constant defined by equation 29

γ Nondimensional flexibility

ϵ_{ij} Strain components

η Dummy variable used in conjunction with ξ

κ_1 Change of curvature

λ Characteristic number of governing equations

ξ Nondimensional axial coordinate

ρ Length to radius ratio

ϕ Rotation of cross-section

ω Characteristic number

INTRODUCTION

The use of pressure stabilized structures for Army field shelters has been the subject of considerable attention over the past decade with emphasis on the single-wall and double-wall type shelters, as indicated by references 1 and 2. A concept that has received little attention is that of constructing tent frames with pressure stabilized beams and arches. In view of this, an investigation of the behavior of these structural elements under load was initiated and this report presents the portion of this work relating to the deformation behavior of the pressurized beams. Specifically, a theoretical study of the problem is presented with the objective being the establishment of a set of equations to be used in the design of tent frame structures using pressurized beams.

The structures under consideration can be described as cylindrical membranes, and considerable theoretical work has been done on such structures. The linear work in this area is a statically determinate problem with the associated limitations regarding the types of boundary condition that can be applied. These limitations are too restrictive for the problem under consideration and the removal of these restrictions requires the use of a nonlinear membrane theory. Since the objective of this work was development of a linear theory, the classical membrane approach was rejected. Most other work regarding the behavior of this type of structure can roughly be divided into two categories. The first of these included work which examined special cases such as beams subjected to uniform moments or statically determinate loadings in which the stresses could be calculated without regard to deformation. In general, the work in this first category, e.g. references 3 and 4, does not constitute attempts to develop a generally applicable theory to be used for the design of pressure stabilized beams. The second category, e.g. references 5, 6 and 7, deals with the wrinkling behavior of pressurized cylinders which relates to the failure criteria for this type of structure. This is an extremely complicated problem and these papers therefore deal with simplified loading problems, typically uniform moments. Thus these results, although useful in developing failure criteria, are not applicable to general loadings and boundary conditions. A theory for the behavior of pressure stabilized beams is presented in reference 8; however, this theory was not used in the present work because it was developed by writing expressions for the 3 components of energy independently and summing them to obtain the total energy. It was felt that a more consistent derivation would result from application of the ideas of superposition of small displacements on large ones, that is, those due to pressurization. In addition, the results in reference 8 deal principally with the buckling of columns.

The present report includes a derivation of the governing equations and boundary conditions for the lateral deformation of pressure stabilized beams and development of Green's Function and uniform load solutions for the simply supported and clamped beam cases. In addition, parametric studies are presented which illustrate the linear behavior of pressure stabilized beams under load.

ANALYSIS

In this section a derivation of a set of governing equations for lateral deformation of pressure stabilized beams and their solution will be carried out. The derivation will utilize the principle of minimum potential energy and begins by establishing an energy principle for deformation about the pressurized state. A simplified displacement approximation is then used to reduce the problem from two dimensions to one dimension. This displacement approximation requires that the cross section remain underformed and includes deformation due to transverse shear. The governing equations based on this approximation are then obtained using variational principles. The fundamental homogenous solution to these equations is obtained in preparation for development of the Green's function solutions for the cases of simply-supported ends and clamped ends. In addition to these Green's function solutions, the solutions for the uniformly loaded beams are obtained directly by finding the appropriate particular solution of the governing equations.

Derivation of the Governing Equations

Fundamental Principles:

The derivation is based on the principle of minimum potential energy which for the membrane state of stress can be written:

$$\delta \int_A \frac{1}{2} \left[\epsilon_{11} N_{11} + \epsilon_{22} N_{22} + 2\epsilon_{12} N_{12} - 2pv_3 \right] dA = 0 \tag{1}$$

The subscripts denote the coordinated directions x_1 and x_2 as shown in Figure 1. In equation (1) the ϵ_{ij} and N_{ij} are respectively the surface strains and the inplane stress resultants; p is the normal load distribution and v_3 is the normal displacement. For the displacements illustrated in Figure 1 the strains are defined as

$$\epsilon_{11} = v_{1,1} + \frac{1}{2} (v_{3,1})^2 \tag{2a}$$

$$\epsilon_{22} = \frac{1}{r} (v_{2,2} + v_3) + \frac{1}{2r^2} (-v_{3,2} + v_2)^2 \tag{2b}$$

$$\epsilon_{12} = \frac{1}{2} \left[v_{2,1} + \frac{1}{r} v_{1,2} + \frac{1}{r} (-v_{3,1})(-v_{3,2} + v_2) \right] \tag{2c}$$

where the commas denote differentiation with respect to the coordinate designated by the subscript following the comma. The stress resultants are given in terms of the strains by the stress-strain relation as:

2
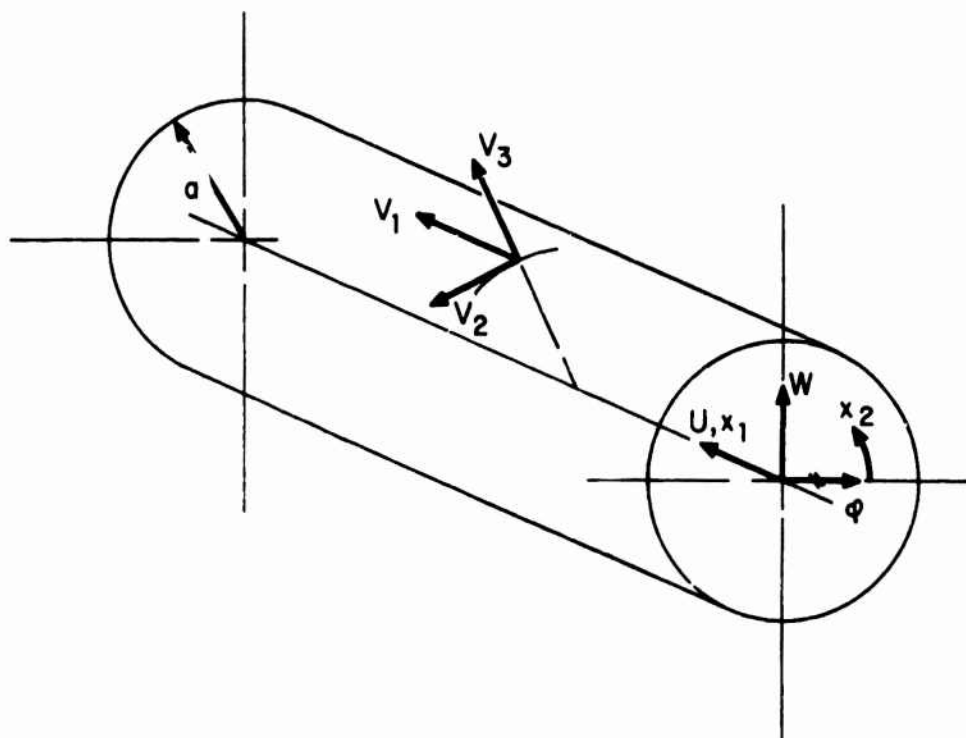
FIGURE 1.   COORDINATES AND DISPLACEMENT
            COMPONENTS.

3

$$N_{11} = C_{11} \, \epsilon_{11} + C_{12} \, \epsilon_{22} \tag{3a}$$

$$N_{22} = C_{12} \, \epsilon_{11} + C_{22} \, \epsilon_{22} \tag{3b}$$

$$N_{12} = 2C_{33} \, \epsilon_{12} \tag{3c}$$

Energy Principle for Deformation about Pressurized State:

To derive the energy principle for deformation about the pressurized state it is assumed that the displacement components v_i can be written as a superposition of displacement components, w_i, of the pressurized state and u_i, describing the deformation about the pressurized state due to applied loads as

$$v_i = w_i + u_i \tag{4}$$

Substituting equation (4) into the strain displacement relation (2) yields

$$\epsilon_{11} = \epsilon_{11}^{\,\circ} + \epsilon_{11}' + \epsilon_{11}'' \tag{5a}$$

$$\epsilon_{22} = \epsilon_{22}^{\,\circ} + \epsilon_{22}' + \epsilon_{22}'' \tag{5b}$$

$$\epsilon_{12} = \epsilon_{12}^{\,\circ} + \epsilon_{12}' + \epsilon_{12}'' \tag{5c}$$

where

$$\epsilon_{11}^{\,\circ} = w_{1,1} + \frac{1}{2} \, (w_{3,1})^2 \tag{6a}$$

$$\epsilon_{11}' = u_{1,1} + w_{3,1} \, u_{3,1} \tag{6b}$$

$$\epsilon_{11}'' = \frac{1}{2} \, (u_{3,1})^2 \tag{6c}$$

$$\epsilon_{22}^{\,\circ} = \frac{1}{r} \, (w_{2,2} + w_3) + \frac{1}{2r^2} \, (-w_{3,2} + w_2)^2 \tag{6d}$$

$$\epsilon_{22}' = \frac{1}{r} \, (u_{2,2} + u_3) + \frac{1}{r^2} \, (-w_{3,2} + w_2)(-u_{3,2} + u_2) \tag{6e}$$

4

$$\epsilon_{22}^{''} = \frac{1}{2r^2} \quad (-u_{3,2} \quad + \quad u_2)^2 \tag{6f}$$

$$\epsilon_{12}^{\circ} = \frac{1}{2} \left[w_{2,1} + \frac{1}{r} w_{1,2} + \frac{1}{r} (-w_{3,1})(-w_{3,2} + w_2) \right] \tag{6g}$$

$$\epsilon_{12}^{'} = \frac{1}{2} \quad [u_{2,1} + \frac{1}{r} u_{1,2} + \frac{1}{r} (-w_{3,1})(-u_{3,2} + u_2)$$

$$+ \frac{1}{r} (-u_{3,1})(-w_{3,2} + w_2)] \tag{6h}$$

$$\epsilon_{12}^{''} = \frac{1}{2r} \quad [-u_{3,1}(-u_{3,2} + u_2)] \tag{6i}$$

By the use of equations (5) in the stress-strain law (3) obtain

$$N_{11} = N_{11}^{\circ} + N_{11}^{'} + N_{11}^{''} \tag{7a}$$

$$N_{22} = N_{22}^{\circ} + N_{22}^{'} + N_{22}^{''} \tag{7b}$$

$$N_{12} = N_{12}^{\circ} + N_{12}^{'} + N_{12}^{''} \tag{7c}$$

where

$$N_{11}^{\circ} = C_{11} \epsilon_{11}^{\circ} + C_{12} \epsilon_{22}^{\circ} \tag{8a}$$

$$N_{11}^{'} = C_{11} \epsilon_{11}^{'} + C_{12} \epsilon_{22}^{'} \tag{8b}$$

$$N_{11}^{''} = C_{11} \epsilon_{11}^{''} + C_{12} \epsilon_{22}^{''} \tag{8c}$$

$$N_{22}^{\circ} = C_{12} \epsilon_{11}^{\circ} + C_{22} \epsilon_{22}^{\circ} \tag{8d}$$

$$N_{22}^{'} = C_{12} \epsilon_{11}^{'} + C_{22} \epsilon_{22}^{'} \tag{8e}$$

$$N_{22}^{''} = C_{12} \epsilon_{11}^{''} + C_{22} \epsilon_{22}^{''} \tag{8f}$$

5

$$N_{12}^{\circ} = 2C_{33} \ \epsilon_{12}^{\circ} \tag{8g}$$

$$N_{12}^{'} = 2C_{33} \ \epsilon_{12}^{'} \tag{8h}$$

$$N_{12}^{''} = 2C_{33} \ \epsilon_{12}^{''} \tag{8i}$$

Substitution of equations (5) and (7) into the energy principle and retaining terms up to those quadratic in the displacement components u_i yields

$$\delta \int_A \frac{1}{2} [(\epsilon_{11}^{\circ} \ N_{11}^{\circ} + \epsilon_{22}^{\circ} \ N_{22}^{\circ} + 2\epsilon_{12}^{\circ} \ N_{12}^{\circ} - 2pw_3) + 2 (\epsilon_{11}^{'} \ N_{11}^{\circ}$$

$$+ \epsilon_{22}^{'} \ N_{22}^{\circ} + 2\epsilon_{12}^{'} \ N_{12}^{\circ} - pu_3) + (\epsilon_{11}^{'} \ N_{11}^{'} + \epsilon_{22}^{'} \ N_{22}^{'} + 2\epsilon_{12}^{'} \ N_{12}^{'}$$

$$+ 2\epsilon_{11}^{''} \ N_{11}^{\circ} + 2\epsilon_{22}^{''} \ N_{22}^{\circ} + 4\epsilon_{12}^{''} \ N_{12}^{\circ} - 2Fu_3)] \ dA = 0 \tag{9}$$

In establishing this relationship the energy due to the applied forces F which do work only through the displacement component u_3 has been added and the following identities which can be proven through the stress-strain law have been used:

$$\epsilon_{11}^{\circ} \ N_{11}^{'} + \epsilon_{22}^{\circ} \ N_{22}^{'} = \epsilon_{11}^{'} \ N_{11}^{\circ} + \epsilon_{22}^{'} \ N_{22}^{\circ} \tag{10}$$

$$\epsilon_{12}^{\circ} \ N_{12}^{'} = \epsilon_{12}^{'} \ N_{12}^{\circ}$$

$$\epsilon_{11}^{\circ} \ N_{11}^{''} + \epsilon_{22}^{\circ} \ N_{22}^{''} = \epsilon_{11}^{''} \ N_{11}^{\circ} + \epsilon_{22}^{''} \ N_{22}^{\circ}$$

$$\epsilon_{12}^{\circ} \ N_{12}^{''} = \epsilon_{12}^{''} \ N_{12}^{\circ}$$

To yield the governing equations for deformation about the pressurized state, which is described by the displacement components u_i, the variation in equation (9) must be interpreted so as to be carried out with respect to the components u_i. Since the expression inclosed in the first set of brackets in equation (9) is independent of the components u_i, its variation must vanish. Carrying out the variation of the second set of brackets and performing the required integration by parts, one finds that the coefficient of the variational displacements, δu_i, are the equilibrium equations governing the pressurized state which must vanish if the stress resultants N_{11}°, N_{22}° and N_{12}° represent an equilibrium

state. Taking this into account, the following is the energy principle for deformation about the pressurized state:

$$\delta \int_A \frac{1}{2} (\epsilon_{11}' N_{11}' + \epsilon_{22}' N_{22}' + 2\epsilon_{12}' N_{12}' + 2\epsilon_{11}'' N_{11}^{\circ} + 2\epsilon_{22}'' N_{22}^{\circ}$$

$$+ 4\epsilon_{12}'' N_{12}^{\circ} - 2Fu_3) \; dA = 0 \tag{11}$$

This is the energy principle to be used to derive the equations governing the deformation about the pressurized state.

Simplified Displacement Approximation:

The problem as presently stated in equations (11), (6) and (8) is a problem in two independent coordinates and will yield partial differential equations. The objective here is to simplify the mathematical form by introducing a displacement approximation which specified the form of the displacement components with respect to the circumferential coordinate. This displacement approximation describes the deformation in terms of rigid body motions of the cross section or stated alternatively, constrains the cross section to retain its shape during deformation. This approximation is the one used in beam theory; here, however, transverse shear deformation is included. This approximation can be stated mathematically by expressing the displacement components u_i in terms of the axial displacement of the cross section, $\bar{U}$, the transverse displacement of the cross section, $\bar{W}$, and the rotation of the cross section about an axis normal to the plane of $\bar{U}$ and $\bar{W}$, ϕ, as:

$$u_1 = \bar{U} - \phi a \; Sin(x_2) \tag{12a}$$

$$u_2 = \bar{W} \; Cos(x_2) \tag{12b}$$

$$u_3 = \bar{W} \; Sin(x_2) \tag{12c}$$

These cross section displacements which are functions of x_1 only are illustrated in Figure 1. These displacement approximations can be used to formulate one-dimensional strain measures, stress resultants and a one-dimensional energy principle. The one-dimensional strain measures are obtained by substituting equations (12) into equations (6). This yields the following non-vanishing strain components:

7

$$\epsilon'_{11} = e_1 - \kappa_1 \; a \; \text{Sin}(x_2) \tag{13a}$$

$$\epsilon''_{11} = \frac{1}{2} \; (\overline{W}')^2 \; \text{Sin}(x_2) \tag{13b}$$

$$\epsilon'_{12} = e_{12} \; \text{Cos}(x_2) \tag{13c}$$

In carrying out this formulation the displacement components due to pressurization w_2 and w_3 have been assumed to vanish and take on a constant value, respectively. The one-dimensional strain measures e_1, κ_1, and e_{12} are defined as:

$$e_1 = \frac{1}{\rho} \; U' \tag{14a}$$

$$\kappa_1 = \frac{1}{l} \; \phi' \tag{14b}$$

$$e_{12} = \frac{1}{2} \left(\frac{1}{\rho} \; W' - \phi \right) \tag{14c}$$

In writing these expressions the axial coordinate and the displacements have been non-dimensionalized as follows:

$$x_1 = l\xi \tag{15a}$$

$$\overline{W} = aW \tag{15b}$$

$$\overline{U} = aU \tag{15c}$$

and the prime denotes differentiation with respect to the non-dimensional variable, ξ, and a and l are respectively the beam cross sectional radius and length measure.

The one-dimensional energy principle is established by substitution of equations (13) into equation (11), taking account of the vanishing strains this yields:

$$\delta \int_\xi \int_0^{2\pi} \frac{1}{2} \left[e_1 r_{11} - \kappa_1 n'_{1} \; a\text{Sin}(x_2) + 2e_{12}n'_{12} \; \text{Cos}(x_2) \right.$$
$$\left. + \frac{1}{\rho^2}(W')^2 n \; \text{Sin}^2(x_2) - 2 \; \frac{1}{\rho} \; \overline{F} \; W \; \text{Sin}(x_2) \right] a l dx_2 d\xi = 0 \tag{16}$$

8

In this expression the stress resultants have been non-dimensionalized by division by a reference value of the elastic modulus $C_{11} = \overline{C}_{11}$. The non-dimensional force parameter is defined by $\overline{F} = F/\overline{C}_{11}$. This energy principle is reduced to a one-dimensional principle by definition of the one-dimensional stress resultants and forces as follows:

$$T = \int_o^{2\pi} n_{11}' a\, dx_1 \tag{17a}$$

$$M = \int_o^{2\pi} n_{11}' a^2 Sin(x_2) dx_2 \tag{17b}$$

$$Q = \int_o^{2\pi} n_{12}' aCos(x^2) dx_2 \tag{17c}$$

$$f = \frac{1}{\pi} \int_o^{2\pi} \overline{F}Sin(x_2) dx_2 \tag{17d}$$

The remaining term appearing in the energy, that containing the stress due to pressurization, n, can be integrated with respect to x_2 since both n and W are functions of x_1 only. The one-dimensional stress-strain law is obtained from equations (17) by substitution of the stress-strain law (8), properly non-dimensionalized, and the strain displacement relations (13) and carrying out the required integration to yield

$$T = 2\pi dae_1 \tag{18a}$$

$$M = -\pi da^3 \kappa_1 \tag{18b}$$

$$Q = 2\pi cae_{12} \tag{18c}$$

The one-dimensional energy principle is put in final form by rewriting equation (16) using the definitions given in (17):

$$\delta \int \frac{1}{2} \left[e_1 T - \kappa_1 M + 2e_{12}Q + \frac{\pi a}{\rho^2} (W')^2 n \right.$$
$$\left. - 2\pi \frac{a}{\rho} fW \right] |d\xi = 0 \tag{19}$$

It should be remembered that the stress resultant due to pressurization, n, is a known quantity in this equation.

9

Governing Differential Equations:

With the preliminaries as developed above, it is possible to derive the equations governing the cross section displacements U, W and ϕ. This is accomplished by writing the energy principle (19) in terms of these displacements by using equations (18) and (14) and carrying out the variation, the integration by parts where required and application of the fundamental lemma of the calculus of variation to yield the differential equations. The energy principle in terms of displacements is given as:

$$\delta \int_\xi \frac{\pi al}{2} \left[\frac{2d}{\rho^2} (U')^2 + \frac{d}{\rho^2} (\phi')^2 + c\left(\frac{1}{\rho} W' - \phi\right)^2 + \frac{n}{\rho^2} (W')^2 \right.$$

$$\left. -2\frac{1}{\rho} fW \right] d\xi = 0 \qquad (20)$$

carrying out the variation

$$\int_\xi \pi al \left(\frac{2d}{\rho^2} U'\delta U' + \frac{d}{\rho^2} \phi'\delta\phi' + \frac{c}{\rho^2} W'\delta W' - \frac{c}{\rho} W'\delta\phi - \frac{c}{\rho} \phi\delta W' \right.$$

$$\left. + c\phi\delta\phi + \frac{n}{\rho^2} W'\delta W' - \frac{1}{\rho} f\delta W \right) d\xi = 0 \qquad (21)$$

Carrying out the required integration by parts with the interval of integration being (ξ_1, ξ_2)

$$\int_{\xi_1}^{\xi_2} \pi al \left[\left(-\frac{2d}{\rho^2} U''\right) \delta U + \left(-\frac{d}{\rho^2} \phi'' - \frac{c}{\rho} W' + c\phi\right) \delta\phi \right.$$

$$\left. + \left(-\frac{1}{\rho^2} (c + n)W'' + \frac{c}{\rho} \phi' - \frac{1}{\rho} f\right) \delta W \right] d\xi \qquad (22)$$

$$+ \left[\frac{1}{\rho^2} (c + n)W' - \frac{c}{\rho} \phi\right]\delta W\Big/_{\xi_1}^{\xi_2} + \left[\frac{2d}{\rho^2} U'\right] \delta U\Big/_{\xi_1}^{\xi_2} + \left[\frac{d}{\rho} \phi'\right] \delta\phi\Big/_{\xi_1}^{\xi_2} = 0$$

According to the fundamental lemma of the calculus of variations satisfaction of this equation requires that the coefficients of ξU, ξW and $\xi\phi$ vanish independently yielding the governing differential equations

10

$$U'' = 0 \tag{23}$$

$$d\phi'' + c\rho W' - c\rho^2 \phi = 0 \tag{24}$$

$$(c + n)W'' - c\rho\phi' + \rho f = 0 \tag{25}$$

and boundary conditions requiring specification of

$$U' \text{ or } U \tag{26a}$$

$$\phi' \text{ or } \phi \tag{26b}$$

$$\frac{1}{\rho^2} \quad (c + n)W' - \frac{c}{\rho} \phi \text{ or } W \tag{26c}$$

at $\xi = \xi_1$ and $\xi = \xi_2$.

It is to be noted that in these euqations the axial deformation specified by the dependent variable U is uncoupled from the transverse deformation specified by the dependent variables W and ϕ. Thus when axial deformation is of no interest, as is the present case, it may be ignored.

Solution of the Governing Equations

The purpose of this section is the formulation of the solutions to the equation obtained in the previous section. This work will be limited to the transverse or bending deformation parameters, W and ϕ. The fundamental homogenous solutions will be obtained first and this will be utilized to develop Green's function solutions for the cases of simply supported ends and clamped ends.

Fundamental Homogenous Solution:

In this section the homogenous solution to the equations governing W and ϕ are found. These equations, (24) and (25), are second order coupled ordinary differential equations with constant coefficients so the solution is of the form:

11

$$W = Ae^{\omega\xi} \tag{27a}$$

$$\phi = Be^{\omega\xi} \tag{27b}$$

Substitution of these expressions into the homogenous differential equations leads to the following matrix equations for the constants A and B and the characteristic roots

$$\begin{bmatrix} (c+n)\omega^2 & -\rho c\omega \\ c\rho\omega & d\omega^2 - c\rho \end{bmatrix} \begin{Bmatrix} A \\ B \end{Bmatrix} = 0 \tag{28}$$

For these equations to have a nontrivial solution the determinate must vanish from which the characteristic roots are found to be $\omega_1 = 0, \omega_2 = 0$, $\omega_3 = \lambda$, and $\omega_3 = -\lambda$. For each of these roots a relation exists between A and B, as follows:

$$B = A_2/\rho$$

$$B_2 = 0$$

$$B_3 = \beta A_4 \tag{29}$$

$$B_4 = \beta A_3$$

$$\beta = c\rho\lambda/(c\rho^2 - \lambda^2 d)$$

$$\lambda = \sqrt{c\rho^2 n/d(c+n)}$$

Thus the homogenous solution is

$$W = A_1 + A_2\xi + A_3 \operatorname{Cosh}(\lambda\xi) + A_4 \operatorname{Sinh}(\lambda\xi) \tag{30a}$$

$$\phi = \frac{1}{\rho} A_2 + A_4\beta \operatorname{Cosh}(\lambda\xi) + A_3\beta \operatorname{Sinh}(\lambda\xi) \tag{30b}$$

12

Green's Function:

The solution obtained in the previous section is not complete in that neither the loading condition nor the boundary conditions have been specified. In this section two sets of boundary conditions will be considered in conjunction with a concentrated loading. This will be handled in the form of a Green's function because the results are applicable not only to the concentrated load but also to the general loading situation by means of direct integration. To carry out the solution, consider a beam of length 2l with the origin of coordinates located at the center of the beam and concentrated load, g, applied at $\xi = \eta$. In a physical situation this load would be applied as a distributed circumferential line load $G(x_2)$ and this is related to the concentrated load, g, by:

$$g = \frac{1}{\pi C_{11}} \int_o^{2\pi} G(x_2) \, Sin(x_2)dx_2 \qquad (31)$$

The Green's function is found (reference 9) by obtaining two sets of functions of the form of equations (30), one of which (W_1, ϕ_1), satisfies the boundary conditions at $\xi = -1$ and the other, (W_2, ϕ_2), satisfies the boundary conditions at $\xi = 1$. The remaining constants of integration are determined by satisfaction of three continuity conditions and a jump condition to account for the concentrated load at $\xi = \eta$. These conditions are:

$$W_1(\eta) = W_2(\eta) \qquad (32a)$$

$$\phi_1(\eta) = \phi_2(\eta) \qquad (32b)$$

$$\phi_1'(\eta) = \phi_2'(\eta) \qquad (32c)$$

$$[(c + n)W_1'(\eta) - c\rho\phi_1(\eta)] - [(c + n)W_2'(\eta) - c\rho\phi_2(\eta)] = \rho g \, . \qquad (32d)$$

Simply Supported Boundary Conditions:

The first case of end restraint to be considered is that of a beam simply supported on both ends. Physically, simply supported means that the transverse displacement and the moment vanish at the end of the beam. These conditions are expressed mathematically as

13

$$W_1(-1) = 0 \tag{33a}$$

$$\phi_1'(-1) = 0 \tag{33b}$$

$$W_2(1) = 0 \tag{33c}$$

$$\phi_2'(1) = 0 \tag{33d}$$

Application of these four conditions in conjunction with the four conditions at the location of the concentrated load, equations (32), is sufficient to determine the eight constants appearing in W_1, ϕ_1, W_2 and ϕ_2, thus completing the solution of this case which is as follows:

for $-1 \leqslant \xi \leqslant \eta$

$$W = \frac{g}{2n} \left[\rho(1-\eta)(1+\xi) + \Gamma_1 \left[Sinh(\lambda\xi) + Tanh(\lambda) \, Cosh(\lambda\xi) \right] \right] \tag{34}$$

$$\phi = \frac{g}{2n} \left[(1-\eta) + \beta\Gamma_1 \left[Cosh(\lambda\xi) + Tanh(\lambda) \, Sinh(\lambda\xi) \right] \right] \tag{35}$$

$$\Gamma_1 = \frac{Sinh(\lambda\eta) - Tanh(\lambda) \, Cosh(\lambda\eta)}{\beta Tanh(\lambda)}$$

for $\eta \leqslant \xi \leqslant 1$

$$W = \frac{g}{2n} \left[\rho(1+\eta)(1-\xi) + \Gamma_2 \left[Sinh(\lambda\xi) - Tanh(\lambda) \, Cosh(\lambda\xi) \right] \right] \tag{36}$$

$$\phi = \frac{g}{2n} \left[-(1-\eta) + \beta\Gamma_2 \left[Cosh(\lambda\xi) - Tanh(\lambda) \, Sinh(\lambda\xi) \right] \right]$$

$$\Gamma_2 = \frac{Sinh(\lambda\eta) + Tanh(\lambda) \, Cosh(\lambda\eta)}{\beta Tanh(\lambda)} \tag{37}$$

In writing these expressions the subscripts on W and ϕ have been dropped and the range of ξ over which the solution is valid has been specified. This solution has the symmetry property required of a Green's function, namely $W(\xi,\eta) = W(\eta,\xi)$.

14

Clamped Boundary Condition:

In this section the Green's function for a beam with clamped or fixed restraints on both ends will be determined. Physically, the clamped end means that both the transverse displacement and the slope vanish at the end of the beam. This is expressed mathematically as

$$W_1(-1) = 0 \tag{38a}$$

$$\phi_1(-1) = 0 \tag{38b}$$

$$W_2(1) = 0 \tag{38c}$$

$$\phi_2(1) = 0 \tag{38d}$$

As with the simple supported case, application of these conditions in conjunction with those specifying the condition at the position of the concentrated load allows all unknown constants of integration to be determined giving the solution as

for $-1 > \xi > \eta$

$$W = \frac{9}{2n\beta} \; [a_3[\rho\beta(1+\xi) \; Sinh(\lambda) - Cosh(\lambda) + Cosh(\lambda\xi)] \tag{39a}$$
$$+ a_4[-\rho\beta(1+\xi) \; Cosh(\lambda) + Sinh(\lambda) + Sinh(\lambda\xi)] \;]$$

$$\phi = \frac{9}{2n} \; [a_3[Sin(\lambda) + Sinh(\lambda\xi)] + a_4 \; [-Cosh(\lambda) + Cosh(\lambda\xi)] \tag{39b}$$

for $\eta > \xi > 1$

$$W = \frac{9}{2n\beta} \; [b_3[\rho\beta(1-\xi) \; Sinh(\lambda) - Cosh(\lambda) + Cosh(\lambda\xi)]$$
$$+ b_4[\rho\beta(1-\xi) \; Cosh(\lambda) - Sinh(\lambda) + Sinh(\lambda\xi)] \;] \tag{40a}$$

$$\phi = \frac{9}{2n} \; [b_3[-Sinh(\lambda) + Sinh(\lambda\xi)] + b_4[-Cosh(\lambda) + Cosh(\lambda\xi)] \;] \tag{40b}$$

15

where

$$a_3 = [1 + Sinh(\lambda) \ Sinh(\lambda\eta) - Cosh(\lambda) \ Cosh(\lambda\eta)] / Sinh(\lambda) \qquad (41a)$$

$$a_4 = [-\rho\beta\eta - \rho\beta \ Sinh(\lambda) \ Sinh(\lambda\eta) + \rho\beta \ Cosh(\lambda) \ Cosh(\lambda\eta) \qquad (41b)$$

$$+ Cosh(\lambda) \ Sinh(\lambda\eta) - Sinh(\lambda) \ Cosh(\lambda\eta)] / [Sinh(\lambda) - \rho\beta \ Cosh(\lambda)]$$

$$b_3 = [1 - Sinh(\lambda) \ Sinh(\lambda\eta) - Cosh(\lambda, \ Cosh(\eta\lambda)] / Sinh(\lambda) \qquad (42a)$$

$$b_4 = [-\rho\beta\eta - \rho\beta \ Sinh(\lambda) \ Sinh(\lambda\eta) - \rho\beta Cosh(\lambda) \ Cosh(\lambda\eta) \qquad (42b)$$

$$+ Cosh(\lambda) \ Sinh(\lambda\eta) + Sinh(\lambda) \ Cosh(\lambda\eta))] / [Sinh(\lambda) - \rho\beta \ Cosh(\lambda)]$$

Although it cannot be seen by inspection, this solution also has the symmetry property associated with a Green's function.

Uniform Load Solution:

A solution that is frequently useful and relatively easy to obtain directly as opposed to integration of the Green's function is that for the uniformly loaded beam. As with the Green's function, solutions will consider a beam of length 2l with the origin of coordinates at the center of the beam as depicted in Figure 1. The solution desired is that for equations (24) and (25) with the loading parameter, f, taken as a constant. This solution can be written as the sum of a homogenous solution and a particular solution. The homogenous solution is given by equations (30) so only the particular solution remains to be determined and this is taken in the form:

$$W = D_1 \xi^2 \qquad (43a)$$

$$\phi = D_2 \xi \qquad (43b)$$

Substitution of these into the differential equations (24) and (25) and solution for the constants D_1 and D_2 yields

$$D_1 = - \frac{\rho f}{2n} \qquad (44a)$$

16

$$D_2 = -\frac{f}{n} \tag{44b}$$

and the full solution is then:

$$W = \frac{f}{n} \left[-\frac{\rho}{2}\xi^2 + A_1 + A_2\xi + A_3 \, Cosh(\lambda\xi) + A_4 \, Sinh(\lambda\xi)\right] \tag{45a}$$

$$\phi = \frac{f}{n}\left[-\xi + \frac{1}{\rho}A_2 + A_4\beta \, Cosh(\lambda\xi) + A_3\beta \, Sinh(\lambda\xi)\right] \tag{45b}$$

The constants A_i are determined by specification of the boundary conditions at $\xi = 1$ and $\xi = -1$.

Simply Supported Boundary Conditions:

Application of conditions (33) in conjunction with this solution (45) gives the solution for the uniformly loaded beam as the simply supported boundary as

$$W = \frac{f}{n}\left[\rho(1+\xi^2)/2 + [Cosh(\lambda\xi) - Cosh(\lambda))]/\lambda\beta Cosh(\lambda)\right] \tag{46a}$$

$$\phi = \frac{f}{n}\left[-\xi + Sinh(\lambda\xi)/\lambda Cosh(\lambda)\right] \tag{46b}$$

Clamped Boundary Conditions:

Application of conditions (38) in conjunction with the solution given by (45) gives the solution for the uniformly loaded beam with clamped boundary conditions as:

$$W = \frac{f}{n}\left[\rho(1-\xi^2)/2 + [Cosh(\lambda\xi) - Cosh(\lambda)]/\beta Sinh(\lambda)\right] \tag{47a}$$

$$\phi = \frac{f}{n}\left[-\xi + Sinh(\lambda\xi)/Sinh(\lambda)\right] \tag{47b}$$

Two observations can be made: 1) the solutions obtained for the uniformly loaded beam, equations (46) and (47) are identical with the solution that would be obtained by integrating the respective Green's functions; 2) the general solutions obtained here (30) and (45) can be used to obtain solutions for other combinations of boundary conditions as allowed by equations (26).

17

DISCUSSION OF RESULTS

The purpose of this section is the presentation of results which describe the behavior of pressure stabilized beams under load as predicted by the theory developed in the preceeding sections. The results to be presented will describe the deformation behavior and the strength behavior for beams subjected to a centrally located concentrated load. This problem is considered here because it represents a characteristic solution of the governing equations and is thus very useful as a tool to illustrate the behavior with respect to the three parameters, pressure, geometry and material, which occur in the equations. These results will be presented for two cases of end restraint, simply supported and fixed ends.

Deformation Behavior

For linear structures the deformation behavior is conveniently characterized by the Green's function $G(\xi, \eta)$ which can be thought of as the deformation at a coordinate ξ due to a unit load at coordinate η or the flexibility function often used in structural mechanics. The flexibility parameter used here is the deformation at the beam midpoint due to a unit load applied at the same point. This flexibility parameter is presented in nondimensional form as a function of the nondimensional pressure, geometry and material parameters.

Pressure Parameter:

The nondimensional flexibility parameter as a function of the pressure parameter is shown in Figures 2 and 3 for the simply supported and fixed end restraints, respectively. For the data shown the geometry parameter $\rho = 100$ and the modulus ratio is $c = 0.1$. The physical pressure range covered in these plots depends on the geometry and elastic modulus of the particular beam, but for a beam having a radius of 3.175 cm (1.25 in) and an elastic modulus of 2.1015×10^8 dyne/cm (1200 lb/in) the pressure range is $0 - 4.236 \times 10^6$ dyne/cm^2 (24.2 lb/in^2). Examination of the plots in Figures 2 and 3 reveals that for very low pressures the flexibility is large and rather large decreases in flexibility result from small increases in pressure. In the higher pressure range the opposite effect is observed, that is, large increases in pressure are required to significantly decrease the flexibility. Thus, in the higher pressure range the benefits to be gained with
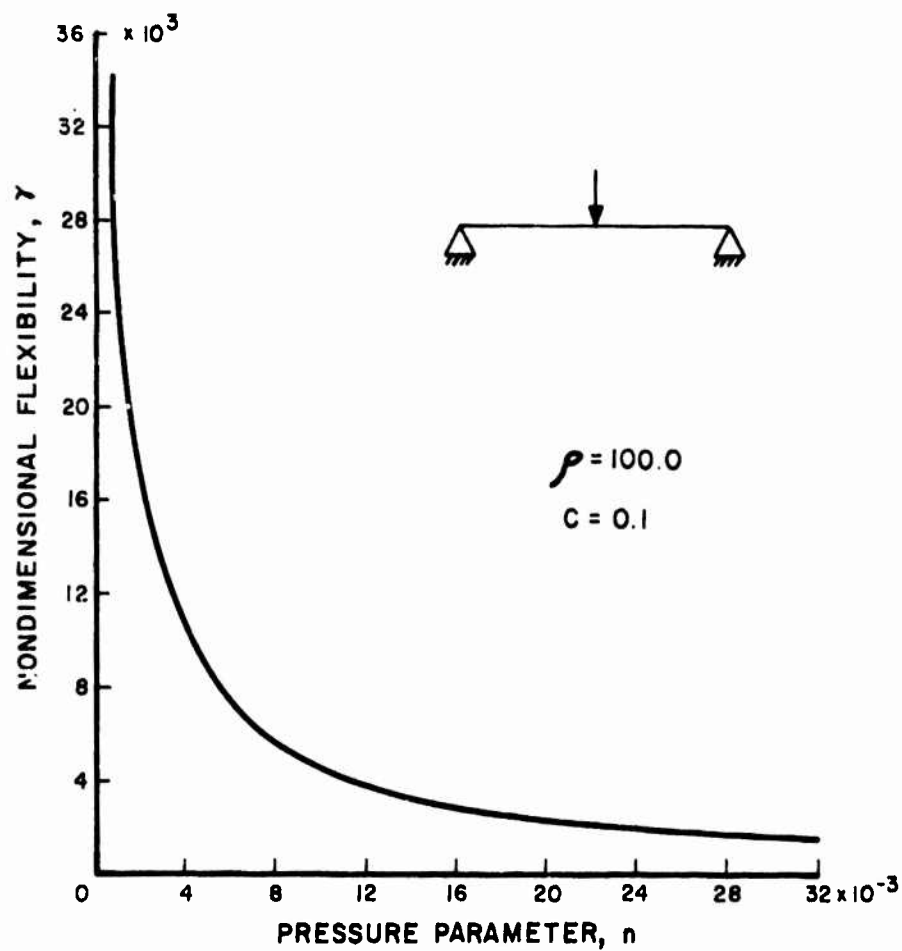
18

FIGURE 2.   FLEXIBILITY AS A FUNCTION OF PRESSURE,
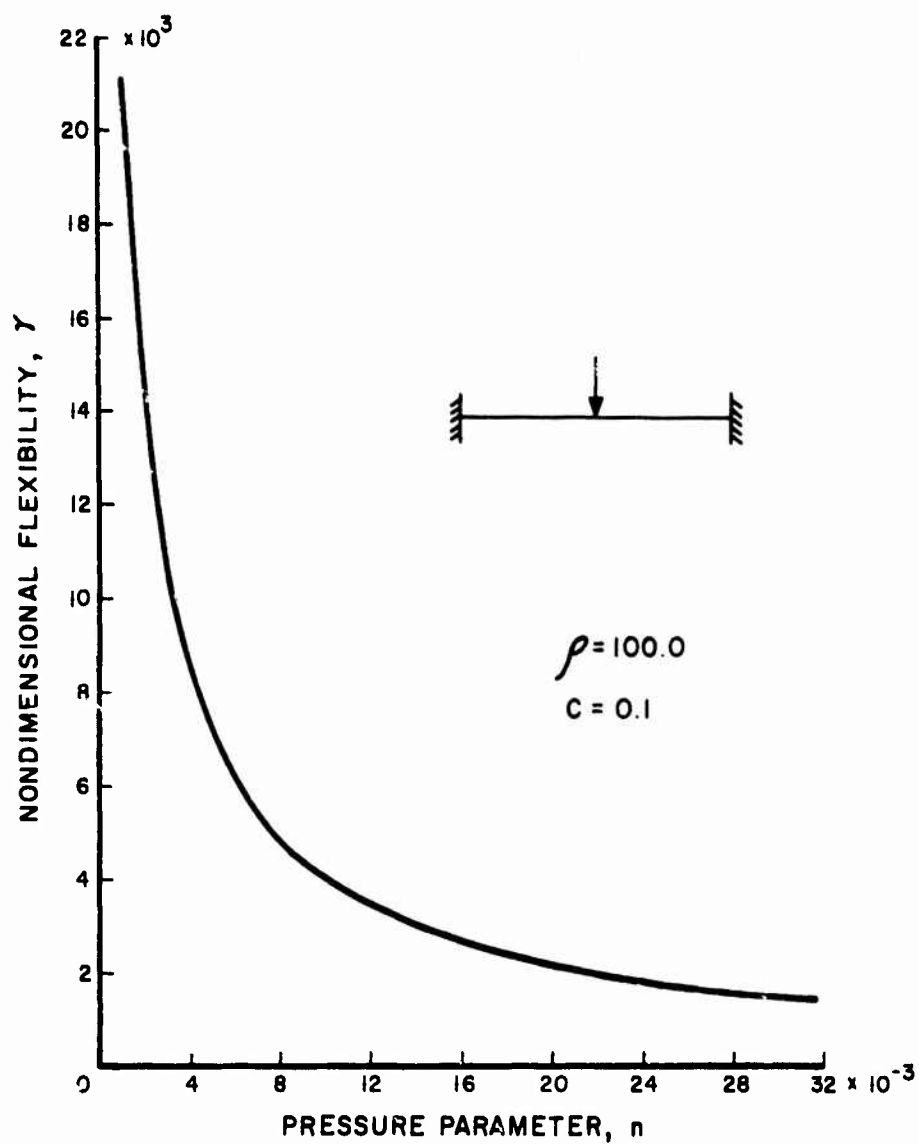SIMPLY SUPPORTED ENDS.

FIGURE 3.   FLEXIBILITY AS A FUNCTION OF PRESSURE,
            FIXED ENDS.

20

respect to decreasing flexibility or increasing stiffness by changes in pressures are not significant. Although it is not exactly the case the flexibility can generally be described as varying inversely with the pressure. The same type of behavior is observed for other geometries and modulus ratios. Comparison of the results for the two types of end restraints reveals that, as would be expected, the simply supported case is more flexible than the fixed case.

Geometry Parameters:

Figures 4 and 5 show respectively the variation of the flexibility parameter with geometry for the simply supported and fixed-end restraints. As indicated above, the geometry is specified by the ratio of the beam length to the cross-section radius. The results presented are for a modulus ratio $c = 0.1$ and for three values of the pressure parameter. These results reveal a linear relationship between flexibility and geometry except in the low range of the geometry parameter.

Modulus Ratio:

The third parameter appearing in the governing equations is the ratio of the shear modulus to the elastic modulus, referred to hereafter as the modulus ratio. The flexibility parameter as a function of the modulus ratio is presented in Figures 6 and 7 for the fixed-end beam and in Figures 8 and 9 for the simply supported beam. For each case results are given for two values of the geometry parameter and three values of the pressure parameter. Examination of the results for the fixed-end case (Figures 6 and 7) reveals that for short beams with large cross sections (Figure 6) the variation of the modulus ratio in the lower range of that parameter can have a significant effect on the flexibility parameter. This effect is more pronounced at low pressures. In the higher range of the modulus ratio for short large-cross-section beams and over the entire range of modulus ratio for long thin beams, the flexibility parameter is nearly constant. Thus, it is apparent that shear deformation is an important consideration for short beams with large cross sections utilizing low-inflation pressures. Similar results are observed for the simply supported case (Figures 8 and 9) although the importance of shear deformation is not as significant as in the fixed-end case.
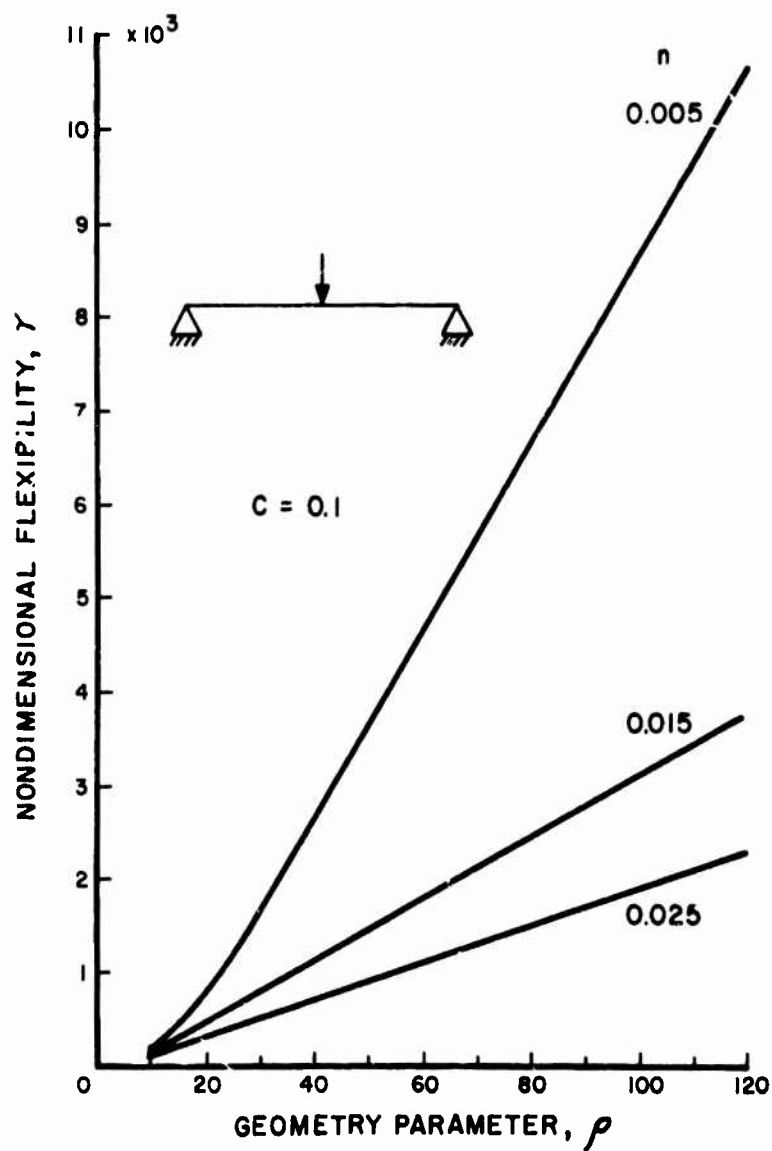
21

FIGURE 4. FLEXIBILITY AS A FUNCTION OF
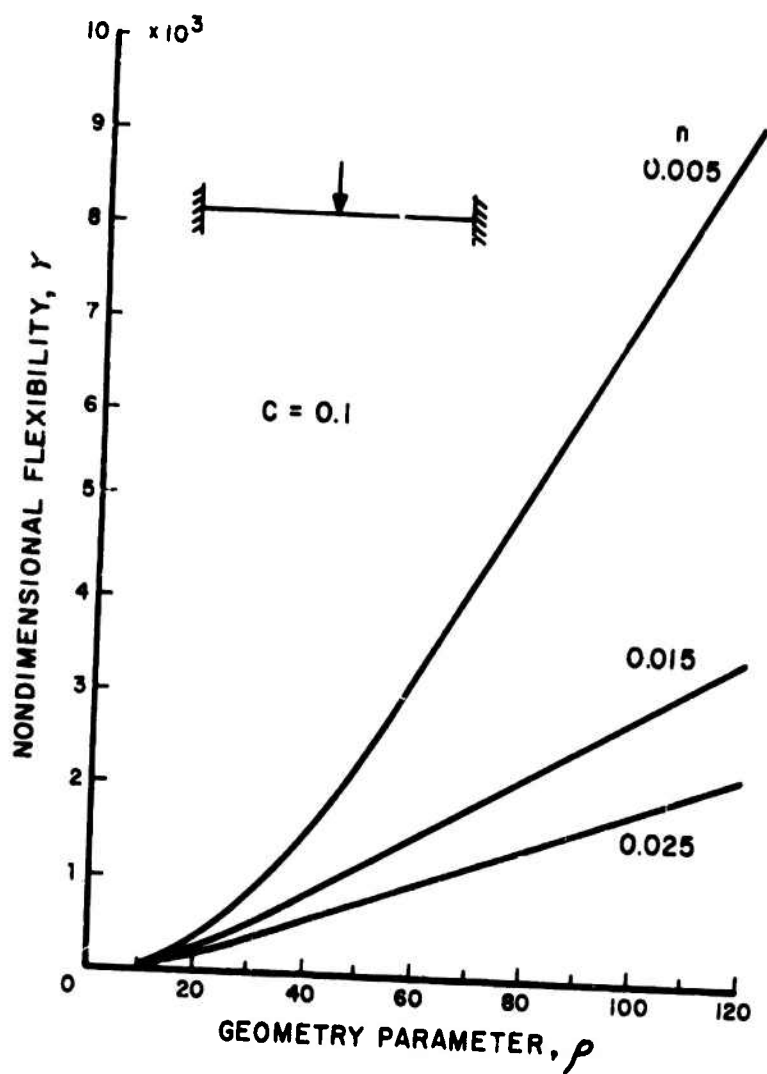GEOMETRY, SIMPLY SUPPORTED ENDS.

FIGURE 5. FLEXIBILITY AS A FUNCTION OF GEOMETRY, FIXED ENDS.
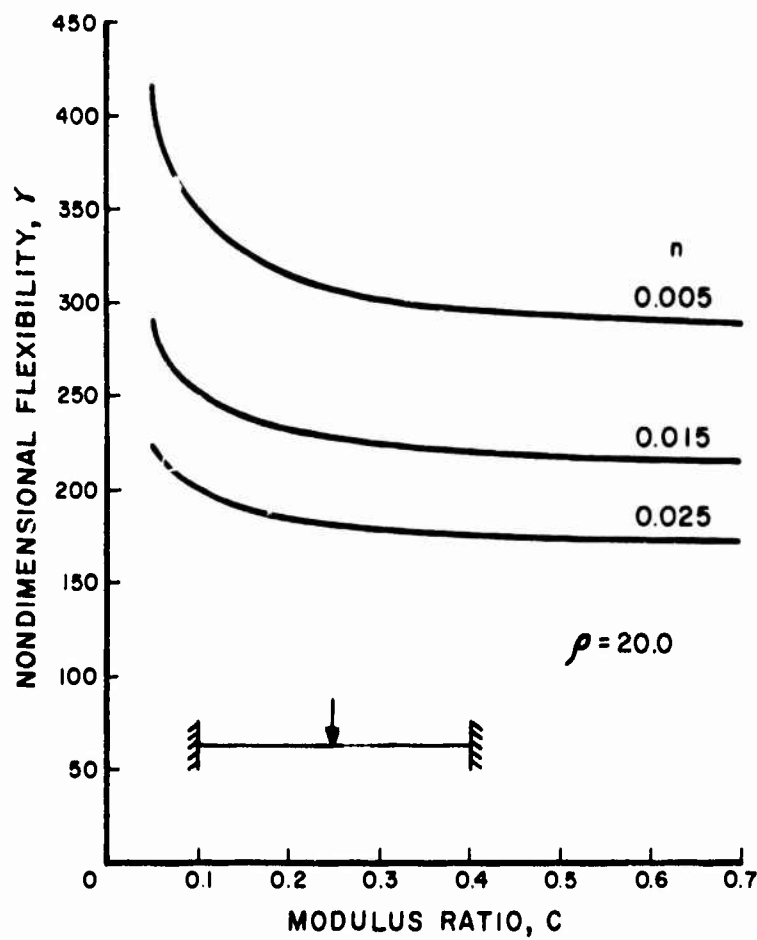
23

FIGURE 6.   FLEXIBILITY AS A FUNCTION OF
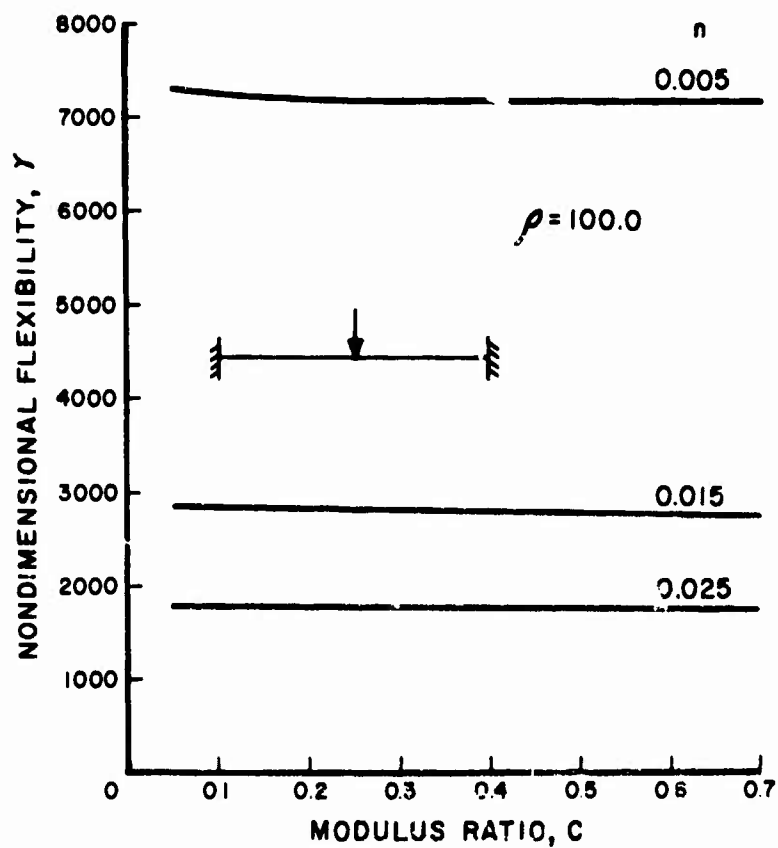MODULUS RATIO, FIXED ENDS.

FIGURE 7. FLEXIBILITY AS A FUNCTION OF
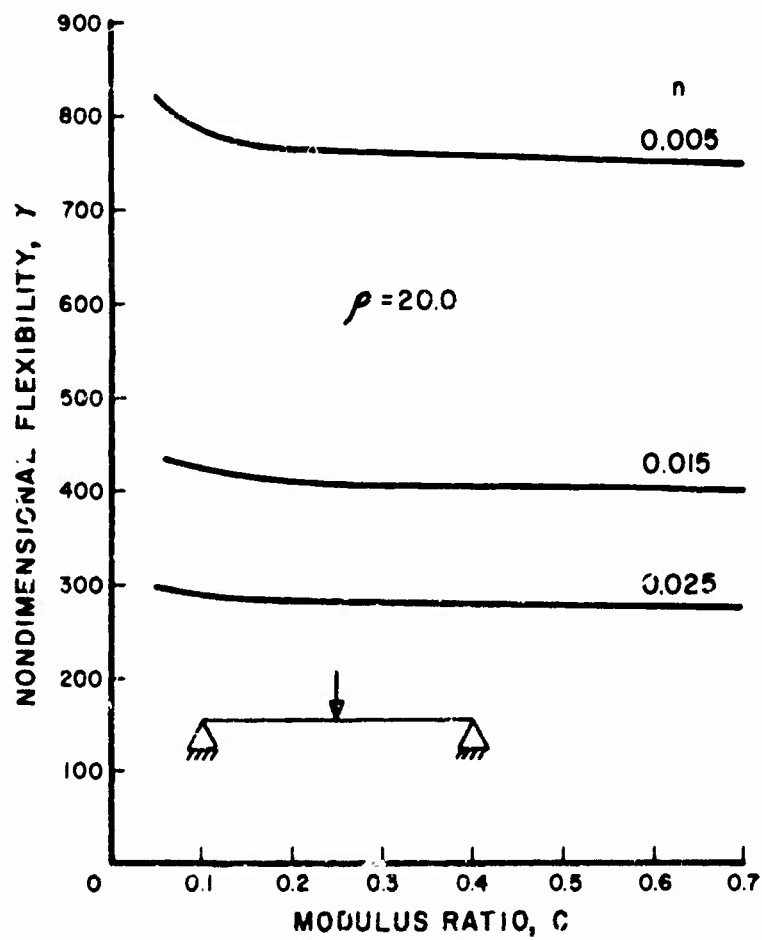MODULUS RATIO, FIXED ENDS.

FIGURE 8. FLEXIBILITY AS A FUNCTION OF
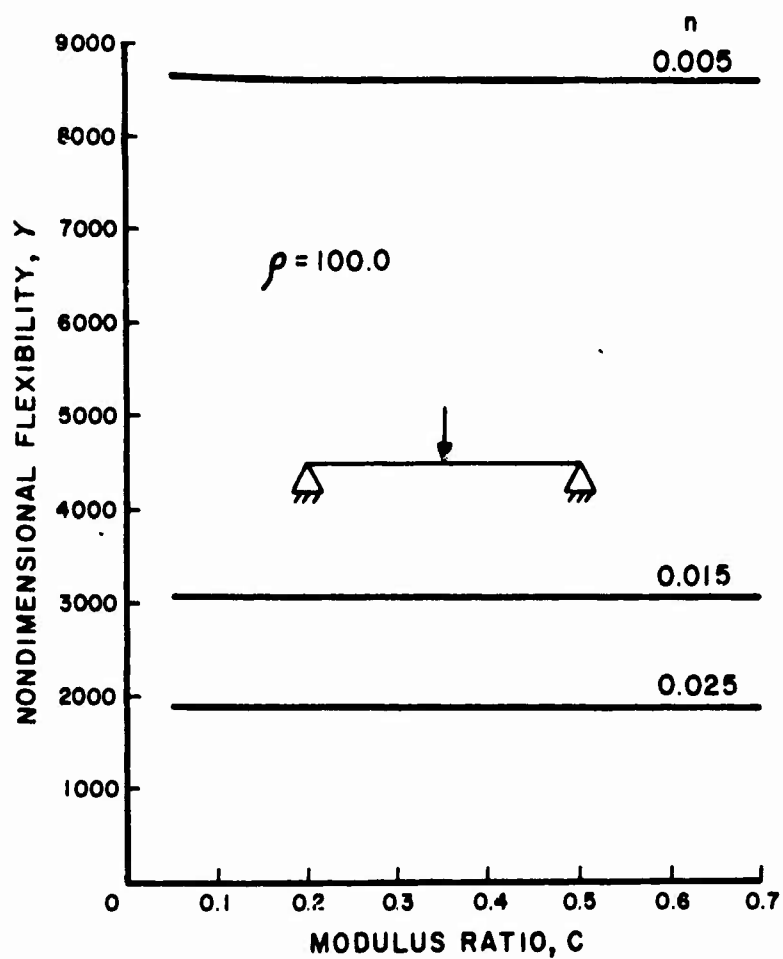MODULUS RATIO, SIMPLY
SUPPORTED ENDS.

26

FIGURE 9.   FLEXIBILITY AS A FUNCTION OF
MODULUS RATIO, SIMPLY
SUPPORTED ENDS.

27

Failure Criteria

The main thrust of this work is the development of design equations for use on structural elements fabricated of materials such as fabric which cannot support compressive stress. For this situation a useful and convenient failure criterion for describing the strength of the structural element is the wrinkling load. Several authors, see for example references 6 and 7, have carried out nonlinear analysis of special loading situations which relate this wrinkling load to the ultimate strength of the member. Thus, this wrinkling load can be used to develop other failure criteria. This wrinkling load is defined as the load which results in a negative stress at some point in the structure equal in magnitude to the positive stress at that point caused by pressure. This results in a stress of zero magnitude at the point in question with any further increase in the load causing a compressive stress which cannot be supported by the material so that wrinkling results. The wrinkling load thus specifies the initiation of failure and can be thought of as analogous to the yield strength.

As was done with the discussion of the deformation behavior, the principal results presented here will be for the case of a concentrated load applied at the center of the beam for which the maximum stress and thus the wrinkling occurs at the center of the beam. Results relative to the behavior of the wrinkling load as a function of the pressure, geometry and material properties will be presented for both the simply supported and fixed-boundary restraints. It is possible to express in a rather concise and useful form the behavior of the wrinkling load for the uniformly loaded beam as a function of the geometry and pressure level and this result will be presented.

Pressure Parameter:

The wrinkling load as a function of pressure level is depicted in Figure 10 for the geometry parameter $\rho = 100$ and the modulus ration $c = 0.1$. The data shown is for the case of clamped restraints as indicated, but the data for the case of simply supported restraints is nearly indistinguishable from the results presented in Figure 10. Examination of these results shows that the wrinkling load, and thus the load supporting capacity of pressure stabilized beams, increase with pressure. The rate of increase is greater than a linear relationship; that is, doubling the pressure results in more than a doubling of the wrinkling load.
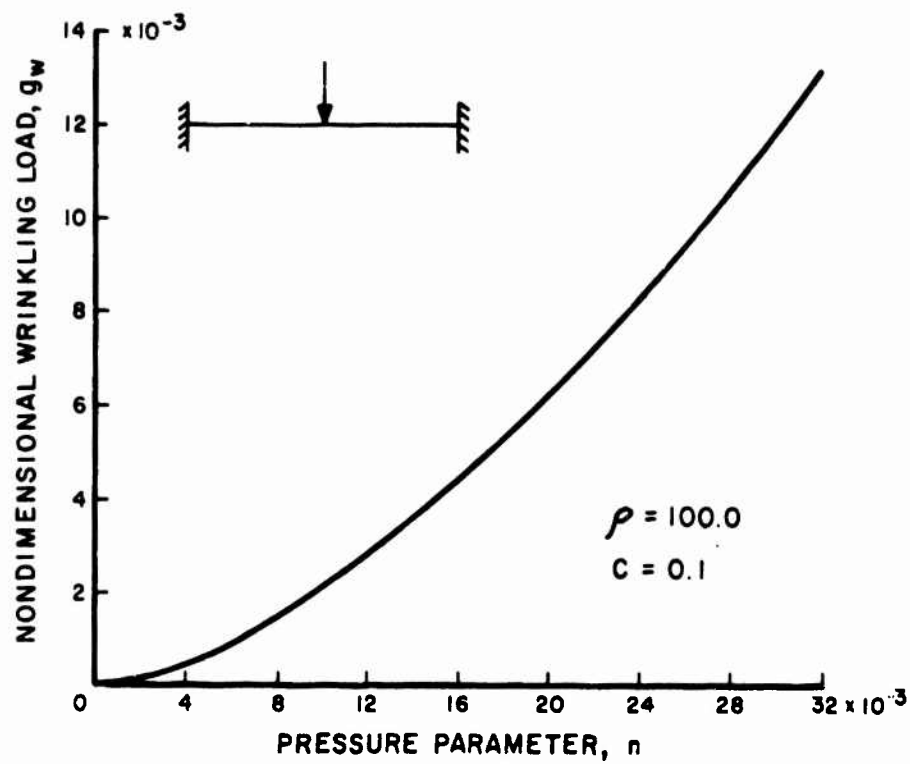
28

FIGURE 10. WRINKLING LOAD AS A FUNCTION OF
PRESSURE, FIXED ENDS.

29

Geometry Parameter:

Figure 11 shows the behavior of the wrinkling load as a function of the geometry parameter for both the clamped and simply supported end restraints. Data is shown for three values of the pressure parameter, n, and all data is for one value of the modulus ratio, c = 0.1. For values of the geometry parameter above 50 for the fixed case and 30 for the simply supported case, the wrinkling load takes on a nearly constant value. Below these values the geometry parameter causes considerable variation in the wrinkling load. The overall variation shown in Figure 11 is that of the negative hyperbolic tangent. These results show that above some value of the geometry parameter the load carrying capacity of the beam is independent of length. Subsequently, similar behavior will be shown for the uniformly loaded beam.

Modulus Ratio:

The variation of the wrinkling load with the modulus ratio is shown in Figure 12 for the case of simply supported end restraints, and Figure 13 for fixed-end restraints. Results are shown for the pressure parameter, n = 0.015 and for two values of the geometry parameter, ρ = 20, 100. These results show that for values of c above 0.2 the modulus ratio has very little effect on the wrinkling load for the values of ρ and n presented. These results are typical for all values of ρ and n. This confirms the finding revealed by examination of the deformation behavior with regard to the significance of shear deformation.

Behavior Under Uniform Load

Since the uniformly loaded beam more closely represents the type of loading situation that would occur in tentage application, some results concerning the magnitude of the wrinkling load for this case are presented. The wrinkling load here is the magnitude of the load per unit length of beam which causes wrinkling to begin. The results presented here are for the load per unit physical length as opposed to the load per unit nondimensional length used in the derivation.

For simply supported end restraints the wrinkling load is given as

$$f_w = \frac{n^2}{a} \left[\frac{\text{Cosh}\,\lambda}{-\text{Cosh}\,\lambda + 1} \right] \tag{48}$$
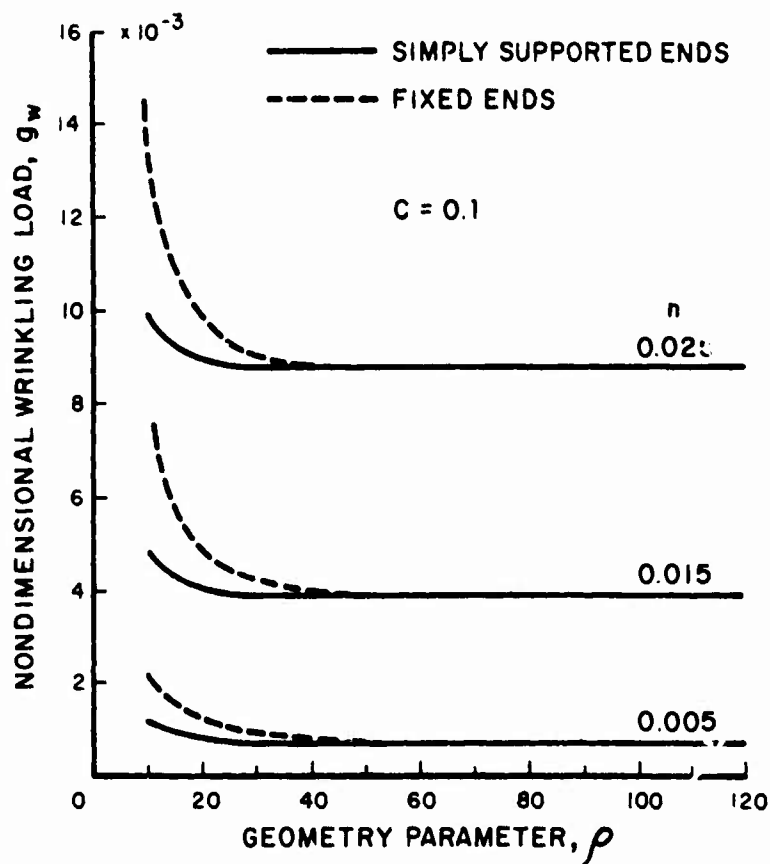
30

FIGURE II. WRINKLING LOAD AS A FUNCTION
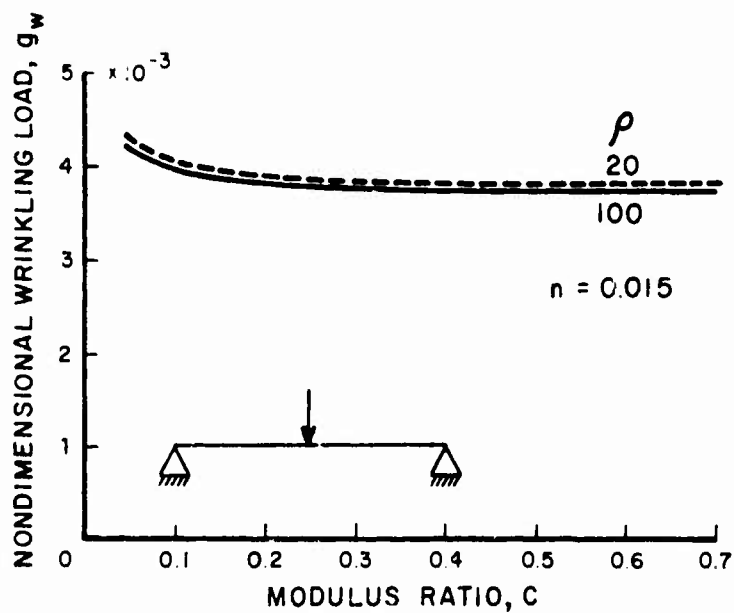OF GEOMETRY.

FIGURE 12.   WRINKLING LOAD AS A FUNCTION
OF MODULUS RATIO, SIMPLY
SUPPORTED ENDS.
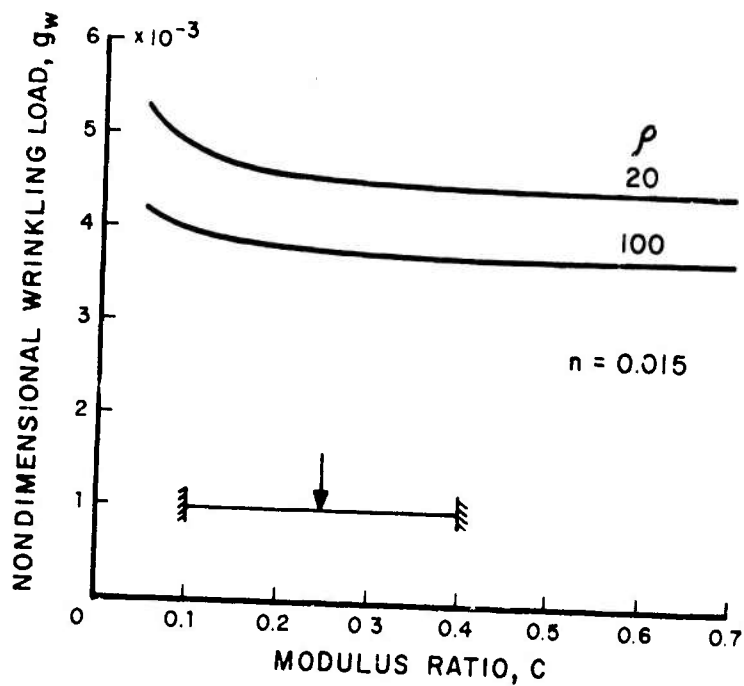
FIGURE 13. WRINKLING LOAD AS A FUNCTION
OF MODULUS RATIO, FIXED ENDS.

and for the fixed-end restraints

$$f_w = \frac{n^2}{a} \left[\frac{\text{Sinh } \lambda}{-\text{Sinh } \lambda + \lambda} \right] \qquad \qquad (49)$$

The parameter λ increases with both n, the pressure parameter and ρ the geometry parameter and the expressions in brackets in equations (48) and (49) rapidly approach unity as n and ρ increase. Thus, over much of the pressure and geometry range, the wrinkling load for both types of end restraint can be approximated by

$$f_w = \frac{\rho a^2}{2\overline{C}_{11}} \qquad \qquad (50)$$

when a is the radius of the tube cross section. From this expression it is seen that the load-carrying capacity increases quadratically with the pressure and linearly with the cross section radius. Thus, if the pressure is doubled the cross section radius may decrease by a factor of four and the same load-carrying capacity is maintained. This is one of the principal advantages of the pressurized rib concept; that is, the use of higher pressure levels allows a significant decrease in the cross section size.

For the case of the concentrated load, it was shown that above some values of the geometry parameter the wrinkling load was independent of the geometry. Examination of equation (50) reveals that the wrinkling load is independent of the beam length as long as the approximation used in setting the terms in equations (48) and (49) to unity are valid.

CONCLUDING REMARKS

The development of a linear theory for the behavior of pressure stabilized beams under static load has been presented and solutions to the resulting governing equations obtained. Results, principally for the case of concentrated loading, are presented which illustrate the behavior of pressure stabilized beams under load and the manner in which that behavior is influenced by pressure level, beam geometry and material properties.

With regard to deformation behavior, the results show that the flexibility of pressure stabilized beams has a variation with pressure that is very nearly an inverse relationship

34

and a linear variation with the length to radius ratio except for small values of that ratio. The material properties enter in two ways, first through the pressure parameter which is the ratio of the axial stress due to pressure to the elastic modulus, and second through the modulus ratio, the ratio of the shear modulus to the elastic modulus. Thus, the elastic modulus has a direct bearing on the behavior by the manner in which it determines the pressure parameter. The results show that the influence of the shear modulus on the deformation behavior is significant only when modulus ratio is less than 0.2 and then only for low pressure levels and length to radius ratio.

The results show that the wrinkling load which is used to characterize the load-carrying capacity of pressure stabilized beams increases with pressure at a rate more rapidly than a linear relationship. The relationship between the wrinkling load and the length to radius ratio used to describe the geometry is that of a negative hyperbolic tangent. As a consequence, the wrinkling load is independent of the geometry for values of the length to radius ratio greater than 50. As with the flexibility, it is found that the shear modulus has very little influence on the wrinkling load. For the case of a uniformly loaded beam it is found that the wrinkling load varies quadratically with the pressure and linearly with the beam cross-section radius.

35

REFERENCES

1. A. E. Dietz, R. B. Proffitt, R. S. Chabot, E. L. Moak and C. J. Monego; Wind Tunnel Tests and Analyses for Ground Mounted, Air Supported Structures (Revised); U.S. Army Natick Laboratories Technical Report 70-7-GP; 1969.

2. A. E. Dietz, R. B. Proffitt, R. S. Chabot and E. L. Moak; Design Manual for Ground Mounted Air Supportes Structures (Single and Double Wall) (revised); U.S. Army Natick Laboratories Technical Report 69-59-GP; 1969.

3. P. Anderson and J. A. Wise; Neoprene-Loaded Air-Retaining Fabric, A Study of its Physical and Structural Behavior, Contract No. DA-49-009 Eng 490 (U.S. Army Engineering Research and Development Laboratory), University of Minnesota.

4. A. F. Foerster; Analytical and Experimental Investigation of Coated Metal Fabric Expandable Structures for Aerospace Applications. AF Flight Dynamics Laboratory Report No. ASD-TOR-63-542 (AD 427763,; 1963.

5. Anon.: Introduction to Structural Analysis of Expandable Structures; Goodyear Aircraft Corp. Report GER-9870, 1960.

6. Stein, Manuel and Hedgepeth, John M; Analysis of Partly Wrinkled Membranes; NASA-TN-D-813; 1961.

7. Webb, J.; Moment-Curvature Relationship of Inflated 'Mattress' Type Beams with Noniinear Material Properties; Cronfield Institute of Technology; Cranfield Memo No. 51; 1971.

8. Fichter, W. B.; A Theory for Inflated Thin-Wall Cylindrical Beams; NASA-TN-D-3466.

9. Courant, R. and Hilbert, D.; Methods of Mathematical Physics; Vol I; Interscience Publishers, Inc., New York; 1953.