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# Continuum Theory of Dislocations And Self-Stresses

By

E. Kröner

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Translated by

I. Raasch

and C.S. Hartley

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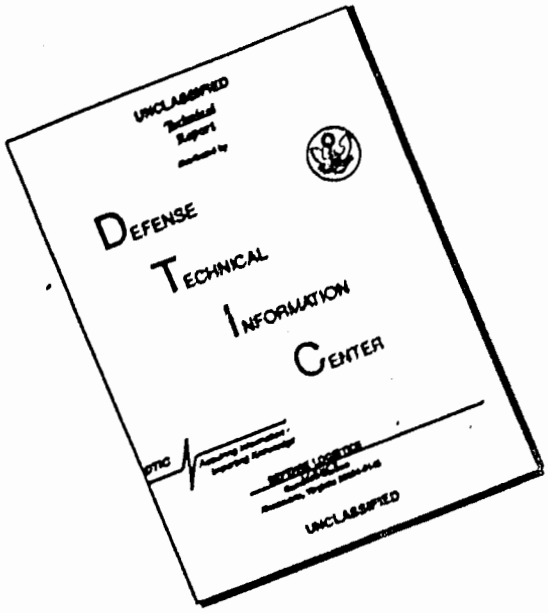


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The translation of any work from one language to another by others than the original author inevitably results in some changes in style, emphasis and occasionally in content. We wish to thank Professor Kröner for permission to distribute this document, however, the translation could not be performed in constant consultation with him. The translators take full responsibility for any inadvertent deviations from the intent of the original. We believe that the technical content is correct, and ask the author's indulgence in those cases where a difference of opinion may occur as to the fidelity with which the translation conveys the meaning of the original text.

I.R.
C.S.H.

## INTRODUCTION

Continuum Mechanics as it was originally developed by Navier, Cauchy, Poisson and Stokes among others, consisted of Elasticity Theory and Hydrodynamics. In the former, one is interested in the response, particularly the stresses, which will arise in a solid body which is subjected to "external" forces--perhaps torques also. In the latter, one analyzes principally the motion of fluids.

Relatively early, thermal stresses were included in the Elasticity Theory by Duhamel and Neumann. However, these have always retained a special position, as they were not subject to Kirchhoff's Uniqueness Theorem of Elasticity Theory, which states that in the domain of linear elasticity the stresses in a simple continuous body due to externally applied forces are uniquely determined. Kirchhoff's theorem is true under the assumption that St. Venant's Compatibility Conditions are fulfilled for the elastic deformation of the whole body. These conditions can not be applied directly to the case of thermal stresses, which explains their special position.

In the second half of the 19th Century, plastic phenomena in continua were investigated by Tresca, St. Venant, Levy and others. This "phenomenological" theory of plasticity, which was further developed later by von Mises, Prandtl, Reuss, Prager, Hencky, Nadai and others, stands to some extent between Elasticity Theory and Hydrodynamics. Thus the resulting deformation (we call it also total deformation $\epsilon^T$)

of the plastic body contains an elastic part, $\epsilon$, which as in the usual Elasticity Theory gives rise to stresses, and a second part, which we call plastic deformation, $\epsilon^P$, which changes the shape of the body but develops no stresses. One has such deformations in pure form in fluids. Therefore

$$\epsilon^T = \epsilon + \epsilon^P \tag{1}$$

Since at least a part of the internal stress remains without external forces after plastic deformation, the elastic strain evidently cannot fulfill the compatibility conditions. There one sees an analogy between thermal stresses and internal stresses after plastic deformation.

In principle, it is possible to scribe a volume element (for instance on the surface of the body) before carrying out the plastic deformation, and to measure the deformation which it has suffered with respect to the initial conditions. This gives the $\epsilon^T$. When the volume element is now cut out and allowed to relax, it takes on not its original shape but retains the plastic strain $\epsilon^P$. Now this element is found as at the beginning in its "natural" state, as it is used in Elasticity Theory by Cauchy, Green and others. The element has indeed changed its shape, but not its state.[1] A function that makes a statement

[1]This statement is rigorously true only when the plastic deformation results without (plastic) volume change. cf. §2.

about a body will be called a "State Function" or "State Quantity" if its values can be measured in an experiment at a certain time without

knowing the previous history of the body. Accordingly, the portion $\epsilon^P$ of the total strain is not a state function, whereas the elastic strain $\epsilon$ is one. The distinction between state functions and functions not changing the state is of great significance and will concern us more often.

It became obvious only in recent years that continuum mechanics with its three branches; Elasticity Theory, Plasticity Theory and Hydrodynamics, each in its existing range, is not sufficient to describe all macroscopically measurable mechanical properties of a body. A simple example may explain this: a beam with its ends fixed in two rigid walls be elastically or partly plastically bent as in Fig. 1. Then let both the walls remain in this position and let the rod become hot. As a result of heating the critical shear stress (defined as that shear stress at which a noticeable flow of the material starts) of the rod decreases, i.e., a flow in the interior of the rod can take place by the gradual replacement of elastic by plastic deformation. After holding the rod sufficiently long at the elevated temperature, the rod is again cooled to room temperature and the restraint at the ends removed. We then observe practically no bending back of the rod; the deformations have become permanent. We can cut out the volume element and find that no (macroscopic) internal stresses are present.[1] Nevertheless, the rod

---

[1]The bending of atomic planes mentioned later is combined with self stresses which change their sign in microscopic domains and hence cannot be found by the above mentioned cutting experiment. Like the macroscopic stresses, these self stresses lead to work hardening. (See below).

responds to subsequent deformation different from a rod which has the same shape "without history." If the critical shear stress of individual volume elements were now measured, it would be found that the body was in a definite work hardened condition. The change of state that has taken place can be characterized in another manner, which can be described more easily by the continuum theory. If the same rod is irradiated with X-rays, or if it is transparent to visible light, then diffraction effects are found that have their origin in the bending of the original atomic lattice planes of the rod. By this experiment the macroscopic curvature can be measured explicitly as a function of position. Therefore, this stress-free curvature of the atomic planes is characteristic of the state of the rod. In the previous continuum mechanics, such curvatures have nowhere been described.

In order to comprehend such geometric changes of the body, one must complete the three deformation tensors of eq. (1) by the addition of rotation tensors $\alpha^T$, $\alpha$, $\alpha^P$ to form the general asymmetric second-order tensors $\beta^T$, $\beta$, $\beta^P$ which we shall denote throughout as "Distortion Tensors."

Work hardening still cannot be correctly treated today in continuum mechanics. Investigations of the last 20 years have shown that it has its origin in the self stresses that develop during the plastic deformation of the material. Hence, a detailed knowledge of the self stresses should, in principle, allow a calculation of the hardening of the material. Furthermore, it was shown that all the self stresses as well as bending of atomic planes (therefore also the work hardening) can be traced back to the same physical entity, the dislocation. However, this is not only responsible for the change of state of the

body but also for that portion of its total deformation which does not alter the state of the body. Accordingly, a continuum theory of dislocations is evidently required in order to close the still wide gaps of continuum mechanics. In addition to the above features, this continuum theory of dislocations should include a theory of self stresses as well as a theory of stress-free bending of atomic planes, as was first formulated by Nye [113]. Also, it must describe the relationship between the dislocation motion and plastic deformation. In this way we are led to phenomenological plasticity theory. Thus one obtains, interrelated and overlapping, Elasicity Theory, Theory of Dislocations and Theory of Plasticity as branches of the comprehensive continuum mechanics, which treats all mechanical phenomena occurring in a solid body.

It remains yet to be said how one has to include in this continuum mechanics thermal stresses and other stresses (we refer to stresses due to magneto- and electrostriction) which arise neither from external forces nor from plastic deformation. On heating a body uniformly to a higher temperature, its material points undergo displacements, without introducing restoring forces. The same fact is also characteristic of plastic deformation. Since it is apparently natural to consider the case of deformation by temperature fields as a kind of plastic deformation, we will call it "quasi-plastic." One can then trace the thermal stresses back to certain "quasi-dislocations" and thus according to Kröner [82] a theory of thermal stresses is obtained, which is to some extent a continuum theory of "quasi-dislocations." This agreement is not only formal but also physically reasonable, and thus it appears quite natural to include thermal stresses (and the other above-mentioned stresses) in the Continuum Theory of dislocations.

The treatment of particularly interesting problems, for example, those in which there exist simultaneously thermal stresses and self stresses after plastic deformation, is made remarkably easy in this way.

The whole continuum mechanics of solid bodies is now contained in a few equations. For the stationary state these are the equations.[1]

---

[1]We consider that the boundary conditions are included in these equations by allowing $\underset{\sim}{F}$ and $\underset{\sim}{\alpha}$ to degenerate surfacewise (and also linewise or pointwise). If one allows external twisting moments also, further equations are added.

---

$$\operatorname{div} \underset{\sim}{\sigma} + \underset{\sim}{F} = 0, \qquad \operatorname{curl} \underset{\sim}{\beta} = \underset{\sim}{\alpha} \tag{2}$$

where $\underset{\sim}{\sigma}$ and $\underset{\sim}{\beta}$ are the stress tensor and the elastic distortion tensor; $\underset{\sim}{F}$ and $\underset{\sim}{\alpha}$, the density of external forces and of dislocations (including quasi-dislocations) respectively. To this are added the equation for the elastic energy density (= strain energy function or elastic potential)[2]

---

[2]We use throughout the text the summation convention where we sum from 1 to 3 over repeated indices. The use of tensor notation is explained in the appendix.

---

$$e = \frac{1}{2} \sigma_{ij} \epsilon_{ij} \tag{3}$$

and the constitutive equations, for instance, Hooke's Law, in the case of small deformations. Under these circumstances the uniqueness theorem

of continuum mechanics of solid bodies can be proved: by specifying the external forces $\underset{\sim}{F}$ and dislocations $\underset{\sim}{\alpha}$ the stresses and elastic distortions of the body are uniquely determined. From this it follows immediately that all self stresses arise from dislocations. However, in the case of larger deformations this is not true, as the example of the invertible hemispherical shell shows [160].

In the beginning of the twenties, the interest of the solid state physicist was concentrated on the crystalline structure which most of our materials, particularly the metals exhibit. These are composed of crystals (polycrystalline) in regions of an average diameter of at least $10^{-3}$ cm ( corresponding to nearly $10^{15}$ atoms). With methods which were developed then and subsequently greatly improved, it is possible to grow a "single crystal" of almost any size of many materials today. Although these are of great significance for experimental and theoretical research, however, they have also found important applications in industry, e.g., in transistors of communication engineering.

The concept of the dislocation was used first in 1928 by Prandtl [108]--still in vague form--to explain anelastic phenomena in metals. In 1929 Dehlinger [29] was able to show by studies of recrystallization, i.e., the formation of new grains which one observes after heavy plastic deformation and which has its origin in the large self stresses developed thereby, that these self stresses are to be traced back to certain defective zones of the otherwise fully regular arrangement of atoms in the crystal, and that these regions can be metastable. Dehlinger named his self stress sources "interlocks"

(Verhakungen) which are nothing but two dislocations of opposite sign lying close together. It was thereby explained why self stresses are generally possible at all in a crystalline medium. As a result of these investigations, attention was particularly directed to disturbances of the regular atomic arrangement. One calls such disturbances "lattice defects"; they play a decisive role in modern solid state physics.

In 1934, the lattice defect which we illustrate with aid of Figs. 2 and 3 was described independently by Orowan [114], Polanyi [118] and Taylor [149]. Figure 2 shows a completely regular crystal, called "Ideal Crystal." Figure 3 shows the same crystal after the invasion of a disturbance from the $x_1$ direction. The disturbance is characterized by the fact that one of the lattice planes terminates in the interior of the crystal. Today, the line of termination of such an extra lattice plane is called an "edge dislocation line" or simply "edge dislocation." Figure 4 shows the same crystals after the disturbance is no longer in the crystal. By the movement of one dislocation through the crystal, the upper and lower halves of the crystal have been displaced relative to each other by one interatomic distance. The vector which specifies the relative displacement in the slip plane is called "the slip vector," $\underset{\sim}{g}$. It is perpendicular to the edge dislocation line. If there was a shear stress applied to the crystal, work was done by passing the dislocation through the crystal. Consequently, such a shear stress constitutes a driving force for dislocation motion. The above mentioned authors now noticed that the movement of an edge dislocation must be

possible under the influence of relatively small stresses. Figure 3 gives a certain optical impression that near the dislocation the adjoining atoms should be more easily movable than the rest.

Already in 1926, Frenkel with the help of an atomic model had computed that slip which produces the transition of the crystal of Fig. 2 to that of Fig. 4 requires a shear stress of the order of magnitude of the shear modulus, $\mu$, if both of the moving lattice planes slip rigidly by an interatomic distance. Experimentally, a critical shear stress more than a thousandfold smaller is measured. The plasticity mechanism proposed by Orowan, Polanyi and Taylor should lead to actually a smaller critical shear stress.[1]

---

[1]According to Dehlinger [31], these purely mechanical considerations are not sufficient to prove that the rigid glide of two adjacent lattice planes cannot actually take place. Therefore, statistical thermodynamic considerations, particularly the theorem that in a solid body only processes of lowest order can take place, must be invoked. Applied to our case it says: it is extremely unlikely that by temperature fluctuations those atoms of a lattice plane simultaneously have such an increased energy that they make a simultaneous slip step, which would be equal to a rigid gliding of the lattice plane in question. Such considerations are essential if we should want to calculate the theoretical critical shear stress under the assumption of the dislocation mechanism. Seeger [137] has shown that the critical shear stress, neglecting temperature fluctuations, which one would calculate purely mechanically, often comes out to more than 100 percent too large.

(Footnote cont'd.)

Because of its importance for such problems, we should mention a new work of Donth [164], who has shown that in a statistical treatment of dislocations one should proceed from Kolmogoroff's equations for statistical processes, since the assumptions for the application of an Arrhenius equation are not satisfied in the case of dislocation.

---

Burgers [12] in 1939 has described an additional lattice defect which causes the original lattice planes now connected in the manner of a screw surface (Fig. 5). The screw axis is called a "screw dislocation (line)." One sees that these screw dislocations also will be relatively easily movable. One can imagine that the screw dislocation in Fig. 5 invaded from the $x_1$ direction. Figs. 6 and 7 show the crystal after the movement of the screw dislocation of Fig. 5 in $x_1$ and $x_3$ directions, respectively. Here again certain crystal parts are displaced relative to each other. But the slip vector here is parallel to the screw dislocation line. Burgers has further shown that there are also dislocations whose slip vector is inclined to the line direction of the dislocation. Such dislocations are appropriately considered as the superposition of one screw and one edge dislocation along the same line, so that such dislocations are not fundamentally new.

However, the possibilities of motion of the dislocation have not yet been completely discussed. There is still the important possibility to consider a motion of the dislocation of Fig. 3 in $x_3$ direction. This means an enlargement of the extra lattice plane, which is possible in practice only if atoms from the neighborhood of the dislocation are added by diffusion. The migration of an atom in a crystal

always occurs over an energy barrier of the order of 1 ev (= 1.63 x $10^{-18}$ kg-m) which cannot be overcome by external applied stresses.[1]

---

[1]Macroscopically 1 eV is a very small energy. However, this must be localized in a space of only some $10^{-24}$ $cm^3$, and this is obviously not possible from externally applied stresses.

---

Rather the temperature fluctuations must make the necessary "activation energy" available. Consequently such a diffusion can take place to a large extent only at elevated temperature. The dislocation motion taking place in this way is called "climb" in contrast to the "slip" described above. Each atom which attaches itself to the extra lattice plane leaves behind a so-called "vacancy." These vacancies are to be counted in the volume we can measure macroscopically, i.e., by the climb of dislocations the volume of the body changes; this kind of motion, after Nabarro [108], is called, therefore, "non-conservative" (with respect to the volume), whereas the glide motion is called "conservative."

If the dislocation were to climb, for example, in the $x_3$ direction completely through the crystal of Fig. 2, this would require that a new lattice plane be formed and causing the crystal to be elongated in the $x_3$ direction. Accordingly, a pure tensile stress ($\sigma_{11} > 0$) should exercise on the dislocation a restraint to climb in $x_3$ direction. A compressive stress, however, may remove the extra lattice plane; however, this is possible only until all the vacancies in the neighborhood of the dislocation are filled up with the atoms

of the extra lattice plane. We see that the volume of a body can even become plastically changed, a possibility which will be included in the theory developed in Chapter I.

The climb of the dislocation plays an important role in many processes in a solid body at temperatures just below the melting temperature, e.g., recrystallization and the formation of casting stresses.

A look at Figs. 2 through 7 shows that we should expect self stresses in the states of Figs. 3 and 5, whereas the crystals in the remaining figures are in the natural state.[1]

---

[1]Strictly speaking, for instance, the state in Figs. 2 and 6 differ from Fig. 3 by the fact that in consequence of the ledge developed the crystal has a changed surface. It is not necessary to consider this for our purpose. See, e.g., the discussion of Nabarro [110], pg. 330.

---

We will show in §1 the close relationship of the states of these self stresses with those of Volterra distortions. Based on the works of Volterra, in 1939 Burgers [13] developed an elasticity theory for a single dislocation in a continuum, from which the self stresses resulting from dislocations could be calculated. This fundamental work was followed by numerous special elasticity theory calculations on dislocations.

Hereafter the following picture of the process of plastic deformation in a metal can be given: under the influence of the externally applied stresses, large amounts of new dislocations are

developed in addition to the dislocations always existing in the crystal.[1] These move according to the forces exerted whereby they produce the macroscopically observed deformations. However, the

---

[1]These originate during the growth of the crystals, which makes them possible in general.

---

increasing number of other dislocations produce self stresses in increasing amounts, which oppose the motion of dislocations, as was first proposed by Taylor [149]. This effect leads to the work hardening of materials.

## CHAPTER I

## DISLOCATIONS IN CONTINUA: GEOMETRY

### §1. Dislocation and Volterra Distortion

In the beginning of this chapter, the close relationship between dislocation and Volterra distortion will be clarified.

Let f be a plane surface ending, at least partly, in the interior of a simply connected continuous medium with (dimensionless) unit normal vector $\bar{\underset{\sim}{n}}(\underset{\sim}{x})$ at position $\underset{\sim}{x}$.[1] Let $\bar{\underset{\sim}{t}}(\underset{\sim}{x})$ be the unit tangent vector of the edge lines of f which are oriented according to the right hand screw rule. We imagine that the stress free initial state of the body under consideration is cut along the surface f, then the positive cut edge of f suffers the infinitesimal plastic displacement $\delta\bar{\underset{\sim}{g}}(\underset{\sim}{x})$ relative to the negative. The displacement $\delta\bar{\underset{\sim}{g}}$ will be carried out in two steps, since the analysis will be in terms of the two components parallel ($\delta\bar{\underset{\sim}{g}}||$) and perpendicular ($\delta\bar{\underset{\sim}{g}}\perp$) to f. After the parallel displacement $\delta\bar{g}||$ both the sides of the surface f are still in touch with each other.

---

[1]The restriction of a plane surface facilitates description, but is not necessary. We see easily that the essential results of this paragraph, particularly the definition of the dislocation are valid also in the case of curved surfaces.

---

For the latter displacement, $\delta\bar{g}_{\perp}$, perpendicular to f, we have two cases to distinguish: (1) Both the sides of f are pushed apart; for this case, we decide that the resulting void is filled with matter identical to the rest of the body. (2) Cases in which $\delta\bar{g}_{\perp}$ signifies a displacement of both the cut edges toward one another; just enough material shall be removed from one of the two cut edges that this displacement becomes possible. After the execution of the operations, we imagine everything to have coalesced and the forces which produced the displacements to have been removed, so that again a united simple coherent body exists, in which, naturally, self stresses remain. Irrespective of the material and shape of the body, these are determined by the position of the surface f, i.e., by $\bar{\underset{\sim}{n}}$, as well as by the resulting "impressed" or "plastic" relative displacement $\delta\bar{g}$.

We remark further after a well-known theorem of Colonnetti [18] that the volume of the body in the final state differs from that in the initial state by the volume of the added or removed material, thus by $\delta V = \iint \underset{\sim}{n} \cdot \delta\underset{\sim}{g}\, df$. This theorem holds only in the domain of linear elasticity theory and there also only for homogeneous bodies (thus, for example, not for bodies which consist of two homogeneous parts with different elastic constants). Along the surface f the elastic deformations and torsions of the volume element of the body are changing discontinuously, which was first investigated by Weingarten [157] and later in detail by Somigliana [147].[1] However, if both cut edges of

[1]The older results of the self stress theory have been reported by Nemanyi [111]. This work contains also many selections valuable even today.

a surface element, $\Delta f$, of f have merely suffered a rigid displacement, then the strains are continuous across $\Delta f$; the rotation is also continuous across $\Delta f$. It is additionally necessary that $\delta\bar{g}$ = constant on $\Delta f$.

At the end of the operation, the body shall be again simply connected, with no cracks, thus the cut edges of the total surface f cannot be rigidly displaced toward one another. Dislocations are formed by the following process. Let $\delta\bar{g}$ be constant on nearly the total surface f, except at the edge of f let it decrease very rapidly to zero. Figure 8 shows the variation of $\delta\bar{g}$ on a plane surface f assumed circular for simplicity. We now define a dislocation line as the boundary of the surface f, or more precisely, the dyadic product $-\underset{\sim}{t}\delta\underset{\sim}{\bar{g}} = -(t_i \delta \underset{\sim}{g}_j)$, where by $\delta\bar{g}$ shall be meant the constant displacement on most of the surface.[1]

---

[1]The minus sign is conventional, in conformity with the usually employed sign convention of Frank [47].

---

To say it more precisely: there is no singular line $-\underset{\sim}{t}\delta\underset{\sim}{g}$, but a quasi-singular band of very small width $2\zeta$ (Fig. 8). Hence we complete the above definition by adding that this shall be valid for the limit $\zeta \to 0$.[2]

---

[2]Then the function $\delta\bar{g}$ assumes the character of a Heaviside step function in the plane in which f lies.

---

A second prescription leads to Volterra distortions. It is required that initially (or at the latest after the cutting along f) the

boundary of f is surrounded by a hollow torus of radius $> \zeta$. Then the body is of course no longer simply connected, and the surface f is bounded everywhere by the surface of the body. Due to this, a rigid displacement of the cut edge of the total surface f is possible. If we set $\delta\bar{g}$ = constant, then we obtain a so-called Volterra distortion state of the first kind, so at large distances we cannot distinguish between the hollow torus and the state developed through a dislocation (Principle of St. Venant). Burgers' investigations on the elasticity theory of dislocations are based on this conclusion.

We shall talk in §7 about the Volterra distortion state of the second kind, in which the rigid relative displacement is a rigid rotation of the cut edge. From our standpoint this is not as important as the state of $\delta\bar{g}$ = constant.

From the definition of the dislocation it follows that:

1. The dislocation, as the boundary of a surface, can only end on the surface of the body.

2. Since the strain and rotation remain continuous across the surface f, it can no longer be found experimentally after the formation of the dislocation line. Thus all surfaces bounded by $\underset{\sim}{t}$ could have served as cut surfaces in order to produce the dislocation or the distortion state, i.e., it is completely determined by the edge line $\underset{\sim}{t}$, and the relative displacement $\delta\bar{g}$.

Assume that a stress resulting from external forces was present in the body during the operation of the relative displacement. Then this stress could do work during the displacement. Consequently, stresses exert forces in the body in the sense of the production and

propagation of dislocations. In particular, if a shear stress acts on a plane, then it creates a tendency for conservative formation and propagation of the dislocation (i.e., $\delta\bar{g}\,||$ to the surface), whereas a normal stress perpendicular to the surface means a tendency for nonconservative formation and propagation of the dislocation ($\delta g \perp$ to the surface). Whether such processes will actually be induced through the application of external stresses alone on the body will depend on the cohesion forces of the material. Especially then, for the nonconservative formation and propagation of a dislocation, diffusion of matter will be necessary. In the introduction it was pointed out that these processes are the fundamental mechanisms of plastic deformation in actual bodies. Therefore, we assume this also for our ideal continuum.

With reference to the explanations in the introduction, we denote the conservative propagation of a dislocation as glide and the associated surface f as the slip plane. The nonconservative propagation we call climb, the associate surface f, the climb plane. In general, we speak also of the motion of the dislocation along its motion surface. We say further that a dislocation has an edge character where $\underset{\sim}{t} \perp \delta\underset{\sim}{g}$ and screw character where $\underset{\sim}{t} \,||\, \delta\underset{\sim}{g}$. Where $\underset{\sim}{t}$ is inclined to $\delta\underset{\sim}{g}$, it has mixed character. Figures 9 and 10 show the formation of a pure edge and screw dislocation. Obviously the dislocation formed purely nonconservatively is an edge dislocation. This corresponds to the statement in the introduction that only edge dislocations climb in the crystal. The conservatively formed dislocation has, in general, mixed character. This corresponds to the result that not only screws but also edge dislocations glide in the crystal.

These ideas clearly show that the dislocation concept employed here is nothing but a transfer of the dislocation concepts from the crystal to the continuum.

## §2. Plastic and Elastic Distortions

First of all, a remark on the ideal continuum. Let it be assumed for simplicity that in the beginning it is in a homogeneous state. On the other hand, it would be a limitation of fundamental importance if we assume isotropy also. Here we do not mean elastic isotropy, this is completely unimportant for the geometric analysis of this section. On the contrary, the possibility that the medium is geometrically anisotropic must be considered. This means that at every point of the medium, three linearly independent, distinct directions exist about which it is assumed that their angles with three normal directions in space can somehow be measured. This geometric structure must therefore be demanded, since the real bodies to which the continuum theory shall hereafter be applied, have this structure. We show it, for example, by means of X-ray techniques.

We assume that this structure is a property of the individual volume elements in the continuum. The stress free state of the medium, in which the distinct directions of all volume elements are parallel to one another, is defined as the initial state. In the final state, we have then a certain orientation distribution which gives proof of the rotations of the volume elements that have taken place (see below). For simplicity, we assume that the distinct directions in the initial state are orthogonal to one another. The reader used to think of crystals

may picture to himself the continuum somewhat like a primitive cubic crystal with vanishingly small lattice constants.

We can now allow the operations described in the last paragraph to take place at very many surfaces f. When these become infinitely dense and the proper relative displacements $\delta\bar{g}$ are continuously distributed, we can perform in this way continuously distributed pure plastic or also mixed plastic-elastic deformations of the body. The first process may be illustrated by Fig. 11. This shows an isolated volume element dV in the initial state (a). This will be cut along surfaces df at distances $dx_2$ perpendicular to $x_2$-direction and afterwards a relative displacement $\delta g$ imposed on every two neighboring layers. We imagine the passage to the limit $\delta x_2 \to 0$, $\delta\bar{g} \to 0$ carried out maintaining $\delta \underset{\sim}{g}/\delta x_2$ constant. In the case of Fig. 11(b), the voids shall be filled with matter of the same volume element in such a way that the density distribution in it remains homogeneous. At the end let all coalesce again. The volume element in Fig. 11(b) is then completely homogeneously plastically stretched (and thereby "thinned") and in Fig. 11(c) and 11(d) homogeneously plastically sheared.

We generally denote by $dg_j$ the relative plastic displacement of the boundary surface of the volume element on the $+x_i$ side with respect to that one on the $-x_i$ side and define the asymmetric tensor of the plastic distortion $\underset{\sim}{\beta}^P = (\underset{\sim}{\beta}^P_{ij})$ through the relation

$$dg_j = \beta^P_{ij}\, dx_i \tag{I.1}$$

where $dx_i$ shall be referred to the relative position of the mentioned boundary surfaces and to the original state. The plastic distortions

which correspond to Fig. 11(b) to 11(d) are accordingly to be denoted by $\beta^p_{22}$, $\beta^p_{23}$ and $\beta^p_{21}$, respectively. The diagonal components of the plastic distortion tensor $\beta^p_{ij}$ are thus plastic elongations, the remaining components are plastic shears whereby the first index indicates the glide planes and the second the glide direction.

It is now particularly important to remark that in the case of plastic distortion, the orientation of the volume element is not changed. We conclude this from the way the distortion in Fig. 11 comes about.[1]

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[1]One may conceive the volume element of Fig. 11(a) somewhat like an infinitely densely packed band of material lines, which lie parallel to the $x_2$-direction. The operations which Fig. 11(b) to 11(d) convey obviously do not alter the direction of these lines. We can also simply postulate the preservation of the orientation, because the real bodies for which the theory is later applied, show this property.

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Hence, the distinction between the shears $\beta^p_{21}$ and $\beta^p_{12}$ is not a rigid rotation but a "plastic rotation" of the volume element while preserving its orientation (Fig. 12(a),(b)). This statement holds for small distortions. Then the symmetric part of $\beta^p_{ij}$ describes a pure plastic strain $\epsilon^p_{ij}$ and the antisymmetric part a pure plastic rotation $\omega^p_{ij}$ both maintaining the orientation. Also for large distortions, the division of distortions into strain and rotation[2] holds

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[2]The additive combination of deformation and rotation holds in the case of large distortions only if $dx_i$ in eq. (1.1) is referred to the initial state. Cf. §10.

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$$\beta^p_{ij} = \epsilon^p_{ij} + \omega^p_{ij} \tag{I.2}$$

Here, however, we have to understand by $\omega^p_{ij}$ the well-known asymmetric tensor for large rotations (Versor) [34].[1] On the other hand, $\epsilon^p_{ij}$ remains symmetric. See also §10.

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[1]Volume 1, p. 78.

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The deformations of the volume element considered hitherto took place without stress. Now we come to the case of the elastic deformation. Let $da_j$ be the elastic relative displacement of the boundary surfaces as before. Then we define the asymmetric tensor of the elastic distortion $\underset{\sim}{\beta} = (\beta_{ij})$ by the equation

$$da_j = \beta_{ij}\, dx_i \tag{I.3}$$

The $\beta_{ij}$ describe the same change of shape and position of the volume element as the $\beta^p_{ij}$, however, an essential geometric difference exists: in the case of the elastic shear, the original right angle beteeen the indicated directions is changed by the shear angle. Consequently, the difference between $\beta_{21}$ and $\beta_{12}$ in the case of smaller distortions is now a rigid rotation of the volume element (Fig. 12(c), (d)). We split $\beta_{ij}$ again into its symmetric and antisymmetric parts

$$\beta_{ij} = \epsilon_{ij} + \omega_{ij}, \tag{I.4}$$

thus $\epsilon_{ij}$ is the ordinary strain tensor of the elasticity theory and $\omega_{ij}$ the tensor which describes the rigid rotation of the volume element. The same holds as before in the case of larger distortions.

There is no basic difficulty in measuring the elastic deformation of a volume element in the final state when we cut it out and let it relax.

Thereafter, its principal directions are again orthogonal to one another and we can measure in addition the orientation of the elements with respect to a normal orientation. If we carry this out for all the elements, we can specify the rotations that took place up to a constant rotation common to all of the elements. This means that the elastic strain is a state function, whereas the same is not true for the rotations but is true for their local derivatives. These describe clearly a bending of the structure. Since the elastic deformations and structure curvatures follow uniquely from the elastic distortion tensor, this characterizes the state of the medium after the deformation. But it is impossible to measure only from the final state the plastic distortions, strains and rotations that took place. This is due to the fact that by a pure plastic distortion, as in Fig. 11, the state of a volume element is not changed. See also the introduction.

In the general case, a volume element will be simultaneously plastically and elastically distorted. Let

$$ds_j^T = da_j + dg_j \tag{I.5}$$

be the total relative displacement of the boundary surfaces of the volume element as before. Then we define the tensor of the total distortion $\underset{\sim}{\beta}^T = (\beta_{ij}^T)$ by the equation

$$ds_{ij}^T = \beta_{ij}^T \, dx_i . \tag{I.6}$$

To begin with, it is sufficiently characterized by the relation

$$\beta_{ij}^T = \beta_{ij} + \beta_{ij}^P \tag{I.7}$$

Equation (I.7) is also correct for larger distortions when one refers $dx_i$ always to the initial state (§10).

## §3. The Geometric Principal Equation of the Continuum Mechanics of Rigid Bodies

We describe in the following a thought experiment, which is taken as a basic experiment in the continuum theory of dislocations.

If we apply a sufficiently large external stress to a plastic medium, it is possible that dislocations are developed, move, and produce plastic distortions of the body's volume elements. It is possible, for instance, that these dislocations leave the body or that dislocations with opposite sign cancel each other in the interior of the body, or that dislocations come to rest after moving in the matter and produce a dislocation density. We assume that these dislocations come to rest not in the interior but between the volume elements. As the size of the volume elements is expected to become zero, we get at least a macroscopically continuous distribution function of dislocations, if the exterior stress is continuous. It is usually assumed in continuum mechanics that the distortions are essentially homogeneous over many volume elements dV, which means that the dislocations move in straight lines in such a region.

As a result of the stresses which are applied to the body, each volume element will experience a certain dislocation motion, and this we will evaluate as a function of the position of the volume element, for instance, relative to the initial conditions (§4). Now we imagine that the body in the initial condition is cut into its volume elements and the dislocation motion associated with the element is carried out in each element independently of the other elements. In other words, we impress on each element a plastic distortion, $\beta^P(\underset{\sim}{x})$. In any case the elements are without stress afterwards and also their orientations are preserved. Now there are two possibilities.

1. The volume elements fit together completely after accomplishing the plastic distortion, and there are no cracks. Then we can imagine that they can coalesce without restraint, and we get the body in the shape that it would be if we had not cut it before the dislocation motion. Especially, the body is without self stress and structure curvature.[1] So the state of the body is not changed but the shape is.

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[1]Experiments of this type in which only the strain (not the rotation) considered, are described often in the literature. See Fopp [44], Reissner [122], von Laue [87].

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2. The volume elements do not fit together after the deformation. Figure 13 shows an example in which the connection of the elements is destroyed, since dislocation, which moved in from top left and whose lines are perpendicular to the plane of the paper, came to rest between the volume elements in such a way that the upper elements have been traversed by more dislocations and therefore are more distorted than the lower ones; meanwhile for the same reason, the elements to the left are more strained than those to the right. If we then try to combine the elements to a compact body, we have to distort them elastically in such a way that they fit together completely. For this we have to use elastic distortions and rotations in general. The former produce stresses, the latter will rotate the orientation. Now we imagine that all is coalesced again, and the forces which produced the elastic strain are removed. In general, then, a relaxation of the body will occur to the state of lowest possible elastic energy. However, the stresses vanish completely

only if rigid rotations would be sufficient to restore the connection which was disturbed by the plastic distortion. Finally, of course, we have the same state that we would have if we did not cut the body before the dislocation motion.

This thought experiment has to be evaluated quantitatively. Both possibilities have in common that the body should be compact and without cracks in the final state; i.e., however, that the total distortion $\underset{\sim}{\beta}^T$ is a function of the position, such that the connection of the volume elements is maintained. This requirement restricts the admissible functions $\underset{\sim}{\beta}^T$ by which the function $\underset{\sim}{\beta}^P$ is also governed in the first case, but not in the second case.

We will show now, that

$$(\epsilon_{ijk} \frac{\partial}{\partial x_j} \beta^T_{kl})^{\mathrm{T}} \mathrm{Curl}\ \underset{\sim}{\beta}^T = 0 \qquad (I.8)$$

is a necessary condition in order that the connection between the volume elements is not changed. Figure 14a shows two elements in the initial state. The connection between the two is maintained if the right boundary surface of the left and the left boundary surface of the right suffers the same displacement. This means that the component $\beta^T_{2j}$ and $\beta^T_{3j}$ have to be the same in both elements, while the components $\beta^T_{1j}$ are allowed to change. Figure 14(b) and (c) shows an example of what it looks like if the elements suffer a different distortion $\beta^T_{21}$ and $\beta^T_{12}$. Hence, it is necessary to maintain the connection that $\partial\beta_{2j}/\partial x_1 = \partial\beta_{3j}/\partial x_1 = 0$.[1] From this we conclude the necessary condition

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[1]It is also sufficient if we assume a continuous total displacement. We will no longer be concerned about this. A non-vanishing

(Footnote contd.)

function Curl $\beta^T$ could be called a crack density. Such things will occur if we take too large a pass during rolling a metal.

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(1.8) at once. Then obviously, $ds_j^T$ in eq. (1.6) is a complete differential, i.e., a function $\underset{\sim}{s}^T$ exists, which measures the total displacement (except for a rigid translation) of the points of the body. So it is

$$\beta_{ij}^T = \partial s_j^T/\partial x_i \equiv (\text{Grad } \underset{\sim}{s}^T)_{ij} \tag{I.9}$$

In the case (1) described above $\underset{\sim}{\beta}^P = \underset{\sim}{\beta}^T$ for $\beta = 0$, so $dg_j$ in eq. (I.1) is a complete integral and

$$\underset{\sim}{\beta}^P = \text{Grad } \underset{\sim}{g} \equiv \text{Grad } \underset{\sim}{s}^P \tag{I.10}$$

In this case we get a pure plastic displacement $\underset{\sim}{s}^P$ of the points of the continuum, by which its state is not changed. This case is of great practical importance for plastic deformation. We will refer to it later on.

Now we define conditionally the asymmetric tensor of the dislocation density $\underset{\sim}{\alpha} \equiv (\alpha_{ij})$ by the expression

$$\underset{\sim}{\alpha} \equiv - \text{Curl } \underset{\sim}{\beta}^P \tag{I.11}$$

and we will show in the next step, that this definition is the same as that of the single dislocation.

Figure 15 shows a body into which a small number of dislocations invaded, of which we assume that they are perpendicular to the plane of the paper. The surfaces of motion of the dislocations are drawn in such a way that they were drawn straight, if the dislocation, which is placed

at the end of a surface of motion, cuts the plane F, which has an arbitrary boundary C. Otherwise they are dotted. The motion surfaces are oriented in such a way that they were cut in the positive direction by C. Now we go along C and add at each motion surface the relative displacement $\delta \underset{\sim}{g}$, resulting from the dislocation motion, considering both the positive and negative side of the surfaces. For simplicity we assume, that $\delta \underset{\sim}{g}$ is the same for all motion planes.[1] As we can see

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[1]This does not mean a restriction of the generality of our consideration, as will be shown in the following calculation.

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at once, the dotted planes do not contribute anything to the sum, as they give two opposite equal values. That is why

$$\underset{\sim}{b} \equiv - \sum_{C} \delta \underset{\sim}{g} \tag{I.12}$$

is a direct measure for the number and kind of the dislocations cut by the plane F. We call $\underset{\sim}{b}$ the "Total Burgers Vector" of those dislocations.[2] In the case that the boundary C encloses only one dislocation

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[2]In honor of J. M. Burgers, who introduced the circuit vector $\underset{\sim}{b}$ to specify a dislocation in a basic paper [12].

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then $\underset{\sim}{b} = - \delta \underset{\sim}{g}$ is the Burgers Vector at this single dislocation.

We showed in §1 that the state of the matter with one dislocation is completely determined by the specification of the curve $\underset{\sim}{t}$ and glide vector $\delta \underset{\sim}{g}$. Now we will see that instead of the glide vector we can use the Burgers vector. Note the important distinction between these

vectors: $\delta \underset{\sim}{g}$ states that due to a dislocation, which moves along a plane, the positive side of the plane is displaced with respect to the negative side by $\delta \underset{\sim}{g}$. However, where no dislocation has moved along $\delta \underset{\sim}{g} = 0$. $\delta \underset{\sim}{g}$ is therefore a vector which is bound to the motion plane and especially preserves its sense in the case that the dislocation has moved out of the matter, i.e., if it no longer exists. Whereas it is only possible to define $\underset{\sim}{b}$ in connection with the curve C and the bounded surface F, respectively, and it tells us something about the distribution of the dislocations in the body.

When the distribution of the dislocations is sufficiently dense, we are allowed to substitute the summation in eq. (I.12) by the integral

$$\underset{\sim}{b} \equiv - \oint_C \delta \underset{\sim}{g} \tag{I.13}$$

If we have infinitesimal areas, $\Delta F$,[1] we call the resulting Burgers

---

[1] $\Delta F$ must be much larger than dF in order to define a dislocation density through $\Delta F$. If first dF goes to zero, then after this you can take the limit for $\Delta F \to 0$.

---

vector $\Delta b$. If we know this for every arbitrarily oriented surface element at each position in the medium, then obviously we know how many dislocations of each kind are at each point of the medium. That is why the expression

$$\alpha_{ij} = \Delta b_j / \Delta F_i \tag{I.14}$$

is defined to be the "tensor of the dislocation density" or (shorter) the "dislocation tensor." As the dislocation density is a tensor field, it is sufficient to know the Burgers vector of three plane elements,

which are oriented in the same manner as the cartesian coordinate axis. If we measure, e.g., the Burgers vector of the plane $\Delta F$ having only the component $\Delta B$, and we have no values for the planes $\Delta F_2$ and $\Delta F_3$, then it is obvious that the line direction and the Burgers vector are parallel; according to §1 the diagonal components of $\alpha_{ij}$ are screw dislocations in the i (=j) direction. Similarly, we will notice that the other components of $\alpha_{ij}$ are edge dislocations in the i-direction with the Burgers vector in the j-direction. In short, the first index of $\alpha_{ij}$ indicates the line direction, the second one the direction of the Burgers vector. We call the total Burgers vector of all dislocations cutting an arbitrary plane F the dislocation flux through F. From eq. (I.14) evidently it is defined as

$$\underset{\sim}{b} = \iint_F d\underset{\sim}{F} \cdot \underset{\sim}{\alpha} \tag{I.15}$$

On the other hand, we can calculate it by eq. (I.13) to be

$$\underset{\sim}{b} = -\oint_C \delta \underset{\sim}{g} = -\oint_C d\underset{\sim}{g} = -\oint_C d\underset{\sim}{x}\, \beta^P = -\iint_F d\underset{\sim}{F}\ \mathrm{Curl}\ \underset{\sim}{\beta}^P \tag{I.16}$$

In this we use the fact that if we integrate $\int \delta \underset{\sim}{g}$ along $dx_i$, $d\underset{\sim}{g}$ results of course (Fig. 11); this is replaced by eq. (I.1) and finally, Stokes' theorem is used. Since the surface F was arbitrary, we conclude eq. (I.11) directly by comparing with eq. (I.15).

Hence, from eq. (I.7), (I.8) and (I.11) follows immediately the "geometrical basic equation of continuum mechanics."[1]

$$\mathrm{Curl}\ \underset{\sim}{\beta} = \underset{\sim}{\alpha} \tag{I.17}$$

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[1]Equation (I.17) or equivalent formulations were independently given by Kondo [73,74], Bilby, Bullough and Smith [3,4,5] and Kröner

(Footnote contd.)

[81,82,84]. The first of the authors mentioned used from the beginning formulations which hold for large distortions (§26 to §28), whereas the present author introduces those distortions later on. The derivation given here was carried out by Kröner and Rieder. Equation (I.17) reads in cartesian coordinates

$$\partial\beta_{31}/\partial x_2 - \partial\beta_{21}/\partial x_3 = \alpha_{11}, \quad \partial\beta_{32}/\partial x_2 - \partial\beta_{22}/\partial x_3 = \alpha_{12}$$

$$\partial\beta_{33}/\partial x_2 - \partial\beta_{23}/\partial x_3 = \alpha_{13}, \text{ etc.}$$

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According to what we mentioned before, it is understood as follows. If dislocation motion or a plastic distortion $\underset{\sim}{\beta}^P$, respectively, occurs in such a way that dislocations with density $\underset{\sim}{\alpha}$ come to rest in the medium, then the distortion $\underset{\sim}{\beta}^P$, if it occurs alone, would destroy the connection of the body. Since the cohesion forces of the medium oppose this, elastic distortions develop simultaneously in such a way that the body remains intact. Equation (I.17) holds also for large distortions, if we refer $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}$ to the initial state and we also differentiate in the initial state. See §10.

Hence, from eq. (I.17) follows the relation first mentioned by Nye [13]

$$(\partial\alpha_{ij}/\partial x_i) \equiv \operatorname{div} \underset{\sim}{\alpha} = 0 \tag{I.18}$$

As the first index of $\alpha_{ij}$ indicates the line direction of the dislocation, obviously this equation means nothing more than the fact that dislocation lines are not allowed to terminate in the interior of a medium. This we emphasized in §1.

From eq. (1.5) follows, since $ds^T$ is a complete differential, the relation $\oint d\underset{\sim}{a} = -\oint d\underset{\sim}{g}$ for an arbitrary closed curve. Hence, it follows from eq. (1.13) also

$$\underset{\sim}{b} = \oint_C d\underset{\sim}{a} = \oint_C d\underset{\sim}{x} \cdot \underset{\sim}{\beta} \tag{I.19}$$

In this form Burgers had introduced the circuit vector $\underset{\sim}{b}$.

Finally, we will mention another analogy, which is related to the theory for the magnetic field of a stationary current and which was extremely useful in finding the geometrical basic equation [81]. The quantities by analogy are: electrical intensity of current i, and Burgers vector $\underset{\sim}{b}$; current density $\underset{\sim}{j}$ and dislocation density $\underset{\sim}{\alpha}$; magnetic field $\underset{\sim}{H}$ and distortion field $\underset{\sim}{\beta}$. For later on we add: magnetic induction $\underset{\sim}{B}$ and stress $\underset{\sim}{\sigma}$. The equations analogous to (I.15), (I.17) and (1.18) are: $i = \iint d\underset{\sim}{F} \cdot \underset{\sim}{j}$, $\text{curl}\ \underset{\sim}{H} = \underset{\sim}{j}$, $\text{div}\ \underset{\sim}{j} = 0$.

## §4. Dislocation Motion and Plastic Distortion

The geometrical basic equation (I.17) contains only state variables, which is why it is useful to describe the state after the plastic deformation is carried out. However, we need a relation which describes quantitatively the motion of the dislocation and the resulting plastic distortions. The equation which will govern this will not contain state variables.

We can imagine that the distortions of Fig. 11 produced dislocations whose line direction was the $x_3$-direction and which moved in the $x_1$-direction. Then the direction of the related Burgers vector was $x_2$, $x_3$, $x_1$, in Fig. 11b,c,d (we do not consider the sign at the

moment). A complete investigation has to consider the nine independent dislocation components and the three independent directions of motion. Therefore, we have to investigate 27 different dislocation motions.

We describe a dislocation motion in general by specifying at every position, $\underset{\sim}{x}$, 27 quantities $N_{ijk}$, which mean the number of $\alpha_{jk}$ dislocations (per unit length measured perpendicular to the direction of the line and of the motion) which moved past $\underset{\sim}{x}$ in the i-direction. For this we assume for simplicity that all dislocations have the same value of $\underset{\sim}{b}$, the Burgers vector, however, it is not difficult to consider also the case in which the Burgers vectors are different.

In the expression $N_{ijk}$ the

first index refers to the direction of motion

second index refers to the direction of the line

third index refers to the direction of burgers vector

of the dislocation; $j = k$ are screw dislocation, $j \neq k$ are edge dislocations. As we concluded from the consideration at the end of §1, the following correspondences hold:

$i \neq j = k$ glide of a screw dislocation

$k = i \neq j$ glide of an edge dislocation

$i \neq j \neq k,\ k \neq i$ climb of an edge dislocation.

$i = j$ means a motion of the dislocation in the direction of its line and causes no distortion. It is not necessary that we investigate this motion.

So we recorded all 27 components of $N_{ijk}$. From the vector property describing the motion direction and the tensor property of the dislocation tensor, we conclude that $N_{ijk}$ are the components of a third-

order tensor, which we call "dislocation motion tensor." Furthermore, we write $\bar{1} = -1$ etc. Therefore, the $N_{123}$ motion causes the same distortion as a $N_{\bar{1}\bar{2}3}$, $N_{1\bar{2}\bar{3}}$ and $N_{\bar{1}2\bar{3}}$ motion. We arrange the choice of the positive side of the motion planes in such a manner that they are $+x_i$ sides. Then the motions which caused the plastic distortions in Fig. 11b to d are the following:

b: $N_{1\bar{3}2}$ and $N_{\bar{1}32}$ respectively or also $N_{312}$ and $N_{3\bar{1}2}$ respectively

c: $N_{1\bar{3}3}$ and $N_{\bar{1}33}$ respectively or also $N_{313}$ and $N_{3\bar{1}3}$ respectively

d: $N_{1\bar{3}1}$ and $N_{\bar{1}31}$ respectively or also $N_{311}$ and $N_{3\bar{1}1}$ respectively

The specification is complete and the reader is advised to check it by considering the sign convention of §1.[1] If we let the edge of the

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[1]The line direction of the dislocation shall be parallel to the boundary of the plane of motion (after the motion) taken in the right hand sense relative to the normal plane.

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volume element have the length $\ell$, then $N_{ijk}\,\delta g$ has the same value as the total glide vector of the dislocations which moved through the volume element, therefore it has the same magnitude as the distortion which caused the motion $N_{ijk}$. This distortion we will temporarily call $\beta^P_{ijk}$. As we can see from our examples the same distortion results from dislocation motions which are anti-symmetrical with respect to the first two indices. As mentioned above, no distortion belongs to the motion $N_{ijk}$ (i=j). We get now (Kröner and Rieder [84] ($\delta g = -\underset{\sim}{b}$)

$$\beta^P_{ijk} = -(N_{ijk} - N_{jik})b \qquad (I.20)$$

as an invariant relation between the dislocation motion and the plastic distortion. As an antisymmetrical third-order tensor, $\beta^P_{ijk}$ has nine independent components, and therefore it is possible to replace it in the usual way by a second-order tensor

$$\beta^P_{ijk} = \epsilon_{ij\ell}\, \beta^P_{\ell k} \quad , \qquad \beta^P_{k\ell} = \frac{1}{2}\, \epsilon_{ijk}\, \beta^P_{ij\ell} \tag{I.21}$$

To verify these relations readily we replace $\beta^P_{ij\ell}$ in the second of eq. (I.21) by eq. (I.20) and then use the resulting equation

$$\beta^P_{k\ell} = -\, \epsilon_{ijk}\, N_{ij\ell}\, b \tag{I.22}$$

for the examples b to d.

Now we assume that the dislocation motion $N_{ijk}$ changes because dislocations come to rest with a constant density. The decrease in the number of dislocations which, for instance, moved a distance $dx_1$ with width $dx_2$ in the $x_1$-direction is, of course, equal to the number of dislocations which cut the plane element $dF = dx_1\, dx_2$ after the motion. That means

$$b\, \frac{\partial N_{ijk}}{\partial x_i} = -\, \alpha_{jk} \tag{I.23}$$

and since $\frac{\partial N_{jik}}{\partial x_i} = 0$ (i.e., also the moving dislocations do not terminate in the interior of the body) it follows with respect to eq. (I.20) that

$$\frac{\partial \beta^P_{ijk}}{\partial x_i} = \alpha_{jk} \tag{I.24}$$

In words: The plastic distortion is changed in the direction of motion if dislocations from out of the moving group come to rest with a density $\alpha_{jk}$. Figure 13 gives an example of this. If we replace $\beta^P_{ijk}$ in eq. (I.24) by $\beta^P_{ij}$, eq. (I.11) follows immediately.

The dislocation motion tensor is connected much more closely with the real process of plastic deformation than the previously used terms. This may be its main importance. We differentiate it with respect to time and define by this a dislocation velocity tensor, which can represent a suitable starting point for future dynamics of dislocations, e.g., it is elementary to formulate a friction law for dislocation motions, since the friction force (which finally balances the driving force, causing a constant dislocation velocity) is proportional to the dislocation velocity tensor.

## §5. The Invariant Elements of the Distortion Fields

In this chapter we assume an infinite medium. The distortions are continuous and twice differentiable and may vanish at infinity. Thus the following decomposition is unique:

$$\underset{\sim}{\beta}^P = \text{grad}\ \underset{\sim}{s}^P + \text{curl}\ \underset{\sim}{\zeta}^P \tag{I.25}$$

$$\beta = \text{grad}\ \underset{\sim}{s} + \text{curl}\ \underset{\sim}{\zeta} \tag{I.26}$$

$\underset{\sim}{\zeta}^P \equiv (\zeta^P_{ij})$ and $\zeta \equiv (\zeta_{ij})$ are symmetrical tensor fields.

According to §3, a distortion, whether plastic or elastic, transforms a compact body into a compact body again if we can derive it from a displacement field by using the gradient. So a plastic distortion, grad $\underset{\sim}{s}^P$, does not require an elastic distortion to maintain

the connection in the body, and so occurs without stress, leaving the orientation unchanged.

In principle, the total distortion $\underset{\sim}{\beta}^T = \underset{\sim}{\beta} + \underset{\sim}{\beta}^P$ has to be a gradient tensor (eq. (I.9)), hence it follows

$$\text{curl } \underset{\sim}{\zeta} = -\text{curl } \underset{\sim}{\zeta}^P \tag{I.27}$$

This means that the destruction of the connection caused by the plastic distortions, curl $\underset{\sim}{\zeta}^P$, is just cancelled by the elastic distortion, curl $\underset{\sim}{\zeta}$. Therefore, it seems that $\underset{\sim}{s}^P$ is completely independent of the functions $\underset{\sim}{\zeta}^P$, $\underset{\sim}{\zeta}$, $\underset{\sim}{s}$. The reason, therefore, is that our consideration is still incomplete. In reality, especially in a real body, and therefore we assume it is also for our continuum, there is a coupling between $\underset{\sim}{s}^P$ and $\underset{\sim}{\zeta}^P$ in such a way that the number of the dislocations which came to rest during the dislocation motion is a function of the number of the dislocations which were moving, and may also depend on the position. Such a relation would mean a restriction for the allowed dislocation motion as a function of position. Thus we are able to separate that part of the total distortion

$$\underset{\sim}{\beta}^T = \text{grad } (\underset{\sim}{s} + \underset{\sim}{s}^P) \tag{I.28}$$

which occurred without changing the state. The meaning of $\underset{\sim}{s}^P$ is obviously the plastic displacement of the points of the medium which belong to the part grad $\underset{\sim}{s}^P$ of $\underset{\sim}{\beta}^P$.

The tensors $\underset{\sim}{\zeta}$ and $\underset{\sim}{\zeta}^P$ have not been interpreted until now. They are a sort of potential from which we can derive the distortions. In spite of this, however, $\underset{\sim}{s}$ is an elastic field of displacement. If we cancel the plastic distortion, curl $\underset{\sim}{\zeta}^P$, by the elastic distortion

according to eq. (I.27), and we remove the forces which produced this, a partial relaxation takes place to a state of the lowest elastic energy, causing the material points to suffer a displacement, $\underset{\sim}{s}$. So we can see how the total displacement $\underset{\sim}{s}^T$ is composed of the plastic and elastic displacements.

In the appendix we will show that, by further decomposition of curl $\underset{\sim}{\zeta}$, we will get for $\underset{\sim}{\beta}$

$$\beta_{ij} = \nabla_i s'_j - \epsilon_{ij\ell}\, \epsilon_{jmn}\, \nabla_k \nabla_m i_{\ell n} + \theta_{ij} \tag{I.29}$$

where $s'_j \equiv s_j + u_j$ and $u_j$ is a vector field with $\operatorname{div} \underset{\sim}{u} = 0$; $i_{\ell n}$ is a symmetric, $\theta_{ij}$ an antisymmetric tensor field. In a similar manner, we define by the equations

$$\theta_{ij} = \epsilon_{ijk}\, \theta_k, \quad \theta_k = \frac{1}{2}\, \epsilon_{ijk}\, \theta_{ij} \tag{I.30}$$

a vector field

$$\theta_k = \epsilon_{ijk}\, \nabla_i u_j + \nabla_k \lambda \tag{I.31}$$

where $\lambda$ is a scalar field. Now we define in general the incompatibility (inc) of a second-order tensor field by the identity

$$\operatorname{inc} \underset{\sim}{\zeta} \equiv (-\epsilon_{ik\ell}\, \epsilon_{jmn}\, \nabla_k \nabla_m \zeta_{\ell n}) \equiv \nabla \times \underset{\sim}{\zeta} \times \nabla \tag{I.32}$$

Its name is derived from the fact [77] that

$$\operatorname{inc} \underset{\sim}{\epsilon} = 0 \tag{I.33}$$

is the condition of compatibility of (small) elastic distortions $\underset{\sim}{\epsilon}$ [86,34]. (The deformations are compatible if their incompatibility vanishes.) As we may easily calculate, the incompatibility of a

symmetric tensor produces a symmetric tensor, and the corresponding result holds for an asymmetric tensor.

Therefore, we can write eq. (I.29) in the form

$$\underset{\sim}{\beta} = \text{grad } \underset{\sim}{s}' + \text{inc } \underset{\sim}{i} + \underset{\sim}{\theta} \tag{I.34}$$

We can write $\beta^P$ in the same form

$$\underset{\sim}{\beta}^P = \text{grad } \underset{\sim}{s}'^P + \text{inc } \underset{\sim}{i}^P + \underset{\sim}{\theta}^P \tag{I.35}$$

Since the total distortion $\underset{\sim}{\beta}^T = \underset{\sim}{\beta} + \underset{\sim}{\beta}^P$ is a gradient tensor, it must be that

$$\text{inc } \underset{\sim}{i} = - \text{ inc } \underset{\sim}{i}^P, \quad \underset{\sim}{\theta} = - \underset{\sim}{\theta}^P \tag{I.36}$$

If we compare this with eq. (I.27), we have to notice that

$$\text{curl } \underset{\sim}{\zeta}^P = \text{inc } \underset{\sim}{i}^P + \underset{\sim}{\theta}^P + \text{grad } \underset{\sim}{u}^P \tag{I.37}$$

So it is sufficient to remove the part inc $\underset{\sim}{i}^P + \underset{\sim}{\theta}^P$ of curl $\underset{\sim}{\zeta}^P$ since the tensor grad $\underset{\sim}{u}^P$ is unimportant for the relations of connection.

If we write ([52], vol. 1, pg. 97)

$$\text{def } \underset{\sim}{s}' \equiv \frac{1}{2} (\nabla_i s'_j + \nabla_j s'_i) \tag{I.38}$$

(read def as "deformation of") then the symmetric part of eq. (I.34) has the form[1]

$$\underset{\sim}{\varepsilon} = \text{def } (\underset{\sim}{s} + \underset{\sim}{u}) + \text{inc } \underset{\sim}{i} \tag{I.39}$$

---

[1]This follows from the theorem that in infinite space it is possible to separate a tensor field which vanishes at infinity uniquely by equations like (I.39)

and the antisymmetric part

$$\omega_{ij} = \frac{1}{2}\,[\nabla_i(s + u)_j - \nabla_j(s + u)_i] + \theta_{ij} \tag{I.40}$$

which we can write by using eq. (I.30,31)

$$\omega_{ij} = \frac{1}{2}\,[\nabla_i(s - u)_j - \nabla_j(s - u)_i] + \epsilon_{ijk}\,\nabla_k\,\lambda \tag{I.41}$$

From the identical relations

$$\begin{aligned} \text{div inc} &\equiv 0 \\ \text{inc def} &\equiv 0 \end{aligned} \tag{I.42}$$

which can easily be checked, it follows that eq. (I.39) shows the decomposition of the elastic deformation field into its compatible and incompatible part. Similarly, eq. (I.41) is the decomposition of the rotation field in its compatible and incompatible part. We can easily prove that only the part with $\lambda$ remains if we substitute $\omega_{ij}$ for $\xi_{ij}$ in eq. (I.32).[1]

---

[1]Every tensor of the form $\epsilon_{ijk}\,\nabla_k\lambda$ can also be written as an antisymmetric incompatibility (Appendix).

---

The incompatible rotation field therefore has the form $\omega_k^{inc} = (\text{grad}\,\lambda)_k$.[2] Especially, notice that the compatible strains and

---

[2]Another view point is: The rotation $\omega_{ij} - \theta_{ij}$ in eq. (I.40) are "incompatible" with the strains def (s + u) in eq. (I.39), so we can also call $\theta_{ij}$ the incompatible rotation [81].

---

rotations are no longer coupled as they are in the classical theory of elasticity. Moreover, there are states possible in which $\underset{\sim}{u} + \underset{\sim}{s}$ is equal to an arbitrary vector $\underset{\sim}{p}$ and $\underset{\sim}{u} - \underset{\sim}{s}$ is equal to an almost[1] arbitrary vector $\underset{\sim}{g}$.

[1] For div u = 0, we can not simultaneously prescribe the part of $\underset{\sim}{p}$ and $\underset{\sim}{g}$ derived from the gradient operation. Anyway the part of $\underset{\sim}{g}$ due to the gradient operation does not contribute to $\omega_{ij}$.

In no case are we allowed to interpret $\underset{\sim}{u}$ as a displacement field $\underset{\sim}{u}$ is not easily interpreted, but is similar to $\lambda$, a sort of a potential from which we can derive the rotation. We also should notice that in eq. (I.41) the rotation of single volume elements was not decomposed, but rather the rotation field, i.e., eq. (I.41) indicates in which way the rotation varies from element to element. $\omega_{ij}$ at the point $\underset{\sim}{x}$ is a rigid rotation of the volume element $dV(\underset{\sim}{x})$ as we defined in §2.

The special importance of eq. (I.34) is that here the distortions which restore the connection, are shown separately in the symmetric and antisymmetric part. If we substitute $\beta_{ij}$ from eq. (I.34) in eq. (I.17), we get by use of

$$\text{curl}\ (\text{inc}\ \underset{\sim}{i} + \underset{\sim}{\theta}) = \underset{\sim}{\alpha} \tag{I.43}$$

the basic equation which the author first derived [81].

The phenomena which are connected with the distortion inc $\underset{\sim}{i}$ and $\underset{\sim}{\theta}$, we will consider in the next two sections. Here we will only count the degrees of freedom which are contained in the plastic and elastic distortions. There are twelve in all, namely, three per

grad $\underset{\sim}{s}^P$ and grad $\underset{\sim}{s}$ and six per curl $\underset{\sim}{\zeta}^P$ and curl $\underset{\sim}{\zeta}$.[1] Of the last six, there are always three for the incompatible strains inc $\underset{\sim}{i}^P$ and inc $\underset{\sim}{i}$ and three for the rotation $\underset{\sim}{\theta}^P$ and $\theta$.

---

[1]For we are allowed to specify three secondary conditions for $\zeta^P$ and $\zeta$. The same holds for $\underset{\sim}{i}^P$ and $\underset{\sim}{i}$ (Appendix).

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## §6. The Geometric Origin of Thermal Stresses, Magnetic Stresses and Stress Concentrations

We will now summarize the important facts of plastic deformation with respect to the macroscopic standpoint. We imagine that the body is cut into its volume elements and we apply to each element by dislocation motion, the desired plastic (stress free) distortion $\beta^P$. After this, in general, the volume elements do not fit together, and elastic distortions (inc $\underset{\sim}{i}$ + $\underset{\sim}{\theta}$) are necessary in order that they be able to fit together again. After this we imagine that all coalesce, and we remove all the forces which caused the elastic deformation. Thereafter, a relaxation (grad $\underset{\sim}{s}'$) into the state of the lowest energy occurs. At the end we observe a dislocation density $\underset{\sim}{\alpha} \equiv -$ curl $\underset{\sim}{\beta}^P$.

We can change the experiment in this way: We apply to the volume element not plastic distortions by dislocation motion but a quasi plastic distortion, e.g., by an increase in temperature. As we know, for a volume element at $\underset{\sim}{x}$ [79]

$$\beta^Q_{ij} = \gamma\, \delta_{ij}\, T \tag{I.44}$$

if $\gamma$ is the thermal expansion coefficient and the reference temperature is zero. Both $T(\underset{\sim}{x})$ and $\underset{\sim}{\beta}^Q(\underset{\sim}{x})$ are continuous functions of position of the volume element. Moreover, $\underset{\sim}{\beta}^Q$ is naturally a spherically symmetric tensor, so it is a pure strain, and we can write $\epsilon^Q$ instead of $\beta^Q$. We call $\beta^Q$ quasiplastic, for these distortions do not cause repulsive forces. Now the equation

$$\delta_{k\ell}\, \nabla_k\, \nabla_\ell\, \Phi = \Delta\, \Phi = \gamma\, T \tag{I.45}$$

always has a solution. So we are able to write $\beta^Q$ by use of eq. (A.2) also in the form

$$\beta^Q_{ij} = \delta_{ij}\, \delta_{k\ell}\, \nabla_k\, \nabla_\ell\, \Phi = \nabla_i\, \nabla_j\, \Phi - \epsilon_{ikm}\, \epsilon_{j\ell m}\, \nabla_k\, \nabla_\ell\, \Phi \tag{I.46}$$

or by use of eq. (1.32) $[\underset{\sim}{I} = \delta_{ij}]$

$$\underset{\sim}{\beta}^Q = \text{def}\ (\text{grad}\ \Phi) + \text{inc}\ (\Phi\ \underset{\sim}{I}) \tag{I.47}$$

where it is possible to substitute grad for def. The second term causes the distortion $\beta^Q$ to occur accompanied by a disturbance of the connection, after which the connection can only be maintained by an elastic distortion of the form $\text{inc}\ \iota = -\ \text{inc}\ (\Phi\ \underset{\sim}{I})$. Now we can define a quasi dislocation density by the equation

$$\alpha^Q = -\ \text{curl}\ \underset{\sim}{\beta}^Q \tag{I.48}$$

The state of elastic distortion which belongs to $\alpha^Q$ is then the same as that which was produced by a dislocation motion during which dislocations

of the density $\alpha = \alpha^Q$ came to rest. Therefore, in a continuum to which a distribution of temperature is applied we can eliminate elastic distortions if the dislocations of density

$$\underset{\sim}{\alpha} = - \underset{\sim}{\alpha}^Q = \gamma \text{ curl } (T \underset{\sim}{I}) \tag{I.49}$$

are introduced. Certainly this process is important when large thermal stresses occur as they do during the cooling of cast iron. Since in this case it is easy to calculate the necessary dislocation arrangement, this is an impressive example of the practical use of the concept that thermal stress is considered as being caused by dislocations.

If we bring a sample of a ferromagnetic metal, but which is non-magnetic as a whole, into a sufficiently strong magnetic field, then all elementary dipoles align themselves parallel to the direction of the field. In many cases a quasiplastic elongation of the sample occurs in the direction of the magnetic field, whereas the volume remains approximately the same. If the magnetic direction changes in the body from place to place, then we can perform again the thought experiment mentioned above. The quasiplastic distortion of the volume elements becomes then a (symmetric) deviator, since the volume remains the same. Also, we can define by the help of eq. (I.48) a quasidislocation density and indicate in which way the dislocations will be arranged in the magnetostrictivly strained body to keep the elastic energy as low as possible. Such investigations are very important in experiments, which are currently being carried out, designed to understand the curve of the magnetization of ferromagnetic metals. For this see [11, 155, 124].

Hereto belongs the case that a crystal of atoms of the sort A dissolve atoms of the sort B so that the concentration $C(\underset{\sim}{x})$ is macroscopically varying. We imagine the pure crystal cut into its volume elements, then we dissolve in each the quantity of the atoms B which belongs to it, which is equivalent to a quasiplastic distortion as above. All the other things are in a similar manner as in both of the former examples.

It is almost not necessary to remark that the methods (which will be discussed in Chapter II) for calculating the stresses which were produced by the dislocation density $\underset{\sim}{\alpha}$ also hold for the above-mentioned cases with a quasidislocation density $\underset{\sim}{\alpha}^Q$. For calculating the stresses it is of great advantage to use another geometrical term which we will discuss now. This is the so-called incompatibility tensor $\underset{\sim}{H} = (H_{ij})$, which is in some respects as important as the tensor $\underset{\sim}{\alpha}$. We define it by

$$\underset{\sim}{H} \equiv - \operatorname{inc} \underset{\sim}{\beta}^P \tag{I.50}$$

therefore, by use of (I.11) it is

$$\underset{\sim}{H} = \underset{\sim}{\alpha} \times \nabla \equiv (\epsilon_{ijk} \nabla_k \alpha_{\ell j}) \tag{I.51}$$

If we substitute it in eq. (I.17), then we get

$$\operatorname{inc} \underset{\sim}{\beta} = \underset{\sim}{H} \tag{I.51'}$$

and the symmetric part of the tensor equation reads with $\underset{\sim}{\eta}$ = symmetric part of $\underset{\sim}{H}$ [1]

[1] Initially we assume small distortions.

$$\text{inc}\ \underset{\sim}{\epsilon} = \underset{\sim}{\eta} \tag{I.52}$$

For $\underset{\sim}{\eta} = 0$ these are the compatibility conditions of de St. Venant. In the case of temperature fields, we get, e.g., by use of eq. (I.51) and (I.49) [79]

$$\underset{\sim}{\eta} = \gamma\ \text{inc}\ (T\ \underset{\sim}{I}) \tag{I.53}$$

i.e., the incompatibility field $\underset{\sim}{\eta}$ which belongs to a temperature field is easy to calculate. If $\underset{\sim}{\eta}$ is known however, then the associated stresses are relatively easy to determine (§13).

Perhaps we can illustrate the importance of eq. (I.52) in elasticity theory as follows: Since it was developed from eq. (I.17) by taking the curl from the right and symmetrizing, it must contain part of the meaning of the equation but the other part must be lost. From the relation inc def $= 0$, we obtain the result that in the case of $\underset{\sim}{\eta} = 0$, the strain $\underset{\sim}{\epsilon}$ can be derived from an elastic displacement field $\underset{\sim}{s}$, if we assume, as was previously always done in the theory of elasticity, that $\underset{\sim}{\theta}$ of eq. (I.40) is zero.[1] The elastic rotation $\frac{1}{2}(\nabla\underset{\sim}{s} - \underset{\sim}{s})$ follows

[1] With $\underset{\sim}{\theta} = 0$, $\underset{\sim}{u} = \text{const}$ by use of eq. (I.31) (for div $\underset{\sim}{u} = 0$).

from the same displacement field. As we know then the displacements are determined by the strain with exception of a rigid rotation of the whole body. In this case the eq. (I.52) are equivalent with curl $\underset{\sim}{\beta} = 0$. The second automatically contains the statement $\underset{\sim}{\theta} = \text{const}$, as will be shown in the next section. Exactly this statement is lost, if we derive the equation inc $\underset{\sim}{\epsilon} = 0$ from the equation curl $\underset{\sim}{\beta} = 0$.

So the classical theory of elasticity is defined by the equation curl $\underset{\sim}{\beta} = 0$ or which is equivalent, by inc $\underset{\sim}{\epsilon} = 0$; $\underset{\sim}{\theta} = 0$.

If $\underset{\sim}{\eta} \neq 0$ the plastic deformation field has the form def $\underset{\sim}{s}^P$ + inc $\underset{\sim}{i}^P$. The second part always causes the plastic or quasi-plastic distortion not to maintain the connection of the body and therefore gives rise to elastic strain and subsequently to self stresses. The existence of an incompatibility field is therefore (in any case in a simply connected body) a presumption that self stresses can appear. It is easy to show that in the region of the linear theory of elasticity the totality of the stresses which are possible in a body are uniquely determined by given external forces and the incompatibilities (§16).

## §7. The Curvature of the Structure without Stresses

The fact that the dislocation causes rotations during its motion, was used for explaining important phenomena in the physics of metals. So first Burgers [12] and Bragg [10] found that the grain boundary between two crystallites (grains) with a small difference in their orientation is built up by a two-dimensional arrangement of dislocations at the grain boundary. If we study the volume shown in Fig. 16a, for instance, a group of $\alpha_{31}$-edge dislocations should run through it in the $x_1$-direction and should come to rest along the indicated plane with a constant density. By cutting previously along this plane, the distortion of Fig. 16b is produced. By rotating each layer through the angle $\delta\,\underset{\sim}{\theta}$ we can restore the destroyed connection. Between each two layers which were separated

by a dislocation wall we have a difference in the orientation by the angle $\delta \underset{\sim}{\theta}$. Figure 16c shows the same for a motion of a $-(\alpha_{22} + \alpha_{33})$ screw dislocation in $x_1$-direction if $\alpha_{22} = \alpha_{33}$. The assignment now is to find a relation between the rotation and the density of the dislocations which come to rest.

In the beginning we can restrict our considerations to the case in which the dislocation density is homogeneous. By eq. (I.51), the incompatibility tensor vanishes, and if also no external force is applied to the volume, then this is free of elastic strain altogether. In §14 we will prove this exactly. This statement holds only for small distortions and dislocation densities, respectively, to which we now restrict our considerations. Therefore, in our case $\beta_{ij} = \theta_{ij}$, where $\theta_{ij}$ are the elastic (= rigid) rotations of the volume elements dV, by which the connection, which was destroyed in Fig. 16b and c, was destroyed. For the Burgers circuit, therefore, we get on the one hand [see eq. (I.15) and (I.19)]

$$\oint_C da_j = \oint_C dx_i \, \beta_{ij} = \iint_F dF_k \, \alpha_{kj} \tag{I.54}$$

and on the other hand

$$\oint_C dx_i \, \beta_{ij} = \oint_C dx_i \, \theta_{ij} = - \oint x_i \, d\theta_{ij} \tag{I.55}$$

as $\oint d(x_i \theta_{ij}) = 0$.[1] If we substitute $d\theta_{ij} = \epsilon_{ijk} \, d\theta_k$, then we get for the right-hand side of eq. (I.55)

---

[1] As proof take $\theta_{ij}$ as a linear function of $x_\ell$.

$$(-\epsilon_{ijk} \oint x_i \, d\theta_k) \equiv \oint_C \underset{\sim}{x} \times d\underset{\sim}{\theta}, \tag{I.56}$$

where $d\theta_k$ is the angle of rotation between two neighboring volume elements. Now we define with Nye [113] the (macroscopic) curvature tensor $\underset{\sim}{K} \equiv (K_{ij})$ by the equation

$$d\theta_k = K_{k\ell} \, dx_\ell \tag{I.57}$$

The diagonal components of $K_{k\ell}$ are twistings (screwings) of the $x_i$-plane, meanwhile the other components are bendings of the $x_i$-plane around the k-direction, as we can easily see, e.g., in Fig. 16. If we put eq. (I.57) in (I.56), we get (Stokes' Theorem)

$$\begin{aligned} - \epsilon_{ijk} \oint_C x_i \, K_{k\ell} dx_\ell &= - \epsilon_{ijk} \, \epsilon_{\ell mn} \iint_F dF_m \, \nabla_n (x_i \, K_{k\ell}) \\ &= - \epsilon_{ijk} \, \epsilon_{\ell mi} \iint_F dF_m \, K_{k\ell} \end{aligned} \tag{I.58}$$

since for a constant dislocation density $K_{k\ell}$ is constant and $\nabla_n x_i = \delta_{ni}$. After comparing with eq. (I.54) and with the decomposing formula (A.2), the relation, which was first derived by Nye [113] using another approach, between dislocation density and curvature of structure follows

$$\alpha_{ij} = \delta_{ij} \, K_{kk} - K_{ij}, \tag{I.59}$$

while the inverse is

$$K_{ij} = \frac{1}{2} \delta_{ij} \, \alpha_{kk} - \alpha_{ij}. \tag{I.60}$$

This equation also holds for small dislocation densities and curvatures, respectively, i.e., the change of the orientation $d\theta_k$ must be small relative to 1 over the distance $dx_\ell$.

For further discussion, we assume a variable dislocation density, and we now call the relative angle of rotation between the volume elements $d\delta_k$ for a reason soon to be obvious.

If we now perform, analogous to the Burgers circuit, a closed circuit C, along which we add the rotations $\delta\delta_i$ and $d\delta_i$ to obtain

$$D_i = \oint_C d\delta_i = \oint_C dx_j \, K_{ij} \tag{I.61}$$

and by Stokes' Theorem

$$\underset{\sim}{D} = - \iint_F (\underset{\sim}{K} \times \nabla) \cdot d\underset{\sim}{F} \tag{I.62}$$

While we will not perform the calculation in detail here, it follows from eq. (I.60), (I.18), and (I.51)

$$\underset{\sim}{K} \times \nabla \equiv (\epsilon_{ijk} \nabla_k K_{\ell j}) = - \underset{\sim}{\eta} \tag{I.63}$$

Therefore, we have for infinite planes

$$\Delta D_i = \eta_{ij} \Delta F_j , \tag{I.64}$$

which we can regard similar to eq. (I.14) as defining equation for $\underset{\sim}{\eta}$ .

According to eq. (I.63), $d\theta_i$ in eq. (I.57) is then only an exact differential if $\underset{\sim}{\eta} = 0$, meanwhile for $dg_i$, the analogous conclusion is $\underset{\sim}{\alpha} = 0$. Also, in the case of $\underset{\sim}{\eta} = 0$, a continuous vector field $\delta_i$ exists, which describes that part of the rotation of the structure (the "part of the grain boundary"),[1] which depends directly on the dislocations, and it is identical with $\theta_i$ of eq. (I.31)(for $\underset{\sim}{\eta} = 0$).

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[1]For this relation also see §23.

This we can see from our previous investigations of the problem: $d\theta_i$ were the rotations which would restore the connection which was disturbed by the dislocation motion. The same holds for $d\delta_i$. The curvatures of the structure related to this occur without stresses because there are no external forces and incompatibilities (§14).

The tensor $\underset{\sim}{K}$ obviously does not contain the elastic rotations, $(\nabla_i s_j - \nabla_j s_i)/2$, which depend on the curvatures. The curvatures of the structure actually observed can be described with another curvature tensor, which is defined instead of eq. (I.57) by

$$d\omega_i = \kappa_{ij}\, dx_j \tag{I.65}$$

However, with a continuously varying dislocation density $\underset{\sim}{\alpha}$, also $\underset{\sim}{\beta}$ and $\underset{\sim}{\omega}$, are continuous functions of position (for $\underset{\sim}{\beta}$, at least in a simply connected body, must be unique), therefore $d\omega_i$ is a total differential. This, however, holds only for small rotations, see for this Bilby and Smith [5]. If there are no elastic deformations $K_{ij}$ become identical with $\kappa_{ij}$ (then, however, $\nabla_i S_j - \nabla_j S_i = 0$).

Equation (I.64) states that the Burgers vector $\underset{\sim}{b}$ has the same relation to the dislocations as the rotation vector $\underset{\sim}{D}$ has to the incompatibilities. Look again at Fig. 15. The original planes of motion of the dislocations will now be dislocation walls with a constant dislocation density in the plane in the manner of those of Fig. 16. Taking a circuit along the boundary of the plane F, we cut dislocation walls, and we always add the relative rotation of two volume elements. Then, however, the dotted dislocation walls do not contribute anything, since they were cut twice but in opposite directions: If we imagine

that we cut a single plane along the dislocation walls, then both of the cut edges would be spread by the angle of rotation $d\delta_i$. With this approach we get directly a method of measuring the incompatibility of a state of stress. We cut a thin as possible closed ring, which represents the boundary of a (macroscopic) plane element $\Delta F_j$. Then we cut this ring open and measure the relative rotation of the cut edge which occurs during the relaxation. The rotation vector is $\Delta D_i$, from which $\eta_{ij}$ follows by eq. (I.64).

For the following reason the ring which is cut out should be so thin as to define effectively only the related surface $\Delta F_j$. For thicker rings there is an additional strain of the cut edge, which interferes with the measurement. In practical cases we will never measure a body in this way. However, we can get a qualitative impression of the "average of the incompatibilities" and therefore of the state of the self stresses (see Chapter II), if we carry out this measurement on several macroscopic planes F.[1]

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[1]The problem of measuring the internal stresses in the interior of a body is presently unsolved. See for this Reimer's [175] work mentioning the magnetic method, which is applicable in some cases.

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From these considerations we approach Volterra's distortions of the second kind (§1). Figure 17a shows a cylinder into which only one dislocation wall penetrates as in Fig. 15. Around the boundary of this wall a hollow torus may be removed. Then the hollow cylinder is in a Volterra state of distortion of the second kind. If we cut open the

cylinder along the dislocation wall or also along another arbitrary plane, both of the cut edges suffer the known discontinuity of rotation, i.e., the state of Fig. 17a can be produced from that of Fig. 17b, which is without stress, by bending together and welding. In contrast to previous opinions, the singular plane of the rotation jump can be found at any time afterwards by experiment. In the case that the body in Fig. 17b is a single crystal, it is evident that we can find at once the jump of the orientation in Fig. 17a by use of x-rays (in many cases much easier). But this is not possible for polycrystals even with more effort.[1]

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[1] See for this also the discussion of Nabarro [110], p. 349.

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Therefore, a complete description of the state of the Volterra distortion of the second kind requires an indication of a singular plane, which we may find in any way. The occasionally used nomenclature "elementary distortion" belongs in our opinion only to the state of the first kind, which is consistent with the fact that we can produce each state of the second kind by a particular arrangement of dislocations.[2,3]

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[2] As we know, Volterra used the word "distortion" in a slightly different context. The above-mentioned statement would be read: The elementary state is that which is caused by a dislocation.

[3] The results which are related to the incompatibilities were found by the author, perhaps first by Moriguti [103] and also by Eshelby [41].

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## §8. The Conditions of the Boundary Planes for the Distortions

Any boundary surfaces of the experimental body have no been considered previously. This we will remedy now. We get at once the conditions associated with the boundary surfaces if we take in eq. (I.11) and (I.17) instead of the curl operation the two-dimensional curl and instead of the three-dimensional dislocation density $\underset{\sim}{\alpha}$ the dislocation density of the plane $\underset{\sim}{\bar{\alpha}}$. If we call the boundary surfaces I and II, and if $\underset{\sim}{u} = (u_i)$ is the dimensionless unit vector of the boundary surface in the direction of I to II, then we get from eq. (I.11) and (I.17)[1]

---

[1] Equation (I.67) was formulated first by Bilby, Bullough and Smith [3]. In many cases these authors consider a two-dimensional dislocation arrangement as an entity called a "surface dislocation" in contrast to the normal line dislocation. In other papers quoted on page 113, these authors mention different applications of the theory of surface dislocations.

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$$\underset{\sim}{n} \times \underset{\sim}{\beta}^P\big|_{II} - \underset{\sim}{n} \times \underset{\sim}{\beta}^P\big|_{I} = -\underset{\sim}{\bar{\alpha}} \tag{I.66}$$

$$\underset{\sim}{n} \times \underset{\sim}{\beta}\big|_{II} - \underset{\sim}{n} \times \underset{\sim}{\beta}\big|_{I} = \bar{\alpha} \tag{I.67}$$

where, if we want, we can take the first equation also as the defining equation for $\underset{\sim}{\bar{\alpha}}$. However, in many cases it is useful to define the dislocation density of the plane $\bar{\alpha}_{ij}$ in the sense of Schwartz's distribution calculus [131] by the equation

$$\alpha_{ij} = \bar{\alpha}_{ij}\, \delta(n) \tag{I.68}$$

where the parameter n characterizes a group of planes in such a way that n = 0 is the boundary plane. $\delta(n)$ is everywhere zero except for n = 0, where it goes to infinity in such a way that $\int_{-\infty}^{\infty} \delta(n)\, dn = 1$. $\bar{\alpha}_{ij}$ is independent of n.

From eq. (I.68) it follows that

$$\int_{-\infty}^{\infty} \alpha_{ij}\, dn = \int_{-\infty}^{\infty} \bar{\alpha}_{ij}\, \delta(n)\, dn = \bar{\alpha}_{ij}. \tag{I.69}$$

Now we consider an infinite body in an initial state without dislocations. By any external forces dislocations may be developed, and they may move. Here we distinguish three cases:

The first group of dislocatons must vanish or be otherwise annihilated at the conclusion of the (continuously distributed) motion.[1]

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[1]This annihilation can be carried out in the infinite body by combining dislocations with opposite sign.

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The second group should come to rest with a continuous density $\underset{\sim}{\alpha}$ in the body. The third group should also come to rest, however as a two-dimensional density $\underset{\sim}{\bar{\alpha}}$, by which the two regions I and II of the body may be bounded.

At the end the connection of the body should also be maintained at the boundary surface. This is required by the boundary condition

$$\underset{\sim}{s}^T\big|_{II} - \underset{\sim}{s}^T\big|_{I} = 0 \tag{I.70}$$

$\underset{\sim}{s}^T$ are total displacements as in section 2. We can differentiate this equation in the boundary surface, and we only lose one irrelevant constant. The equations

$$\underset{\sim}{n} \times \nabla \underset{\sim}{s}^T\big|_{II} - \underset{\sim}{n} \times \nabla \underset{\sim}{s}^T\big|_{I} = 0 \tag{I.71}$$

and therefore practically equivalent to eq. (I.70). Instead of (I.71) we write with eq. (I.9)

$$\underset{\sim}{n} \times \underset{\sim}{\beta}^T\big|_{II} - \underset{\sim}{n} \times \underset{\sim}{\beta}^T\big|_{I} = 0 \tag{I.72}$$

e.g., we obtain for a location where $\underset{\sim}{n}$ points in the $x_1$-direction that $\beta^T_{21}$, $\beta^T_{22}$, $\beta^T_{23}$ and $\beta^T_{31}$, $\beta^T_{32}$, $\beta^T_{33}$ must be the same on both sides of the boundary surface.[1] Equation (I.72) is the sum of eqs. (I.66) and (I.67). Also it follows, formally, of course, from eq. (I.8).

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[1]The same idea led us previously to eq. (I.8).

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Again we consider the three dislocation groups and assume that they occurred consecutively. The first group, of course, produced a plastic distortion grad $\underset{\sim}{s}^P$, which is continuous in the whole body. The second group causes a distortion grad $\underset{\sim}{s}^P_2$ + curl $\underset{\sim}{\zeta}^P_2$, which is also continuous in the whole body (that this contains a part grad $\underset{\sim}{s}^P_2$ follows from the fact that the distortions which were caused by the 2nd group also depend on the path of the dislocation).

The distortions depending on group 3 are discontinuous on the boundary surface, but are continuous in the partial bodies I and II. Which shape do they have? The dislocations are neither annihilated

in the body nor come to rest. Then we will remember the well-known fact that the decomposition of the tensor field

$$\underset{\sim}{\beta}_3^P \quad \text{grad } \underset{\sim}{s}_3^P + \text{curl } \underset{\sim}{\zeta}_3^P \tag{I.73}$$

is not unique in a body with one or more boundary surfaces, but there is a distortion which can be described either as vector gradient or as a curl tensor. If we write it as grad $\underset{\sim}{s}_3^P$, then since it is simultaneously a curl tensor, it follows that,

$$\text{div grad } \underset{\sim}{s}_3^P = \Delta \underset{\sim}{s}_3^P = 0 \tag{I.74}$$

In reverse, we can, of course, describe every gradient of a harmonic vector field as a curl tensor.

From these considerations we can assume[1] that we can write the boundary conditions in the form

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[1]An exact proof can be derived in addition to §4.

---

$$\underset{\sim}{n} \times \nabla \underset{\sim}{s}_3^P\Big|_{II} - \underset{\sim}{n} \times \nabla \underset{\sim}{s}_3^P\Big|_{I} = - \underset{\sim}{\bar{\alpha}} \tag{I.75}$$

or if we take

$$\underset{\sim}{g} \equiv \underset{\sim}{s}_3^P\Big|_{II} - \underset{\sim}{s}_3^P\Big|_{I} \tag{I.76}$$

as the plastic displacement jump in the boundary surface, we also can write it in the form

$$\underset{\sim}{n} \times \nabla \underset{\sim}{g} = - \underset{\sim}{\bar{\alpha}}$$

This equation tells us that dislocations must be on a surface and in what arrangement on the surface which had a nonconstant displacement

jump g. For application this arrangement is very important. It applies also for large distortions, as does eq. (I.17), if we relate all values to their initial condition (§10). The boundary conditions (I.66) and (I.67) become very simple, if one of the partial bodies becomes infinitely soft (air) or infinitely rigid. In the first case the boundary conditions are fulfilled identically, as we easily can see from (I.70). In the second case, one term of the sum cancels in eq. (I.66) and (I.67), since the distortion is zero in a rigid medium (in a rigid medium it is obvious that no dislocation can move). In reality there is no rigid medium, but it often happens that, e.g., a soft metal contains an inclusion of hard metal, the deformation of which we can neglect. In this case the boundary value problem which must be solved is greatly simplified. Notice that the situation for stresses in the case of known boundary conditions is just opposite (they are fulfilled identically at a boundary surface with a rigid body).

## §9. The Boundary Surface Conditions for the Strain, Two-Dimensional Incompatibility Distribution

In §8 we assumed that the spatial density $\underset{\sim}{\alpha}$ of the dislocations which came to rest is a continuous function in the whole body. This restriction we will drop now, by allowing an additional jump of $\underset{\sim}{\alpha}$ in the boundary surface. We will easily understand that this leads to a jump in the plastic displacement; e.g., it is possible that in I, edge dislocations climb by enlarging the volume of I. meanwhile in II a dislocation motion occurs without change in the volume. The related

displacement jump similarly would be described by

$$\underset{\sim}{g}_2 \quad \underset{\sim}{s}_2^P\Big|_{II} - \underset{\sim}{s}_2^P\Big|_{I} \tag{I.78}$$

In §7 we emphasized the importance of the incompatibilities with respect to the self stresses. We may expect that two-dimensional distributions of incompatibilities also may play a role in determining self stresses. Not only the surface dislocations contribute to them, but also the jump in $\underset{\sim}{\alpha}$. To study this, we take the case in which plastic distortions $\underset{\sim}{\beta}_I^P$ and $\underset{\sim}{\beta}_{II}^P$ are continuously distributed in I and II, where the passage between I and II can be arbitrarily discontinuous. We assume that the function of $\underset{\sim}{\beta}_I^P$ which is continuous and has two derivatives is continued in II, as well as $\underset{\sim}{\beta}_{II}^P$ into I. We write the distortions, summarizing for the whole regions I and II

$$\underset{\sim}{\beta}^P = \underset{\sim}{\beta}_I^P + (\underset{\sim}{\beta}_{II}^P - \underset{\sim}{\beta}_I^P)\ \delta^\circ(n) \tag{I.79}$$

where $\delta^\circ(n)$ is Heaviside's step function, i.e., $\delta^\circ(n) = 0$ in I and 1 in II. We will use the following rule for the calculation [131]

$$\frac{\partial}{\partial n}\delta^\circ(n) = \delta^1 n, \qquad \frac{\partial}{\partial n}\delta^1(n) = \delta^2(n)$$

where $\delta^1$ is the Dirac Delta function and $\delta^2$ is the distribution which describes a doublet function (Dopplebelegung). As all $\delta$ depend only on n, it further holds that $\nabla\delta^\circ = \underset{\sim}{n}\delta^1$ and $\nabla\delta^1 = n\delta^2$. Finally, if f is a continuous function of $\underset{\sim}{n}$ and eventually of two other coordinates, we obtain, $\frac{\partial}{\partial n}(f\delta^1) = f\delta^2$. In the following, we derive the asymmetric incompatibility,

$$\underset{\sim}{H} \equiv -\nabla \times \underset{\sim}{\beta}^P \times \nabla. \tag{I.80}$$

Initially we have[1]

$$-\underset{\sim}{\alpha} = \nabla \times \underset{\sim}{\beta}^P = \nabla \times \underset{\sim}{\beta}^P + \nabla \times (\overset{\downarrow}{\underset{\sim}{\beta}}{}^P_{II} - \overset{\downarrow}{\underset{\sim}{\beta}}{}^P_{I})\delta^\circ + \underset{\sim}{n} \times (\underset{\sim}{\beta}^P_{II} - \underset{\sim}{\beta}^P_{I})\delta^1 \qquad (I.81)$$

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[1] In cases of doubt the arrow indicates which term was differentiated.

---

The first two terms are the space density and the last is the surface density of the dislocations according to §8. For further differentiation of the last term, we will use a decomposition into parts which allows differentiation perpendicular and parallel to the surfaces:

$$\nabla = \underset{\sim}{n} \frac{\partial}{\partial u} + \overset{n}{\nabla}, \qquad \overset{n}{\nabla} \equiv - \underset{\sim}{n} \times (\underset{\sim}{n} \times \nabla). \qquad (I.82)$$

Then we get

$$\underset{\sim}{H} \equiv \underset{\sim}{\alpha} \times \nabla = - \nabla \times \underset{\sim}{\beta}^P_I \times \nabla - \nabla \times (\underset{\sim}{\beta}^P_{II} - \underset{\sim}{\beta}^P_I) \times \nabla \delta^\circ$$

$$- [\nabla \times (\overset{\downarrow}{\underset{\sim}{\beta}}{}^P_{II} - \overset{\downarrow}{\underset{\sim}{\beta}}{}^P_{I}) \times \underset{\sim}{n} + \overset{\downarrow}{\underset{\sim}{n}} \times (\underset{\sim}{\beta}^P_{II} - \underset{\sim}{\beta}^P_{I}) \times \overset{n}{\nabla}] \delta^1$$

$$- \underset{\sim}{n} \times (\underset{\sim}{\beta}^P_{II} - \underset{\sim}{\beta}^P_{I}) \times \underset{\sim}{n} \delta^2 \qquad (I.83)$$

The first two terms represent the space, the next two the ordinary two-dimensional incompatibility density. Furthermore, the first part of this is caused by the jump of the dislocation density $\underset{\sim}{\alpha}$, and the second follows from the surface density $\bar{\underset{\sim}{\alpha}}$. Finally, the last term corresponds to a two-dimensional doublet of incompatibilities.

If we carried out the operation first from the left-hand side instead of (I.81) so we would get a factor of $\delta^1$ in eq. (I.83),

$$- \underset{\sim}{n} \times (\overset{\downarrow p}{\underset{\sim}{\beta}}_{II} - \overset{\downarrow p}{\underset{\sim}{\beta}}_{I}) \times \nabla - \overset{n}{\nabla} \times (\overset{\downarrow p}{\underset{\sim}{\beta}}_{II} - \overset{\downarrow p}{\underset{\sim}{\beta}}_{I}) \times \underset{\sim}{n} \tag{I.84}$$

otherwise eq. (I.83) would be the same. We can show that this expression is identical to the coefficient of $\delta^1$ in eq. (I.83), as it must be. (For it is $\underset{\sim}{n} \times \underset{\sim}{\beta} \times \overset{n}{\nabla} = \nabla \times \underset{\sim}{\beta} \times \underset{\sim}{n}$, as we easily can show by writing $\overset{n}{\nabla} = \underset{\sim}{v}\, \partial/\partial v + \underset{\sim}{w}\, \partial/\partial w$, where $\underset{\sim}{v}$ and $\underset{\sim}{w}$ are the principal curvature directions. Therefore $\partial \underset{\sim}{n}/\partial v \sim \underset{\sim}{v}$ and $\partial \underset{\sim}{n}/\partial w \sim \underset{\sim}{w}$).

For that part of $\underset{\sim}{H}$ which lies in the boundary surface, we now obtain

$$\underset{\sim}{\bar{H}}\, \delta^1 + \underset{\sim}{\bar{\bar{H}}}\, \delta^2 \tag{I.85}$$

with

$$\bar{H} \equiv - \overline{\text{inc}}\, (\underset{\sim}{\beta}^P_{II} - \underset{\sim}{\beta}^P_{I}), \quad \bar{\bar{H}} \equiv - \overline{\overline{\text{inc}}}\, (\underset{\sim}{\beta}^P_{II} - \underset{\sim}{\beta}^P_{I}). \tag{I.86}$$

where the operations $\overline{\text{inc}}$ and $\overline{\overline{\text{inc}}}$ are defined by

$$\overline{\text{inc}}\, \underset{\sim}{\beta} \equiv \nabla \times \overset{\downarrow}{\underset{\sim}{\beta}} \times \underset{\sim}{n} + \overset{\downarrow}{\underset{\sim}{n}} \times \overset{\downarrow}{\underset{\sim}{\beta}} \times \overset{n}{\nabla} \tag{I.87}$$

$$\overline{\overline{\text{inc}}}\, \underset{\sim}{\beta} \equiv \underset{\sim}{n} \times \underset{\sim}{\beta} \times \underset{\sim}{n} \tag{I.88}$$

Therefore it is obvious that (see eq. (I.51))

$$\underset{\sim}{\bar{H}} = \underset{\sim}{\alpha} \times \underset{\sim}{n}\, \Big|_{II} - \underset{\sim}{\alpha} \times \underset{\sim}{n}\, \Big|_{I} + \bar{\alpha} \times \overset{n}{\nabla}, \quad \underset{\sim}{\bar{\bar{H}}} = \underset{\sim}{\bar{\alpha}} \times \underset{\sim}{n}\,. \tag{I.88a}$$

Of course it holds that $\overline{\text{inc}}\, (\underset{\sim}{\beta}^P_{II} - \underset{\sim}{\beta}^P_{I}) = 0$ and $\overline{\overline{\text{inc}}}\, (\underset{\sim}{\beta}^T_{II} - \underset{\sim}{\beta}^T_{I}) = 0$.

The conditions of the boundary surface can be written as

$$\overline{\text{inc}}\, (\underset{\sim}{\beta}_{II} - \underset{\sim}{\beta}_{I}) = \underset{\sim}{\bar{H}}\,, \quad \overline{\overline{\text{inc}}}\, (\underset{\sim}{\beta}_{II} - \underset{\sim}{\beta}_{I}) = \underset{\sim}{\bar{\bar{H}}}\,. \tag{I.88b}$$

We can easily show that

$$(\overline{\text{inc}}\, \underset{\sim}{\beta})^S = \overline{\text{inc}}\, \underset{\sim}{\beta}^S, \quad (\overline{\overline{\text{inc}}}\, \underset{\sim}{\beta})^S = \overline{\overline{\text{inc}}}\, \underset{\sim}{\beta}^S. \tag{I.88c}$$

With $\underset{\sim}{\mathfrak{E}}^S = \underset{\sim}{\varepsilon}$ and $\underset{\sim}{H}^S = \underset{\sim}{\Upsilon}$, you get then

$$\overline{\text{inc}}\,(\epsilon_{II} - \epsilon_I) = \bar{\eta}\,, \quad \overline{\text{inc}}\,(\epsilon_{II} - \epsilon_I) = \bar{\bar{\eta}}\,. \tag{I.89}$$

These are the boundary conditions for the strains. From eq. (I.88a) the important practical relations follow,

$$\underset{\sim}{\bar{\eta}} = (\underset{\sim}{\alpha} \times \underset{\sim}{n}\,|_{II} - \underset{\sim}{\alpha} \times \underset{\sim}{n}\,|_I)^S + (\underset{\sim}{\bar{\alpha}} \times \overset{n}{\nabla})^S : \quad \underset{\sim}{\bar{\bar{\eta}}} = (\underset{\sim}{\bar{\alpha}} \times \underset{\sim}{n})^S \tag{I.90}$$

The eqs. (I.30) and (I.51) show, after a simple calculation, whether a body has or has not self stresses under the given conditions (dislocation density or applied distortions). We easily can show, that in the region where the linear theory of elasticity holds, with given $\underset{\sim}{\bar{\eta}}$, $\underset{\sim}{\bar{\bar{\eta}}}$ and $\underset{\sim}{\eta}$ the self stresses are determined uniquely (§14). In particular the self stresses vanish with simultaneously vanishing $\underset{\sim}{\bar{\eta}}$, $\underset{\sim}{\bar{\bar{\eta}}}$ and $\underset{\sim}{\eta}$. It may be possible to calculate relatively easily the self stresses, which belong to the given incompatibilities $\bar{\eta}$, $\bar{\bar{\eta}}$ and $\Upsilon$, (§13-15). That is why it should be mentioned at this point that we easily can formulate the First Boundary Value Problem (1BVP) of the theory of elasticity (boundary displacement given) by "$\underset{\sim}{\bar{\eta}}$, $\underset{\sim}{\bar{\bar{\eta}}}$ given." Then we imagine that the edge dislocation is maintained by the fact that the body adheres to a rigid surrounding, and then we can interpret the edge displacement $\underset{\sim}{s}$ as the displacement jump $-\underset{\sim}{g}$ as in §8. From the assumed $\underset{\sim}{g}$ the surface dislocation density is given by $-\underset{\sim}{n} \times \nabla\underset{\sim}{g}$, and from eq. (I.90) the associated $\underset{\sim}{\bar{\eta}}$ and $\underset{\sim}{\bar{\bar{\eta}}}$ follow in a simple way. For the solution of the problem "$\bar{\eta}$, $\bar{\bar{\eta}}$ given" see §15.

As the only application we take the case that along a plane boundary surface between two materials, the temperature, which is

constant in each, has a jump $\Delta T = T_2 - T_1$. Then we have to substitute only $\underset{\sim}{\beta}^Q$ according to eq. (I.44) into eq. (I.88), and after a simple calculation we obtain (because of the symmetry of $\underset{\sim}{\beta}^Q$) the surface incompatibility $\underset{\sim}{\bar{\bar{\eta}}}$, while $\underset{\sim}{\bar{\eta}}$ vanishes because of the constancy of $\beta$. An obvious explanation of the doublet of the incompatibilities $\bar{\bar{\eta}}$ follows in §23.

## §10. Some Problems of Large Distortions

As we mentioned during the derivation of the geometric basic equation, it holds for arbitrarily large distortions if we refer $\alpha_{ij}$ and $\beta_{ij}$ to the initial state. Perhaps we imagine some resistance which initially prevented the body from distorting during the development of the dislocation and its motion, respectively. The distortions which the volume element suffers after removing the resistance obviously can be related completely to the initial state. Only if we interpret it in this way does the geometric basic equation have the simple form (I.17). As we see at once, the distortions which are defined by

$$da_j = \beta^{\nu}_{ij}\, dx_i \tag{I.91}$$

referred to the initial state are additive (but not the strain and rotation by themselves). If we sum over a number of sequential distortions $\beta_{ij}$, then we obtain

$$d\,A_j \equiv \sum_{\nu} d\,a^{\nu}_j = \sum_{\nu}(\beta^{\nu}_{ij}\, dx^{\nu}_i) = (\sum_{\nu} \beta^{\nu}_{ij})\, dx_i \tag{I.92}$$

where the last equation only holds, if we always take for $dx_i$ the (constant) distance of the respective point in the initial state.

In this case also the distortion is composed additively from the strain and rotation, i.e.,

$$\beta_{ij} = \epsilon_{ij} + \omega_{ij} \tag{I.93}$$

where, as we know, $\omega_{ij}$ has the form [34] Vol. I, pg. 78, [86]

$$\omega_{ij} = (1 - \cos q)(k_i k_j - \delta_{ij}) + \sin q \; \epsilon_{ijk} \, k_k \tag{I.94}$$

if $k_i$ is a unit vector in the direction of the rotation axis and q the magnitude of the rotation angle. A given distortion can easily be decomposed according to eq. (I.93), since $\epsilon_{ij}$ is symmetric and $\omega_{ij}$ is completely determined by its antisymmetric part. Then eq. (I.93) is not a decomposition of the tensor $\beta_{ij}$ into its symmetric and antisymmetric part. So it is understandable that all equations in which $\epsilon_{ij}$ is assumed to be the symmetric part and $\omega_{ij}$ to be the antisymmetric part of $\beta_{ij}$, hold only for small distortions. The symmetric equations of incompatibility are affected by this in particular through $\underset{\sim}{\bar{\eta}}$, $\underset{\sim}{\bar{\bar{\eta}}}$ and $\underset{\sim}{\eta}$, while the asymmetric equations of incompatibility as eq. (I.51) also hold for large distortions. However, their importance has not been clarified sufficiently till now. If we relate all values to the final state, then we get a complicated form for the geometrical basic equation, which we will derive in Chapter IV. However, on the other hand, the statical basic equation (condition of equilibrium of all forces) assumes its known simple form only in the final state, while it becomes very complicated for the initial state. This means that we cannot use a simple form of the geometrical and statical basic equation simultaneously.

There is an important exception: When the rotations $\omega_{ij}$ (and especially their grain boundary parts $\theta_{ij}$) are large, but the strain

$\epsilon_{ij}$ small. This case is of such great importance because the rotation $\theta_{ij}$ [1] occurs without stresses, so there is no energy used and the

[1]This statement is only approximately true in the crystal (§23) in contrast to the continuum.

dislocations arrange themselves primarily in such a way that $\epsilon_{ij}$ is as small as possible, whereas $\epsilon_{ij}$ may be larger.

However, in the case of pure rotations $\theta_{ij}$, the total distortion $\theta^P_{ij} + \theta_{ij} = 0$ (see §5); i.e., all volume elements remain in their place, only the orientation of the lattice is rotated.[2] If there are

[2]This case corresponds to Fig. 16c, but not to Fig. 16b, where the layers $\delta x_1$ did not suffer a pure (plastic) rotation, but were simultaneously plastically strained, which is the reason for the total displacement of the layer. If we reestablish the connection with elastic rotations, then the body seems to be bent, whereas it is (in the case of small plastic distortions) still without stresses.

simultaneously small strains $\epsilon_{ij}$, then the volume elements only suffer small displacements, and it is not necessary that we distinguish between the initial and final state, e.g., the conditions of equilibrium remain in their simple form if they were taken in the coordinates of the initial state. In the next section we will show how we can decompose the dislocation density into a part which causes the rotation $\theta_{ij}$ and one which

causes the strain $\epsilon_{ij}$ by calculating the rotation of the lattice. To determine the elastic strain and the self stresses, respectively, is only a problem of the linear theory of elasticity.

## §11. Determination of the Distortions of A Body Containing Dislocations

The main problem of the theory of dislocations is to calculate the elastic distortions, self stresses and rotations and curvature of the lattice, respectively, corresponding to a certain distribution of dislocations $\underset{\sim}{\alpha}$, $\underset{\sim}{\bar{\alpha}}$. $\underset{\sim}{\alpha}$, $\underset{\sim}{\bar{\alpha}}$ are not entirely arbitrary functions of position since they must satisfy the condition that it is not possible for dislocations to end in the interior of a body or at the boundary surface. However, dislocations $\underset{\sim}{\bar{\alpha}}$, which are in the boundary surface, may move out of this and contribute in this way to the space density $\alpha$. Equation (I.18)

$$\nabla_i \alpha_{ij} = 0 \tag{I.95}$$

is a necessary and sufficient condition for the dislocation $\underset{\sim}{\alpha}$ not to end. Additionally, we have for each boundary surface the equation[1]

$$\overset{n}{\nabla}_i \bar{\alpha}_{ij} + n_i(\alpha_{ij}\,|_{II} - \alpha_{ij}\,|_{I}) = 0 \tag{I.96}$$

which says that where the dislocations $\underset{\sim}{\bar{\alpha}}$ have divergences, the dislocations $\underset{\sim}{\alpha}$ meet a boundary surface and join it.

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[1] For $\overset{n}{\nabla}$ see eq. (I.82).

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If we substitute in this equation $\underset{\sim}{\bar{\alpha}} = \underset{\sim}{n} \times (\underset{\sim}{\beta}_{II} - \underset{\sim}{\beta}_{I})$ and $\underset{\sim}{\alpha} = \nabla \times \underset{\sim}{\beta}$, then it is satisfied identically, which we can take as a proof for eq. (I.96).

The given dislocation distribution must always satisfy the conditions (I.95) and (I.96).

The simplest problem in which $\underset{\sim}{\alpha}$ and $\underset{\sim}{\bar{\alpha}}$ are important is that of a body which adheres to a rigid surrounding. We will restrict the following considerations to this (at a free edge $\underset{\sim}{\bar{\alpha}} = 0$).

The above mentioned problem can be solved at present for small distortions most simply by first calculating the stresses (and so also the strains) and then the rotations. For this we have to calculate the incompatibilities $\underset{\sim}{\eta}$, $\underset{\sim}{\bar{\eta}}$, $\underset{\sim}{\bar{\bar{\eta}}}$ from $\underset{\sim}{\alpha}$, $\underset{\sim}{\bar{\alpha}}$, which is very simple according to §9. After calculating the stresses belonging to $\underset{\sim}{\eta}$, $\underset{\sim}{\bar{\eta}}$, $\underset{\sim}{\bar{\bar{\eta}}}$ we get the strain $\underset{\sim}{\epsilon}$ from Hooke's Law. If we now write the basic equation (I.17) in the form

$$\text{curl } \underset{\sim}{\omega} = \underset{\sim}{\alpha} - \text{curl } \underset{\sim}{\epsilon} \tag{I.97}$$

then the right-hand side is now known. So after an easy calculation[1] eq. (I.97) becomes

$$\delta_{k\ell} \nabla_m \omega_m - \nabla_\ell \omega_k = (\underset{\sim}{\alpha} - \text{curl } \underset{\sim}{\epsilon})_{k\ell} \tag{I.98}$$

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[1]For curl $\underset{\sim}{\omega}$ we can write $\epsilon_{kji} \nabla_j \omega_{i\ell} = \epsilon_{kji} \epsilon_{i\ell m} \nabla_j \omega_m$. (I.98) follows according to the decomposing formula (A2).

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and by contracting

$$2 \nabla_m \omega_m = (\underset{\sim}{\alpha} - \text{curl } \epsilon)_{mm} \tag{I.99}$$

If we substitute this in eq. (I.98), we get

$$\nabla_\ell \omega_k = \frac{1}{2} \delta_{k\ell} (\underset{\sim}{\alpha} - \text{curl } \underset{\sim}{\epsilon})_{mm} - (\underset{\sim}{\alpha} - \text{curl } \underset{\sim}{\epsilon})_{k\ell} \tag{I.100}$$

where the right-hand side is still a known function. By a simple integration we obtain from this the rotation of the structure up to a constant.

It is remarkable that the BVP which is to be solved for the stresses contains the form "$\bar{\eta}$ $\bar{\eta}$ given" initially, see §9.

Now we will describe a method for calculating the rotation of the structure before determining the stresses, which is presently derived only for infinite bodies [81], however it may be possible to extend it without much trouble for finite bodies. The starting point is the basic equation in the form (I.43)

$$\text{curl}\ (\text{inc}\ \underset{\sim}{i} + \underset{\sim}{\theta}) = \underset{\sim}{\alpha} \tag{I.101}$$

It is easy to show that $(\text{curl inc}\ \underset{\sim}{i})_{ii}$ vanishes identically because inc $\underset{\sim}{i}$ is a symmetrical tensor. Thereafter it follows similar to before

$$(\text{curl inc}\ \underset{\sim}{i})_{k\ell} - \nabla_\ell\ \theta_k = \alpha_k - \frac{1}{2}\alpha_{mm}\ \delta_{k\ell} \tag{I.102}$$

The left-hand side is (for small distortions) according to eq. (I.60) equal to minus the curvature tensor $K_{k\ell}$. By taking the rotation from the right formerly we obtained $\underset{\sim}{\eta}$, see eq. (I.63). Now we take the divergence from the right where the first term vanishes (since inc $\underset{\sim}{i} = \nabla \times \underset{\sim}{i} \times \nabla$):

$$\Delta\ \theta_k = \nabla_\ell\ (\frac{1}{2}\delta_{k\ell}\ \alpha_{mm} - \alpha_{k\ell}) \tag{I.103}$$

Here the right-hand side is known. $\theta_k$ follows by integration, undetermined up to a harmonic vector. The lack of uniqueness arises from the fact that the decomposition of $\underset{\sim}{\beta}$ into grad $\underset{\sim}{s}$ + curl $\underset{\sim}{\zeta}$ in a finite

body is not unique; moreover, we can add an arbitrary part of the distortion which is related to the surface density $\bar{\underset{\sim}{\alpha}}$, in the form grad $\underset{\sim}{s}$ or curl $\underset{\sim}{\zeta}$. On the contrary in the infinite body $\theta_k$ is determined uniquely by eq. (I.103) (assuming $\underset{\sim}{\alpha}$ vanishes at infinity).

As mentioned in the discussion following eq. (I.96), a rotation tensor is already determined by its antisymmetric part. So the integration of eq. (I.103) gives after a short calculation the rotation tensor with $\underset{\sim}{\theta}$ as antisymmetric part, which we will call $\tilde{\underset{\sim}{\theta}}$. This is the grain boundary part of the rotation of the structure. So it is

$$\epsilon^{inc} = \text{inc}\ \underset{\sim}{i} + \underset{\sim}{\theta} - \tilde{\underset{\sim}{\theta}} \tag{I.104}$$

where $\epsilon^{inc}$ obviously is the incompatible part of the strain. The basic equation assumes the form

$$\text{curl}\ \epsilon^{inc} = \underset{\sim}{\alpha} - \text{curl}\ \tilde{\underset{\sim}{\theta}} \tag{I.105}$$

where the right-hand side is known. Then we can calculate the incompatibility tensor $\underset{\sim}{\eta}$ which belongs to the elastic deformation $\underset{\sim}{\epsilon}$, by taking a further curl from the right, which permits the calculation of the stresses in a simple way if $\underset{\sim}{\epsilon}$ is small. At least in the case of infinite bodies, this method allows the determination of the strain from the dislocation density if the rotations are large and the strains small.[1] During these investigations we no longer need to distinguish between the initial and final state.

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[1] We did not consider the case where the component $(\nabla_i s_j - \nabla_j s_i)/2$ of $\omega_{ij}$ is large. This case is known from the ordinary theory of elasticity. See e.g., Truesdell [153], who mentions the methods which we then have to use. We will not add anything to that there.

The case of large rotations and small strains cannot be treated according to the first method. The symmetric part of the rotation tensor, however, contributes to the values $\underset{\sim}{\eta}$, $\underset{\sim}{\bar{\eta}}$, $\underset{\sim}{\bar{\bar{\eta}}}$, which were determined from $\underset{\sim}{\alpha}$ $\underset{\sim}{\bar{\alpha}}$ by the use of the formulas of §9, and we are not allowed to neglect this part relative to the strains for large deformations. In eq. (I.105), however, the initially calculated part of the rotation tensor is considered exact.

The particular case investigated above is at least as important as the case of stresses in connection with large distortions, which has not been treated. However, during metal manufacturing, we often find plastic deformations of 10 and 100%, but these must be mostly in the form of grad $\underset{\sim}{s}^{p}$. Deformations of the type curl $\underset{\sim}{\zeta}^{p}$ cause simultaneous elastic distortions, curl $\underset{\sim}{\zeta}$, the symmetric part of which cause stresses. With the relatively weak forces with which we deform plastically, we never can produce elastic deformations of 10 and 100%; i.e., in most cases we can consider the strain part of curl $\underset{\sim}{\zeta}$ as small and so also that $(\underset{\sim}{\epsilon})$ of $\underset{\sim}{\beta}$.

## CHAPTER II

## DISLOCATIONS IN THE CONTINUUM: STATICS

### Preface

Elastostatics is the theory of the forces which are applied to a medium, and the problems which are investigated by this theory especially consider the calculation of the internal forces (stresses) in a body which result from any external sources. In our considerations the sources are mostly dislocations, also quasi-dislocations, according to §6. In the previous literature we will find almost entirely calculations which investigate singular lines of dislocation or at most perhaps calculations about two-dimensional arrangements of dislocations. We can treat these problems comprehensively with methods of the classical theory of elasticity. The reason is that the elastic strain field has the simple form $\epsilon_{ij} = (\nabla_i s_j + \nabla_j s_i)/2$ beyond the dislocation as in the classical theory of elasticity. However, in the case of dislocations which are distributed three-dimensionally in the whole body it is not possible to derive the elastic strain from a field of displacements, and in principle a new method is necessary, e.g., for calculating the self stresses $\sigma_{ij}$ which belong to a distribution of dislocations $\alpha_{ij}$. Of course, also in a body with dislocations, the conditions of equilibrium of elastostatics must be satisfied, which can be written in the form

$$\operatorname{div} \underset{\sim}{\sigma} = 0 \qquad \text{(II.1)}$$

when there are no body forces. The equation states that the tensor of self stresses is a special tensor, for it is a tensor of incompatibility

$$\underset{\sim}{\sigma} = \text{inc}\ \underset{\sim}{\chi} \tag{II.2}$$

which follows at once from eq. (I.42).[1] The symmetrical tensor $\chi \equiv (\chi_{ij})$ is called "2nd order stress function tensor," since its components are the stress functions of Maxwell and Morera.[2] In

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[1]Equation (II.2) was written first by Beltrami [161] but investigated no further. See, for example, (II.2), also Finzi [43].

[2]The addition "2nd order" should remind us that in order to get stresses, we have to differentiate the stress functions twice. We need this addition sometimes to distinguish these stress functions from others as we will see later.

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contrast to previous opinions, stress functions are also useful aids for three-dimensional problems of the theory of elasticity. Moreover, in the case of three-dimensionally distributed dislocations, where the method based on the displacement field fails, stress functions are necessary. But we should not only consider them as a convenient aid to calculation, for their role in the continuum theory of dislocations is of profound importance. This can be clarified best by the remark that the stress function tensor is the analogue of the often used vector potential $\underset{\sim}{A}$ in electrodynamics. By its use we satisfy the Maxwell equation div $\underset{\sim}{B} \doteq 0$ identically; in a similar manner the conditions of equilibrium (II.1) are satisfied by the use of the stress function tensor.

## §12 The Stress Function Tensor

Usually we define the stress tensor $\underset{\sim}{\sigma}$ by the differential form

$$dp_j = \sigma_{ij}\, d\, F_i, \tag{II.3}$$

where $dp_j$ is the force which is applied on the cut surface $dF_i$ if no displacement with respect to the cut should occur.[1]

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[1] In contrast to isotropic bodies, in a crystal asymmetric stress tensors also play a certain role, which we will consider in §19. In all the other sections we assume that the stress tensor is symmetrical in order to avoid useless difficulties. However, it is useful even then to maintain the order of the indices defined by eq. (II.3). (The 1st index shows the surface element, the 2nd the applied force.)

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Time independent continuum mechanics of solids, as far as we consider the state of the body, is governed by the equations

$$\text{curl}\ \underset{\sim}{\beta} = \underset{\sim}{\alpha} \tag{II.4}$$

$$\text{div}\ \underset{\sim}{\sigma} = -\underset{\sim}{\mathcal{F}} \tag{II.5}$$

and in addition we have the equation of energy density

$$e = \frac{1}{2}\sigma_{ij}\,\epsilon_{ij} \tag{II.6}$$

The plastic deformation grad $\underset{\sim}{s}^P$, which does not change the state of the body, is not contained in these equations. In addition to eq. (II.4) to eq. (II.6), we have the constitutive equation, for which

we always take Hooke's Law

$$\sigma_{ij} = c_{ijk\ell}\, \epsilon_{k\ell} \tag{II.7}$$

As mentioned in §11, generally we can consider it to be satisfied with metals even for large plastic deformations.

$c_{ijk\ell}$ is the Hooke's tensor of the elastic moduli with the symmetry properties

$$c_{ijk\ell} = c_{jik\ell} = c_{ij\ell k} = c_{k\ell ij} \tag{II.8}$$

In the case of elastic isotropy, we have[1]

$$c_{ijk\ell} = \lambda\, \delta_{ij}\, \delta_{k\ell} + \mu(\delta_{ik}\, \delta_{j\ell} + \delta_{i\ell}\, \delta_{jk}) \ , \tag{II.9}$$

where $\lambda$, $\mu$ are the Lame's constants. The tensor of the elastic coefficients $s_{ijk\ell}$, which is reciprocal to $c_{ijk\ell}$ is defined by

$$c_{ijk\ell}\, s_{k\ell mn} = \frac{1}{2}\,(\delta_{im}\, \delta_{jn} + \delta_{in}\, \delta_{jm}) \tag{II.10}$$

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[1] See, e.g., [34], Vol. III, pg. 60.

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For isotropy it holds

$$s_{ijk\ell} = \underset{\sim}{\lambda}'\, \delta_{ij}\, \delta_{k\ell} + \underset{\sim}{\mu}'(\delta_{ik}\, \delta_{j\ell} + \delta_{i\ell}\, \delta_{jk}) \tag{II.11}$$

with

$$\underset{\sim}{\lambda}' = \frac{(1/2)G}{m+1} \ , \qquad \underset{\sim}{\mu}' = (1/4)G \tag{II.12}$$

where $G = \mu$, the shear modulus, m is the Poisson's ratio. Then the Hooke's Law takes the form

$$2G\,\epsilon_{ij} = \sigma_{ij} - \frac{1}{m+1}\sigma_{kk}\,\delta_{ij}, \quad \sigma_{ij} = 2G(\epsilon_{ij} + \frac{1}{m-2}\epsilon_{kk}\,\delta_{ij}) \tag{II.13}$$

We showed in §6 that it is easy to calculate the incompatibility tensor which belongs to a distribution of dislocations. So we now consider eq. (I.52) instead of eq. (II.4)

$$\text{inc}\ \underset{\sim}{\epsilon} = \underset{\sim}{\eta} \tag{II.14}$$

In this the rotations of the structure are initially omitted according to §6. Now the conditions of equilibrium are identically satisfied by the stress function (II.2) and it is unnecessary to consider them further. In Cartesian coordinates, eq. (II.2) can be written

$$\begin{aligned} \sigma_{11} &= \frac{\partial^2\chi_{22}}{\partial x_3^2} - \frac{\partial^2\chi_{33}}{\partial x_2^2} + 2\frac{\partial^2\chi_{33}}{\partial x_2\,\partial x_3} \\ \sigma_{12} &= -\frac{\partial}{\partial x_3}\left(-\frac{\partial\chi_{12}}{\partial x_3} + \frac{\partial\chi_{23}}{\partial x_1} + \frac{\partial\chi_{31}}{\partial x_2}\right) + \frac{\partial^2\chi_{33}}{\partial x_1\,\partial x_2} \end{aligned} \tag{II.15}$$

in addition to these we have the four equations obtained by cyclic change of the indices. If we take $\chi_{11} = \chi_{22} = \chi_{33} = 0$, we have the well-known equation of Morera [102]; with $\chi_{12} = \chi_{23} = \chi_{31} = 0$, we obtain Maxwell's equation [99] if we assume $\partial/\partial x_3 = 0$, i.e., two-dimensional state of stress, we will get from eq. (II.15)

$$\sigma_{11} = \frac{\partial^2\chi_{33}}{\partial x_2^2}, \quad \sigma_{22} = -\frac{\partial^2\chi_{33}}{\partial x_1^2}, \quad \sigma_{12} = \frac{\partial^2\chi_{33}}{\partial x_1\,\partial x_2} \tag{II.16}$$

$$\sigma_{23} = -\frac{\partial}{\partial x_1}\left(-\frac{\partial \chi_{23}}{\partial x_1} + \frac{\partial \chi_{31}}{\partial x_2}\right), \quad \sigma_{31} = \frac{\partial}{\partial x_2}\left(-\frac{\partial \chi_{23}}{\partial x_1} + \frac{\partial \chi_{31}}{\partial x_2}\right) \tag{II.16'}$$

$$\sigma_{33} = -\frac{\partial^2 \chi_{11}}{\partial x_2^2} - \frac{\partial^2 \chi_{22}}{\partial x_1^2} + 2\,\frac{\partial^2 \chi_{12}}{\partial x_1\,\partial x_2}\ . \tag{II.16''}$$

Equations (II.16) exactly represent Airy's stress function for a two-dimensional problem of stresses.[1] If we set the terms in parentheses in eqs. (II.16') equal to the function $\underset{\sim}{\Phi}$, we have the well-known stress function of torsion.[2] Notice that every stress function $\chi_{ij}$ appears

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[1] In most cases $\chi = -\chi_{33}$ is called Airy's stress function.

[2] See, e.g., Love [95] or Biezeno-Grammel [1].

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only in one of the three lines (II.16), which means that the related states of stress are independent of each other, at least with respect to the equilibrium conditions.

Maxwell [99] and Morera [102] showed that it is possible to describe with their functions every state of stress by div $\underset{\sim}{\sigma} = 0$. That the symmetric tensor $\underset{\sim}{\chi}$ has only three degrees of freedom is caused by the fact that according to eq. (I.42) $\underset{\sim}{\chi}^{o} = \text{def}\ \underset{\sim}{q}$ contributes nothing to $\underset{\sim}{\sigma}$. Therefore, a stress function tensor of the form def $\underset{\sim}{q}$ is called "tensor of the zero stress functions" [126]. So we can subject the tensor $\underset{\sim}{\chi}$ to certain secondary conditions; those of Maxwell are $\chi_{12} = \chi_{23} = \chi_{31} = 0$; those of Morera, $\chi_{11} = \chi_{22} = \chi_{33} = 0$; but in all cases we first have to prove that these secondary conditions are "permitted." We say a secondary condition is permitted if it is

possible to describe every arbitrary state of stresses which satisfies the equation div $\underset{\sim}{\sigma} = 0$ by use of the totality of the stress function which is restricted by the secondary conditions.

The conditions of the equilibrium are fulfilled by the stress function. Further, the conditions of compatibility place additional restrictions on the stress function; we will obtain those if we substitute $\underset{\sim}{\sigma}$ of eq. (II.2) into eq. (II.14) and use Hooke's Law (II.7)

$$\text{inc}\,[s_{ijk\ell}\,(\text{inc}\,\underset{\sim}{\chi})_{,k\ell}] = \underset{\sim}{\eta} \tag{II.17}$$

It is not worth while to write these equations in detail. Even in the case of Maxwell and Morera they are very complicated; therefore, these functions are never used.

For the following treatment of eq. (II.17), we define--restricted to elastic isotropy--the symmetrical tensors $\chi'_{ij}$, $\eta'_{ij}$ by the equations

$$2G\,\chi'_{ij} = \chi_{ij} - \frac{1}{m+2}\,\chi_{kk}\,\delta_{ij}, \quad \chi_{ij} = 2G(\chi'_{ij} + \frac{1}{m-1}\,\chi'_{kk}\,\delta_{ij}) \tag{II.18}$$

$$\eta'_{ij} = 2G(\eta_{ij} + \frac{1}{m-1}\,\eta_{kk}\,\delta_{ij}), \quad 2G\,\eta_{ij} = \eta'_{ij} - \frac{1}{m+2}\,\eta'_{kk}\,\delta_{ij} \tag{II.18'}$$

With the secondary condition

$$\nabla_i\,\chi'_{ij} = 0 \tag{II.19}$$

The equations (II.17) get the simpler form [77]

$$\Delta\Delta\,\chi'_{ij} = \eta_{ij} \tag{II.20}$$

or simultaneously

$$\Delta\Delta\,\chi_{ij} = \eta'_{ij} \tag{II.20'}$$

as we will show now.

First, by use of the formulas of decomposition (A.1)

$$\eta_{ij} = (\text{inc}\ \underset{\sim}{\epsilon})_{,ij} = -\epsilon_{jk\ell}\,\epsilon_{jmn}\,\nabla_k\,\nabla_m\,\epsilon_{\ell n}$$

$$= \Delta\,\epsilon_{ij} - (\nabla_i\,\nabla_k\,\epsilon_{kj} + \nabla_j\,\nabla_k\,\epsilon_{ki}) + \nabla_k\,\nabla_\ell\,\epsilon_{k\ell}\,\delta_{ij}$$

$$+ \nabla_i\,\nabla_j\,\epsilon_{kk} - \Delta\,\epsilon_{kk}\,\delta_{ij} \qquad \text{(II.21)}$$

and by use of the Hooke's Law (II.13) and the condition of equilibrium $\nabla_i\,\sigma_{ij} = 0$, it follows easily

$$\Delta\,\sigma_{ij} + \frac{m}{m+1}\,(\nabla_i\,\nabla_j\,\sigma_{kk} - \Delta\,\sigma_{kk}\,\delta_{ij}) = 2G\,\eta_{ij} \qquad \text{(II.22)}$$

These equations are known in the case of $\eta_{ij} = 0$ (from which it follows that $\sigma_{kk} = 0$) as the Beltrami's equation. Now we let $\sigma_{ij} = (\text{inc}\ \chi_{ij})$ and assume that these equations are written in a form similar to eq. (II.21) and substitute $\chi'_{ij}$ according to eq. (II.18). Then we get, by use of (II.19)

$$\sigma_{ij}/2G = \Delta\,\chi'_{ij} + \frac{m}{m-1}\,(\nabla_i\,\nabla_j\,\chi'_{kk} - \Delta\,\chi'_{kk}\,\delta_{ij}) \qquad \text{(II.23)}$$

This equation substituted in eq. (II.22) gives us eq. (II.20) directly as we easily can check. The secondary conditions of (II.19) are sufficient but not necessary for us to get eq. (II.17) from eq. (II.20). We get the necessary and sufficient conditions if we substitute $\sigma_{ij}$ into eq. (II.22), which is calculated without the conditions (II.19). Then we get

$$\Delta\Delta\,\chi'_{ij} - \Delta(\nabla_i\,\nabla_k\,\chi'_{kj} + \nabla_j\,\nabla_k\,\chi'_{ki}) + \frac{m}{m+1}\,\nabla_i\,\nabla_j\,\nabla_k\,\nabla_\ell\,\chi'_{k\ell}$$

$$+ \frac{1}{m+1}\,\Delta\,\nabla_k\,\nabla_\ell\,\chi'_{k\ell}\,\delta_{ij} = \eta_{ij}\,, \qquad \text{(II.24)}$$

and the necessary and sufficient conditions are obviously

$$- \Delta(\nabla_i \nabla_k \chi'_{kj} + \nabla_j \nabla_k \chi'_{ki}) + \frac{m}{m+1} \nabla_i \nabla_j \nabla_k \nabla_\ell \chi'_{k\ell}$$

$$+ \frac{1}{m+1} \Delta \nabla_k \nabla_\ell \chi'_{k\ell} \delta_{ij} = 0 \qquad \text{(II.25)}$$

which is identical to

$$\text{def}\ [(m+1)\ \Delta \underset{\sim}{p} - m\nabla\nabla \underset{\sim}{p}] - \Delta\ \nabla\ \underset{\sim}{p} \cdot \underset{\sim}{I} = 0\ ;\ \underset{\sim}{p} \equiv \nabla \cdot \underset{\sim}{\chi}'$$

They are satisfied by the stronger conditions (II.19). For the proof that the conditions (II.19) are allowed, which naturally contains the conditions (II.25), we refer to the original paper [77] in which the author initially presented the conditions (II.19). This was also found independently by Marguerre [98].

## §13 Solution of the Superposition Problem by Self Stresses

The first problems which were solved by three-dimensional stress functions were related to an infinite body. Its volume is V. Then we have only a superposition problem, but no boundary value problem.

The stress function tensor $\underset{\sim}{\chi}$ of the related problem must satisfy the necessary and sufficient conditions (II.17). Substituted these by the sufficient conditions (II.20) and (II.20'), respectively, and (II.19). The two first mentioned equations are satisfied by the expressions[1]

$$\chi_{ij}(\underset{\sim}{x}) = - \frac{1}{8\pi} \iiint_V \eta'_{ij}(\underset{\sim}{x}') \ |\underset{\sim}{x}-\underset{\sim}{x}'| \ dV' \qquad \text{(II.26)}$$

$$\chi'_{ij}(\underset{\sim}{x}) = - \frac{1}{8\pi} \iiint_V \eta_{ij}(\underset{\sim}{x}') \ |\underset{\sim}{x}-\underset{\sim}{x}'| \ dV' \qquad \text{(II.26')}$$

[1] If we substitute eq. (II.26) into (II.27), then we get the stresses as a function of the distribution of the incompatibilities after carrying out the differentiation. These formulas were first mentioned by Moriguti [102] (without the use of the stress function tensor), who proved it by direct verification. I thank Dr. J. D. Eshelby for calling my attention to the paper of Moriguti (March 1957).

as we know from the theory of the equations of the bipotentials. From the identity $\nabla_i \eta_{ij} = 0$, it easily follows that the secondary equations are also fulfilled.

We have seen in the case of Maxwell's and Morera's functions that the tensor inc $\underset{\sim}{\chi}$ has only three, not six, degrees of freedom. That holds for every tensor of incompatibility, so also for $\underset{\sim}{\eta}$ and $\underset{\sim}{\eta}'$. So the six integrations can be reduced to three as follows.

As mentioned above, $\underset{\sim}{\chi}'$ of eq. (II.26) is a tensor of incompatibility (div $\underset{\sim}{\chi}' = 0$), since $\underset{\sim}{\eta}$ is such a tensor. Also it is easy to show that $\underset{\sim}{\chi}$ in eq. (2.26) becomes a deformator if we substitute it for $\eta'$. However, a deformator does not contribute to the stresses, since $\underset{\sim}{\sigma}$ = inc $\underset{\sim}{\chi}$ according to eq. (I.42). So obviously we get the same stresses if we add an arbitrary deformator to the real $\underset{\sim}{\eta}'$ and we replace $\underset{\sim}{\eta}'$ in eq. (2.26) by the resulting tensor ($\underset{\sim}{\eta}''$). However, we can choose $\underset{\sim}{\eta}''$ in such a way that, e.g., $\eta''_{11} = \eta''_{22} = \eta''_{33} = 0$ or $\eta''_{12} = \eta''_{23} = \eta''_{31} = 0$. Accordingly, it becomes $\chi_{11} = \chi_{22} = \chi_{33} = 0$ or $\chi_{12} = \chi_{23} = \chi_{31} = 0$. So we get a form in Maxwell's or Morera's functions, respectively, which is very useful for determining the states of self stresses.

Now the calculation of $\underset{\sim}{\eta}''$ is very easy. If we let be

$$\underset{\sim}{\eta}'' = \text{def } \underset{\sim}{a} + \underset{\sim}{\eta}' \tag{II.27}$$

then, e.g., we have

$$\partial a_1/\partial x_1 = -\eta'_{11}\ , \quad \partial a_2/\partial x_2 = -\eta'_{22}, \quad \partial a_3/\partial x_3 = -\eta'_{33} \tag{II.28}$$

from which we get useful functions $a_1$, $a_2$, $a_3$ after ordinary integration. We obtain $\underset{\sim}{\eta}''(= \eta''_{ij})$, $i \neq j$, if we substitute this in (II.27). From this follows that Morera's stress function of the self stresses is

$$\chi_{ij}(\underset{\sim}{x}) = -\frac{1}{8\pi}\iiint_V \eta''_{ij}(\underset{\sim}{x}')\ |\underset{\sim}{x}-\underset{\sim}{x}'|\ dV' \qquad i \neq j \tag{II.29}$$

Then the simple formulas of Morera hold for the stresses

$$\sigma_{11} = 2\,\frac{\partial^2\chi_{23}}{\partial x_2\,\partial x_3}\ , \text{ etc.}$$

$$\sigma_{23} = -\frac{\partial}{\partial x_1}\left(-\frac{\partial\chi_{23}}{\partial x_1} + \frac{\partial\chi_{21}}{\partial x_2} + \frac{\partial\chi_{12}}{\partial x_3}\right) \tag{II.30}$$

In a similarly easy way we can determine such values for $a_1$, $a_2$, $a_3$

$$\frac{1}{2}\left(\frac{\partial a_1}{\partial x_2} + \frac{\partial a_2}{\partial x_1}\right) = -\eta'_{12}\ , \text{ etc.} \tag{II.31}$$

and from eq. (II.27) we get Maxwell's stress functions of the self stresses to be

$$\chi_{ij}(\underset{\sim}{x}) = -\frac{1}{8\pi}\iiint_V \eta''_{ij}(\underset{\sim}{x})\ |\underset{\sim}{x}-\underset{\sim}{x}'|\ dV' \qquad i = j \tag{II.32}$$

The stresses follow from these

$$\sigma_{11} = -\frac{\partial^2 \chi_{22}}{\partial x_3^2} - \frac{\partial^2 \chi_{33}}{\partial x_2^2}, \quad \text{etc.}$$

$$\sigma_{23} = \frac{\partial^2 \chi_{11}}{\partial x_2 \, \partial x_3}, \quad \text{etc.} \tag{II.33}$$

With finite media, a boundary value problem always follows after the problem of superposition. Before we treat this, we must first of all investigate whether or not we are allowed to apply to finite media those methods which we have derived for calculating the particular integral of the differential equation (II.17) in the case of infinite media.

We can easily show that in the finite medium the $\underset{\sim}{\chi}'$ of eq. (II.26') does not generally satisfy the secondary condition div $\underset{\sim}{\chi}' = 0$. So it is obviously not certain that this $\underset{\sim}{\chi}'$ represents a solution of eqs. (II.17). However, since div $\underset{\sim}{\chi}' =$ is a permitted condition, there must be a solution of $\Delta\Delta \, \underset{\sim}{\chi}' = \underset{\sim}{\eta}$ for which we have div $\underset{\sim}{\chi}' = 0$ and so fulfills eqs. (II.17).

To get such a solution, we must look for a continuation of the function $\underset{\sim}{\eta}'$ in the volume external to the medium, which matches $\underset{\sim}{\eta}'$ at the boundary surface and is continuous and differentiable across it, and which vanishes very strongly at infinity. It is not difficult to get such a continuation. We call the function which we find by this method and which agrees with $\underset{\sim}{\eta}'$ at the surface of the body, $\underset{\sim}{\eta}'_f$. We substitute this function for $\underset{\sim}{\eta}'$ into eq. (II.26) and integrate over the infinite volume. So we get a field of stress functions which fulfills the differential equation (II.20') and the secondary conditions (II.19)

in the whole body; thus it represents the desired particular solution of eq. (II.17). Now we will add a deformator, def $\underset{\sim}{a}$, to $\underset{\sim}{\eta}'_f$ as above so that we again obtain a Maxwell's or Morera's function. If this method succeeds, the deformator does not contribute to the stresses. It can easily be shown that this holds, but we dispense with the proof. So it is possible to calculate a tensor $\underset{\sim}{\eta}''_f = \underset{\sim}{\eta}'_f + \text{def } \underset{\sim}{a}$ even in a finite medium, where only three components of $\underset{\sim}{\eta}''_f$ are different from zero. Then it follows that $\Delta\Delta\, \underset{\sim}{\chi} = \underset{\sim}{\eta}''_f$ and eqs. (II.29) and (II.32) with $\underset{\sim}{\eta}''_f$ instead of $\underset{\sim}{\eta}''$ are particular integrals of these equations, which simultaneously satisfy the equation (II.17).

It should be mentioned that in the case of given distributions of dislocations and incompatibilities, respectively, the methods which are derived in this section are in practice the only ones available to solve the superposition problem.[1] In the case of quasi-dislocations (§6)

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[1]For this, see also, Eshelby [41], S. 91 ff.

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where primarily the applied strains $\underset{\sim}{\epsilon}^T$ are given, there still exists the method which is known from the old theory of thermal stresses of Duhamel [33] and Neumann [112]. By use of the expression

$$\nabla_i\, c_{ijk\ell}\, \epsilon^Q_{k\ell} = \mathcal{F}^Q_j \tag{II.34}$$

the related "quasi forces" $\mathcal{F}^Q_j$ and the displacement field associated with these are calculated according to the well-known methods. From this the total strain $\underset{\sim}{\epsilon}^T$ follows by taking the deformator. Then the elastic strain is $\underset{\sim}{\epsilon} = \underset{\sim}{\epsilon}^T - \underset{\sim}{\epsilon}^Q$ from which the stresses follow by use

of Hooke's Law. This method might require the same effort as that of the stress function.

Finally, we will mention a further method which is only applicable to infinite media at this time. We take instead of (II.2)[1]

$$\underset{\sim}{\sigma} = \text{curl}\ \underset{\sim}{\varphi} \tag{II.35}$$

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[1]Guenther [61] and Schaefer [126] use a stress function tensor $\gamma_{ijk} = \epsilon_{ij\ell}\,\varphi_{\ell k}$ for other cases.

---

where $\underset{\sim}{\varphi} \equiv (\varphi_{ij})$ is the asymmetrical 1st order stress function tensor (because it will be differentiated only once to get the stresses). Then it is obvious that $\underset{\sim}{\varphi} = \underset{\sim}{\chi} \times \nabla$, from which it follows

$$\varphi_{ii} = 0 \qquad\qquad \nabla_j\,\varphi_{ij} = 0 \tag{II.36}$$

By taking the curl from the right hand side we get with help of (II.18) for the secondary condition (II.19) which contains $\underset{\sim}{\chi}$

$$\nabla_i\,\varphi_{ij} = 0 \tag{II.37}$$

as we can easily check. Since $\chi$ has three degrees of freedom by the restriction (II.19), $\underset{\sim}{\varphi}$ also has three by the restrictions (II.36) and (II.37). With Hooke's Law, it follows from eq. (II.35)

$$\epsilon_{ij} = s_{ijk\ell}\,\epsilon_{kmn}\,\nabla_m\,\varphi_{n\ell} \tag{II.38}$$

and by taking the curl from the left hand side, we get

$$\epsilon_{ghi}\,\nabla_h\,\epsilon_{ij} \equiv \alpha'_{gj} = s_{ijk\ell}\,\epsilon_{ghi}\,\epsilon_{kmn}\,\nabla_n\,\nabla_m\,\varphi_{n\ell} \tag{II.39}$$

which becomes in the case of isotropy, according to eq. (II.11),

$$\alpha'_{gj} = \lambda' \, \epsilon_{ghj} \, \epsilon_{\ell mn} \, \nabla_h \, \nabla_m \, \varphi_{n\ell}$$

$$+ \mu'(\epsilon_{ghk} \, \epsilon_{kmn} \, \nabla_h \, \nabla_m \, \varphi_{nj} + \epsilon_{gh\ell} \, \epsilon_{jmn} \, \nabla_h \, \nabla_m \, \varphi_{n\ell}) \, . \qquad \text{(II.40)}$$

After multiplying with $\epsilon_{fgj}$, we obtain by use of eq. (A.2,3) and (II.36)

$$\epsilon_{fgj} \, \alpha'_{gj} = - \, 2(\lambda' + \mu') \, \epsilon_{\ell mn} \, \nabla_f \, \nabla_m \, \varphi_{n\ell} \, . \qquad \text{(II.41)}$$

This we substitute into eq. (II.40) after replacing the indices f,g,j by hpq and finally, if we also use the decomposition formulas (A.2) and consider (II.36,37)[1] we get

$$\Delta \, \varphi_{ij} = - \frac{2G}{m-1} \, (m\alpha'_{ij} - \alpha'_{ji}) \, . \qquad \text{(II.42)}$$

---

[1]For (II.36,37), $\epsilon_{gh\ell} \, \epsilon_{jmn} \, \nabla_h \, \nabla_m \, \varphi_{n\ell} = - \, \Delta \, \varphi_{gj}$ holds, since it follows from (A.1).

---

What does $\alpha'_{ij}$ mean? We take the decomposition, which is unique in the infinite medium,

$$\underset{\sim}{\epsilon} = \text{def} \, \underset{\sim}{s} + \text{inc} \, \underset{\sim}{i} \qquad \text{(II.43)}$$

and express the equation div $\sigma = 0$ by use of Hooke's Law in terms of $\underset{\sim}{s}$ and $\underset{\sim}{i}$.[1] With (1.42), we get

$$\Delta s_i + \frac{m}{m-2} \, \nabla_i \nabla_j s_j + \frac{2}{m-2} \, \nabla_i (\text{inc} \, \underset{\sim}{i})_{jj} = 0 \qquad \text{(II.44)}$$

---

[1]We restrict ourselves to small deformations and rotations.

---

and by taking the curl it follows that

$$\Delta \text{ curl } \underset{\sim}{s} = 0 \ . \tag{II.45}$$

That is why curl $\underset{\sim}{s}$ = const. in the infinite medium. From this is follows that

$$\text{curl def } \underset{\sim}{s} \equiv \frac{1}{2} \nabla \times (\nabla \underset{\sim}{s} + \underset{\sim}{s} \nabla) = \frac{1}{2}(\text{curl } \underset{\sim}{s}) \nabla = 0 \tag{II.46}$$

and it holds that

$$\alpha' \equiv \text{curl } \underset{\sim}{\varepsilon} = \text{curl inc } \underset{\sim}{i} \tag{II.47}$$

I.e., according to §5, $\underset{\sim}{\alpha}'$ is that part of the total dislocation density which caused the stresses. In the case of quasi-dislocations $\underset{\sim}{\alpha}^Q$ of §6, $\underset{\sim}{\alpha}'$ is often directly $\underset{\sim}{\alpha}^Q$. Furthermore, we have shown in §11 that we can calculate $\underset{\sim}{\alpha}'$ when we determine the rotations ($\underset{\sim}{\alpha}'$ is equal to the right hand side of eq. (I.105))). So we are still allowed to consider $\underset{\sim}{\alpha}'$ as a given function. From eq. (II.47) we can easioy see that $\underset{\sim}{\alpha}'$ is governed by the same conditions, (II.36) and (II.37), as $\underset{\sim}{\varphi}$, i.e., that also the particular integral of eq. (II.42) satisfied these conditions:

$$\varphi_{ij}(\underset{\sim}{x}) = \frac{G/2\pi}{m-1} \iiint_V [m\alpha'_{ij}(\bar{\underset{\sim}{x}}') - \alpha'_{ji}(\underset{\sim}{x}')]/|\underset{\sim}{x} - \underset{\sim}{x}'| \, dV' , \tag{II.48}$$

thus it gives the correct stress function tensor corresponding to $\alpha'_{ij}$, from which the stresses follow according to eq. (II.35).

It is easy to reduce the nine integrations of eq. (II.48) to six, similar to the reduction of the six integrations of eq. (II.26) to three. But we do not know if it is possible to get only three integrations of (II.48). In general the stress functions $\varphi_{ij}$ have not been explored previously, but we believe for other reasons that it is worth

while to explore them. If we write, analogous to previous derivation, $\underset{\sim}{n} \times \underset{\sim}{\varepsilon} = - \bar{\underset{\sim}{\alpha}}'$ for a body which adjoins rigid surroundings, then the elastic energy of the body takes the form, since div $\underset{\sim}{\sigma} = 0$,

$$E = \frac{1}{2} \iiint_V \alpha'_{ij} \varphi_{ij} \, dV + \frac{1}{2} \iint_F \bar{\alpha}'_{ij} \varphi_{ij} \, dF \tag{II.49}$$

or also

$$E = \frac{1}{2} \iiint_V \alpha_{ij} \varphi_{ij} \, dV + \frac{1}{2} \iint_F \bar{\alpha}_{ij} \varphi_{ij} \, dF \tag{II.50}$$

as we will show in the next section. E is expressed by the dislocations which cause the self stresses.

E.g., if $\varphi^1_{ij}$ and $\varphi^2_{ij}$ are fields of stress functions caused by two single dislocations $\alpha^1_{ij}$ and $\alpha^2_{ij}$ in the infinite medium,[1] the energy can be written

$$E = \frac{1}{2} \iiint_V \alpha^1_{ij} \varphi^1_{ij} \, dV + \frac{1}{2} \iiint_V \alpha^2_{ij} \varphi^2_{ij} \, dV + \frac{1}{2} \iiint_V \alpha^1_{ij} \varphi^2_{ij} \, dV$$
$$+ \frac{1}{2} \iiint_V \alpha^2_{ij} \varphi^1_{ij} \, dV \tag{II.50'}$$

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[1] In this case we have to consider $\alpha^1_{ij}$ and $\alpha^2_{ij}$ as distributions.

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Obviously the third and fourth integral mean the potential energy of one dislocation in the field of the other dislocation and vice versa. In this way we obtain an interpretation of the stress function $\varphi_{ij}$. This represents a dislocation potential. (The circumstances are analogous to electrostatics, where we have the energy $E \equiv \frac{1}{2} \iiint_V \rho \, U \, dV$, if $\rho$ is the density of the charge. U is called the potential of the charge.)

The 2nd order stress functions $\chi_{ij}$, are analogously the elastic potential of the incompatibilities.

## §14. The Elastic Energy and the Variational Problem in a Medium with Self Stresses

Now we will calculate the expression for the elastic energy of a medium with self stresses in terms of stress functions and incompatibilities. The starting point is the formula

$$E = \frac{1}{2} \iiint_V \sigma_{ij} \, \epsilon_{ij} \, dV \tag{II.51}$$

which can be written with eq. (II.2)

$$E = -\frac{1}{2} \epsilon_{ijk} \, \epsilon_{\ell mn} \iiint_V \epsilon_{i\ell} \, \nabla_j \, \nabla_m \, \chi_{kn} \, dV \tag{II.52}$$

Partial integration yields

$$E = -\frac{1}{2} \epsilon_{ijk} \, \epsilon_{\ell mn} \Big[\iint_F n_j \, \epsilon_{i\ell} \, \nabla_m \, \chi_{kn} \, dF$$

$$- \iiint_V (\nabla_j \epsilon_{i\ell}) \, \nabla_m \, \chi_{kn} \, dV\Big] \tag{II.53}$$

which is identical to eq. (II.49).[1] Since in the initial equation (II.51) we can also write the distortion $\beta_{ij}$ instead of the strains (because of the symmetry of $\sigma_{ij}$), eq. (II.50) is proved.

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[1] It holds that

$$\epsilon_{\ell mn} \, \nabla_m \, \chi_{kn} = -\varphi_{k\ell}, \quad \epsilon_{ijk} \, \nabla_j \epsilon_{i\ell} = -\alpha'_{k\ell}, \quad \epsilon_{ijk} \, n_j \, \epsilon_{i\ell} = \bar{\alpha}'_{k\ell}$$

After partial integration of the volume integral in (II.53), we get

$$E = -\frac{1}{2}\,\epsilon_{ijk}\,\epsilon_{\ell mn}\,[\iint_F n_j\,\epsilon_{i\ell}\,\nabla_m\,\chi_{kn}\,dF$$

$$-\iint_F (\nabla_j\,\epsilon_{i\ell})\,n_m\,\chi_{kn}\,dF] + \frac{1}{2}\iiint_V \chi_{kn}\,\eta_{kn}\,dV \qquad \text{(II.54)}$$

where the relation (II.21) is used. Here we decompose $\nabla_m$ in the first integral according to the formula

$$\nabla_m = n_m\,\nabla_p\,\overset{\downarrow}{n}_p + \epsilon_{mpq}\,\epsilon_{rsq}\,n_s\,\nabla_r\,\overset{\downarrow}{n}_p \qquad \text{(II.55)}$$

which can easily be verified by the use of eq. (A.2), and where the arrow indicates that in addition to the function which is influenced by $\nabla_m$, $n_p$ is also differentiated. The integral which is produced by the second term of (II.55) is integrated by parts again by the use of Stokes' theorem. The related line integral vanishes since F is a closed surface. With the abbreviation $\overset{n}{\nabla}_m = \epsilon_{mpq}\,\epsilon_{rsq}\,n_s\,n_p\,\nabla_r$ of eq. (I.82), the non-vanishing terms are

$$E = -\frac{1}{2}\,\epsilon_{ijk}\,\epsilon_{\ell mn}\,[\iint_F n_j\,\epsilon_{i\ell}\,n_m\,\nabla_p(n_p\,\chi_{kn})\,dF$$

$$-\iint_F \chi_{kn}\,\{n_m\,\nabla_j\,\epsilon_{i\ell} + \overset{n}{\nabla}_m\,(n_j\,\epsilon_{i\ell})\}dF]$$

$$+\frac{1}{2}\iiint_V \chi_{kn}\,\eta_{kn}\,dV \qquad \text{(II.56)}$$

The comparison with eqs. (I.87) and (I.90) gives in the case of the body with a rigid surface

$$E = \frac{1}{2}\iiint_V \chi_{ij}\,\eta_{ij}\,dV + \frac{1}{2}\iint_F \chi_{ij}\,\bar{\eta}_{ij}\,dF$$

$$+\frac{1}{2}\iint_F \nabla_k(n_k\,\chi_{ij})\bar{\bar{\eta}}_{ij}\,dF \qquad \text{(II.57)}$$

In this equation $n_k$ are the Cartesian components of the normal vector $\underset{\sim}{n}$ of the family of surfaces (§8) of which one is the boundary surface F. This interpretation of $n_k$ is necessary to carry out the differentiation $\nabla_k n_k$ properly, for now $\underset{\sim}{n}$ has also a meaning off of F (it is sufficient to define $\underset{\sim}{n}$ in an infinitesimal neighborhood of F).

Equation (II.57) states that the elastic energy, hence the self stresses, of a body which has a rigid surface vanishes if there is no body force and the incompatibilities vanish.

Now we will treat a body with a free surface. According to the well-known theorem of Colonnetti [17], the elastic energy of a body to which external forces and self stresses are applied simultaneously is obtained gy adding the elastic energies of both parts; in our nomenclature

$$E(\underset{\sim}{\eta},\underset{\sim}{\mathfrak{F}}) = E(\underset{\sim}{\eta}) + E(\underset{\sim}{\mathfrak{F}}), \tag{II.58}$$

where $\underset{\sim}{\mathfrak{F}}$ represents the body and surface forces.[1] The step from

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[1]This theorem also holds in the case of a body with a rigid surface (then we can include the surface incompatibilities in $\underset{\sim}{\eta}$). If we let $\underset{\sim}{\sigma} = \underset{\sim}{\sigma}^L + \underset{\sim}{\sigma}^S$, $\underset{\sim}{\epsilon} = \underset{\sim}{\epsilon}^L + \underset{\sim}{\epsilon}^S$ where L indicates the stresses caused by the load and S the self stresses, then the $E(\underset{\sim}{\sigma})$, which is calculated according to eq. (II.51), differs from $E(\underset{\sim}{\sigma}^L) + E(\underset{\sim}{\sigma}^S)$ by the interaction energy, $\iiint \sigma^S_{ij}\, \epsilon^L_{ij}\, dV$ (theorem of Betti), and since $\nabla_i\, \sigma^S_{ij} = 0$ this can be written in a similar form to (II.57)

$$E^{LS} = \iiint_V \chi^S_{ij}\, \eta^L_{ij}\, dV + \iint_F \chi^S_{ij}\, \bar{\eta}^L_{ij}\, dF + \iint_F \nabla_k(n_k\, \chi^S_{ij})\bar{\bar{\eta}}^L_{ij}\, dF$$

Since $\bar{\eta}^L_{ij}$, $\bar{\bar{\eta}}^L_{ij}$ and $\underset{\sim}{\eta}^L$ vanish, we have $E^{LS} = 0$.

eq. (II.51) to (II.52) holds if there no body forces. However, eq. (II.52) and following still contain terms due to the boundary forces. However, as shown in §8 at a free surface the surface incompatibilities are zero. We can show that the surface integral in eq.(II.54) vanishes if there are no surface forces and $n_i \eta_{ij} = 0$ so that only

$$E = \frac{1}{2} \iiint_V \chi_{ij} \eta_{ij} \, dV \quad \text{if } n_i \eta_{ij} = 0 \tag{II.59}$$

remains. This equation contains the theorem that in a simply connected body and in the region of the linear theory of elasticity all self stress can be derived from incompatibilities. Eq. (II.57) contains the same theorem for a body with a rigid surface. This holds even if we create new boundary surfaces by allowing $\eta_{ij}$ to degenerate in the interior of the body to a plane (or even to a line). According to this it is obvious that generally in the region of the linear theory of elasticity the theorem holds that all self stresses are caused by incompatibilities. Furthermore, the reverse holds, that all (symmetric) incompatibilities cause self stresses, which is obvious from the meaning of incompatibilities as derivatives of the elastic strains.

The question of the uniqueness of the solution is of great importance. Thanks to the uniqueness theorem of Kirchhoff of the classic theory of elasticity and to the theorem of Colonnetti it is sufficient to clarify that in the absence of external forces the self stresses (which follow from the stress functions) are uniquely determined by given incompatibilities. It can readily be proved that for

an infinite medium eq. (II.26) is a unique solution of (II.19) and (II.20), since there is no incompatibility at infinity. In a finite medium, however, it must be shown that the additional boundary value problem has a unique solution. We will show in the next section that with self stresses we always get both of the well-known boundary problems of the theory of elasticity, for which the proof of uniqueness is given in literature, so it generally holds in the region of linear theory of elasticity that the stresses of a body are uniquely determined by the external forces and the incompatibilities.

All the previously considered bodies were simply connected bodies even if we allow that the incompatibilities are out of the body. (This procedure is known from hydrodynamics. We calculate a flow around a body as if there were sources and vortices in the body.) However, in the region of the nonlinear theory of elasticity, we cannot refer all stresses to external forces or incompatibilities, as the example of the invertible hermisphere shows [160].

The variational problem of self stresses has already been formulated by Colonnetti [19]. In our nomenclature the expression of the variation of the energy should vanish

$$\iiint_V \left(\frac{1}{2}\,\epsilon_{ij}\,\sigma_{ij} + \epsilon^P_{ij}\,\sigma_{ij}\right) dV \qquad \text{(II.60)}$$

where $\epsilon^P_{ij}$ is the imposed (plastic or quasi-plastic) strain. The second term of eq. (II.60) Colonnetti called the "potential of the applied strain." If we substitute $\epsilon^P_{ij} = \epsilon^T_{ij} - \epsilon_{ij}$ according to eq. (I.1) of page 1 into eq. (II.60), we obtain

$$-\frac{1}{2}\iiint_V \epsilon_{ij}\,\sigma_{ij}\,dV \qquad \text{(II.61)}$$

for

$$\iiint_V \epsilon^T_{ij}\, \sigma_{ij}\, dV = 0 \qquad \text{(II.62)}$$

as was first found by Rieder [125] for the case of the body with a free surface. In this case

$$\iiint_V \epsilon^T_{ij}\, \sigma_{ij}\, dV = \iiint_V (\nabla_i\, s^T_j)\sigma_{ij}\, dV$$

$$= \iint_F n_i\, s^T_j\, \sigma_{ij}\, dF - \iiint_V s_j\, \nabla_i\, \sigma_{ij}\, dV = 0 \qquad \text{(II.63)}$$

since $n_i\sigma_{ij}$ and $\nabla_i\sigma_{ij}$ vanish if there are no external forces. According to Rieder [125], eq. (II.17) are the Euler-Lagrange equations of the problem of variation of eq. (II.61) (if there are no external forces; i.e., the incompatibilities are given).

In the case of a rigid surface, it is possible to transform the left-hand side of eq. (II.62), multiplied by 1/2, to the expression (II.57) where $\eta_{ij}$, $\bar{\eta}_{ij}$, and $\bar{\bar{\eta}}_{ij}$ are the incompatibilities which belong to $\epsilon^T_{ij}$. These vanish according to the physical meaning of $\epsilon^T_{ij}$ ($\underset{\sim}{\epsilon}^T$ is a deformator, so $\eta^T_{ij} = 0$) furthermore, $s^T_i$ is zero on the rigid boundary of the body. Also, $\bar{\eta}^T_{ij} = \bar{\bar{\eta}}^T_{ij} = 0$ (see eqs. (I.87) to (I.89)). The variational problem which refers to the energy expressions (II.61) and (II.57), respectively, should include in addition to the differential eq. (II.17) also the boundary condition (I.89) which is in terms of $\chi_{ij}$. This has not been calculated previously.

For solving the variational problem of the body with a free surface by direct methods, we must take account of the fact that there are stress functions for which the related $\eta_{ij}$ vanishes, but not the

related boundary forces $n_j \sigma_{ij}$. Such stress functions do not contribute to the integral (II.59) but do contribute to the integral (II.61). In order to get the correct solution, therefore, the integral (II.61) must become an extremum by constant $\underset{\sim}{\eta}$.[1] In the case of the body with a rigid surface the integral (II.57) is equivalent to (II.61). One of these must become an extremum by constant $\eta_{ij}$, $\bar{\eta}_{ij}$, and $\bar{\bar{\eta}}_{ij}$.

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[1]Even Föppl [44] mentioned that (II.61) should become an extremum. The improvement is the addition "$\eta$ = const."

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We will shortly consider both of the boundary value problems for the body with a free surface (the superposition problem must be solved); according to §8 the problem "boundary displacement given" is the same as the problem "$\bar{\underset{\sim}{\eta}}$, $\bar{\bar{\eta}}$ given" for the body with a rigid surface (see also the following section).

If we take the equation which was first mentioned by Schaefer [127][2]

$$\chi_{ij} = \omega_{ij} - \omega_{kk}\,\delta_{ij} + \Omega\,\delta_{ij} \tag{II.64}$$

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[2]This $\omega_{ij}$ is not the same as that formerly used.

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where $\Delta\,\omega_{ij} = 0$, then in addition to the conditions of equilibrium, eqs. (II.17) are also automatically satisfied by $\underset{\sim}{\eta} = 0$, if

$$\Delta\,\Omega = \frac{m}{m-1}\,\nabla_i\,\nabla_j\,\omega_{ij}. \tag{II.65}$$

$\omega_{ij}$ is similar to $\chi_{ij}$, a 2nd order tensor. If we take as the solution of (II.65)

$$\Omega = \frac{m/2}{m-1} x_i \nabla_j \omega_{ij} + \omega_{kk} + v \tag{II.66}$$

then we obtain

$$\chi_{ij} = \omega_{ij} + H \delta_{ij} \tag{II.67}$$

where

$$H = v + \frac{m/2}{m-1} \left( x_1 \frac{\partial \omega_{11}}{\partial x_1} + x_2 \frac{\partial \omega_{22}}{\partial x_2} + x_3 \frac{\partial \omega_{33}}{\partial x_3} \right) \tag{II.68}$$

and $\Delta v = 0$. We can show [77] that we are allowed to let $v = \omega_{12} = \omega_{23} = \omega_{31} = 0$. With

$$\chi_{ij} = \omega_{ij} + H \delta_{ij} \qquad i = j \tag{II.69}$$

the Maxwell's functions in the case of $\underset{\sim}{\eta} = 0$ are reduced to three harmonic functions. We can easily show that these relations fulfill (II.25) but not (II.19). For the functions $\omega_{ij}$ we take series of harmonic functions and determine the coefficients by usual methods to that the boundary conditions, which are expressed in terms of $\chi_{ij}$, are matched very closely. If we add the stress functions derived in such a way to the particular Maxwell's function, which were obtained according to §13, then we obtain the resulting Maxwell's functions for the associated state of stress.

## §15. Boundary Value Problems Which Arise with Self Stresses and Their Treatment with Stress Functions

The particular integral (II.29) and (II.32) of the differential equation (II.17) which governs the self stresses does not fulfill in general the boundary conditions in a body with a free surface, since the tractions $n_i \sigma_{ij} \neq 0$ on the boundary, whereas in the body with a rigid surface it does not satisfy the strains (I.89). In the first case the boundary value problem of the form

$$n_i \sigma_{ij} = A_j \tag{II.70}$$

remains, whereas in the second case, the form

$$\overline{\text{inc}}\ \underset{\sim}{\varepsilon} = \bar{\underset{\sim}{\eta}}\ , \qquad \overline{\overline{\text{inc}}}\ \underset{\sim}{\varepsilon} = \bar{\bar{\underset{\sim}{\eta}}} \tag{II.71}$$

must be solved. In both cases the stress functions used have to satisfy eqs. (II.17) with $\underset{\sim}{\eta} = 0$. We replace these according to §12 by the equations

$$\Delta\Delta\ \underset{\sim}{\chi}' = 0, \qquad \nabla_i\ \chi'_{ij} = 0 \tag{II.72}$$

In practice problems occur where portions of a body separated by a dislocation wall have different elastic moduli. In this case we must consider the boundary conditions in terms of both the stresses and strains. (II.70) and (II.71) indicate the limiting cases (modulus = 0 and $\infty$, respectively, in the partial volume) of this problem, to which it always can be reduced. This remark should explain why we are concerned with the body with a rigid surface.

We will show now that we also can replace the boundary value problem (II.71) by the problem "boundary displacement given." $\eta_{ij}$, $\bar{\eta}_{ij}$, and $\bar{\bar{\eta}}_{ij}$ may be prescribed. Then the elastic strain is composed of two parts. The particular solution $\epsilon^{O}_{ij}$ and a second part $\epsilon^{H}_{ij}$ which satisfies the homogeneous equations

$$\text{inc}\ \underset{\sim}{\epsilon} = 0 \qquad \text{div}\ \underset{\sim}{\sigma} = 0 \qquad \sigma_{ij} = c_{ijk\ell}\,\epsilon_{k\ell} \tag{II.73}$$

We can formally associate surface incompatibilities $\bar{\underset{\sim}{\eta}}^{O}, \bar{\bar{\underset{\sim}{\eta}}}^{O}$ and $\bar{\underset{\sim}{\eta}}^{H}, \bar{\bar{\underset{\sim}{\eta}}}^{H}$ with both $\underset{\sim}{\epsilon}^{O}$ and $\underset{\sim}{\epsilon}^{H}$ according to eqs. (II.71). We have $\bar{\underset{\sim}{\eta}}^{H} = \bar{\underset{\sim}{\eta}} - \bar{\underset{\sim}{\eta}}^{O}$ since $\underset{\sim}{\epsilon}^{H} = \underset{\sim}{\epsilon} - \underset{\sim}{\epsilon}^{O}$, etc. The problem is to determine a strain $\underset{\sim}{\epsilon}^{H}$ which satisfies eqs. (II.72) and at the same time the boundary conditions (II.71) which are written with the index H. For simplicity we omit the index H in the following. Because of the first equation of (II.73) $\underset{\sim}{\epsilon}$ has the form def $\underset{\sim}{s}$. We can easily show, but we will not at this time, that in this case the boundary conditions can be integrated over the boundary surface and can be cast in the form

$$\underset{\sim}{s} = \underset{\sim}{g} \tag{II.74}$$

where $\underset{\sim}{g}$ is derived from $\bar{\underset{\sim}{\eta}}$ and $\bar{\bar{\underset{\sim}{\eta}}}$. $\underset{\sim}{g}$ is that displacement which will occur at the surface of the medium if the restraint of the rigid surrounding suddenly vanished.

So it is obvious that we can treat the self stresses with the previously known boundary value problems. For these there exist numerous solution methods, which is why we will discuss shortly the application of three-dimensional stress functions to solve these problems.

The stress function tensor does not represent a unique system of functions, thus it represents a greater variety of physical situations than, e.g., the displacement vector, which is expressed by the secondary conditions. So we have the possibility of adjusting ourselves to given problems by choosing the secondary conditions. Furthermore, Airy's stress functions are so convenient for two-dimensional problems that we will at least try to obtain a similar method for three-dimensional problems.

The goal is not reached. However the simple form of the energy equation (II.57) leads us to believe that the boundary value problem defined by eqs. (II.71) and (II.74), respectively, can be successfully treated by stress functions. If we take the approach of the classical Green's method, then we have to start with the theorem of Betti, which we can write in terms of stress functions after comparing with eq. (II.57)

$$\iiint_V \chi^1_{ij}\, \eta^2_{ij}\, dV + \iint_F \chi^1_{ij}\, \bar{\eta}^2\, dF + \iint_F \nabla_k(n_k\, \chi^1_{ij})\bar{\bar{\eta}}^2_{ij}\, dF$$
$$= \iiint_V \chi^2_{ij}\, \eta^1_{ij}\, dV + \iint_F \chi^2_{ij}\, \bar{\eta}^1_{ij}\, dF + \iint_F \nabla_k(n_k\, \chi^2_{ij})\bar{\bar{\eta}}^1_{ij}\, dF\,, \tag{II.75}$$

Then we identify $\eta_{ij}$, $\bar{\eta}^1_{ij}$ and $\bar{\bar{\eta}}^1_{ij}$ with the given incompatibilities, $\chi^1_{ij}$ with the desired stress function tensor, whereas we substitute for $\chi^2_{ij}$ the fundamental solution (main solution) of the differential equations (II.17). Naturally we want to use the fact that these equations assume the form $\Delta\Delta\chi_{ij} = 0$ with the related secondary conditions, because the fundamental integral of this equation is given by the form $|\underset{\sim}{x} - \underset{\sim}{x}'|/8\pi$. Then we hope to reduce the whole boundary value problem to the determination of a biharmonic Green's function for the region

considered. Among other things, the difficulty is now that we do not know how we can satisfy the secondary conditions which at least guarantee that the differential equation (II.17) is satisfied. Written in terms of $\chi_{ij}$ the boundary conditions (II.70) are

$$\epsilon_{ijk}\,\epsilon_{\ell mn}\,n_i \nabla_j \nabla_m \chi_{kn} = A_\ell \tag{II.76}$$

which we have to fulfill simultaneously with eqs. (II.17). If we replace these by eqs. (II.72), then we must be sure that $\nabla_i\,\chi'_{ij} = 0$ is fulfilled. We can do this if we prescribe

$$\nabla_i\,\chi'_{ij} = 0, \qquad \frac{\partial}{\partial n}(\nabla_i\,\chi'_{ij}) = 0 \tag{II.77}$$

on the boundary in addition to the boundary conditions. For with (II.72) the following holds:

$$\Delta\Delta(\nabla_i\,\chi'_{ij}) = 0, \tag{II.78}$$

from which it is clear that with the boundary values (II.77) $\nabla_i\,\chi'_{ij}$ vanishes simultaneously in the whole volume. The biharmonic problem defined by the boundary conditions (II.76) and (II.77) has not been investigated until now.

On the other hand, it is very easy to find a field $\chi_{ij}$ which satisfies the boundary conditions (II.76) and the differential equations $\Delta\Delta\chi_{ij} = 0$. For this it is only necessary to set up the Green's function. It is not impossible to find without great difficulty the field which must be added to $\chi_{ij}$ so that the secondary conditions are satisfied, i.e., also the differential equations (II.17).

## §16 Extension for Elastic Anisotropy, Double Forces

The crystals of metals, to which we will apply the theory of distortions, are in many cases very elastically anisotropic, which can not always be neglected. Therefore, even Burgers [13] applied the anisotropic theory of elasticity to dislocations. Now we will gather important formulas, which not only allow us to treat the dislocation by elastic anisotropy, but also give a basis for the treatment of other important elasti ingularities.

First of all $\epsilon_{ij} = \frac{1}{2}(\nabla_i s_j + \nabla_j s_i)$ in the region without singularities. If we introduce this by use of the Hooke's Law into the equations of equilibrium (II.5), then we get

$$D_{j\ell}\, s_\ell + \mathcal{F}_j = 0, \qquad D_{j\ell}(\nabla) \equiv c_{ijk\ell}\, \nabla_i \nabla_k \tag{II.79}$$

Let $f(\nabla)$ be the determi nt $|D_{ik}|$ and $D^*_{ij}(\nabla)$ be the symmetric tensor of the subdeterminant of f, i.e., $D_{j\ell}\, D^*_{ik} = f\delta_{jk}$. With

$$s_\ell = D^*_{k\ell}\, h_k \tag{II.80}$$

equation (II.79) becomes

$$f\, h_j + \mathcal{F}_j = 0\,. \tag{II.81}$$

In the case of a point force $P_j$ at the point $\underset{\sim}{x}'$ we can write

$$\mathcal{F}_j(\underset{\sim}{x}) = P_j \delta(\underset{\sim}{x}-\underset{\sim}{x}'), \quad \delta(\underset{\sim}{x}-\underset{\sim}{x}') \equiv \delta(x_1-x_1')\, \delta(x_2-x_2')\, \delta(x_3-x_3') \tag{II.82}$$

then

$$f\, h_j + P_j(\underset{\sim}{x}')\, \delta(\underset{\sim}{x}-\underset{\sim}{x}') = 0 \tag{II.83}$$

By the equation

$$f\,U + \delta(\underset{\sim}{x}) = 0 \tag{II.84}$$

we define the fundamental solution $U(\underset{\sim}{x})$ of the linear homogeneous differential equation of order 6, $fu = 0$, in the infinite medium (so it is unique except for a negligible function of the 5th power of $\underset{\sim}{x}$). If we know U, then we also know the particular integral of (II.81)

$$h_j(\underset{\sim}{x}) = \iiint_V U(x)\, \mathcal{F}_j\,(\underset{\sim}{x}')dV', \qquad x \equiv |x - x'|. \tag{II.85}$$

Therefore, for the point force in the infinite medium it holds that

$$h_j(\underset{\sim}{x}) = U(x)\; P_j(\underset{\sim}{x}') \tag{II.86}$$

The related displacement field is, according to eq. (II.80)

$$s_j(\underset{\sim}{x}) = S_{ij}(x)\; P_i(\underset{\sim}{x}')\;; \quad S_{ij} = D^*_{ij}\; U\,. \tag{II.87}$$

The symmetric tensor $S_{ij}$ is the fundamental solution of the elastic differential equations (II.79) for the displacement. By use of it the particular solution of eq. (II.79) becomes

$$s_j(\underset{\sim}{x}) = \iiint_V S_{ij}(x)\; \mathcal{F}_i(\underset{\sim}{x}')dV'$$

The physical meaning of the components of $S_{ij}$ can easily be shown from eq. (II.87) if we assume $|P_i| = 1$. Then $S_{ij}$ is the j-component of the related displacement.

We say,

$$s_k(\underset{\sim}{x}) = P_{ij}(\underset{\sim}{x}')\; \nabla_i\; S_{jk}(x) \tag{II.89}$$

is the displacement at the point $\underset{\sim}{x}$, which is caused by a double force $P_{ij}$ at the point x'. We call the tensor, $P_{ij}$, which is not necessarily

symmetric, the force dipole. The second index shows the direction of opposite and equal point forces. The first index shows the direction of the connection of the points of application; this is also the direction in which the two forces move together as we pass to the limit

$$P_{ij} = \lim_{\ell_i \to 0,\, P_j \to \infty} \ell_i P_j \tag{II.90}$$

The diagonal components of $P_{ij}$ are double forces without moments; the other components have moments about an axis perpendicular to the i- and j-direction. The entire torque is described by the antisymmetric part of $P_{ij}$. For further information about double forces, see Love [95].

The displacement

$$s_\ell(\underset{\sim}{x}) = P_{ijk}(\underset{\sim}{x}') \nabla_i \nabla_j S_{k\ell}(x) \tag{II.91}$$

is caused by the quadrupole forces $P_{ijk}$, and in a similar way we can define poles of higher order.

In the case of elastic isotropy we can calculate with eq. (II.9)

$$D_{ij} = (\lambda+\mu) \nabla_i \nabla_j + \mu \Delta \delta_{ij} \tag{II.92}$$

$$D^*_{ij} = [- \mu(\lambda+\mu) \nabla_i \nabla_j + \mu(\lambda+2\mu) \Delta \delta_{ij}] \Delta \tag{II.93}$$

$$f = \mu^2(\lambda+2\mu) \Delta\Delta\Delta, \quad U = \frac{1}{96\pi \mu^2(\lambda+2\mu)} x^3 \tag{II.94}$$

therefore with $\Delta x^3 = 12 x$

$$S_{ij} = \frac{1}{8\pi\mu} \left(- \frac{\lambda+\mu}{\lambda+2\mu} \nabla_i \nabla_j + \delta_{ij} \Delta\right) x \tag{II.95}$$

Here the components of $S_{ij}$ are elementary functions of x. The same remark holds only in the case of hexagonal symmetry [76,180].

Now we will describe briefly the stress function method for anisotropy [80], which again has the advantage that it can be applied to continuously distributed dislocations. Let

$$\chi_{ij} = X_{ijk\ell}\, \psi_{k\ell}, \quad \nabla_i\, \psi_{ij} = 0 \tag{II.96}$$

where $X_{ijk\ell}$ is a 2nd order differential operator which is known only for the case of isotropy and cubic symmetry. It has the same symmetry as the Hooke's tensor of the related crystal and can be written in the case of isotropy

$$X_{ijk\ell} = \mu^3 [2\lambda\, \delta_{ij}\, \delta_{k\ell} + (\lambda+2\mu)(\delta_{ik}\, \delta_{j\ell} + \delta_{i\ell}\, \delta_{jk})]\, \Delta \tag{II.97}$$

The $\psi_{ij}$ satisfy the differential equation

$$f\, \psi_{ij} = \eta_{ij} \tag{II.98}$$

which is solved by

$$\psi_{ij}(\underset{\sim}{x}) = -\iiint_V \eta_{ij}(\underset{\sim}{x}')\, U(x)\, dV' \,. \tag{II.99}$$

For the stress function $\chi_{ij}$ we get

$$\chi_{k\ell} = -\iiint_V \eta_{ij}(\underset{\sim}{x}')\, X_{ijk\ell}\, U(x)\, dV' \tag{II.100}$$

In the case of isotropy with (II.97) and (II.94)

$$X_{ijk\ell}\, U = \frac{\mu}{8\pi}\left[\frac{2\lambda}{\lambda+2\mu}\, \delta_{ij}\, \delta_{k\ell} + \delta_{ik}\, \delta_{j\ell} + \delta_{i\ell}\, \delta_{jk}\right] x \tag{II.101}$$

by which eq. (II.100) can be transformed into (II.26). The treatment of the plane problem is of interest in order to apply it to straight dislocation lines. We assume $\partial/\partial x_3 = 0$. The functions we will get now should be signified by a bar. It can be shown that $\bar{f}$, a function of order 6, can be decomposed into a triple product of terms which are of order 2 and homogeneous (for isotropic and hexagonal crystals this is possible for the three-dimensional f). Thus, the characteristic 6th order equation related to $\bar{f}$ has elementary solutions and the related (two-dimensional) fundamental solution $\bar{U}$ becomes an elementary function, which can be derived very easily in all cases. Therefore we assume them to be known. It holds that

$$\bar{U}(\bar{x}) = \int_{-\infty}^{\infty} U(x)\, dx_3 \qquad \text{(II.102)}$$

as is well known from the theory of differential equations. Now we have gathered the basic techniques to treat dislocations and other singularities with sufficient completeness.

## §17. The Treatment of Singular Dislocations According to the Theory of Elasticity

For applications, the single dislocation is an important factor. In Fig. 9a,b the development of a single edge dislocation was shown. We imagine that the cut cylinder of Fig. 9a is pressed together to become the complete cylinder of Fig. 9b so that the relative displacement $\underline{g}$ of the plane A with respect to B has only an $x_1$ component; afterwards A and B coalesce to a plane f. Figure 10 shows the development of a screw dislocation: We imagine that the cylinder of Fig. 10

is produced by cutting a complete cylinder open and by a relative displacement of the cut edges in the direction of the cylinder axis.

In general the dislocation will occur along an arbitrary three-dimensional curve L with the unit tangent vector $\underset{\sim}{t}$ which surrounds its surface of development (of the dislocation) f with the unit normal vector $\underset{\sim}{n}$. $\underset{\sim}{q}$ is the unit vector $\underset{\sim}{n} \times \underset{\sim}{t}$. We assume that $\zeta$ is so small that we can consider the "dislocation strip" of width $2\zeta$ to be locally linear. Then q is the shortest distance between the curve L and a point which lies on f.

According to eq. (I.77) the dislocation density in the strip is $-\underset{\sim}{n} \times \nabla \underset{\sim}{g}$, which is equal to $\underset{\sim}{t}(\partial \underset{\sim}{g}/dq)$, because $\underset{\sim}{g}$ changes only in the q-direction. The distribution of $\underset{\sim}{g}(q)$ is naturally unknown to us; in Fig. 9, 10 it is shown linearly changing. More generally $\partial|\underset{\sim}{g}|/\partial q$ may be an arbitrary curve, which we write $-\gamma(q)|\underset{\sim}{g}^{o}|$, where $\underset{\sim}{g}^{o}$ is the constant displacement along most of the surface f. According to §1,2, $\underset{\sim}{b} = -\underset{\sim}{g}^{o}$ is the Burgers vector of the dislocation. Thus we obtain the two-dimensional dislocation density

$$\bar{\alpha}_{i\ell} = t_i \, b_\ell \, \gamma(q) \tag{II.103}$$

After this calculation the main calculation follows. The starting point is eq. (II.26') in which we substitute $\eta_{ij}$ according to eq. (I.51)[1]

$$\eta_{ij} = -(\varepsilon_{jk\ell} \, \nabla_k \, \alpha_{i\ell})^S \tag{II.104}$$

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[1] S means "symmetric part of"

After partial integration we obtain, since the surface integral vanishes with $\nabla'_k = \partial/\partial x'_k$

$$\chi'_{ij}(\underset{\sim}{x}) = \frac{-1}{8\pi}\left(\epsilon_{jk\ell}\iiint_V \alpha_{i\ell}(\underset{\sim}{x}')\nabla'_k \times dV'\right)^S, \quad x \equiv |\underset{\sim}{x}-\underset{\sim}{x}'|. \qquad \text{(II.105)}$$

Here we have evidently to substitute $\alpha_{i\ell}\, dV' = \bar{\alpha}_{i\ell}\, dq'\, dL'$, where $dL'$ is the magnitude of the line element $dL'_i$ of the curve L. With (II.103) we obtain

$$\alpha_{i\ell}\, dV' = t_i\, b_\ell\, \gamma(q')dq'\, dL' = b_\ell\, dL'_i\, \gamma(q')\, dq' \qquad \text{(II.106)}$$

If we substitute this in eq. (II.105) and $\gamma(q')$ by the delta function $\delta(q')$, then we can carry out the integration with respect to q' and we obtain for the singular dislocation line

$$\chi'_{ij} = -\frac{1}{8\pi}\left(\epsilon_{jk\ell}\, b_\ell \oint_L \int_{-\zeta}^{\zeta} \nabla'_k \times dL'_i\, \delta(q')dq'\right)^S$$

$$= -\frac{1}{8\pi}\left(\epsilon_{jk\ell}\, b_\ell \oint \nabla'_k \times dL'_i\right)^S \qquad \text{(II.106')}$$

If we take $\nabla'_k$ in front of the integral, where the sign will be reversed, since we not differentiate with respect to $x_i$, then we obtain the final formula, which was first mentioned by the author [78]

$$\chi'_{ij} = \frac{1}{8\pi}\left(\epsilon_{jk\ell}\, b_\ell\, \nabla_k \oint_L \times dL'_i\right)^S \qquad \text{(II.107)}$$

By this technique the state of stress which is caused by a dislocation along the curve L, is reduced to the relatively simple integral $\oint \times dL'$. It will be shown that the stresses at the curve itself diverge, which obviously arises from the limit $\gamma(q) \to \delta(q)$. If we are interested in the state in the immediate surrounding of the curve L, then we are not

allowed to take this limit, but we have to integrate the equation with $\gamma(q)$ of (II.106').[1]

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[1]At some distance from L the principle of de St. Venant becomes effective: the result no longer depends on the exact distribution function $\gamma(q)$.

---

If the dislocation line is a straight line and is along the $x_3$ axis, then $dL_i$ becomes $dx_3$. We can easily check the formula

$$\int_{-L/2}^{L/2} x\, dx_3' = -\rho^2 \ln(\rho/L) + L^2/4, \quad \rho^2 \equiv x_1^2 + x_2^2 \qquad \text{(II.108)}$$

which holds everywhere, where $\rho$, $x_3 \ll L$.[2] In this expression L is the

---

[2]The exact expression (II.108) is

$$(\rho^2/2)\ln\left[\left(\frac{L}{2} + x_3 + \sqrt{+}\right)\left(\frac{L}{2} - x_3 + \sqrt{-}\right)/\rho^2\right] + \frac{1}{2}\left(\frac{L}{2} + x_3\right)\sqrt{+}$$

$$+ \frac{1}{2}\left(\frac{L}{2} - x_3\right)\sqrt{-}\ ; \ \sqrt{\pm} \equiv \sqrt{\rho^2 + \left(\frac{L}{2} \pm x_3\right)^2}\ .$$

---

length of the dislocation, which now shall go to infinity. According to eq. (II.107) and (II.2) the expression (II.108) is to be differentiated three times in order to get the stresses. $\rho^2 \ln L$ does not contribute to these and we write by substituting (II.108) into (II.107)

$$\chi_{3j}' = \frac{-1/2}{8\pi}(\epsilon_{jk\ell} + \epsilon_{3k\ell}\,\delta_{3j})b_\ell\,\nabla_k\,\rho^2 \ln\rho \qquad \text{(II.109)}$$

From the similarity of eq. (II.26') to (II.99) we conclude the corresponding formulas for anisotropy, hence, eq. (II.107) corresponds to

$$\psi_{ij} = (\epsilon_{jk\ell}\, b_\ell\, \nabla_k \int_L U(x)\, dL_i')^S \tag{II.110}$$

and eq. (II.109) to

$$\psi_{3j} = -\frac{1}{2}(\epsilon_{jk\ell} + \epsilon_{3k\ell}\, \delta_{3j}) b_\ell\, \nabla_k\, \bar{U}(\bar{x}) \tag{II.111}$$

from which we calculate the stress functions $\chi_{ij}$ according to eq. (II.96).

The calculation of $X_{ijk\ell}$, which is necessary only once for each crystal, may be not very extensive in the two-dimensional case, but it is in the three-dimensional case [80]. We can see that in the case of straight dislocation lines and anisotropy, the stresses are found to be elementary functions (since $\bar{U}$ is an elementary function according to §16). Eshelby pointed out how we can obtain these also in the complex plane, $x_1 + ix_2$, starting with a complex displacement field, $s_1 + is_2$. Previously Burgers noticed [13] that the anisotropic formulas became very simple in the cubic crystals if the dislocations occur in a special crystallographic direction. Especially, we obtain the same displacements for a screw dislocation in the $\langle 001 \rangle$ direction as for isotropy.

Now we will distinguish two cases:

1. The straight edge dislocation in the $x_3$-direction. Then $b_3 = \nabla_3 = 0$ and in eq. (II.109) $j = 3$, i.e., only $\chi'_{33}$ is different from zero. If we choose, e.g., $x_1 \parallel b_\ell$ (i.e., $\ell = 1$), then

$$\chi'_{33} = \frac{1}{8\pi}\, b_1\, \frac{\partial}{\partial x_2}\, (\rho^2 \ln \rho) \tag{II.112}$$

According to eq. (II.16) and (II.18), the relation between Airy's stress function and $\chi'_{33}$ is given[1] by $-\frac{2Gm}{m-1}\chi'_{33} = \chi$; with this we can

---

[1] We can obtain also $\chi_{11}$ and $\chi_{22}$ different from zero according to

(Footnote Continued)

eq. (II.18). By use of eq. (II.16) we can easily show that from this the relation, $m\,\sigma_{33} = \sigma_{11} + \sigma_{22}$, follows which is known from the theory of two-dimensional state of strain [1]; $\sigma_{11}$, $\sigma_{12}$ and $\sigma_{22}$ are determined alone by $\chi$, thus we need not consider $\chi_{11}$ and $\chi_{22}$.

---

write eq. (II.112)

$$\chi = -\frac{A}{2}\frac{\partial}{\partial x_2}(\rho^2 \ln \rho), \quad A \equiv \frac{b_1}{2\pi}\frac{mG}{m-1} \tag{II.113}$$

This equation was obtained first by Koehler [111]. According to the normal rules of the theory of elasticity the stresses follow

$$\begin{aligned}
\sigma_{11} &= \partial^2\chi/\partial x_2^2 = -A\frac{x_2}{\rho^2}\frac{3x_1^2 + x_2^2}{\rho^2} \\
\sigma_{12} &= -\partial^2\chi/\partial x_1\partial x_2 = A\frac{x_1}{\rho^2}\frac{x_1^2 - x_2^2}{\rho^2} \\
\sigma_{22} &= \partial^2\chi/\partial x_1^2 = A\frac{x_2}{\rho^2}\frac{x_1^2 - x_2^2}{\rho^2}
\end{aligned} \tag{II.114}$$

and the displacements are[1]

$$\begin{aligned}
S_1 &= \frac{b_1}{2\pi}\left[(\varphi - \frac{\pi}{2}) + \frac{m/2}{m-1}\frac{x_1 x_2}{\rho^2}\right], \quad \varphi = \arc\tan\frac{x_2}{x_1} \\
S_2 &= -\frac{b_1}{4\pi(m-1)}\left[(m-2)\ln\frac{\rho}{\zeta} + m\frac{x_2^2}{\rho^2}\right]
\end{aligned} \tag{II.115}$$

---

[1]These equations were first derived in a different way by Burgers (see below) [12]. Taylor was the first who applied the theory of elasticity to crystal dislocations and particularly he demonstrated

(Footnote Continued)

their relation to Volterra's work, but he was not exact in his detailed calculation.

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For the calculation of the displacement a special consideration is necessary since $s_2$ diverges logarithmically with $\rho$ [71].

2. Screw dislocations in the $x_3$ direction. Then eq. (II.109) becomes

$$\chi'_{31} = -\frac{1/2}{8\pi} b_3 \frac{\partial}{\partial x_2} (\rho^2 \ln \rho), \chi'_{32} = \frac{1/2}{8\pi} b_3 \frac{\partial}{\partial x_1} (\rho^2 \ln \rho) \tag{II.116}$$

After multiplication with 2G, these are simultaneously the $\chi_{ij}$ values. According to eq. (II.16) the stress function for torsion is then

$$\Phi = -\frac{Gb_3}{8\pi} \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) (\rho^2 \ln \rho) = -\frac{Gb_3}{2\pi} (\ln \rho + 1) \tag{II.117}$$

the stresses are

$$\sigma_{31} = \frac{Gb_3 x_2}{2\pi \rho^2} \quad , \quad \sigma_{32} = \frac{Gb_3 x_1}{2\pi \rho^2} \tag{II.118}$$

and the displacements

$$s_1 = s_2 = 0, \quad s_3 = \frac{b_3}{2\pi} \varphi \qquad \varphi = \arctan \frac{x_2}{x_1} \tag{II.119}$$

In this case it is extremely instructive to use the displacements, since we can immediately see from eq. (II.119) the transformation of the $x_3$ plane into the screw surface.[1]

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[1]Because of $s_1 = s_2 = 0$, it is very convenient to derive these equations starting with the displacement $s_3$, as was originally done by Burgers [12] and as can be read in all books concerning dislocations.

Though the integral (II.107) looks very simple, its integration is only possible in a closed form in the simplest cases. This happens with dislocations which are piecewise straight lines; also with dislocations which are plane quadratic curves. In the last case we get elliptic integrals.

Originally, instead of the formula (II.107), the displacement field of the general dislocation line was represented as a surface integral (by Burgers [12]). By use of the Green's method in the case $\underset{\sim}{\eta} = 0$ we can express the displacements in a body in terms of the volume and surface forces and the surface displacements. We obtain[1] (for the explicit calculation, see, e.g., Seeger [134])

$$s_h(\underset{\sim}{x}) = \iiint_V S_{i} (x)\, \mathcal{F}_i(\underset{\sim}{x})dV' + \iint_F S_{ih}(x)\, \bar{\mathcal{F}}_i(\underset{\sim}{x}')dF + \iint_F c_{ijk\ell}\, n_i\, s_j(\underset{\sim}{x}')\, \frac{\partial}{\partial x_k} S_{\ell h}(x)\, dF' \tag{II.120}$$

[1]This formula was first mentioned by Fredholm [57] and discuassed in Gebbia [58].

We can apply this formula, e.g., to the cut cylinder in Fig. 9a if we bend it together without welding initially (Fig. 9b). However, $\mathcal{F}_i = 0$, also the second integral of eq. (II.120) vanishes since the surface forces which cause the planes A and B to be bent together are equal and opposite at the same points on these planes. Thus if we call the displacement jump, $s_i^A - s_i^B, g_i$ (§8), we obtain from eq. (II.120)

$$s_h(\underset{\sim}{x}) = \iint_f c_{ijk\ell}\, n_i g_j\, \frac{\partial}{\partial x_k} S_{\ell k}(x)\, df' + \iint_F c_{ijk\ell}\, n_i s_j(\underset{\sim}{x}')\, \frac{\partial}{\partial x_k} S_{\ell k}(x) dF' \tag{II.121}$$

where f is the coalesced surface AB with A as the positive part and F is the surface of the body after welding. If we neglect an exact description of the very near neighborhood of the dislocation, which means that we assume $\underset{\sim}{g}$ on f is constant up to the curve L, and simultaneously we restrict ourselves to an infinite medium, then we obtain

$$s_h(\underset{\sim}{x}) = - b_j \iint_f c_{ijk\ell}\, n_i \frac{\partial}{\partial x_k} S_{\ell h}(x)\, df' \,. \qquad \text{(II.122)}$$

According to Burgers this is the displacement field, to a good approximation, caused by a displacement which occurred along an arbitrary curve L. Similar to eq. (II.107), this does not hold in the near neighborhood of the curve. In eq. (II.122), the glide vector is replaced by the Burgers vector.

From the comparison with eq. (II.89) we conclude that $-c_{ijk\ell}\, n_i b_j$ is the surface density of force dipoles. After welding, these dipoles have obviously taken the place of the external forces which bent the cylinder together. It is possible to imagine that the fields which were caused by a dislocation are either produced by a dislocation line (eq. (II.110)) or by a surface density of force dipoles, corresponding exactly to the fact in the theory of magnetic fields that a line current and a magnetic double layer are equivalent.[1] However, the direct proof of the equivalence of eq. (II.110) and (II.122) is very hard and is only carried out for isotropy [83].

The integral (II.122) contains the part

$$\frac{b_h}{4\pi}\, \Omega(\underset{\sim}{x}) = \frac{b_h}{4\pi} \iint_f n_k\, \nabla_k \frac{1}{x}\, df' \qquad \text{(II.123)}$$

---

[1] See also Nabarro for this [109].

in the case of isotropy, as we can easily verify, e.g., by substituting eq. (II.9) into (II.95); $\Omega(\underset{\sim}{x})$ is the three-dimensional solid angle by which the dislocation line is seen from the point $\underset{\sim}{x}$. This part causes an ambiguity of the displacement.[1] The remaining part was represented

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[1]That the displacement field is not unique in the case of a dislocation follows from the existence of the Burgers vector $\underset{\sim}{b}$, which says that the displacement increases by $\underset{\sim}{b}$ if we go once around the dislocation. The surface of the dipoles is the branch surface. The corresponding behavior of the electric potential around a linear electric current is well known.

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as a line integral by Burgers [12]. Peach and Koehler [115] also described $\Omega$ by a line integral and thus obtained the whole displacement field as a line integral, however, their formula does not have the simplicity of our line integral (II.107). For anisotropy the displacement field was first calculated as a line integral by Leibfried [90]. This method has a special importance, as the transformation of eq. (II.122) into a line integral is the mathematical proof of the fact that the position of the surface f, on which we imagine the force dipoles are distributed, is arbitrary if the boundary is the dislocation line. Naturally this proof can be also derived from eq. (II.107).

The second integral in eq. (II.121) represents the displacement which must be added to that of (II.122) to match the boundary conditions. Moreover, eq. (II.121) holds also in the case that $\underset{\sim}{g}$ is an arbitrary function of the surface f. This is the case of the "Somigliana dislocation." This has been investigated previously by Mann [97] and Bogdanoff [9].

## §18. The Elastic Energy of the Singular Dislocation

We define the elastic (self) energy of a dislocation to be the increase of the potential energy which the medium, which was originally in a natural state, suffered during the invasion or development of a dislocation. If the dislocation invades from the exterior, then we often have an edge on the surface (Fig. 10) through which the surface stresses of the continuum are locally changed. This part of the change of the potential energy can be neglected in most cases, and therefore we will no longer be concerned with it. See Nabarro [110], p. 332.

We restrict our considerations to the infinite medium. In the center of the singular dislocation, the stresses become infinite according to eq. (II.114). Therefore the energy of the dislocation per unit length (line energy) diverges. This is the big problem with all problems concerning the energy of dislocations. The real dislocations have all a certain finite "width" $2\zeta$ and so finite self energy. Fortunately $\zeta$ is only contained logarithmically in the formula of the energy, so that it is not necessary for practical use to have the exact distribution of $\underset{\sim}{g}(q)$ of the plastic relative displacement in the dislocation strip $2\zeta$.

In the following we will talk of the dislocation "line," and where a finite width is important, we will emphasize this.

According to Cottrell [122] we imagine that a dislocation lies along a curve $L^B$ in an infinite medium without external forces, and a second dislocation is developed by cutting along a surface f with the boundary curve $L^A$, and both of the edges are plastically displaced by the glide vector $g_i^A = - b_i^A$. The work done in this process is

$$W = b_i^A \iint_f (\sigma_{ij}^B + \frac{1}{2}\sigma_{ij}^A) df_j \,, \tag{II.124}$$

[1] In the finite body we have, according to Bilby [2] additional to eq. (II.124) a term $\iint_F s_i(\sigma_{ij}^B + \frac{1}{2}\sigma_{ij}^A) dF_j$ which has a meaning similar to the second integral of eq. (II.121). Another starting point of the following investigation, which would bring the same results, is eq. (II.59).

where $\sigma_{ij}^B$ is the stress field caused by the dislocation $L^B$, $\sigma_{ij}^A$ is the stress field which is developed during the procedure (therefore 1/2) caused by the curve $L^A$. By replacing the stresses by the stress function according to eq. (II.2) and using Stokes' theorem, we get

$$W = b_i^A \, \epsilon_{ijk} \oint_{L^A} \nabla_j (\chi_{k\ell}^B + \frac{1}{2}\chi_{k\ell}^A) dL_\ell^A \tag{II.125}$$

or according to eq. (II.96) expressed in terms of $\psi_{ij}$

$$W = - b_i^A \, \epsilon_{ijk} \oint_{L^A} \nabla_j \, \chi_{k\ell mn} (\psi_{mn}^B + \frac{1}{2}\psi_{mn}^A) dL_\ell^A \quad . \tag{II.126}$$

Here we substitute $\psi_{mn}^B$ of eq. (II.110) and get with

$$E^{AB} = b_i^A \, b_q^B \, M_{iq}^{AB} \tag{II.127}$$

the following

$$M_{iq}^{AB} = - \epsilon_{ijk} \, \epsilon_{npq} \oint_{L^A} \oint_{L^B} \nabla_j \, \nabla_p \, \chi_{k\ell mn} \, U(x) dL_m^B \, dL_\ell^A \tag{II.127'}$$

$$x \equiv |\underset{\sim}{x}^A - \underset{\sim}{x}^B|$$

This is the part of the work whish was caused by the presence of the dislocation $L^B$ during the development of the dislocation line $L^A$, and we call it the interaction energy of the dislocation $L^A$ and $L^B$. It holds that $E^{AB} = E^{BA}$. In the case of elastic isotropy with eq. (II.94) and (II.97) for eq. (II.127'), we obtain

$$M_{iq}^{AB} = -\frac{G}{8\pi}\,\epsilon_{ijk}\,\epsilon_{npq}\oint_{L^A}\oint_{L^B}(\nabla_j\nabla_p x)\left[\frac{2}{m-1}\,dL_n^B\,dL_k^A + dL_k^B\,dL_n^A + dL_\ell^B\,dL_\ell^A\,\delta_{nk}\right]. \tag{II.128}$$

The generally asymmetric tensor $M_{iq}^{AB}$ is analogous to the well-known mutual inductance in the theory of linear currents.[1]

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[1] Blin [7] gave an equivalent formula to (II.128) and Stroh [148] gave an additional expression for the case that the dislocations are in a plane. Eq. (II.127') was first mentioned by the author [80]. The formula (II.128) in the same paper contains a calculation error. For special arrangements of the dislocation the dislocation mutual inductance can be equal to the magnetic inductance of the similar arrangement of the current, as Hart [170] shows.

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Here we cannot substitute $\psi_{mn}^A$ according to eq. (II.110), for then we get a double line integral along the same line, which diverges. Therefore, we do not get a simple formula such as (II.127') for the self energy of a dislocation starting with (II.127'). We can approach the problem as follows: We imagine that the dislocation strip of width $2\zeta$ consists of "dislocation filaments" infinitely close together,

with the infinitesimal glide vector $d\underline{g}(q) = -\underline{b}^A \gamma(q)dq$ (see §17) and we calculate the interaction energy of all these filaments according to eq. (II.127). So we get (for isotropy)[1]

$$E^{AA} = b_i^A b_q^A M_{iq}^{AA} \tag{II.129}$$

$$M_{iq}^{AA} = -\frac{G}{16\pi} \epsilon_{ijk} \epsilon_{npq} \int_{-\zeta}^{\zeta} dq\gamma(q) \int_{-\zeta}^{\zeta} dq'\gamma(q') \oint\oint (\nabla_i \nabla_p x) \left[\frac{2}{m-1} dL'_n dL_k + dL'_k dL_n + dL'_\ell dL_\ell \delta_{nk}\right] \tag{II.129'}$$

---

[1]The superscript A is left off on the left-hand side of eq. (II.129). The integration $\oint\oint$ is taken along two dislocation filaments.

---

Here the interaction energy of each line element with all elements of all other filaments is considered, however, not the interaction energy with the elements of its own filament and not the self energy of the element of the filament. However, we recognize, that with increasing number of dislocation filaments the latter part loses importance with respect to the former part, and it becomes infinitely small with respect to the former part if the filaments are infinitely near together. Therefore eq. (II.129) shows the correct value of the self energy of the dislocation strip indeed, and the symmetric tensor $M_{iq}^{AA}$ is the analogy to the self inductance.

By use of the concepts of self and mutual inductance the energy of an arrangement of many dislocations can be written in the form

$$E = \sum_{A,B} b_i^A b_q^B M_{iq}^{AB} \tag{II.130}$$

where A and B take the value of each dislocation line.

Now we show an application of the important equation (II.128), e.g., a straight dislocation line $L^B$ lies along the $x_3$-direction. Then the integration along $L^B$ in eq. (II.128) can be carried out in an elementary fashion. We use eq. (II.108) and write

$$\int_{L^B} (\nabla_j \nabla_p x) dL_s^B = \nabla_j \nabla_p \int_{L^B} x \, dL_s^B = - \nabla_j \nabla_p \rho^2 \ln \frac{\rho}{L} i_s \qquad \text{(II.131)}$$

Hence,

$$M_{iq}^{AB} = - \frac{G}{8\pi} \epsilon_{ijk} \epsilon_{npq} \int_{L^A} \nabla_j \nabla_p (\rho^2 \ln \frac{\rho}{L}) \left[ \frac{2}{m-1} dL_k^A i_n + dL_n^A i_k + dL_\ell^A i_\ell \delta_{nk} \right] . \qquad \text{(II.132)}$$

We can easily check that the following relation holds

$$\nabla_j \nabla_p \rho^2 \ln \frac{\rho}{L} = 2 \left[ (\ln \frac{\rho}{L} + \frac{1}{2}) \delta_{jp} + \frac{x_j x_p}{\rho^2} \right] , \quad j,p = 1,2 \quad . \qquad \text{(II.133)}$$

Now we restrict ourselves to the case that the dislocation $L^A$ is in the $x_2 = 0$ plane and we obtain for the right-hand side of eq. (II.133) $2[(\ln \frac{x_1}{L} + \frac{1}{2}) \delta_{jp} + \delta_{jp}^1]$, where $j,p = 1,2$ and $\delta_{jp}^1 = 1$ for $j = p = 1$, otherwise vanishes. This we substitute in eq. (II.132) and consider simultaneously that there we have $i_n = i_k = i_\ell = i_3$ and furthermore, $dL_2^A = 0$. For $j = p = 1$, we get the part

$$\frac{G}{8\pi} \left[ \frac{4}{m-1} \epsilon_{i13} \epsilon_{31q} + 2\epsilon_{i13} \epsilon_{31q} + 2\epsilon_{i1k} \epsilon_{k1q} \right] \int_{L^A} (\ln \frac{x_1}{L} + \frac{3}{2}) dx_3 \qquad \text{(II.134)}$$

to be $M^{AB}$ and for $j = p = 2$, the part

$$\frac{G}{8\pi}\left[\frac{4}{m-1}\,\epsilon_{i2k}\,\epsilon_{32q} + 2\epsilon_{i23}\,\epsilon_{k2q}\right]\int_{L^A}\left(\ln\frac{x_1}{L} + \frac{3}{2}\right)dx_k$$

$$+ \frac{G}{4\pi}\,\epsilon_{i2k}\,\epsilon_{k2q}\int\left(\ln\frac{x_1}{L} + \frac{1}{2}\right)dx_3 \,. \qquad \text{(II.135)}$$

From this it easily follows that:

$$M_{11}^{AB} = -\frac{G}{2\pi}\frac{m}{m-1}\int_{L^A}\left(\ln\frac{x_1}{L} + \frac{1}{2}\right)dx_3$$

$$M_{22}^{AB} = -\frac{G}{2\pi}\frac{m}{m-1}\int_{L^A}\left(\ln\frac{x_1}{L} + \frac{1}{2}\right)dx_3$$

$$M_{33}^{AB} = -\frac{G}{2\pi}\int_{L^A}\left(\ln\frac{x_1}{L} + 1\right)dx_3$$

$$\text{(II.136)}$$

$$M_{31}^{AB} = \frac{G}{2\pi}\frac{1}{m-1}\int_{L^A}\left(\ln\frac{x_1}{L} + \frac{1}{2}\right)dx_1$$

$$M_{13}^{AB} = \frac{G}{2\pi}\int_{L^A}\left(\ln\frac{x_1}{L} + \frac{1}{2}\right)dx_1$$

$$M_{23}^{AB} = M_{32}^{AB} = M_{12}^{AB} = M_{21}^{AB} = 0 \,.$$

For the further treatment of the integration, we have to substitute in the logarithm the equation, $x_1(x_3)$, of the dislocation line $L^A$ where $dx_3$ appears. If $L^A$ is also a straight line and parallel to $L^B$ we have $M_{31}^{AB} = M_{13}^{AB} = 0$, since $dx_1 = 0$. Therefore, only the diagonal components of $M_{iq}^{AB}$ remain, i.e., parallel dislocation lines whose Burgers vector are perpendicular do not have any influence on each other in isotropic medium (this conclusion follows from eq. (II.127). Now we write down

the components of $M_{iq}^{AB}$ for $x_1 = d$ (= parallel dislocation lines with the distance d)

$$M_{11}^{AB} = -\frac{G}{2\pi}\frac{m}{m-1}L'(\ln\frac{d}{L}+\frac{1}{2})$$

$$M_{22}^{AB} = -\frac{G}{2\pi}\frac{m}{m-1}L'\ (\ln\frac{d}{L}+\frac{3}{2}) \qquad \text{(II.137)}$$

$$M_{33}^{AB} = -\frac{G}{2\pi}L'\ (\ln\frac{d}{L}+1)$$

We carried out the second integration with the limits $-L'/2 \to L'/2$ where $L' \ll L$ is assumed, since the presumption of eqs. (II.136) (validity of eq. (II.108)) only holds then. However, we will show later on that eqs. (II.137) are only slightly changed for the exact calculation $L' \to L$, as long as $L \gg d$.

The interaction energy of two straight parallel dislocations separated by a distance, d, was investigated by various authors. By differentiating it, the attractive ($b^A$, $b^B$ antiparallel) or repulsive ($b^A$, $b^B$ parallel) force can be derived. Perhaps those authors got the formulas for the interaction energy which can be obtained by multiplying eqs. (II.137) with $b^A b^B$, in an easier way than we did, for the simplifications of this special case can be used at the very beginning. However, we used our derivation to get the equations (II.136), which are relatively simple and govern a group of problems which are of some importance in practice.[1] We will discuss an application of these equations in §29.

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[1] In eqs. (II.136) we can immediately see an important part of the result for dislocations whose lines are perpendicular to each other, which were discussed by Nabarro [110], §309. Thus $L^A$ has the direction

(Footnote Continued)

$x_1$. Then $dx_3 = 0$ in the equations (II.136) and the interaction is only present in the case $M_1^{AB}$ ($L^A$ and $L^B$ are screw dislocations) and in the case $M_{31}^{AB}$ ($L^A$ and $L^B$ are edge dislocations with parallel Burgers vector).

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Our formulas (II.137) are distinguished from those of other authors in two ways.[1] Cottrell [22], who starts from eq. (II.124), does not change the surface integral into a line integral and so in his final formula, instead of the dislocation length L, the dimension R of the

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[1]Koehler was the first to calculate the energy of a straight dislocation and the interaction energy of two dislocations.

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medium which is perpendicular to the dislocation line and also goes to infinity, appears in the logarithm. Eshelby (see [110], pp. 305-306), who used different approaches to the problem, e.g., the use of bipolar coordinates, obtained the same. In all cases we get a logarithmic divergence of the interaction energy per unit length of parallel, straight dislocation lines, thus we are forced in reality to calculate in the finite body. We will show that the same is true for the self energy of a straight dislocation line. Often this complication can be avoided by taking for R or L the approximate value of the average distance between dislocations with different sign (dislocation network §29)(e.g., $10^{-4}$ cm in undeformed metal). This procedure is not sufficient. However, practical problems often occur in such a way (§29) that either L or R, respectively, cancels or is known from the beginning, whereas, in many cases, the approximate value of this term is sufficient

because of the logarithmic dependence of the energy on L and R. Second, our formulas are distinguished from those of other authors by the term in addition to ln(d/L). This is small with respect to ln(d/L)) according to the assumption ($d \ll L$). The difference is caused by the different treatment of dislocation center. Also, it plays a role if we have R or L in the logarithm.

Next we apply eq. (II.128) to two straight dislocations,[1] which

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[1]This will lead us to a method which abbreviates the calculation of the self energy very much.

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are parallel and separated by the distance $\rho$ along the $x_3$-direction (both of them have the finite length L, Fig. 18) and especially $\rho \ll L$. We carry out the differentiation $\nabla_j \nabla_p x$ before integrating and neglect then all terms of this expression which have the factor $x_1^A - x_1^B$ or $x_2^A - x_2^B$.[2]

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[2]These terms are less by a factor $\rho/L$ than the remaining terms, as a more intensive investigation shows.

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As we can easily see, only the differentiations $\partial^2/\partial x_1^2$ and $\partial^2/\partial x_2^2$ will give a contribution $1/x$ and all other terms vanish. Furthermore, we assume that the Burgers vectors of the two dislocations are equal and the dislocations lie in the plane $x_2 = 0$. Their angle with respect to the line direction is $\beta$, thus the components of $\underset{\sim}{b}$ in the $x_1$- and $x_3$-direction are $b \sin \beta$ and $b \cos \beta$, respectively. So we easily calculate the interaction energy $E^{AB}$ according to eq. (II.127) and (II.128) to be

$$E^{AB} = \frac{Gb^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\iint \frac{dx_3^A\, dx_3^B}{x} \tag{II.138}$$

We calculate the integral more generally than necessary here, i.e., for the limits

$$\int_{-L/2}^{L/2} dx_3^A \left(\int_{-L/2}^{x_3-\epsilon} dx_3^B \ldots + \int_{x_3+\epsilon}^{L/2} dx_3^B \ldots\right) \tag{II.139}$$

and we get exactly

$$\iint \frac{dx_3^A\, dx_3^B}{x} = 2L\, \ln \frac{L + \sqrt{\rho^2 + L^2}}{\epsilon + \sqrt{\rho^2 + \epsilon^2}} - 2\sqrt{\rho^2 + L^2} \quad . \tag{II.140}$$

With $\epsilon = 0$ we get, if we neglect $\rho^2$ in comparison to $L^2$ and divide by L

$$T^{AB} = \frac{Gb^2}{2\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\left(\ln \frac{2L}{\rho} - 1\right) \quad . \tag{II.141}$$

This is the interaction energy per unit length of the dislocation line. It differs only slightly from the results of eqs. (II.137) for large L, as we can see if we substitute $\ln(2L/\rho) = \ln(L/\rho) + \ln 2$ into eq. (II.141).

Now we assume that both of the dislocations are filaments of a dislocation line, whose self energy we will now determine. Then we can use the results obtained above directly in eq. (II.129)

$$T^{AA} = \frac{Gb^{A2}}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\int_{-\zeta}^{\zeta} dx_1\, \gamma(x_1) \int_{-\zeta}^{\zeta} dx_1'\, \gamma(x_1') \times$$

$$\times \left(\ln \frac{2L}{|x_1 - x_1'|} - 1\right) \tag{II.142}$$

This simplest case is $\gamma = \text{const} \cdot (\frac{1}{2})_\zeta$, which is the same as a linear increase of the relative displacement in the strip $2\zeta$ (Fig. 9). Then we carry out the integration in (II.142) in an elementary fashion, and we obtain

$$E^{AA} = \frac{G(b^A)^2}{4\pi} \left(\frac{m}{m-1} \sin^2\theta + \cos^2\theta\right)\left(\ln \frac{L}{\zeta} + \frac{1}{2}\right) \tag{II.143}$$

or also

$$E^{AA} = \frac{G(b^A)^2}{4\pi} \left(\frac{m}{m-1} \sin^2\theta + \cos^2\theta\right)\left(\ln \frac{L}{\zeta e^{3/2}} - 1\right) \tag{II.144}$$

for the energy of the dislocation line. This formula is exact for $L \to \infty$ if Hooke's Law also holds in the strip $2\zeta$, which is correct for sufficiently small $b^A$. From the logarithmic dependence of $\zeta$ we see that the energy is not very sensitive to small changes of $\zeta$ and $\gamma(x_1)$.

We can obtain exactly the same formula (II.144) if we substitute the integral (II.140) into eq. (II.138) with $\varepsilon = 0$ and $\varepsilon = \zeta e^{3/2}$ and multiply the result by 1/2. I.e., if we substitute

$$M^{AA}_{iq} = -\frac{1}{2} \varepsilon_{ijk} \varepsilon_{npq} \oint_L \oint_{L'_\varepsilon} (\nabla_j \nabla_p x) \left[\frac{2}{m-1} dL'_n dL_k + dL'_k dL_n + dL'_\ell dL_\ell \delta_{nk}\right] \tag{II.145}$$

instead of (II.129') into eq. (II.129), where $L'_\varepsilon$ means that in the integral considered, the piece $x - \varepsilon \ldots x + \varepsilon$ is excluded from the integration, we obtain, at least in the case of a straight dislocation line, the exact self energy. We can justify that this is also true to a good approximation for a curved dislocation.[1] However, the integrals

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[1]The most important point is that the main contribution of a line

(Footnote Continued)

element is its long range stress field. However, in (II.145) all parts of the interaction effects of an element with other than nearest neighbor elements are exactly included. However, the part of the effect due to nearest neighbors does not depend on the curvature of the dislocation if the radius of curvature is sufficiently large with respect to $\epsilon$. But (II.145) is exact for straight dislocations.

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(II.145) can be evaluated in many cases where we cannot do the integrations of (II.129') with a reasonable amount of work. This is the practical importance of the calculations shown here.

A remark to eq. (II.144). The energy $T^{AA}$ per unit length of the dislocation depends only logarithmically on L. E.g., in many cases we have to consider the bending of an initially straight dislocation line during which the length of the dislocation is not greatly changed. Then we can neglect the dependence of $T^{AA}$ on L and $\beta$ to a good approximation, and we find the energy proportional to its length. $T^{AA}$ is often called the "line tension" of the dislocation by analogy with the conditions of a stretched string. E.g., we can derive a differential equation for the vibration of such a dislocation which has the exact form of the equation for a vibrating string. For this see the papers of Eshelby [35] and Koehler [72].

In the last two sections we considered dislocations in an infinite medium. In practice we always have finite bodies, and in some cases the results for the infinite medium does not represent a reasonable approximation to reality. (This is especially true for problems with

straight dislocations.) Then we have additionally to solve the boundary value problem. These problems were treated very successfully by Dietze [163] and Leibfried and Dietze [171]. Seeger reported about this in the appendix, see [134], S 560. These are dislocations in bodies which are bounded by plane or circular cylindric surfaces. In these cases, generally, solutions in closed form were found. Here the reflection procedure used by Leibfried and Dietze [171] is of special interest. Furthermore, see also Eshelby and Stroh [167].

## §19. Forces on Dislocations and Other Elastic Singularities. The Dislocation as an Elementary Source of Self Stresses

The problem of the forces which a stress field exerts on elastic singularities, especially dislocations, is completely unknown in the classic theory of elasticity. However, it has its analogy in electrodynamics, where we have simple formulas for the forces on linear current elements, magnetic dipoles, etc. The formulas which we will derive in the following are almost as simple.

The great importance of such considerations is obvious for the theory of dislocations: The motion of the dislocations, therefore the plastic deformation of the material, occurs under the influence of externally applied stresses. This influence must attain a certain level first to develop dislocations and second to maintain their motion.

After preparations of Mott and Nabarro [105] and Leibfried [88], Peach and Koehler [115] succeeded in deriving the general expression for the force which the line element of the dislocation experiences at the point $\underset{\sim}{x}$ in the stress field $\underset{\sim}{\sigma}(\underset{\sim}{x})$. This often used formula is the

fundamental equation of the dislocation theory. It is somehow similar to the formula of the Lorentz force on an electric current line element in the magnetic field.

We define the force $d\underset{\sim}{K} \equiv (dK_i)$ on the line element, $dL_i$ of a dislocation of the Burgers vector $b_i$ as follows: $-dW^a$ is the work done by the external forces during a displacement, $dx_i$, of the dislocation element; $dW^i$ is the simultaneously occurring increase of the elastic energy of the body. Then $d\underset{\sim}{K}$ is defined by the equation

$$- (dW^a + dW^i) = d\underset{\sim}{K} \cdot d\underset{\sim}{x} \qquad \text{(II.146)}$$

Now we imagine that the displacement is carried out as follows: The surface region along which $d\underset{\sim}{L}$ will move is $d\underset{\sim}{f} = d\underset{\sim}{x} \times d\underset{\sim}{L}$. (Then relative to the motion $d\underset{\sim}{L}$ is a right-hand screw boundary of $d\underset{\sim}{f}$.) We cut open along $d\underset{\sim}{f}$ and in order to avoid a displacement, apply the forces $d\underset{\sim}{f} \cdot \underset{\sim}{\sigma}$ to the cut edge and $- d\underset{\sim}{f} \cdot \underset{\sim}{\sigma}$ on the other cut edge, respectively. Now we consider both of the cut edges as part of the surface of the body. (The remaining part is the initial surface, i.e., now the body is doubly connected, the internal forces become external through this operation.)

Now let the relative displacement of the glide vector $\underset{\sim}{g} = - \underset{\sim}{b}$ of the cut edges. This means a motion of $d\underset{\sim}{L}$ along $d\underset{\sim}{x}$. If we consider this displacement to be virtually infinitesimal in the sense of the principle of the virtual displacement, then, since the body is in equilibrium, there is no net work done, i.e.,

$$dW^a + dW^i + d\underset{\sim}{f} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{b} = 0 \qquad \text{(II.147)}$$

By comparison with eq. (II.146) if we consider that $(d\underset{\sim}{x} \cdot d\underset{\sim}{L}) \cdot \underset{\sim}{b} = d\underset{\sim}{x} \cdot (d\underset{\sim}{L} \cdot \underset{\sim}{\sigma})$, it follows that

$$d\underset{\sim}{K} = d\underset{\sim}{L} \cdot \underset{\sim}{\sigma} \cdot \underset{\sim}{b} \qquad \text{(II.148)}$$

or

$$dK_k = \epsilon_{ijk}\, dL_i\, b_\ell\, \sigma_{j\ell} \qquad \text{(II.148')}$$

This is the formula of Peach and Koehler. The only assumption we used was that the displacement $-\underset{\sim}{b}$ was virtually infinitesimal in the sense of the principle of virtual displacement. This assumption is not exactly satisfied for finite $|\underset{\sim}{b}|$; thus, we have to consider eq. (II.148) as an approximation (however in most cases sufficient). $\sigma_{ij}$ is that stress we will measure at the point $\underset{\sim}{x}$ of the line element if we carry out the known cut. This $\sigma_{ij}$ includes besides the stresses caused by the external forces and other sources of internal stresses, also those stresses caused by all the other line elements of the dislocations at the point $\underset{\sim}{x}$; furthermore, it includes the part of the stresses with which the line element reacts, e.g., due to the free surface or other boundary surfaces in the body to itself. Finally, eq. (II.148) holds for arbitrary inhomogeneity of the elastic constants in the body. By a method similar to that of Eshelby (see below) Rieder [124] showed that eq. (II.148) also is true for quasi displacements. The extensive applicability of eq. (II.148) is based on the fact that it is only a consequence of the very general principle of virtual displacement. Equation (II.148) is sometimes mentioned in context with the theorem of Colonetti [134,108], in §14, which reads for this case, that under certain conditions (Hooke's Law; no other source of internal stresses than the considered dislocation;

elastic homogeneity) $dW^i = 0$. But we should notice that the truth of the theorem of Colonetti for the medium considered is no assumption of eq. (II.148).

The derivation of eq. (II.148) does not require the stress tensor to be symmetric. We will show an important result of this fact. We ask which stresses are able to bring a planar distribution of crossed screw dislocations[1] as in Fig. 16c, into motion as a whole perpendicular

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[1]Such an arrangement of dislocations has a great stability in practice for it completely cancels the long range stress fields of the contributing dislocations (§23).

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to their plane (e.g., $x_1 = 0$)? I n this case the dislocation tensor $dL_i\, b_\ell = \alpha^o_{i\ell}$ in eq. (II.148) has only the components $\alpha^o_{22} = \alpha^o_{33} = \alpha \equiv \alpha^o$. So we obtain

$$dK_1 = \alpha^o_{22}\,\sigma_{32} - \alpha^o_{33}\,\sigma_{23} = \alpha^o(\sigma_{32} - \sigma_{23}) \qquad \text{(II.149)}$$

I.e., the dislocation motion similar to that of Fig. 16c can only occur if the stress tensor is anti-symmetric. We will now mention examples of the fact that asymmetric stress tensors really occur in crystals.

In ferromagnetic crystals we have a spontaneous magnetization in a favored crystallographic direction which minimizes the free energy of the crystal. An external magnetic field can rotate this magnetization into a more energetically unfavorable direction. One of the magnetically favored directions of the crystal tries to rotate into the new direction of magnetization, therefore the external magnetic

field causes torques on the volume elements, the result of which are asymmetric stresses. According to a remark of Rieder [124], it would be possible in the most favorable case for the force to develop to start a motion of a grain boundary of the kind mentioned earlier.

We are also able to produce such torques with currents, which may be very small, in a crystal, in the case where strong anisotropy of electrical conduction exists. However, we believe that large differences between the free energy of two crystallites separated by a grain boundary (e.g., such as during recrystallization, phase changes and similar procedures) such strong asymmetric stresses can be produced that the grain boundary moves into the crystallite with the higher free energy. Recently, Rieder [124] described briefly how we can treat such asymmetric stresses in the theory of elasticity. However, we assume that Cosserat's strain torques, neglected by Rieder, have an important influence on the real circumstances. Investigations of the above phenomena in crystals seem to be worthwhile problems.

Now we will describe another important application of the Peach-Koehler formula. L is a small, plane dislocation loop, not necessarily circular, with the Burgers vector $b_i$. In the region containing the loop there are no body forces, thus $\nabla_i \sigma_{ij} = 0$. We obtain the total force on this loop in the stress field $\sigma_{ij}$ by integrating eq. (II.148) along L. Applying Stokes' theorem and expanding $\sigma_{ij}$ in a Taylor series about the center $\underset{\sim}{x} = 0$, we easily obtain

$$K_k = \nabla_k \sigma_{j\ell}\Big|_o \iint_f n_j b_\ell df + \nabla_k \nabla_m \sigma_{j\ell}\Big|_o \iint_f x_m n_j b_\ell df + \ldots \qquad \text{(II.150)}$$

Let $f \to 0$ and simultaneously increase $b_\ell$ so that the integrals in (II.150) remain finite. Thus f becomes a point and $n_j b_\ell df$ means that both of the points on the positive and negative side of f suffer the relative displacement $-b_\ell = g_i$. So we define the "displacement dipole" $Q_{j\ell}$ by

$$Q_{j\ell} \equiv \lim_{f \to 0} \iint_f n_j g_\ell df \tag{II.151}$$

and similarly the displacement quadrupole $Q_{mj\ell}$ by

$$Q_{mj\ell} \equiv \lim_{f \to 0} \iint_f x_m n_j g_\ell df \tag{II.152}$$

The sign in eq. (II.151) is so defined that a positive dipole $Q_{11}$ causes the cut edges to be drawn apart. Symbolically <- - - - ->. Previously we called a force dipole $P_{11}$ positive if it was derived from the limit of two point forces < - - - - - - - - > .

Now we will compare eq. (II.151) and eq. (II.122). There we recognized that the expression $-c_{ijk\ell}\, n_i b_j df$ is an infinitesimal force dipole. An infinitesimal displacement dipole can be written according to eq. (II.151) $-n_j b_\ell df$. I.e., for the force dipole and the displacement dipole we have the relation[1]

$$P_{ij} = c_{ijk\ell}\, Q_{k\ell} \tag{II.153}$$

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[1] Summarizing, we call $P_{ij}$ and $Q_{ij}$ an "elastic dipole."

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However, it should be noticed that this is only true if the elastic constants of the body are homogeneous in the surroundings of the dipole and also at its point of application. This is included in eq. (II.122)

[illegible] quadrupoles. Now we can write the total force which acts on a dislocation loop (II.150)[1]

$$K_k = Q_{j\ell} \nabla_k \sigma_{j\ell} + Q_{mj\ell} \nabla_k \nabla_m \sigma_{j\ell} + \ldots \qquad (II.154)$$

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[1] The form of this equation given in the author's original paper [82], $K = \text{grad}\,(\sigma_{j\ell} Q_{j\ell} + \nabla_m \sigma_{j\ell} Q_{mj\ell} + \ldots)$ is not very convenient in the following respect. It must be indicated that before carrying out the multiplication in the parentheses the differentiation, grad $\sigma_{j\ell}$ has to be done. Also the form of eq. (I.154) could lead to erroneous conclusions about the physical meaning of the terms in parentheses. (See below.)

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In the following discussion we will no longer consider quadrupoles. Thus the force on a displacement dipole itself becomes

$$K_k = Q_{j\ell} \nabla_k \sigma_{j\ell} \qquad (II.155)$$

Such a force also produces antisymmetric displacement dipoles. However, we also will not consider this further, as these are not as important as symmetric displacement dipoles [82]. In the following, $Q_{ij}$ is a symmetric tensor. By substituting $\epsilon_{ij}$ for $\sigma_{ij}$ in eq. (II.155) according to Hooke's Law and considering (II.153), we obtain for the force on a force dipole

$$K_k = P_{ij} \nabla_k \epsilon_{ij} \qquad (II.156)$$

Also this equation only holds if the elastic constants are homogeneous within the region of the dislocation loop,[1] which is equivalent to the dipole. An example should clarify this:

(Footnote for preceding page)

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[1] The homogeneity is violated if, as the limit of (II.151 says, the relative displacement $\underset{\sim}{g}$ goes to infinity. I.e., initially eq. (II.156) are only true for dipoles of infinitesimal magnitude. Also the Peach-Koehler formula holds exactly only for this case. The conclusion as to the truth of eq. (II.156) with a dipole of finite magnitude follows below.

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In a body II a small region I of another material is constrained so that we can relate the boundary surface f between I and II with its normal vector $\underset{\sim}{n}$ to the displacement jump $\underset{\sim}{g}$ according to eq. (I.77). In this case $n_i g_j$ is the related density of the displacement dipoles on the boundary surface, and $\iint n_i g_j df = Q_{ij}$ represents the total displacement dipole. We let the volume of the inclusion decrease to zero, while $b_j$ simultaneously increases so that $Q_{ij}$ remains finite. The related force dipole (we assume that the higher order poles all vanish) indicates the forces which the constrained region I exerts on its surrounding. For a given displacement dipole, these will become larger as the inclusion becomes harder. It is true that we can write down an eq. (II.153), but in this we are not allowed to identify $c_{ijk\ell}$ as the elastic moduli of I or II, but we have first of all to solve a boundary value problem on the boundary surface f to get the right value for $c_{ijk\ell}$.

For practical applications of these considerations (§31) the case in which elastic homogeneity is violated at the point of the inclusion is of special interest. Then eq. (II.156) but not eq. (II.155)

holds. This can be clarified best in connection with a method with which Eshelby determined the force on elastic singularities in an elastic field.

A body with the surface $S_o$ is stressed by surface forces $n_i\sigma^a_{ij}$, which cause displacements $s^a_i$ and stresses $\sigma^a_{ij}$ in it. Furthermore, it contains a singularity[1] at the point $\underset{\sim}{x}'$, which causes an

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[1]The term "singularity" is used here in a general sense, e.g., it can be a number of singularities or an arbitrary distrubance which is located in a partial region of the body.

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additional displacement $s^s_i$ and stresses $\sigma^s_{ij}$, and another singularity with the related values $s^f_i$ and $\sigma^f_{ij}$. The force on the singularity is defined by a relation equivalent to eq. (II.146)

$$K_\ell = -(dW^a/dx'_\ell + dW^i/dx'_\ell) \equiv K^a_\ell + K^b_\ell + K^t_\ell \tag{II.157}$$

where $dW^i/dx'_\ell$ has two parts: $-K^b_\ell \equiv dW^b/dx'_\ell$ is the so-called "image force," i.e., the force which the singularity puts on itself via the surface; $-K^t_\ell = dW^t/dx'_\ell$ is the force resulting from the change of the interaction energy between both of the singularities. In addition, the truth of the previously mentioned theorem of Colonetti is assumed for the medium considered. First of all we have

$$dW^a = dx'_\ell \iint_{S_o} \sigma^a_{ij} \frac{\partial s^s_i}{\partial x'_\ell} dS_j \tag{II.158}$$

$$K^a_\ell = \iint_{S_o} \sigma^a_{ij} \frac{\partial s^s_i}{\partial x'_\ell} dS_j \tag{II.159}$$

Therefore it is important for this part to know which additional displacement the surface of the body suffers during a displacement of $dx'_\ell$ due to the singularity. The related expression for $K^b_\ell$ was derived by Eshelby

$$K^b_\ell = \iint_{S_o} \sigma^b_{ij} \frac{\partial s^s_i}{\partial x'_\ell} dS_j \tag{II.160}$$

This force is interpreted as the action of the image force on the whole surface. Since the expressions (II.159) and (II.160) are completely independent of the existence of the second singularity, we can contract the integration surfaces $S_o$ without changing the value of the integral to a closed surface s, which only surrounds the movable singularity. After some calculation we obtain ( $,\ell$ means differentiation with respect to $x_\ell$)

$$K^a_\ell + K^b_\ell = \iint_S [(s^a_i + s^b_i)\sigma^\infty_{ij,\ell} - (\sigma^\infty_{ij} + \sigma^b_{ij})s^\infty_{i,\ell}]dS_j \tag{II.161}$$

where $s^\infty_i$, $\sigma^\infty_{ij}$ are the displacement and stresses of a singularity of an infinite medium. Eshelby succeeded in proving that the force $K^t_i$ can be represented in the similar form; thus the total force becomes

$$K_\ell = \iint_S (s_i \sigma^\infty_{ij,\ell} - \sigma_{ij} s^\infty_{i,\ell})dS_j \tag{II.162}$$

where $s_i \equiv s^a_i + s^b_i + s^t_i$ and similarly $\sigma_{ij}$. Equation (II.162) can be written after some calculation in the form

$$K_\ell = \iint_S M_{j\ell}\, dS_j, \quad M_{j\ell} \equiv -\sigma_{ij}\, s_{i,\ell} + \frac{1}{2}\sigma_{ik}\, \epsilon_{ik}\, \delta_{j\ell} \tag{II.163}$$

By analogy to electrostatics, Eshelby calls the asymmetric tensor $M_{j\ell}$ the Maxwell tensor of elasticity.

The eqs. (II.162) and (II.163) are generally true. Their application to point singularities are of special importance. However, it does not cover the line element of the dislocation since we can not have a closed surface $s_1$ which lies completely in a region without self stress sources. For our use the importance of Eshelby's equations is that the singularities used need not necessarily be defined at the point of application but at some distance from it (by the displacements which they cause there if it is an infinite medium), where clearer conditions are given than at the center of the singularity. E.g., it is of no significance if the force dipole (this governs the displacement $s_i^\infty$ according to eq. (II.89)) is caused by a soft or a hard inclusion. Only the magnitude of the displacment $s_i^\infty$ which it produces on the surface S is important. However, this is proportional to the force dipole according to eq. (II.89). Hence, we can conclude at once that eq. (II.156) holds generally, for if it holds for one case (no inhomogeneity at the location of the inclusion), then it also must hold for an arbitrary inhomogeneity.

Furthermore, in eq. (II.162) we can expand $\sigma_{ij}$ and $s_i$ in a Taylor series and obtain our formula (II.156) in the case of the dipole after some calculation [83]. Simultaneously this is the proof that eq. (II.156) is also true for a finite dipole force. Eshelby [38] carried out the calculation for the special case of the so-called dilatation center (see Love [95]) $P_{ij} \equiv P_{ii} > 0$, which is of special interest, since atoms of type B can be described as such centers in the lattice of atoms of type A (§31).

Now we can determine the work done by the force $K_k$ in eq. (II.156) during a change of position of the dipole from a point where the strain is zero to a point where the strain is $\epsilon_{ij}$ and we obtain

$$\int_{\epsilon_{ij}=0}^{\epsilon_{ij}} K_k dx_k = \int P_{ij}\nabla_k \epsilon_{ij} dx_k = P_{ij} \int \nabla_k \epsilon_{ij} dx_k = P_{ij}\epsilon_{ij} \qquad \text{(II.164)}$$

Obviously this work is independent of the path. Therefore the expression,

$$U = - P_{ij}\ \epsilon_{ij} \qquad \text{(II.165)}$$

can be taken to be the potential energy of the dipole $P_{ij}$ in the strain field $\epsilon_{ij}$. In case $P_{ij}$ is the only self stress source, $P_{ij}\ \epsilon_{ij}$ is the change of the potential energy of the boundary forces plus the change of the self energy of the dipole caused by the changed interaction between it and the surface. The sign in eq. (II.165) reasonably indicates, that e.g., a compressed inclusion ($P_{ii} > 0$) in a compressed part of the body ($\epsilon_{ii} < 0$) causes a positive energy. Furthermore, we can see that we will not get such simple formulas as (II.156) in the case of volume forces, since such forces do not have potentials in general, and so we will not get an integral $\int K_k\ dx_k$, which is independent of the path.

From eq. (II.165) it is simple to derive formulas for the torque $\underset{\sim}{L}$ on a symmetric dipole $P_{ij}$. If we write, corresponding to eq. (II.90), $P_{ij} = \ell_i P_j$, because $d\underset{\sim}{L} = d\underset{\sim}{\delta} \times \underset{\sim}{\ell}$ and $d\vec{P} = d\underset{\sim}{\delta} \times \vec{P}$,[1]

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[1]Here we assumed that the self energy of a dipole does not change during the rotation.

then the change of $P_{ij}$ for a rotation through an angle $d\delta_k$ is

$$dP_{ij} = d\ell_i P_j + \ell_i dP_j = (\epsilon_{ik\ell}\,\ell_\ell P_j + \epsilon_{jk\ell}\,\ell_i P_\ell)d\delta_k$$

$$= (\epsilon_{ik\ell}\,P_{\ell j} + \epsilon_{jk\ell}\,P_{i\ell})d\delta_k \qquad \text{(II.166)}$$

On the other hand, the change of the potential energy of the dipole for an infinitesimal rotation is

$$dU = -L_k d\delta_k = -\epsilon_{ij} dP_{ij} = -\epsilon_{ij}(\epsilon_{ik\ell}P_{\ell j} + \epsilon_{jk\ell}P_{i\ell})d\delta_k \qquad \text{(II.167)}$$

Because of the symmetry of $\epsilon_{ij}$ and $P_{ij}$ at once we conclude

$$L_k = 2\epsilon_{ik\ell}\,P_{\ell j}\,\epsilon_{ij} \qquad \text{(II.168)}$$

The conditions under which this formula is true are the same as for eq. (II.156).

In the case of an inhomogeneous inclusion (i.e., the elastic homogeneity is disturbed at the point of the inclusion) another effect appears, which was first properly investigated by Eshelby [38] and Crussard [27], the "polarization" of the singularity.[1] The simplest

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[1]See also additional papers of Crussard [28], Eshelby [40] and Teltow [151].

---

example is a stress free body with an inclusion of another material which fits in without any restraint. If we now apply surface forces to the body, then a force dipole $P_{ij}^{ind}$ is induced in the inclusion, for which we also can use eq. (II.156), since Eshelby was able to show that we again end up with an expression like (II.163).

It is convenient to define the polarizability, $R_{ijk\ell}$, by the equation

$$P_{ij}^{ind} = R_{ijk\ell}\, \epsilon_{k\ell} \tag{II.169}$$

for determination of which a boundary value problem with respect to the boundary surface must be solved, even for the smallest inclusions. For the case of spherical inclusions and elastic isotropy, we get elementary solutions according to Eshelby [38], [40].[1] Now it is obvious that

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[1]Eshelby was now able to show that the stress field induced by an ellipsoidal inclusion is homogeneous if the induced stress field is homogeneous at a large distance from the inclusion (holds also for elastic anisotropy). The similar result for the polarization of an ellipsoidal dielectric is well known.

---

we can represent the elastic displacement field of an arbitrary point source of internal stresses in the infinite medium by equation

$$s_i = P_{jk} \nabla_j S_{ki} + P_{jk\ell} \nabla_j \nabla_k S_{\ell i} + \dots \tag{II.170}$$

from which we can determine the internal stresses by the usual method. A quadrupole is nothing other than two dipoles close together; a similar analogy holds with poles of higher order, i.e., we can describe every self stress source by a suitable combination of force dipoles. But these are nothing other than infinitesimal dislocation loops. Therefore, the theorem mentioned in the preface holds: All self stresses in the continuum are caused by dislocations. In §14 we stated all self stresses are caused by incompatibilities. This holds even now and is compatible with the theorem mentioned above. For according to

eq. (II.51) incompatibilities have their origin in dislocations. Thus we can call either the dislocations or the incompatibilities the elementary self stress source. However, the statement identical to eq. (II.17) is more consequent: Dislocations are the vortexes of elastic distortions as forces are the causes of stresses.

On the other hand we can also describe continuous distributions of self stress sources, dislocations, by a three-dimensional distribution of such point sources. For the same reason we could declare the force dipole to be the elementary component in the theory of self stresses. The circumstances are similar to (stationary) Maxwell's theory. There infinitesimal current loops and magnetic dipoles are equivalent. But normally the electric current is preferred, and only this appears in Maxwell's equations. In this sense we also preferred the dislocation, and thus, in our opinion, we get an impressive representation of the continuum mechanics of solid bodies.

## CHAPTER III

## DISLOCATIONS IN A CRYSTAL

### §20 General Statements

In this chapter we will discuss applications of the continuum theory of dislocations to real bodies, the most important of which are crystalline. Physical problems of this kind deal almost entirely with single crystals, but when the results are transferred to polycrystals as generally happens in the technique, there is almost no success. So we will restrict ourselves to the investigation of single crystal problems, however, we will emphasize that there are no difficulties in principle in applying the continuum theory of dislocations to polycrystalline bodies, and we will return to this soon.

The main difference between a continuum and a body composed of single mass points, as, e.g., a crystal, is that in the latter a volume element is defined. In the continuum the distortion of the volume element is the determining geometrical quantity; displacements were mentioned comparatively little. In a system of points we primarily measure displacements, so we are tempted to relate all to this. However, it is soon obvious that in general we will not succeed with this approach, since we cannot get the required degree of freedom. Instead of this we have to consider the relative displacement of two neighboring atoms. From the considerations of §21 it can easily be seen that the distribution of such relative displacements has three

times as many degrees of freedom as the distribution of dislocations. Then we can develop exactly the theory of dislocations in the crystal (§21), where, however, we do not obtain differential equations but difference equations. However, since the number of points (atoms) in a crystal is immensely large, in many cases we can replace these difference equations to a very good approximation by differential equations, and in general we have to do this as we will not succeed in solving the problem numerically.

This approach is especially reasonable for many of the more "microscopic" problems of the physics of crystals, in which we investigate the behavior and the properties of a single dislocation. With macroscopic problems, however, we are interested in the combined effect of many dislocations. In this case it is obvious that we must consider certain "physical" volume elements having properties which we will now discuss.

The assumption for the application of continuum mechanics to real bodies is that the deformation of the volume elements of the body can be measured as a macroscopic ally continuous function of position. For this the volume elements must be sufficiently small with respect to the external size of the body, for otherwise we cannot formulate the differential equation. On the other hand, the glide and climb planes are discrete and microscopically considerably far apart, whereas the distance between them and the magnitude of the glide or climb occurring is subject to random fluctuations. If a distortion should change from volume element to volume element, then it is only possible to speak of an average distortion and, furthermore, this will change continuously

only if each volume element is hit by a sufficient number of dislocations. E.g., if the distance between the glide plane is $10^{-5}$ mm and if we consider 1,000 glide planes necessary for averaging the fluctuations, then the physical volume element must have a linear size of at least $10^{-2}$ mm. In general we can consider this to be sufficiently small with respect to the dimensions of the body. However, with much bigger distances between the glide plane in small experimental bodies as happens in special cases sometimes, it may occur that the volume element calculated according to the rules above is no longer sufficiently small with respect to the size of the body and then obviously the application of continuum mechanics is no longer reasonable. We realize that the dislocation theory in the crystal is in general "less exact" than the continuum theory of the dislocation; however, it is sufficiently exact to justify its application and even to require it. The inaccuracy of this calculation consists in the fact that we assume that the physical volume elements are mathematically infinitesimal which means that we can apply all formulas of the continuum theory to a real body.

This approach is very simple and corresponds to the concept of this book. Another viewpoint is that we derive the macroscopic equations obtained in continuum theory from the equations of the microscopic problem by adding the interaction of many dislocations and taking an appropriate average. With this procedure we always remain in the crystal. The solid state physicist is used to thinking of crystals, and for him the crystal is an easy body to imagine. Therefore, the transformation from microscopic quantities to macroscopic ones will be presented shortly [22].

In polycrystals, strains generally change discontinuously from one crystal to another, first because of elastic anisotropy and second because of the plastic anisotropy of the crystallites originating from the fact that in each crystallite there are only a few discrete glide systems (= group of glide planes and related glide directions), which become effective at a certain shear stress.[1] If we want to have a continuously changing strain from volume element to volume element, then

[1]This leads to the conclusion that generally not all crystallites start to flow simultaneously. Among others Greenough successfully investigated the problems arising from this.

it can only be an average strain, and we need a physical volume element consisting of many crystallites.

It is more difficult to answer the question of the structure curvature. It is true that we can define the rigid (elastic) rotation of volume elements with reference to the initial condition of an "ideal polycrystal," but the statement of the orientation in this polycrystal has no meaning. The problem now is whether the rigid rotation of the volume element which occurred in the polycrystal (e.g., in the absence of elastic strain) changes the state of the body. Then we must be able to prove this experientially. In investigations of this question, §23, lead to the result that indeed curvatures of the structure can be shown in polycrystals. Accordingly, it is obviously possible to apply the continuum theory of dislocations in its previously developed form to polycrystals.[2]

[2]Prof. U. Dehlinger pointed this out to me.

## §21 The Geometrical Basic Equation in the Crystal. The Microscopic Theory

We start with a definition of the dislocation in the crystal, which originated with Frank [47]. Figure 19a shows the lattice plane of the ideal crystal of Fig. 2, Fig. 19b shows the same for the disturbed or real crystal of Fig. 3. The difference of the position vector of two neighboring atoms in Fig. 19a and 19b is $\delta \underset{\sim}{x}$ and $\delta \underset{\sim}{x}'$. Now we take the sum $\sum_{\underset{\sim}{\kappa}} \delta x'$ along an arbitrary closed path $\underset{\sim}{\kappa}$ in the real crystal. We may proceed from atom to atom starting at the point $P'$, going 7 steps in the $x_3$ direction, then 4 steps in the $x_1$ direction, etc., and on an arbitrary path back to $P'$. Then we repeat the same procedure (i.e., 7 steps in the $x_3$ direction, 4 steps in the $x_1$ direction, etc.) starting in the ideal crystal from the point P, corresponding to $P'$ (circuit $\underset{\sim}{\kappa}$). With the corresponding step with which we reach $P'$ in the circuit $\underset{\sim}{\kappa}'$, we do not reach P in $\kappa$ if the circuit was then around a dislocation line as the figure shows. We now state that the path should go around the dislocation line in the right-hand screw-sense. Then the vector $\vec{OP} \equiv \delta \underset{\sim}{b}$ from the end point Q of the path corresponding to its starting point is characteristic of the dislocation surrounded by the cycle $\underset{\sim}{\kappa}'$. So we can define the dislocation by use of the "Frank-Burgers circuit";[1]

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[1]The Burgers vector of a single dislocation is indicated by $\underset{\sim}{b}$ in previous literature (also in Fig. 19a). From the standpoint of the continuum the nomenclature $\delta \underset{\sim}{b}$ is more convenient (see below). Do not look for a special secret behind this nomenclature.

$\delta \underset{\sim}{b}$ is called the Burgers vector of the crystal dislocation. Further considerations will show that it corresponds to the Burgers vector in the continuum.

Imagine that the dislocation in Fig. 19b moved into the crystal from the right. During this, two neighboring atoms between which the dislocation moved suffer a plastic relative displacement $\delta \underset{\sim}{g} = - \delta \underset{\sim}{b}$.[1]

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[1] It is necessary to mention that for all the other atom pairs $\delta g = 0$ is required. The minus sign goes along with the convention that on the one hand the direction of the dislocation line is chosen in such a way that $\underset{\sim}{\mathcal{L}}$ and $\mathcal{L}'$, respectively, become right-hand screw circuits and that on the other hand $\delta \underset{\sim}{g}$ is the relative displacement of the atoms on the positive side of $\mathcal{L}$ with respect to those on the negative side.

---

If we take the sum along $\underset{\sim}{\mathcal{L}}$, $\Sigma\, \delta \underset{\sim}{g}$, then we obtain

$$\sum_{\mathcal{L}} \delta \underset{\sim}{g} = - \delta \underset{\sim}{b} \, . \qquad \text{(III.1)}$$

This equation corresponds to eq. (I.12) of §3. However, we have to notice the following: from the way we defined the cycle, $\delta \underset{\sim}{b}$ is a so-called lattice vector from the very beginning, i.e., a vector which (in the ideal crystal) points from one atom to another. This is a physical requirement. The relative displacement of the atoms during the motion of atoms must occur such that the regular arrangement of the atoms remains except the center of the dislocation: An irregular arrangement of the atoms over a surface or even more in a three-dimensional region would cause a considerable increase in the internal energy of the crystal and is therefore "forbidden."

On the other hand, we can not initially exclude a certain three-dimensional extension of the disturbed region in the immediate neighborhood of the dislocation. I.e., we can assume that the atoms displaced by a dislocation at a larger distance from the dislocation center had suffered the whole relative displacement $\delta \underset{\sim}{g}$, but for the atoms near the center this is not necessarily true; i.e., the transfer to $\delta \underset{\sim}{g}$ from $-\delta \underset{\sim}{g}$ (in Fig. 19b) on the right side of the dislocation to zero (on the left side of the dislocation) need not occur abruptly from one atom to the next, but, e.g., can appear over a region of two or three atom distances. This coresponds to our concept of a dislocation width $2\zeta$. To include this possibility, we imagine that the dislocation is composed of filaments two-dimensionally arranged over the infinitesimal thickness $\partial \underset{\sim}{b}$, where the dislocation should be: $\int \partial \underset{\sim}{b} = \delta \underset{\sim}{b}$. Each filament gives them a relative displacement of $-\partial b$ of the atoms between which it moved.[1]

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[1]According to Frank the circuit $\mathfrak{L}'$ must be carried out at a sufficient distance from the dislocation center so that all dislocation filaments are within the circuit. We will not require this in our investigations.

---

We can consider a crystal in its ideal initial condition, its atoms numbered and the relative position of two neighboring atoms indicated by $\delta x_1$. We call $\delta u_j$ initially the imagined relative displacement of atoms on the positive side of $\delta x_i$ with respect to those of the negative side. So $\delta u_j(x_i)$ is given in the whole crystal, where $x_i$ is the

position of that atom which lies on the negative side of $\delta x_i$ in the initial condition. We allow $\delta u_j$ to be arbitrarily discontinuous, especially since the crystal is no longer necessarily connected after carrying out the relative displacement. Only it is not allowe that one or more atoms occupy the same place.

Now the question arises: Is it possible at all to bring the crystal into such a condition that two atoms always suffered the displacement difference assigned by $\delta u_j$. The answer is: In general this is not possible. The most important facts we can see on a "crystal" consisting of only 4 atoms 1, 2, 3, 4. If we prescribe $\delta u_j$ for the pairs of atoms 12, 23, 34, then the position relative to each other is completely determined, and the assignment for the lost pair of atoms is no longer arbitrary but must be compatible with the first three statements (Fig. 20).

The next question is: What are the restrictions on $\delta u_j$ in order to produce a state which can be described by $\delta u_j$? Immediately we see that the sum of $\delta u_j$ along an arbitrary path from an atom (a) to another (b) must be independent of the path, i.e.,

$$\Sigma\, \delta u_j = 0 \qquad \text{(III.2)}$$

for an arbitrary closed path carried out in an ideal crystal. From eq. (III.2) follows the existence of a function of $\delta u_j(x_i)$ which can be arbitrarily discontinuous till now. Obviously, $u_j$ is the displacement of the atom, unique up to a rigid displacement of the crystal.

By the equation

$$\delta u_j = \gamma_{ij}\, \delta x_i \qquad \text{(III.3)}$$

the "microscopic" distortion tensor, $\underset{\sim}{\gamma} \equiv (\gamma_{ij})$, is defined. We explain it in the following way: A certain atom is at the position $x_i$. Its three neighboring atoms in the direction of the positive $x_i$-axis have the position (in the perfect crystal) $x_i + \delta x_i$. These four atoms which are the basic triad of a lattice, are enough to define and explain conveniently a distortion at the point $x_i$. E.g., according to eq. (III.3) the distortion $\gamma_{11}$ means an extension of the triad in the $x_1$-direction (Fig. 21b). Similarly, we notice that, e.g., $\gamma_{21}$ is a shearing of the triad as shown in Fig. 21c. In the case of small distortions, the symmetric part of $\gamma_{ij}$ is a pure strain; the antisymmetric part is a pure rotation of the triad.

If we substitute eq. (III.3) into eq. (III.2), then by Stokes' theorem it follows[1]

$$({}^{\delta}\mathrm{curl}\ \underset{\sim}{\gamma})_{i\ell} \equiv \epsilon_{ijk} \frac{\delta\gamma_{k\ell}}{\delta x_j} = 0 \qquad \text{(III.4)}$$

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[1]"${}^{\delta}$curl" should indicate that it is a difference equation in regions of atomic dimensions.

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Only distortions which satisfy eq. (III.4), are indeed possible in the Euclidean space.[2]

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[2]However, there is a non-Euclidean space for every arbitrary distribution $\delta u_j$; $x_i$ are then the coordinates of this space; e.g., a curved surface in the case of a two-dimensional crystal.

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The application of Stokes' theorem is also meaningful in the case of discretely distributed points and arbitrarily discontinuous relative

displacements $\delta u_j$, as we can show even in the example of four atoms in Fig. 20. Let all points remain in the plane $x_3 = 0$. Then according to eq. (III.3), the following distortions are defined. For point 1, $\beta_{11}$, $\beta_{12}$, $\beta_{21}$, $\beta_{22}$, for point 2, $\beta_{21}$, $\beta_{22}$ for point 3, none, for point 4, $\beta_{11}$, $\beta_{12}$. Formally we have

$$(\text{curl } \underset{\sim}{\beta})_{31} = \frac{\delta\beta_{21}}{\delta x_1} - \frac{\delta\beta_{11}}{\delta x_2} \; ; \quad (\text{curl } \underset{\sim}{\beta})_{32} = \frac{\delta\beta_{22}}{\delta x_1} - \frac{\delta\beta_{12}}{\delta x_2} \tag{III.5}$$

are different from zero or written as difference equations

$$\begin{aligned} (\text{curl } \underset{\sim}{\beta})_{31} &= \frac{\beta_{21}(2) - \beta_{21}(1)}{\delta x_1} - \frac{\beta_{11}(4) - \beta_{11}(1)}{\delta x_2} \\ (\text{curl } \underset{\sim}{\beta})_{32} &= \frac{\beta_{22}(2) - \beta_{22}(1)}{\delta x_1} - \frac{\beta_{12}(4) - \beta_{12}(1)}{\delta x_2} \end{aligned} \tag{III.6}$$

Thus only the distortion components mentioned above appear. After multiplying with $\delta x_1 \; \delta x_2$, we get $\Sigma \; \delta x_i \; \beta_{ij}$. Totally if we write $\delta F_k$ instead of $\delta x_i \; \delta x_j$

$$\delta F_k (\text{curl } \underset{\sim}{\beta})_{kj} = \Sigma \; \delta x_i \; \beta_{ij} \tag{III.7}$$

i.e., Stokes' theorem.

The previous considerations were purely geometric in nature. We have not discussed whether there are reaction forces associated with the relative displacement between two atoms. Now we consider the procedure of the invasion of a dislocation into the perfect crystal of Fig. 2 through which this is transformed into the state of Figs. 3 and 5. If we make a circuit $\underset{\sim}{\Sigma}$ in the ideal crystal and we add all the above mentioned relative displacements $\delta g_j$ between two atoms, we obtain

$$\sum_{\underset{\sim}{\Sigma}} \delta g_j = - \delta b_j \tag{III.8}$$

if $\mathcal{L}'$ is the circuit in the crystal corresponding to $\mathcal{L}'$ after the motion of the dislocation, otherwise we get $\Sigma\, \delta g_i = 0$.[1] It follows

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[1]Instead of the ideal and the real crystal we can consider only an ideal crystal into which a dislocation invaded, however, due to a restraint, initially there is no distortion. In this sense we can have a dislocation in the ideal crystal. With this consideration, we get simple equations also for arbitrary large distortions. See §10 beginning.

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from eq. (III.8) that the plastic relative displacement $\delta g_j$ in a crystal with dislocations does not satisfy the condition (III.2). However, Figs. 3 and 5 show that the dislocation in a crystal is surrounded by an elastic strain field. We call the relative displacement of two neighboring atoms $\delta a_j$. The total relative displacement which we will now call $\delta \underset{\sim}{s}^T$ is composed of the elastic and plastic displacement

$$\delta s_j^T = \delta a_j + \delta g_j \tag{III.9}$$

and it holds for every closed path that

$$\Sigma\, \delta s_j^T = 0\,. \tag{III.10}$$

From this the existence of a function $s_j^T(x_i)$ follows indicating the difference of the position of a certain atom in its ideal state and its "dislocated" state except for a constant displacement common to all atoms. The existence of this function, which need not necessarily be continuous, is a consequence of the fact that the procedure which transformed the crystal from its state of Fig. 2 into that of Figs. 3 and 5, is possible in Euclidean space.

Furthermore, we define the general asymmetric tensors of the (microscopic) total distortion $\underset{\sim}{\beta}^T = (\beta^T_{ij})$, of the elastic distortion $\underset{\sim}{\beta} = (\beta_{ij})$ of the plastic distortion $\underset{\sim}{\beta}^P = (\beta^P_{ij})$ by the equations

$$\delta s^T_j = \beta^T_{ij}\, \delta x_i, \qquad \delta a_j = \beta_{ij}\, \delta x_i, \qquad \delta g_j = \beta^P_{ij}\, \delta x_i \tag{III.11}$$

Then first of all we have to investigate to what extent we can connect a meaning to the elastic and plastic distortion, since these represent procedures similar to $\delta u_j$ and $\delta g_j$, each of which is not possible independently in Euclidean space. Now we notice that each procedure $\delta u_j$, defined only relative to a triad, can be carried out in the Euclidean space, but if we have additional atoms, e.g., a cube of eight atoms, the restriction (III.2) becomes effective. Obviously this is related to the fact that a triad cannot have a distortion in its interior, while it is possible, e.g., for a cube (see below).

If we substitute the third eq. (III.11) into eq. (III.8), then with Stokes' theorem it follows that

$$\sum_F \delta F_i ({}^{\delta}\mathrm{curl}\ \underset{\sim}{\beta}^P)_{ij} = -\,\delta b_j \tag{III.12}$$

where F is the surface bounded by $\underset{\sim}{\Sigma}$. This result is then meaningful even if the circuit touches only four atoms. Then we write eq. (III.12)

$$({}^{\delta}\mathrm{curl}\ \underset{\sim}{\beta}^P)_{ij} = -\,\delta b_j/\delta F_i \tag{III.13}$$

if we assume that all filaments of the dislocation cut the surface element $\delta F_i$ bounded by the four atoms.[1]

---

[1]According to footnote 1 of Pg. 151.

Apparently $\delta b_i/\delta F_i$ is the average surface density of dislocation filaments in the region $\delta F_i$. This statement holds even then if a part of the dislocation filaments are outside of $\delta F_i$ if we assume that $\delta b_j$ is the total Burgers vector of the filaments which cut $\delta F_i$.[1]

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[1] In §25 it is shown that we can approximately calculate the distribution of the filaments of a dislocation. The result is that most of the filaments of a dislocation are within the cross section of magnitude $|\delta F|$.

---

In order words, $\delta b_j/\delta F_j$ is the (microscopic) dislocation density; we call it $\alpha_{ij}$ and write for eq. (III.12)

$$^{\delta}\mathrm{curl}\ \underset{\sim}{\beta}^P = -\underset{\sim}{\alpha} \tag{III.14}$$

If many dislocations cut the surface F of Eq. (III.12), then the right-hand side of the equation is the total Burgers vector of all dislocations cutting F.

Because of eq. (III.10) for the total distortion, it holds that

$$\underset{\sim}{\beta}^T = \underset{\sim}{\beta} + \underset{\sim}{\beta}^P \tag{III.15}$$

$$^{\delta}\mathrm{curl}\ \underset{\sim}{\beta}^T = 0 \tag{III.16}$$

and thus we obtain the (microscopic) geometric basic equation of the crystal to be

$$^{\delta}\mathrm{curl}\ \underset{\sim}{\beta} = \underset{\sim}{\alpha} \tag{III.17}$$

This equation states that the existence of dislocations in crystal is always connected with elastic distortions since plastic distortions,

by definition appearing during the invasion or development of the dislocation are not possible themselves in the Euclidean space.[1]

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[1] I will thank Prof. E. Fues for his critical remarks to one of my previous papers which leads me to the description given above.

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## §22 The Geometric Basic Equation in the Crystal: Transfer to the Macoscopic Theory

In this section we will carry out the transfer of the microscopic quantities to the macroscopic ones. For this we define first all the macroscopic (= physical) volume elements, $\Delta V$, which should be an ideal crystal of the minimum size required in §20 in its initial state. By the term "element" we will imply that the experimental body has a very large number of such volume elements.

After this we will consider the dislocation tensor. It is obvious that we have no longer to consider the filament structure of the single dislocation, but we describe it as formerly by its unit tangent vector and its Burgers vector; i.e., by $t_i \delta b_i$. Because of the huge amount of dislocations appearing in all crystals, we no longer can state the direction and the Burgers vector of each line.

Now we are concerned with how to describe as simply and as completely as possible a state with very many dislocations lying close together. Apparently this is possible if we indicate at each point $\underset{\sim}{x}$ of the crystal how many ($N^{ab}$) dislocations of direction $\underset{\sim}{t}^a$ and of Burgers vector $\delta b^b$ pass the oriented surface element $\Delta \underset{\sim}{F}$ at $\underset{\sim}{x}$. The surface element should be of such a size that it is crossed by many

dislocations so that an averaging is meaningful. In other words, the number $N^{ab}$ only changes slightly from element to element. Then we can say that dislocations cross the surface element with a certain density.

Now we call the total Burgers vector of all dislocations intersecting $\Delta \underset{\sim}{F}$, $\Delta \underset{\sim}{b}$. It is

$$\Delta \underset{\sim}{b} = \sum_{a,b} N^{ab} \, \delta b^{b} \, . \qquad \text{(III.18)}$$

We notice that $\Delta \underset{\sim}{b}$ does not change if we double the number $N^{ab}$ and divide the related Burgers vector by two. The limiting process to the continuous distribution must be carried out so that we let the number $N^{ab}$ increase and decrease the Burgers vector to zero simultantously so that the total Burgers vector $\Delta \underset{\sim}{b}$ remains constant.[1]

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[1]This limit was first derived by Nye [13].

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In a macroscopic theory we can not handle a discrete distribution, since it is represented by the number $N^{ab}$; i.e., we must restrict ourselves to stating the total Burgers vector $\Delta \underset{\sim}{b}$. With crystals this is a real loss; however, in general, this can be compensated for by additional crystallographic considerations. This is a consequence of the fact that in this case the Burgers vector can have only discrete values and thus dislocations indeed appear as separate phenomena.[2]

---

[2]With only a few exceptions the theorem holds that the Burgers vector not only must be a lattice vector (§21), but also must be the smallest possible lattice vector, thus the disturbance in the dislocation center does not require too much energy (which is proportional to

(Footnote Continued)

$\underset{\sim}{b}^2$ §18). Then in the primitive cubic lattice three Burgers vectors are possible; in the face centered cubic lattice six are possible.

---

Therefore in many cases it is of special interest to know the character and distribution of dislocation lines as they were described by eq. (III.18).

If we know the total Burgers vector of all dislocation lines crossing $\Delta\underset{\sim}{F}$ for each surface element $\Delta\underset{\sim}{F}$, then we are well informed about the dislocation state with the above mentioned restriction. Now we can define the macroscopic tensor of dislocation density $\underset{\sim}{\alpha} = (\alpha_{ij})$ by use of the equation

$$\Delta b_j = \alpha_{ij}\, \Delta F_i \qquad \text{(III.19)}$$

Since by assumption the dislocation curve must be quite continuous in the neighborhood of the surface element, we can assume that the dislocation is straight within the volume element $\Delta V$. Furthermore, we assume that the dislocations $\alpha_{ij}$ cut the surface $\Delta F_i$ perpendicularly. Real dislocation lines do not do this in general, but we made the distribution of the total Burgers vector of the single dislocation lines of different types according to eq. (III.18) to be a matter of crystallography. Our macroscopic dislocation lines $\alpha_{ij}$ are all directed in the i-direction and have all the Burgers vectors in the j-direction. Therefore, the diagonal components of $\alpha_{ij}(\underset{\sim}{x})$ at the position $\underset{\sim}{x}$ represent, as previously, screw dislocations while the other components represent edge dislocations.

The macroscopic dislocation density in the crystal is a very well-known quantity; it represents directly a group of lines as in Figs. 3 and 5, whose flux through an arbitrary surface $\underset{\sim}{F}$, i.e., the dislocation flux, is equal to the total Burgers vector of all dislocation lines passing F and according to eq. (III.19)

$$\underset{\sim}{b} = \iint_{\underset{\sim}{F}} \Delta \underset{\sim}{F} \cdot \underset{\sim}{\alpha} \qquad \text{(III.20)}$$

Now we will investigate the relation between microscopic and macroscopic distortions. For this purpose we imagine that each pair of neighboring atoms in the crystal in its initial state suffers a relative displacement $\delta u_j$ similar to the cases mentioned above. Now let $\delta u_j$ be distributed continuously in the interior of a volume element, dV (in contrast to $\Delta V$ we used before) which contains many atoms, however, it may change discontinuously from element to element. Figure 22 shows a simple example.

We can describe the geometric position of all atoms of Fig. 22b by the microscopic $\gamma_{ij}$ defined by eq. (III.3) as a function of the position of the atoms in their initial state. Then we get $\gamma_{21}$ and $\gamma_{11}$ different from zero, where $\gamma_{21}$ only depends on $x_1$, whereas $\gamma_{11}$ is only different from zero for those atoms which bound the element dV on the $+x_1$ side, and there it only depends on $x_2$. The condition (III.4), which must be fulfilled everywhere, can be written

$$\frac{\delta\gamma_{21}}{\delta x_1} - \frac{\delta\gamma_{11}}{\delta x_2} = 0 \qquad \text{(III.21)}$$

From eq. (III.3) we can derive the function $u_j$ from the known $\gamma_{ij}$ up to a rigid displacement of the whole crystal.

Another description of the state of Fig. 22b would be always to give the distortion for the whole element where it is constant according to the assumption. We can consider this as a definition of the macroscopic distortion. The diagonal components are measured as functions of position in the initial state by the ratio of the extension of the element to its initial length, the remaining components are measured by the tangent of the shear angle. In the interior of a homogeneously distorted volume element each atomic triad has the same magnitude of (microscopic) distortion as the element (macroscopic). For an element of Fig. 22b (macroscopic) only $\gamma_{21}$ is different from zero. E.g., $dx_i = n\,\delta x_i$ is the distance of the center of mass of two neighboring elements dV in the initial state. Then the magnitudes of $d\gamma_{21}$ and $\delta\gamma_{21}$ on the boundary surface are equal, so the magnitude of $d\gamma_{21}/dx_1$ is equal to the value $\delta\gamma_{21}/n\delta x_1 = \delta\gamma_{11}/n\delta x_2$ on the boundary surface. With a given macroscopic $\gamma_{21}$ we can determine the microscopic $\gamma_{11}$ on the boundary surface up to a constant. Simultaneously it follows that the statement

$$(\mathrm{curl}\ \underset{\sim}{\gamma})_{i\ell} \equiv \epsilon_{ijk} \frac{\partial \gamma_{k\ell}}{\partial x_j} \neq 0 \qquad \text{(III.22)}$$

is in general true for the macroscopic $\gamma_{ij}$, in contrast to the microscopic $\gamma_{ij}$. The condition that curl $\underset{\sim}{\gamma} = 0$, reads in the example above obviously $\delta\gamma_{11}/\delta x_2 = 0$; in other words, a varying displacement jump on the boundary surfaces of the volume elements is not allowed. We will refer to this again soon.

The previously assumed homogeneity of the distortion within a volume element consisting of very many atoms does not occur in the

real process of plastic ceformation. However, we can assume for a sufficiently large volume element dV that at least an average homogeneity exists, which can be shown by the fact that the surface of a marked volume element does not change very much during deformation or during relaxation after cutting the volume element (otherwise the assumption mentioned in §20 for the application of the continuum theory is not satisfied).

Consequently we can identify the physical volume element dV with the mathematical volume element dV used in §3, which is the cause of the uncertainty we discussed in §20. At this element we define, as previously done in §21, the tensors of macroscopic total distortion, plastic and elastic distortion $\beta^T_{ij}$, $\beta^P_{ij}$, $\beta_{ij}$. We found above that the equation

$$\text{curl } \underset{\sim}{\beta}^T = 0 \qquad \text{(III.23)}$$

is necessary, in order that the body not have cracks after the deformation.[1] We see that the macroscopic equation states something quite different than the microscopic eq. (III.16)

$$^{\delta}\text{curl } \underset{\sim}{\beta}^T = 0 \qquad \text{(III.24)}$$

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[1]The function $\underset{\sim}{\beta}^T$ restricted by eq. (III.23) is so defined that it never requires two material points to occupy the same position. If we call a region in which parts of two volume elements exist simultaneously (which is only possible in the imagination) such that they overlap each other, a "negative crack," then we need not discuss this case extensively in addition to the normal "positive crack."

This makes the operation possible in Euclidean space. There the development of cracks is not mentioned, for indeed the expression "crack" is not defined in a system of discrete points. In contrast the way in which the distortions are defined in the macroscopic theory ensures from the very beginning that the procedure is possible in Euclidean space,[1] and this not only holds for the total distortion but also for the elastic and plastic distortions. Additionally, we have the restriction (III.23) which prevents the development of cracks.

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[1] In other words, a deformation which can be described by a macroscopic distortion tensor field can generally be carried out in the Euclidean space.

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We will notice that also the macroscopic plastic distortion is a procedure which can be carried out by itself in the Euclidean space, however, then in general the connection of the body is destroyed, since curl $\underset{\sim}{\beta}^P \neq 0$. From the remarks following eq. (III.22) it follows that where curl $\underset{\sim}{\beta}^P \neq 0$, a linearly varying plastic displacement jump occurs between the volume elements, and according to §8 this is always the case if dislocations come to rest with constant density. If in the initial state we carry out a circuit around the surface element $\Delta F$, which is composed of many elements dF, then we obtain similar to §3 the total Burgers vector of the dislocations which came to rest

$$\Delta b_j = -\oint_{\gamma} dx_i\, \beta^P_{ij} = \iint_{\Delta F} dF_k\, (\mathrm{curl}\, \underset{\sim}{\beta}^P)_{kj} = -\Delta F_k (\mathrm{curl}\, \underset{\sim}{\beta}^P)_{kj} \qquad \text{(III.25)}$$

if we assume a homogeneous distribution of curl $\underset{\sim}{\beta}^P$ in the region $\Delta F$. By comparison with (III.19), we obtain again

$$\text{curl}\ \underset{\sim}{\beta}^{P} = -\underset{\sim}{\alpha} \tag{III.26}$$

and after combination with eq. (III.23) the macroscopic basic equation

$$\text{curl}\ \underset{\sim}{\beta} = \underset{\sim}{\alpha} \tag{III.27}$$

Here we made the assumption that the dislocations would come to rest between the volume elements. In reality they came to rest in a three-dimensional manner rather than two-dimensionally. By taking the limit $dV \to 0$, the two-dimensional arrangement between the volume elements becomes a three-dimensional arrangement.

We discussed the case of linearly varying plastic displacement jumps, but not the case of constant displacement jumps. With such jumps the volume elements are displaced macroscopically relative to each other; i.e., the displacement of the point of the medium becomes macroscopically discontinuous. This case may not be of any significance so we will not consider it any longer.

## §23 Plane Dislocation Arrangements in Crystals

This section will consider the application of the boundary equation of §8 and 9 to a crystalline body. The boundary between two crystallites of different orientations is called a "grain boundary." Such grain boundaries are developed, e.g., during the growth of crystallites in the melt. The growth is initiated from randomly developed nuclei. If two neighboring nuclei with different orientations grow, then they will finally touch each other with different orientations, and along the boundary a region of atomic disturbance is created. If the difference in the orientation is less than 20°, then we can distinguish single dislocations, which make the transition from one orientation to

the other. Figure 23 shows a simple example. The related dislocation arrangement is schmatically shown beside it.[1] Here it is a wall of

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[1]Often an edge dislocation perpendicular to the paper plane is designated by the symbol $\perp$. The bars indicate the glide plane and the extra half plane introduced by the dislocation in an obvious manner. Therefore the dislocation in Fig. 19b would be indicated by $\perp$.

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edge dislocations whose Burgers vector is perpendicular to the wall plane. However, a wall of edge dislocations with the Burgers vector in the plane of the wall does not give a difference in orientation (Fig. 24). Such dislocation walls occur in phase boundaries (§32) and with variable density as a barrier in the glide plane[2] if a resistance blocks the further motion of the dislocations.

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[2]This we call a "pileup." The calculation of the dislocation distribution in the pileup is an interesting mathematical problem which was solved with different assumptions by Eshelby, Frank and Nabarro [166] and by Leibfried [189]. Besides others Leibfried showed that we can take instead of a discrete dislocation distribution a continuous one to a good approximation even if the pileup contains only a few dislocations. Then we get the equilibrium-dislocation distribution with respect to the externally applied stresses from a linear integral equation. The problem following from this, to calculate the self stresses related to this equilibrium distribution, was generally solved by Haasen and Leibfried [169] by integrating in the complex plane. The stresses can be obtained essentially by differentiating the dislocation distribution.

(Footnote Continued)
Leibfried [172] furthermore investigated pileups of circular dislocations in the glide plane. The applications of all these calculations deal with the hardening of metals.

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First we will consider grain boundaries. From §7 we know that walls of intersecting screw dislocations can also be used as grain boundaries. Unfortunately, these are very hard to draw. Practically two problems occur:

1. Given the difference in the orientation, find the dislocation arrangement which is developed along the grain boundary.
2. Given a dislocation wall, find the difference in the orientation of two neighboring grains.[1]

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[1]The general solution of this problem is due to Frank [46].

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We decompose this problem into a continuum theoretical part and a crystallographic part. In the former we determine the macroscopic dislocation density $\underset{\sim}{\alpha}$; in the latter by use of this and the crystallographic circumstances we calculate the microscopic dislocation arrangement.

We can at once write down the solution of the continuum theoretical part. A grain boundary is only defined by the difference of the orientation of the grains according to the facts mentioned above. No macroscopic deformations are implied by this. (Microscopic elastic deformations caused by the finite magnitude of the Burgers vectors in the crystal need not be considered in the continuum part of the problem.)

In our case the elastic distortion $\beta_{ij}$ is simply a rigid rotation $\omega_{ij}$ of the volume elements and eq. (I.67) becomes

$$\epsilon_{ijk}\, n_i\, \omega_{j\ell}\Big|_{II} - \epsilon_{ijk}\, n_i\, \omega_{j\ell}\Big|_{I} = \bar{\alpha}_{k\ell} \tag{III.28}$$

In the case of large differences in the orientation, we use for $\omega_{j\ell}$ a rotation tensor according to eq. (I.94). Its symmetric part is negligible for small orientation differences, and we can use instead the rotation vector related to its antisymmetric part

$$\omega_i = \frac{1}{2}\, \epsilon_{ijk}\, \omega_{jk}, \qquad \omega_{ij} = \epsilon_{ijk}\, \omega_k \tag{III.29}$$

Equation (III.28) then becomes

$$\epsilon_{ijk}\, \epsilon_{j\ell m}\, n_i\, \omega_m\Big|_{II} - \epsilon_{ijk}\, \epsilon_{j\ell m}\, n_i\, \omega_m\Big|_{I} = \bar{\alpha}_{k\ell} \tag{III.30}$$

and with the decomposition formula (A.2)[1]

$$(\delta_{k\ell}\, n_m\, \omega_m - n_\ell\, \omega_k)\Big|_{II} - (\delta_{k\ell}\, n_m\, \omega_m - n_\ell\, \omega_k)\Big|_{I} = \bar{\alpha}_{k\ell} \tag{III.31}$$

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[1]This is obviously the boundary surface form of eq. (I.59).

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In Fig. 23 only the components $n_1$ and $\omega_3$ were different from zero, therefore the related dislocation density $\bar{\alpha}_{ij}$ has only the component ($n_1 = 1$)

$$\bar{\alpha}_{31} = -\,\omega_3\Big|_{II} + \omega_3\Big|_{I} \tag{III.32}$$

which are exactly the dislocations formed if the rotation between two crystallites is about an axis perpendicular to the boundary surface (twist boundary in contrast to the above mentioned tilt boundary),

then we get intersecting screw dislocations. E.g., if the rotation axis is parallel to the $x_1$-direction (only $n_1$ and $\omega_1$ are different from zero), then we obtain from eq. (III.31) only

$$\alpha_{22} = \alpha_{33} = \omega_1\Big|_{II} - \omega_1\Big|_{I} \qquad \text{(III.33)}$$

different from zero.

We will show now that indeed these grain boundaries do not cause macroscopic stresses. For this it is necessary and sufficient that the surface incompatibilities vanish. From eq. (I.87) at once follows that because of the constant surface density of the dislocations in the boundary surface $\bar{\underset{\sim}{\eta}} = 0$. For $\bar{\bar{\underset{\sim}{\eta}}}$ we have from eq. (I.90) $\bar{\bar{\underset{\sim}{\eta}}} = (\bar{\underset{\sim}{\alpha}} \times \underset{\sim}{n})^S$. If $\underset{\sim}{i}_i$ are the cartesian unit vectors, then we have for the grain boundary of Fig. 23 with $\underset{\sim}{n} = \underset{\sim}{i}_1$ and $\bar{\underset{\sim}{\alpha}} = \bar{\alpha}_{31}\, \underset{\sim}{i}_3\, \underset{\sim}{i}_1$

$$(\bar{\alpha}_{31}\, \underset{\sim}{i}_3\, \underset{\sim}{i}_1 \times \underset{\sim}{i}_1)^S = 0 \qquad \text{(III.34)}$$

i.e., $\bar{\bar{\underset{\sim}{\eta}}} = 0$, no stresses. For the intersecting screw dislocations it is $\bar{\underset{\sim}{\alpha}} = \alpha_{22}\, \underset{\sim}{i}_2\, \underset{\sim}{i}_2 + \alpha_{33}\, \underset{\sim}{i}_3\, \underset{\sim}{i}_3$ thus for $\alpha_{22} = \alpha_{33} = \alpha_o$

$$[\alpha_o(\underset{\sim}{i}_2\, \underset{\sim}{i}_2 + \underset{\sim}{i}_3\, \underset{\sim}{i}_3) \times \underset{\sim}{i}_1]^S = \alpha_o(-\underset{\sim}{i}_2\, \underset{\sim}{i}_3 + \underset{\sim}{i}_3\, \underset{\sim}{i}_2)^S = 0 \qquad \text{(III.35)}$$

again $\bar{\bar{\underset{\sim}{\eta}}} = 0$.

However, for the dislocation wall of Fig. 24, we find $\bar{\underset{\sim}{\alpha}} = \bar{\alpha}_{32}\, \underset{\sim}{i}_3\, \underset{\sim}{i}_2$ and calculate from this

$$(\bar{\alpha}_{32}\, \underset{\sim}{i}_3\, \underset{\sim}{i}_2 \times \underset{\sim}{i}_1)^S = -\bar{\alpha}_{32}\, i_3\, i_3 \qquad \text{(III.36)}$$

i.e., $\bar{\bar{\eta}}_{33} = -\bar{\alpha}_{32}$. Therefore in this case we have macroscopic elastic deformations and stresses.

We can make the doublet $\bar{\bar{\eta}}_{33}$ understandable by taking the circuit $\oint_{\mathcal{L}} K_{ij}\, dx_j\, dF$ (87), which gives the incompatibilities. For this we imagine that the wall of Fig. 24 is produced by taking the limit of a group of grain boundaries of infinitesimal width (Fig. 25). The circuit $\mathcal{L}$ apparently is zero but not the circuits $\mathcal{L}'$ and $\mathcal{L}''$; these contribute an opposite and equal rotation angle $\underset{\sim}{D}$. This indicates that the dislocation wall of Fig. 24 is nothing more than a doublet of surface incompatibilities.

We can change the sign of one group of intersecting screw dislocations in the grain boundary and then obtain a surface incompatibility $\bar{\bar{\eta}}_{23}$ with eq. (III.35).

However, a single group of parallel screw dislocations contributes only as one part to $\bar{\bar{\underset{\sim}{\eta}}}$, since the other causes a difference of orientation. So all dislocation walls of constant density have been discussed.

We summarize the results of the dislocation walls of constant density: There are mainly four different arrangements:

1. Edge dislocations ⊢ ⊢ ⊢ ⊢ ⊢

   Grain boundary of the 1st kind (tilt)

2. Intersecting screw dislocations. Burgers vector and line direction in both groups parallel or antiparallel

   } No far-reaching self stresses

   Grain boundary of the 2nd ki ist)

3. Edge dislocations ⊥ ⊥ ⊥ ⊥ ⊥ ⊥

4. Intersecting screw dislocations. Burgers vector and line direction in one group parallel and in other antiparallel

   } Sources of far-reaching self stresses, no difference of orientation.

For non-constant density of the dislocations in a wall there is always a surface incompatibility $\bar{\eta}$ which is always connected with stresses.[1]

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[1]Furthermore, the papers of Read and Shockley [174] and van der Merwe [100] should be mentioned, in which the energy of a small angle grain boundary is calculated according to the theory of elasticity (as sum of the self energy and interaction energy of the dislocations developed in the grain boundary) as a function of the difference in the orientation. Concerning interesting applications of the theory of two-dimensional arrangements of dislocations see Bilby [2], Bilby and Christian [6], Bullough and Bilby [14]. The latter two papers contain applications to the important phase change of the martensitic type.

Furthermore, Bullough [162] explained by use of the theory mentioned the observed twin structure in lattices of the diamond type.

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Finally we will consider the problem of the curvature of the structure in polycrystals, mentioned previously in §20. Let the body of Fig. 14 be an ideal polycrystal which suffers dislocation motions transforming it into the shape of Fig. 14b, if the boundary between the two parts is not initially connected. In reality it remains continuous and obtains the state of Fig. 1. The question arises whether the crystal of Fig. 1 is still an ideal polycrystal. If not we have to show it by x-rays.[2]

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[2]See for this also Seeger [139].

We saw that we can consider grain boundaries to be walls of dislocations. This holds only for small differences in the orientation, since only these have the special property that we can distinguish single dislocations. However, we can imagine that more and more dislocations approach and join the grain boundary, thus we get any arbitrarily large difference in the orientation. If we use the theorem that a grain boundary is always a two-dimensional dislocation arrangement,[1] then the statement "ideal polycrystal" in its stress-free state implies a certain requirement for the dislocations of the

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[1]In crystal physics this theorem is generally not used, as the "rough grain boundary" often can be described more simply. However, sometimes its description as a two-dimensional dislocation array is more convenient (p. 40, Bullough [162]).

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body. It can only be the following requirement: The Burgers circuit around an arbitrary surface element $\Delta F$, which intersects very many crystallites has to be zero.[2] By this prescription we can define an

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[2]We imagine the volume element $\Delta V$ to be an ideal crystal in its initial state, which becomes the state of a polycrystal by dislocation motions. Then the Burgers circuit should be carried out in the ideal crystal.

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ideal polycrystal. Hence, it follows that the body of Fig. 1 is no longer an ideal polycrystal, and it must be possible to show this with x-rays.[3]

(Footnote #3 for page 168)

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[3]This conclusion also holds for amorphous bodies, which therefore can be included in the continuum theory of dislocations.

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For large curvature the distance between the dislocation walls in Fig. 1 is smaller than the average linear dimension of a crystallite. Then there are a number of such dislocation walls in most of the crystallites, which results in an average curvature of the lattice planes. This can be seen as asterism in x-ray pictures. Thus it is sufficiently explained why we also can show macroscopic curvatures in the structure of polycrystals.

## §24 The Types of Dislocations of the Face Centered Cubic Crystal

Previously we always considered the simple cubic crystal since by this we get a simplified view of the real circumstances, which is sufficient in many cases. Indeed, there is no metal which crystallizes in the simple cubic lattice. However, it is typical for metals to aspire to a high spatial density, thus it never happens that neighboring lattice planes are in opposition as in the simple cubic lattice, but they are arranged to fill the space between the atoms as the example of the face centered cubic lattice shows in Fig. 26. So it occurs that most of the metals only crystallize in three different kinds of lattice, the hexagonal close packed, the face centered cubic and the body centered cubic lattice of which the second is most common. In all these crystals

we do not have such simple relations as in the simple cubic lattice; we will shortly take account of this at least in the case of the face centered cubic lattice, for some important problems follow from this.[1]

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[1]The important cubic face centered metals are gold, silver, copper, aluminum, brass, nickel, and certain iron alloys. However iron is body centered cubic at room temperature.

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To describe the crystal lattice, we introduce three cartesian unit vectors $\underset{\sim}{a}_1$, which as in the example of Fig. 26 are directed from the left, lower, front corner atoms to the right lower ($\underset{\sim}{a}_1$) and the left, upper, front atom ($\underset{\sim}{a}_2$) and it should be $\underset{\sim}{a}_3 = \underset{\sim}{a}_1 \times \underset{\sim}{a}_2$. To indicate a group of parallel lattice planes, we write in parentheses the components of its normal vector simplified by multiplying by a common factor, so that they have the smallest whole numbers that are possible.[2] So (100) are the planes $\perp$ to $\underset{\sim}{a}_1$, (010) $\perp$ $\underset{\sim}{a}_2$, (001) $\perp$ $\underset{\sim}{a}_3$. These groups of lattice planes are equivalent crystallographically, if we wish to indicate them all together, then we use {100} for all plane crystallographically equivalent to the plane (100).

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[2]This simplified representation holds only for cubic crystals, where it is not necessary to distinguish covariance and contravariance. For the rather complicated relations in the general case see, e.g., Jagodziński [68].

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To indicate a direction, we write in brackets the least integer components of one vector in the same direction. Therefore, [100] is the

direction of $\underset{\sim}{a}_1$, etc. All directions such as [010], [001], furthermore, $[\bar{1}00]$, $[0\bar{1}0]$, $[00\bar{1}]$, where $\bar{1} = -1$ are indicated by ⟨100⟩.

The most important planes of the face centered cubic lattice are the planes {111}, since these are the only possible glide and climb planes under normal conditions. The {111} plane is the densest plane, one of which is shown in Fig. 27. A second plane now can be located in B or C. A stacking sequence ABABAB ... ("two layer sequence") results the hexagonal close packed packing, a sequence ABC ABC ABC ... ("three layer sequence") is the face centered cubic lattice.

For the internal energy of a crystal the forces between neighboring atoms are most important. If now the stacking sequence contains an error, in such a way, that we have ABC AB ABC ... so that each atom is surrounded in the same way by 12 nearest neighbor atoms as before, but the arrangement with the farther neighbors is no longer the same. The increase of energy caused by this is very small because of the short range of atomic forces, therefore such stacking faults occur relatively often.

As noticed in §22 the Burgers vector of a dislocation should be the smallest possible lattice vector, therefore as we can easily verify, it is directed always along a ⟨110⟩ direction in the face centered cubic lattice. Simultaneously it is always the glide direction. Now the (110) plane is perpendicular to the [110] direction. However, an edge dislocation with the Burgers vector in the [110] direction is not the edge of one inserted in the (110) plane but two, as the "thickness" of one (110) plane is the half of the interatomic distance as can be seen, e.g., from Fig. 26. This is represented schematically in Fig. 28a.

Now we have in practice the important procedure of the splitting of such a "complete" dislocation into two so-called "Shockley partial dislocations," Fig. 28.[1] The nomenclature should indicate, that the

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[1]The stacking fault and the split dislocation were described first by Heidenreich and Shockley [64]. For comprehensive discussions of this, see beside others, Frank [45], Frank and Nicholas [50], Read [121], Thompson [152], Seeger [134], [136], [140].

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Burgers vectors of both half dislocations are no longer whole lattice vectors. Usually the procedure of splitting is written as a "reaction equation"

$$\frac{1}{2}[110] = \frac{1}{6}[211] + \frac{1}{6}[12\bar{1}], \qquad \text{(III.37)}$$

where [110] 2 stands for the Burgers vector of the total dislocation, the other expressions on the right hand side of eq. (III.37) stand for the Burger vectors of the partial dislocations. Eq. (III.37) is simply an addition equation for these vectors. As we easily see, a stacking fault remains in the plane between the partial dislocations, which now increases the internal energy by the "stacking fault energy," thus, we obtain an equilibrium distance $2\bar{\eta}$ between the partial dislocations.

The Burgers vector [211]/6 shows (Fig. 27), e.g., an atom is moved from the B position to the C position, if a partial dislocation moved between it and neighboring lattice plane A. If both partial dislocations moved through, then it is again the B position. Then we can no longer determine if the dislocation which caused the relative displacement was split or not. In many cases, especially for macroscopic problems, we can neglect the splitting of the dislocation.

In general the theorem holds that each stacking fault terminating in the interior of a crystal is bounded by a partial dislocation. It is not necessary that complete dislocations and partial dislocations be straight; they can also produce closed loops in the {111} plane. Therefore, portions can be in the screw orientation, whereby in such cases the width $2\zeta$ is a little bit smaller.

We will state without proof the following results for dislocations in face centered cubic metals in a simplified representation:

1. Complete dislocations. They are almost exclusively in the {111} plane and there they are always split into Shockley partial dislocations. Where they go e.g., from a {111} plane to a neighboring plane the splitting must become zero again. Such a location is called a "jog." The complete dislocation can only glide in its stacking fault plane and cannot climb at all. A pure screw dislocation always lies along the ⟨110⟩ direction, since its line is parallel to the Burgers vector. This direction is the line of intersection of two {111} planes. The screw dislocation can split in both planes. Under appropriate forces it can go from one glide plane to another, therefore it has more possibilities for motion than an edge dislocation, which is always bound to its glide plane.

2. Frank's partial dislocation [45] is the edge line of an inserted (or an extracted) {111} lattice plane; its Burgers vector is ⟨111⟩/3. It also bounds a stacking fault. This dislocation, in contrast to Shockley partial dislocations which normally occur in pairs is stable alone. It can climb in its {111} plane, but it has no other possibility for motion. It is almost complementary to those of point 1 above.

3. The compound Lomer-Cottrell dislocation [94], [24]. If along an intersection line of two {111} planes, two split dislocations meet each other, then both of the partial dislocations next to each other can "react," thus we obtain a compound dislocation of great stability. Then we have a stacking fault which turns from one {111} plane into another. Such a Lomer-Cottrell dislocation can neither glide nor climb and is therefore completely immovable. Therefore, it is a very efficient barrier against motion of further dislocations in the related glide planes and plays an important role in the theory of hardening.[1]

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[1] See for this, papers of Mott [104], Leibfried and Haasen [92], Cottrell and Stokes [26], Friedel [56], Seeger, Diehl, Mader and Rebstock [143].

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Other dislocations which may be possible in cubic face centered crystals are of less importance relative to those described in 1 to 3.

## §25 The Nonlinear Treatment of Singular Dislocations According to Peierls

A look at the dislocation of Figs. 3 and 5 shows that in the center of the dislocation the elastic deformation is certainly far too large to be calculated according to a linear theory.[2] Indeed these

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[2] The same holds also in the case of the split dislocations.

deformations have not been calculated exactly up to now. Initially there was no theory of the extension $2\zeta$ of the dislocation center, which is the governing value in the equation of the energy of the dislocation (§18). Peierls succeeded in an approximate calculation of the extension of the dislocation center by a very interesting combination of microscopic and macroscopic methods.

The basic idea of Peierls [116][1] is to accommodate himself to the nonlinearity of the circumstances at least in glide plane. We imagine that the crystal is divided into two parts by a cut in the glide plane. Each part will then be treated as a half space $\underline{A}: x_2 > a/2$ and $\underline{B}: x_2 < -a/2$ (Fig. 29); additionally, a nonlinear theorem of elasticity is used, which adjusts both half spaces.

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[1]Especially Nabarro [106] developed the very short paper of Peierls further. Therefore some authors refer to the Peierls-Nabarro model.

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The half spaces are bounded by the lattice planes A and B. During this the special atomic arrangements in the glide plane are taken into account. The simplest circumstances are obtained for the simple cubic lattice with {100} glide planes, which was investigated by Peierls. However, Leibfried and Dietze [91] succeeded also with the face centered cubic crystal.[2]

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[2]Nabarro [106] investigated the interaction of two dislocations in the simple cubic crystal and van der Merwe [100] treated plane arrangements of many dislocations. Seeger and Schöck [141] were

(Footnote Continued)

able to determine, among other things, the energy recovered when a dislocation dissociates into partial dislocations, according to Peierls. A comprehensive summary of all results can be found in Seeger [134].

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The nomenclature is explained in Fig. 29. Initially we have the (simple cubic) ideal crystal. The tangential displacements of two atoms lying opposite in the lattice planes A and B are called $u^A$ and $u^B$, respectively, and their relative displacement is

$$u^{AB}(x_1) = u^A(x_1) - u^B(x_1) \qquad \text{(III.38)}$$

where $x_1$ indicates the initial position. Let the crystal be infinite in all directions; therefore, $\partial/\partial x_3 = 0$. The calculation according to the theory of elasticity shows, if it is right at all, that for the edge dislocation in the glide plane, $x_2 = 0$, at least up to the very near neighborhood of the dislocation center, only the stress $\sigma_{21}$ is different from zero (eq. (II.114)). Also, in the dislocation center all the other stresses may be smaller than $\sigma_{21}$. Therefore, they are assumed to be zero in the whole glide plane. We call this the Peierls assumption, in which we also will include the theorem of elasticity used by Peierls to adjust the planes A and B. We obtain it by the following consideration:

If we displace the upper half space with respect to the lower tangentially by the interatomic distance a, then the whole crystal is in equilibrium again, i.e., there are no repulsive forces. Hence, it follows that the reaction to a relative displacement $u^{AB}$, i.e., the

stress $\sigma_{21}$, is a periodic function of $u^{AB}$ with period a. The simplest assumption is that used by Peierls

$$\sigma_{21} = \frac{G}{2\pi} \sin \frac{2\pi\, u^{AB}(x_1)}{a} \tag{III.39}$$

where the constants are chosen in such a way that we get Hooke's law for small displacements.

For cutting along a glide plane in a state with dislocations, we have to apply the two-dimensional force density $\sigma_{21}$ at A and of $-\sigma_{21}$ on B, in order that no displacements of the atoms in A and B occur. The state of strain in the interior of the half space due to these "surface forces" on the half space $\underset{\sim}{A}$ is known in the theory of elasticity from Boussinesq and others. For the plane A it holds (see Leibfried and Lücke [93], eq. (12))

$$\frac{du^{A}(x_1)}{dx_1} = \frac{m-1}{\pi m G} \int_{-\infty}^{\infty} \frac{\sigma_{21}(x_1')}{x_1 - x_1'}\, dx_1' \tag{III.40}$$

A similar equation with changed signs on one side holds for B, i.e.,[1]

$$\frac{du^{AB}(x_1)}{dx_1} = \frac{2(m-1)}{\pi m g} \int_{-\infty}^{\infty} \frac{\sigma_{21}(x_1')}{x_1 - x_1'}\, dx_1' \tag{III.41}$$

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[1]Therefore, $d(u^A + u^B)/dx_1 = 0$, we take the free integration constants to be zero, i.e., $u^A = -u^B$.

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On the right hand side we substitute eq. (III.39)

$$\frac{du^{AB}}{dx_1} = \frac{m-1}{\pi^2 m} \int_{-\infty}^{\infty} \frac{\sin\,(2\pi\, u^{AB}/a)}{x_1 - x_1'}\, dx_1' \qquad \text{(III.42)}$$

and we obtain according to the integral theorem of Hilbert[1] the

---

[1] This is

$$f(\xi) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{g(x)}{\xi - x}\, dx; \; g(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(\xi)}{\xi - x}\, d\xi \qquad [128]$$

---

so-called Peierls' integral equation

$$\int_{-\infty}^{\infty} \frac{du^{AB}(x_1')/dx_1'}{x_1 - x_1'}\, dx_1' = -\frac{m-1}{m} \sin \frac{2\pi\, u^{AB}(x_1)}{a} \qquad \text{(III.43)}$$

with which we can determine the displacements $u^A$ and $u^B$ of the plane A and B.

As Eshelby [37] showed, we can obtain a similar integral equation with an equivalent assumption for a screw dislocation in the $x_3$-direction and with $x_2 = 0$ as its glide plane

$$\int_{-\infty}^{\infty} \frac{dw^{AB}(x_1')/dx_1'}{x_1 - x_1'}\, dx_1' = -\sin \frac{2\pi\, w^{AB}(x_1)}{a} \qquad \text{(III.44)}$$

where $w^{AB}$ is the relative displacement of the atoms as above, only in the $x_3$-direction instead of in the $x_1$-direction.

Equation (III.43) has the exact solution

$$u^A = -\frac{a}{2\pi} \arctan \frac{x_1}{\zeta'}, \qquad \zeta' \equiv \frac{a}{2}\,\frac{m}{m-1} \approx \frac{3}{4}\, a \qquad \text{(III.45)}$$

according to Peierls, as we can check easily (see footnote 1), corresponding to a static edge dislocation with Burgers vector of magnitude $b = a$ in the $x_1$-direction (see below). With $u^A$ and $u^B$ given according to eq. (III.39), immediately $\sigma_{21}$ is known for the atomic plane, i.e., the surface forces on the half spaces. The following classic problem, to find the corresponding Airy's stress functions in the whole half spaces has the solution

$$\chi = -\frac{G\zeta'}{2\pi}\left(x_2 - \frac{a}{2}\right) \ln[x_1^2 + (x_2 - a/2 + \zeta')^2] \qquad \text{(III.46)}$$

according to Leibfried and Lücke [93] for the half space A.
This equation contains in contrast to the previous solution (II.113) the additional term $-a/2$ and $-a/2 + \zeta' \approx a/4$ and therefore shows that at a distance of a few atoms from the dislocation center Peierls' solution practically does not differ from that of (II.113).

As Eshelby [37] emphasized and as it follows from our eq. (I.77) we can consider

$$\frac{du^{AB}}{dx_1} = \frac{a}{\pi}\,\frac{\zeta'}{x_1^2 + \zeta'^2} \qquad \text{(III.47)}$$

to be the two-dimensional dislocation distribution in the glide plane, where

$$\frac{a}{\pi}\int_{-\infty}^{\infty} dx_1\, \zeta'/(x_1^2 + \zeta'^2) = a \qquad \text{(III.48)}$$

i.e., apparently the total Burgers vector of the surface dislocation is equal in magnitude to a. The calculation of Peierls obtains, therefore, the following results: The edge dislocation in the simple cubic crystal has a two-dimensional extension. We can consider it as previously to be

composed out of filaments of magnitude $du^{AB}$. The distribution function of the filaments is a Gaussian distribution (III.47).[1] $2\zeta'$ is its half-width.[2]

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[1]That we get the extension of the dislocation in two but not three dimensions is caused by the initial assumptions of the calculation. Whether the dislocation has a three-dimensional extension in reality can not be said with certainty. However, even the two-dimensional extension leads to a finite self energy.

[2]In the literature $\zeta'$ is called the dislocation width in most cases.

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According to Eshelby [37] the solution of eq. (III.44) which describes the static screw dislocation in the simple cubic lattice, is exactly the same as that according to the theory of elasticity. Apparently in this case Peierls' method is less efficient than with the edge dislocation, so we get no finite self energy of the dislocation line.

Now we will report briefly the most important result of Leibfried and Dietze [91] for dislocations in the most densely packed plane of the cubic face centered and hexagonal crystals. Here the simple assumption (III.39) is no longer sufficient. We need a theorem of elasticity which contains $\sigma_{21}$ and $\sigma_{23}$ as a periodic function of $u^{AB}$ and $w^{AB}$. With this we get two simultaneous integral equations of the Peierls type, which we will not write down. These equations do not have elementary solutions. According to Leibfried and Dietze, we obtain simple and useful approximate solutions if we let the total elastic energy per unit length

$$T = T^A + T^B + T^{AB} \tag{III.49}$$

be a minimum, where $T^A$ and $T^B$ are related to both half spaces and $T^{AB}$ is the interaction energy per unit length in the strained state. Leibfried and Dietze succeeded in proving that those displacements $u^A$ and $w^A$, which cause the energy to be a minimum, satisfy the Peierls' integral equation, thus the variational method mentioned represents a proper approximation method. Leibfried and Dietze gave solutions for some special dislocation types, particularly for the important partial dislocation.

We now give the most general (approximate) solution for the partial dislocation in the most densely packed planes [83]. The glide plane is $x_2 = 0$. The Burgers vector also lies in this plane (§24). $\beta$ is its angle with the line direction. We find by assumption corresponding to (III.45)

$$u^A = -\frac{b}{2\pi} \sin\beta \arctan\frac{x_1}{\zeta'}, \quad w^A = -\frac{b}{2\pi} \cos\beta \arctan\frac{x_1}{\zeta'} \tag{III.50}$$

where $\zeta'$ is a free parameter, the minimum energy per unit length is

$$T = \frac{Gb^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\left(\ln\frac{R}{2\zeta'} + 1\right) \tag{III.51}$$

with

$$\zeta' = \frac{\pi\sqrt{2}}{3\sqrt{3}}\, b\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right) \tag{III.52}$$

There 2R is the linear dimension of the medium going to infinity in the $x_1$-direction. The formula for the partial dislocation given by Leibfried and Dietze follows from this for a particular angle $\beta$.

Eq. (III.51) can not be compared with the previously found eq. (II.144), since there is no relation between the L of the latter and R of the former. However, we can easily calculate a two-dimensional dislocation distribution in the glide plane corresponding to eq (III.47). If we identify $2u^A$ and $2w^A$ with the components $g_1$ and $g_3$ of the displacement jump in eq. (1.77), then we easily obtain the dislocation density $\bar{\gamma}$

$$\bar{\gamma} = i_3 \left( \frac{du^A}{dx_1} i_1 + \frac{dw^A}{dx_1} i_3 \right)$$

$$i_3 (\sin \beta \, i_1 + \cos \beta \, i_3) \frac{b}{2\pi} \frac{\zeta'}{x_1^2 + \zeta'^2} \tag{III.53}$$

Here $i_3$ indicates the line direction, the term in parentheses the direction of the Burgers vector of the half dislocation. Then we consider the density (III.53) to be an arrangement of dislocation filaments with infinitesimal magnitude

$$db = \frac{b}{\pi} \frac{\zeta'}{x_1^2 + \zeta'^2} \cdot b_r(x_1) \, dx, \tag{III.54}$$

Thus, we obtain from eq. (II.142)

$$T = \frac{G}{4\pi} \left( \frac{m}{m-1} \sin^2\beta + \cos^2\beta \right) \frac{b^2 \zeta'^2}{\pi^2} \int_{-\infty}^{\infty} \frac{dx_1}{x_1^2 + \zeta'^2} \int_{-\infty}^{\infty} \frac{dx_1'}{x_1'^2 + \zeta'^2} \tag{III.55}$$

Now we approximate these integrals by taking instead of (III.54) for db

$$db = \frac{b}{\pi\zeta'} dx_1 \quad \text{for} \quad -\frac{\pi\zeta'}{2} < x_1 < \frac{\pi\zeta'}{2} \; ; \text{ otherwise} = 0 \tag{III.56}$$

which satisfies the condition $\int_{-\infty}^{\infty} d\underline{b} = \underline{b}$ as does (III.54). (III.54) means a constant distribution of dislocation filaments in the given region. $(b\pi)\zeta'$ is the magnitude of the maximum of the Gaussian distribution $b_y(x_1)$. The resulting integral we calculated in §18 if we substitute $\zeta$ by $\zeta'/2$. Then we get for the energy per unit length

$$T = \frac{Gb^2}{4\pi}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right)\left(\ln\frac{2L}{\pi\zeta' e^{3/2}} - 1\right) \qquad \text{III.57)}$$

This formula follows from eq. (II.145) for which we noticed that it holds even for curved dislocation lines, as long as we take the right core radius $\epsilon$. For $\epsilon$ we calculated the value $\epsilon = \zeta e^{3/2}$ in §18. With $\zeta = \pi\zeta'/2$ and eq. (III.52), we obtain

$$\epsilon\, b = \frac{\pi^2}{\sqrt{54}\, e^{3/2}}\left(\frac{m}{m-1}\sin^2\beta + \cos^2\beta\right) \qquad \text{(III.58)}$$

where the factor in front of the parentheses is exactly 0.3. Thus it is possible for us to calculate the energy of curved partial dislocations in the most densely packed plane of the face centered cubic and hexagonal crystals in certainly not too bad an approximation.[1]

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[1]The calculation contains some approximations (i.e., Peierls' assumption) on the other hand the energy is not very sensitive to small changes of $\epsilon$, since the energy depends logarithmically on $\epsilon$. We can improve the calculation by determining the integral (III.55) exactly, even further if we also consider the elastic anisotropy in the half spaces A and B. In the case of straight dislocations, such calculations were carried out by Seeger and Schöck [141].

We restricted ourselves in these calculations to Shockley partial dislocation. These play the principal roles in face centered cubic and hexagonal crystals at room temperature. In other crystals, e.g., the body centered cubic, we have other types of dislocations and we have to carry out for each type a single Peirels calculation.

Summarizing, we can say the following about Peierls' method: The dislocation width $\zeta'$ gives a measure of the extension of the dislocation center about whose magnitude it happens that the calculation according to the theory of elasticity gives negligible results at distances of a few interatomic spacings. Peierls' calculation gives an idea of in which cases the treatment of dislocations according to the theory of elasticity will lead to meaningtul results, and it gives even then an indication of when that treatment cannot be applied because of large complications. For the investigation of the dislocation center itself (e.g., calculation of its energy) the Peierls' calculation gives a first approximation.

## CHAPTER IV

## NON-RIEMANNIAN GEOMETRY OF DISLOCATIONS[1]

Kondo, also Bilby, Bullough and Smith, realized independently the close relation between the geometrical problems of plasticity and those of the non-Euclidean geometry. Accordingly, we can use the highly developed methods of differential geometry to treat such problems, especially the concept of torsion. This is due to Cartan, whose papers are a very nice application to real bodies. The relation between the dislocation tensor $\alpha^{\lambda\kappa}$ and the torsion tensor $L^{\kappa}{}_{[\mu\nu]}$ is given by the equation

$$\alpha^{\lambda\kappa} = \epsilon^{\lambda\mu\nu} L^{\kappa}_{[\mu\nu]} \tag{IV.1}$$

The difference between the theories of Kondo and Bilby, Bullough and Smith is similar to that between our Chapters I and III: The theory of Kondo is a continuum theory, whereas Bilby, Bullough and Smith developed their theory in the crystal. We will discuss in more detail differences between those theories in section 28.

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[1] I am very grateful to Prof. K. Kondo and Dr. B. A. Bilby for the discussion about this point. I thank Dr. J. D. Eshelby very much for giving me Kondo's book [76] from which I first learned about the efforts of Kondo. (Dec. 1956.)

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## §26 The Theory of Kondo and Collaborators[1]

First we review familiar concepts using the nomenclature of Kondo [74]. In the ordinary theory of elasticity we are not particularly interested in rotations, as long as they do not cause elastic forces. The important quantity is then the elastic deformation. By the natural state of a volume element we mean the stress-free state which occurs if it suffers restraint neither due to external forces nor due to neighboring elements. If self-stresses are present, then the volume elements can acquire their natural state only after cutting, since the elements are bound to Euclidean space. However, we can imagine having non-Euclidean space, in which the volume elements can relax without being cut, if the restraints causing them to remain in a Euclidean space were suddenly to vanish. Such an imagined stress-free state in the non-Euclidean space can also be called a natural state. We can consider the cut elements in their natural state as the (material) Euclidean space, which is tangent to the non-Euclidean space at the related points. Finally we have to define the final state to be the (Euclidean) state of the body with the stresses which we want to investigate.

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[1] The results mentioned here are all summarized in the book [74]. Most of them were reported first at the 2nd Nat. Congress of Appl. Mech., Japan 1952, by Kondo [73].

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In sections 26 to 28 we distinguish between co- and contravariance. If $ds_F^2$ is the square of the distance between two arbitrary

points of a volume element in its final state, and $ds_N^2$ the square of the distance between some points in their natural state, then we have

$$dS_F^2 - dS_N^2 = (\delta_{Fij} - g_{Nij})\, dx^i\, dx^j \tag{IV.2}$$

where $x^i$ are the space fixed orthogonal Cartesian coordinates of the points in the final state. $g_{Nij}(x^i) = g_{Nji}(x^i)$ is the metric tensor of the natural state. The metric of the final state is obviously completely defined by the quantity

$$\epsilon_{ij} = (\delta_{ij} - g_{ij})\, 2 \tag{IV.3}$$

where we omit the subscript F and N, since we are not afraid of confusion. From the theory of large deformations we know that in the case of small compatible deformations, the $\epsilon_{ij}$ of eq. (IV.3) have the form

$$\epsilon_{ij} = \left(\frac{\partial s_i}{\partial x^i} + \frac{\partial s_j}{\partial x^j}\right)\Big/2 \tag{IV.4}$$

therefore $\epsilon_{ij}$ is identical with the previously used deformation tensor.

The vanishing of the Riemann-Christoffel curvature tensor $R^i_{jk\ell}$, which is derived from the Christoffel symbol related to $g_{ij}$

$$\left\{ {j \atop jk} \right\} = \frac{1}{2}\, g^{i\ell} \left(\frac{\partial g_{k\ell}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^\ell}\right) \tag{IV.5}$$

is apparently equivalent to satisfying the conditions of compatibility for the deformation. By

$$R^i_{jk\ell} = \frac{\partial}{\partial x^k} \left\{ {i \atop j\ell} \right\} - \frac{\partial}{\partial x^\ell} \left\{ {i \atop jk} \right\} + \left\{ {i \atop mk} \right\} \left\{ {m \atop j\ell} \right\} - \left\{ {i \atop m\ell} \right\} \left\{ {m \atop jk} \right\} = 0 \tag{IV.6}$$

the classic theory of elasticity is distinguished from the theory of self-stresses. This statement also holds for large deformations, whereas the form of the compatibility conditions, $\eta^{hm} = 0$, only holds for small deformations. The total covariant curvature tensor

$$R_{ijk\ell} = g_{ih} R^{h}_{jk\ell} \tag{IV.7}$$

is antisymmetric in both its first and last indices and symmetric in the pairs ij and $k\ell$. For small deformation the following holds (see McConnell [171])

$$R_{ijk\ell} = \varepsilon_{hij}\, \varepsilon_{k\ell m}\, \eta^{hm}, \qquad \eta^{hm} = 4\varepsilon^{hij}\, \varepsilon^{k\ell m} R_{ijk\ell} \tag{IV.8}$$

as we can easily verify if we introduce $\varepsilon_{ij}$ of eq. (IV.4) into eq. (IV.7). Then we get

$$\eta^{hm} = \varepsilon^{hij}\, \varepsilon^{k\ell m} \frac{\partial^2 \varepsilon_{j\ell}}{\partial x^i\, \partial x^k} \tag{IV.9}$$

and with (IV.6) it follows that $\eta^{hm} = 0$.

In a theory of elasticity which also considers self-stresses $R^{i}_{jk\ell} \neq 0$, the description of the deformations is then a problem of Riemannian geometry. Since problems solved by such a theory are mostly geometric in nature, we can state as Kondo and his collaborators did: "The theory of elasticity is a Riemannian geometry and vice versa."

However, in previous considerations we succeeded in treating sufficiently the geometric problem--at least for small distortions--without Riemannian geometry, so we will not consider the above-mentioned theorem as a rule for theorists in elasticity but as a helpful hint for readers who are especially familiar with geometry. There is no doubt

that the study of Riemannian and, as we will see later on, the non-Riemannian geometry, will give us further understanding of the geometry of the deformation of the solid body.

Kondo associates with each (Euclidean) volume element in its natural state its own local Cartesian coordinate system with unit vectors $\underset{\sim}{e}_\lambda$ ($\lambda = 1,2,3$), which have no restrictions. Then it follows

$$d\underset{\sim}{x} = \underset{\sim}{e}_x \, d\omega^\kappa \tag{IV.10}$$

$$d\omega^\kappa = A^\kappa_i \, dx^i, \qquad dx^\ell = A^\ell_\mu \, d\omega^\mu \tag{IV.11}$$

$$\underset{\sim}{e}_\lambda = A^j_\lambda \underset{\sim}{i}_j, \qquad \underset{\sim}{i}_j = A^\kappa_j \underset{\sim}{e}_\kappa \tag{IV.12}$$

$$de_\lambda = e_\kappa \, \Gamma^\kappa_{\lambda\mu} \, d\omega^\mu \tag{IV.13}$$

$$A^j_\kappa \, A^\kappa_i = \delta^j_i, \qquad A^j_\kappa \, A^\lambda_j = \delta^\lambda_\kappa \, . \tag{IV.14}$$

where $d\underset{\sim}{x} = dx^i \, \underset{\sim}{i}_i$ is the difference of the position of two material points in the final state, and $d\omega^\kappa$ are the components of the corresponding vectors related to the same material points) in its natural state. $\underset{\sim}{e}_\kappa$, $A^\ell_\mu$, $A^\kappa_i$, $\Gamma^\kappa_{\lambda\mu}$ are functions of $x^i$. For the natural system we use Greek, for the (Cartesian) final system we use Latin indices. For abbreviation we use

$$\partial_i \equiv \partial/\partial x_i \qquad \partial_\nu \equiv A^i_\nu \, \partial/\partial x_i . \tag{IV.15}$$

Then apparently the first condition of integrability must be satisfied

$$(\partial_j \partial_i - \partial_i \partial_j)\underset{\sim}{x} = 0 \tag{IV.16}$$

Since $\partial_i x = e_\kappa A^\kappa_i$, $\partial_j x = e_\kappa A^\kappa_j$,

$$\partial_j \partial_i x = A^\lambda_i \partial_j e_\lambda + e_\kappa \partial_j A^\kappa_i = e_\kappa (\Gamma^\kappa_{\lambda\mu} A^\lambda_i A^\mu_j + \partial_j A^\kappa_i)$$

$$\partial_i \partial_j x = e_\kappa (\Gamma^\kappa_{\lambda\mu} A^\lambda_j A^\mu_i + \partial_i A^\kappa_j) \tag{IV.17}$$

and

$$\Gamma^\kappa_{\lambda\mu} (A^\lambda_i A^\mu_j - A^\lambda_j A^\mu_i) + \partial_j A^\kappa_i - \partial_i A^\kappa_j = 0 \tag{IV.18}$$

from which

$$\Gamma^\kappa_{[\lambda\mu]} = \frac{1}{2} A^i_\lambda A^j_\mu (\partial_j A^\kappa_i - \partial_i A^\kappa_j) \tag{IV.19}$$

follows by multiplication with $A^i_\nu A^j_\pi$ and eq. (IV.14). We always indicate with brackets that we take the antisymmetric part of the enclosed indices. The antisymmetric part of an affine connection is called torsion according to Cartan [15]. Where it does not vanish in the region, we have a non-Riemannian geometry. Apparently $\Gamma^\kappa_{[\lambda\mu]}$ is a third order tensor, whose Cartesian components are

$$\Gamma^i_{[km]} = A^i_\kappa A^\lambda_k A^\mu_m \Gamma^\kappa_{[\lambda\mu]}, \tag{IV.20}$$

and with the relation (IV.14) it follows ([174] pg. 461)

$$\Gamma^i_{[km]} = \frac{1}{2} A^i_\kappa (\partial_m A^\kappa_k - \partial_k A^\kappa_m) . \tag{IV.21}$$

The eqs. (IV.10) and (IV.11), which define the coordinate system $e_\lambda$ are generally Pfaffian differential forms, i.e.,

$$(\partial_i \partial_j - \partial_j \partial_i) e_\lambda \neq 0. \tag{IV.22}$$

After a simple calculation, the left-hand side of this equation gives the Riemann-Christoffel curvature tensor related to the natural state,

corresponding to $\Gamma^{\varkappa}_{\lambda\mu}$, which we will not write down explicitly, as

$$R^{\varkappa}_{\lambda\mu\nu} \neq 0 \tag{IV.23}$$

Now we can displace parallel (w.r. to the connection $\Gamma^{\varkappa}_{\lambda\mu}$) a vector $c^{\lambda}$ around an infinitesimal surface element $\Delta F^{\mu\nu} = \epsilon^{\mu\nu\rho}\,\Delta F_{\rho}$ according to the rules of the differential geometry,[1] so we take the

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[1]See, e.g., Schouten [130].

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integral

$$-\oint \Gamma^{\varkappa}_{\lambda\mu}\, c^{\lambda}\, d\omega^{\mu} \tag{IV.24}$$

and we obtain in this manner (see Kondo) for the change of $c^{\lambda}$

$$\Delta c^{\varkappa} = \left(\frac{1}{2} R^{\varkappa}_{\lambda[\mu\nu]}\, c^{\lambda} + \Gamma^{\varkappa}_{[\mu\nu]}\right)\Delta F^{\mu\nu} . \tag{IV.25}$$

Kondo compared the above circuit with the Frank-Burgers circuit (§21). The torsion $\Gamma^{\varkappa}_{[\mu\nu]}$ is the reason for a translation

$$\Delta b^{\varkappa} = \Gamma^{\varkappa}_{[\mu\nu]}\, \Delta F^{\mu\nu} = \alpha^{\rho\varkappa}\, \Delta F_{\rho} \tag{IV.26}$$

where $\Gamma^{\varkappa}_{[\mu\nu]}$ is expressed by the corresponding 2nd order tensor. By comparing with eq. (I.14), we see that $\Gamma^{\varkappa}_{[\mu\nu]}$ is connected with our previous dislocation tensor $\underset{\sim}{\alpha}$ according to

$$\Gamma^{\varkappa}_{[\mu\nu]} = \frac{1}{2}\, \epsilon_{\mu\nu\rho}\, \alpha^{\rho\varkappa}, \qquad \alpha^{\lambda\varkappa} = \epsilon^{\lambda\mu\nu}\, \Gamma^{\varkappa}_{[\mu\nu]} \tag{IV.27}$$

if $\Delta b^{\varkappa}$ is really the Burgers vector. This problem will be discussed in section 28.

It is difficult to discuss the second part of eq. (IV.25). For this see Kondo [74], pg. 466 et seq. First we will mention the following additional fact: The tensor $R^{\varkappa}_{\lambda\mu\nu}$ corresponding to $R_{\iota\lambda\mu\nu} = g_{\iota\varkappa} R^{\varkappa}_{\lambda\mu\nu}$ ($g_{\iota\varkappa} \equiv \underset{\sim}{e}_{\iota} \cdot \underset{\sim}{e}_{\varkappa}$) is antisymmetric only with respect to its last two indices since $\Gamma^{\varkappa}_{\lambda\mu}$ no longer has the from $\left\{ {\lambda \atop \varkappa\mu} \right\}$. But we can split a part of the above-mentioned symmetry of $R_{\iota\lambda\mu\nu}$ and obtain

$$\Delta C_{\iota} = \frac{1}{2} R_{[\iota\lambda][\mu\nu]} C^{\lambda} \Delta F^{\mu\nu} . \tag{IV.28}$$

With $R_{[\iota\lambda][\mu\nu]}$ substituted according to eq. (IV.8) we get

$$\Delta C_{\iota} = \epsilon_{\iota\lambda\pi} \eta^{\pi\rho} C^{\lambda} \Delta F_{\rho} \tag{IV.29}$$

or written as vectors

$$\Delta \underset{\sim}{C} = \underset{\sim}{C} \times \underset{\sim}{\eta} \cdot \Delta \underset{\sim}{F} \tag{IV.30}$$

Accordingly, $\Delta \underset{\sim}{C} \perp \underset{\sim}{C}$, i.e., the vector suffers (in the case of small distortions) a pure rotation

$$\Delta \underset{\sim}{D} = - \underset{\sim}{\eta} \cdot \Delta \underset{\sim}{F} \tag{IV.31}$$

This result shows (see eq. (I.64)) that we obtain by the use of non-Riemannian geometry not only our previous results concerning the dislocations (see below) but also the results related to the incompatibilities.[1]

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[1]The difference in the sign is purely conventional.

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Beside Cartan's torsion and Riemann's curvature, there is another quantity of importance in this context, which we will discuss

now. We can see the important properties very easily on a two-dimensional material, e.g., a bent membrane. Generally we can restrain it by two rigid plane walls and so force it into a two-dimensional Euclidean space, in which it exhibits "self stresses." But if the membrane makes a thin circle hollow cylinder, then this is no longer possible without making a cut first. For this membrane, although it is bent, the Riemannian curvature is zero. Its curvature, which is evident in the three-dimensional Euclidean space in which it is inserted, is described by $\partial^2 \underset{\sim}{x}/\partial x^i \, dx^i$, if for a moment $\underset{\sim}{x}$ is the position in the three-dimensional space and $x^i$ are the coordinates on the surface.

Now we can imagine that our three-dimensional body in its natural state is represented by a "three-dimensional membrane" of six-dimensional Euclidean space. $X^\Lambda$ ($\Lambda = 1,\ldots,6$) are its Cartesian coordinates. Then the "Euler-Schouton's curvature tensor" is defined by[1]

$$H^\Lambda_{ij} = \frac{\partial^2 X^\Lambda}{\partial x^i \partial x^j} \qquad \text{(IV.32)}$$

where $x^i$ has the previous meaning.

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[1]For this nomenclature see [130], p. 256.

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The relation between $R_{ijk\ell}$ and $H^\Lambda_{ij}$ is given by

$$R_{ijk\ell} = \sum_{\Lambda=1}^{6} (H^\Lambda_{ik} H^\Lambda_{j\ell} - H^\Lambda_{i\ell} H^\Lambda_{jk}) \qquad \text{IV.33)}$$

([74], p. 468) from which we find that when $H^\Lambda_{ij} = 0$, $R_{ijk\ell}$ always vanishes, but the reverse does not hold. Also the last case can be included in the complete theory.

Now Kondo classifies lattice defects which occur in the crystal as follows:

1. Lattice defects with incompatible metric, indicated by a non-vanishing Riemannian curvature tensor in the natural state ("curvature defects").

2. "Non-Riemannian" lattice defects, indicated by a non-vanishing torsion tensor in the natural state ("torsion defects").

3. Lattice defects connected with a non-vanishing Euler-Schouten tensor.

According to Kondo, it is possible that of the three quantities $\Gamma^{i}_{[jk]}$, $R_{ijk\ell}$, $H^{i}_{ij}$, only $\Gamma^{i}_{[jk]}$ is different from zero but also it may be that only $H^{i}_{ij}$ is zero, and it may be that $\Gamma^{i}_{[jk]} = 0$, but $R_{ijk\ell} \neq 0$. Kondo concludes from the fact that $\Delta c^{i}$, which is the same as the curvature tensor according to eq. (IV.25), is proportional to the vector $c^{i}$, that the curvature tensor describes defects which are spread over a larger volume, whereas the torsion tensor corresponds to dislocation in a small (almost microscopic) region. As defects which can be described by the use of the curvature tensor, we should especially mention pileups of dislocations in a slip plane, interstitial atoms distributed in a volume, and lattice distortions caused by changes of temperature. According to eq. (IV.32), all curvature defects can be described with the Euler-Schouten tensor.

If we compare these statements with our previous discussions, then we notice that two points need further explanation.

(a) The Riemannian curvature should also be finite if no torsion is present.[1] We can show that this is not compatible with the condition

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[1]The two following sentences are in anticipation of section 27.

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(IV.46) of BBS, which reads that $L^{\alpha}_{\beta\gamma}$ has the form (IV.37). Indeed, Kondo allows a more generalized form of the connection. The case considered here extends the theory of BBS and our theory, in which all could be reduced to dislocations. It will be the duty of future research to clarify which phenomena are governed by such a curvature tensor with vanishing torsion.

(b) $R_{ijk\ell}$ vanishes, then if no external forces are present, at least in the case of small distortions, we have no elastic deformations and the crystal is free of self stresses. Additionally, it should be that $\Gamma^{i}_{[jk]} = 0$. Then, according to our previous considerations, the only possible distortion is that by which the body remains intact, the plastic distortion grad $\underset{\sim}{s}^{p}$, through which dislocations deform the body, but at the end of the procedure they are no longer in the interior of the body. It is obvious to connect the distortion with the tensor $H^{\Lambda}_{ij}$ in the case $R_{ijk\ell} = 0$, $\Gamma^{i}_{[jk]} = 0$, especially (analogous to the above-mentioned example of the cylinder) two-dimensional cuts are necessary to transform the body which is in the Euler-Schouten state of curvature into the three-dimensional Euclidean state. Each cut corresponds to a motion of a dislocation through the body.[1]

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[1]A two-dimensional membrane only can have two-dimensional stresses. Therefore it can bend without stresses to the mentioned cylinder.

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Certainly the final explanation of these two points will complete the picture developed here. Thus it is completely compatible with our previous results. Even now the conformity is very impressive. Finally, we mention that eq. (IV,17) is nothing other than our geometric basic equation now only written in (space fixed) Cartesian coordinates of the final state. We will prove this in the next section.

## §27 The Theory of Bilby, Bullough and Smith [3,4,5]

The states considered in the theory of Bilby and collaborators are the ideal crystal as the reference state described by a Cartesian coordinate system with unit vectors i, which should be lattice vectors (§21), and the final state, called the dislocated state by the authors. To describe this we choose at each point three independent unit vectors $e_a(P)$, which are everywhere the same lattice vectors.[1,2]

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[1]During the real plastic deformation, small regions bounded by the glide and climb planes are only elastically deformed. In these regions neighboring atoms remain neighboring during this procedure. Since these "elementary regions" are very small with respect to the physical volume elements (§20) their elastic distortions can be considered to be homogeneous. Then we can imagine that $e_a$ is the atomic triad (perhaps in the center of mass of the elementary region). Thus $e_a$ describes directly the lattice in its dislocated state.

[2]We use now Greek indices corresponding to the Cartesian reference system.

We can imagine that these are derived from the unit vector of the reference system by a distortion

$$e_a = D^{\alpha}_{a}\, i_{\alpha} \quad ; \quad i_{\alpha} = E^{a}_{\alpha}\, e_a \tag{IV.34}$$

where $D^{\alpha}_{a}$ is the related tensor of the affine transformation and $E^{a}_{\alpha}$ is its reciprocal tensor, thus

$$D^{\alpha}_{a}\, E^{b}_{\alpha} = \delta^{b}_{a} \quad , \quad E^{a}_{\alpha}\, D^{\beta}_{a} = \delta^{\beta}_{\alpha} \quad . \tag{IV.35}$$

BBS now define a new theorem of parallel transport in Euclidean space by stating that vectors which have the same components in the $e_a$ system should be parallel. The actual difference of two parallel vectors c at two neighboring points P and Q is expressed as a vector originating at the point P. After simple calculation, the authors obtain

$$dC^{\mu} = F^{a}_{\lambda} \frac{\partial D^{\mu}_{a}}{\partial x^{\nu}} C^{\lambda}\, dx^{\nu} \quad . \tag{IV.36}$$

Then they consider a Euclidean space with the linear connection

$$L^{\alpha}_{\beta\nu} = - E^{a}_{\beta} \frac{\partial D^{\alpha}_{a}}{\partial x^{\nu}} = D^{\alpha}_{a} \frac{\partial E^{a}_{\beta}}{\partial x^{\nu}} \tag{IV.37}$$

and the related theorem of parallel transport (IV.36). The torsion tensor follows

$$L^{\alpha}_{[\beta\nu]} = \frac{1}{2} D^{\alpha}_{a}\left(\frac{\partial E^{a}_{\beta}}{\partial x^{\nu}} - \frac{\partial E^{a}_{\nu}}{\partial x^{\beta}}\right) , \tag{IV.38}$$

which corresponds to eq. (IV.21) of Kondo. Now the Frank-Burgers' circuit is carried out. The typical element of the circuit in the

dislocated crystal is[1]

$$dx^{\alpha}\, \underset{\sim}{i}_{\alpha} = dx^{\alpha}\, E^{a}_{\alpha}\, \underset{\sim}{e}_{a} \,. \tag{IV.39}$$

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[1] In the sense of footnote (2 of page 132) we imagine that we do not go from atom to atom but from elementary region to elementary region. This corresponds to the fact that in the case of a macroscopically continuous dislocation distribution the dislocations are arranged between the elementary regions. For a comprehensive discussion of the generalization of the Burgers circuit of §21 to continuous dislocation distributions see the papers of BBS [3,4].

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The corresponding step in the reference lattice has numerically the same component in the $\underset{\sim}{i}_{\alpha}$ system, according to §21, the step (IV.39) in the $\underset{\sim}{e}_{a}$ system which is

$$dx^{\lambda}\, E^{a}_{\lambda}\, \underset{\sim}{i}_{a} \quad \text{for} \quad \alpha = a \,. \tag{IV.40}$$

The circuit around a surface F with boundary curve C gives the related Burgers vector (= dislocation flux) according to §21

$$\underset{\sim}{B} \overset{n}{=} -\oint_{C} dx^{\lambda}\, E^{a}_{\lambda}\, \underset{\sim}{i}_{a} \quad \text{for} \quad \alpha = a \,. \tag{IV.41}$$

For the symbol $\overset{n}{=}$ see below. The application of Stokes' theorem and the use of infinitesimal surfaces leads to

$$dB^{\alpha} \overset{n}{=} -\frac{1}{2}\left(\frac{\partial E^{\alpha}_{\mu}}{\partial x^{\lambda}} - \frac{\partial E^{a}_{\lambda}}{\partial x^{\mu}}\right) dF^{\lambda\mu} \quad \text{for} \quad \alpha = a. \tag{IV.42}$$

In eq. (IV.41) $dx^\lambda$ means the difference of two points in their final state, also $E^a_\lambda$ is taken in the final state. Therefore, also the right-hand side of eq. (IV.42) is related to the final state. Therefore, eq. (IV.42) (also the equation before) is no simple vector equation, the sign $\overset{n}{=}$ means that the components on both sides of eq. (IV.42) are numerically the same for $\alpha = a$. Because of (IV.34) then the right-hand side of eq. (IV.42) is equal to the final state, which we obtain if we map $dB^\alpha$ to the final state.

$$dL^\alpha = D^\alpha_a \, dB^a \tag{IV.43}$$

or

$$dL^\alpha \overset{!}{=} \frac{1}{2} D^\alpha_a \left( \frac{\partial E^a_\beta}{\partial x^\gamma} - \frac{\partial E^a_\gamma}{\partial x^\beta} \right) dF^{\beta\gamma} \, . \tag{IV.44}$$

$dL^\alpha$ is called the "local Burgers vector" according to BBS whereas $dB^a$ is the "true Burgers vector." Notice: from the point of view of an invariant representation, there exists only one Burgers vector, $dB^a$ and $dL^\alpha$ are only different characterizations of this vector. For small distortions $D^\alpha_a$ in eq. (IV.44) can be replaced by $\delta^\alpha_a$ and it is not necessary to distinguish local and true Burgers vectors. By comparison with eq. (I.19), we find the relation between the torsion tensor and the dislocation density to be

$$L^\varkappa_{[\mu\nu]} = \frac{1}{2} \epsilon_{\mu\nu\rho} \, \alpha^{\rho\varkappa}, \quad \alpha^{\lambda\varkappa} = \epsilon^{\lambda\mu\nu} L^\varkappa_{[\mu\nu]} \, . \tag{IV.45}$$

BBS are talking of local dislocation density.

The Reimann-Christoffel curvature tensor

$$L^\alpha_{\beta\gamma\delta} = \frac{\partial L^\alpha_{\beta\delta}}{\partial x^\gamma} - \frac{\partial L^\alpha_{\beta\gamma}}{\partial x^\delta} + L^\alpha_{\lambda\gamma} L^\lambda_{\beta\delta} - L^\alpha_{\lambda\delta} L^\lambda_{\beta\gamma} \tag{IV.46}$$

with the connection (IV.37) vanishes identically. According to BBS this is the condition that we can define the local basis $e_a$ everywhere uniquely, and it means parallelism at infinity. This corresponds to the fact that the reference and final state are in the Euclidean space. However, Kondo took the curvature tensor (IV.23) in the non-Riemannian (natural) state and therefore it did not vanish.[1] We

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[1] The two important curvature tensors are defined according to our previous point of view by $\Delta^2 T_{\alpha\beta} = L_{\alpha\beta\gamma\delta}\, \Delta F^{\gamma\delta}$, $\Delta\beta_{\alpha\beta} \equiv R_{\alpha\beta\gamma\delta}\, \Delta F^{\gamma\delta}$, where $\Delta$ has the same meaning as in eq. (IV.31).

---

obtain the same result if we take the curvature tensor according to the $\underset{\sim}{e}_a \cdot \underset{\sim}{e}_b$ metric (called by BBS the "local metric"). Thus this is not the same curvature tensor as (IV.46). The vanishing of the curvature tensor taken with the local metric means that the crystal is free of self stresses.

It was shown that the theory reported above is very useful to investigate the pure rotation states of Nye (§7), which are similar to states of the classic theory of elasticity characterized by the vanishing of the incompatibilities (§7). We cannot discuss in detail calculations of great interest as applications of non-Riemannian geometry to real bodies, but we have to restrict ourselves to the most important point, the relation between the torsion tensor and Nye's curvature tensor.

The starting point is the theorem of parallel translation (IV.36) which we write in the form

$$dC_a = - L_{abc}\, C^b\, dx^c \,. \tag{IV.47}$$

We split $L_{abc}$ into its symmetric part $L_{(ab)c}$ and the antisymmetric part $L_{[ab]c}$:

$$dC_a = - (L_{(ab)c} + L_{[ab]c}) C^b\, dx^c. \tag{IV.48}$$

If we assume to be taken in sequence the lattice vector $\underset{\sim}{e}_b$ instead of $\underset{\sim}{C}^b$, then $d\underset{\sim}{C}_a$ is the change as we go to a neighboring point. As we see immediately, $L_{(ab)c}$ means a pure deformation; $L_{[ab]c}$, a pure rotation of the lattice (putting sequentially the vector $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ instead of $\underset{\sim}{C}^b$). In case $L_{[ab]c}$ is $d\underset{\sim}{C} \perp \underset{\sim}{C}$, the angle between $\underset{\sim}{e}_1$, $\underset{\sim}{e}_2$ remains unchanged, whereas for $L_{(ab)c}$ generally the length and the angle will be changed. If we take

$$L_{[ab]c} \equiv \epsilon_{abd}\, \varkappa^d_c, \quad \varkappa^d_c = \frac{1}{2}\, \epsilon^{abd}\, L_{abc} \tag{IV.49}$$

then the rotation part of $d\underset{\sim}{C}$ becomes

$$dC^{Rot}_a = - \epsilon_{abd}\, \varkappa^d_c\, C^b\, dx^c \tag{IV.50}$$

or written using vectors as $d\underset{\sim}{C}^{Rot} = - \underset{\sim}{C} \times \underset{\sim}{\varkappa} \cdot d\underset{\sim}{x}$ from which we obtain for the step from one point to another

$$d\vec{\underset{\sim}{\omega}} = \underset{\sim}{\varkappa} \cdot d\underset{\sim}{x} \,. \qquad \text{IV.51)}$$

By definition then $\underset{\sim}{\varkappa}$ is Nye's curvature tensor and (IV.49) is the relation between this and the torsion. $\underset{\sim}{\varkappa}$ is associated with the first two indices of the torsion and the dislocation tensor with the last two indices. In the case of a pure state of rotation we can easily obtain

Nye's relation (I.59) by comparison with eq. (IV.45) [5].[1] In this case also the tensors $\alpha$ and K of section 7 become identical.[2]

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[1] For this write eq. (IV.49) with Greek indices.

[2] For a discussion of the various curvature tensors, see also Eshelby [11].

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Furthermore, BBS show that the equation

$$\underset{\sim}{\kappa} \times \nabla = 0 \qquad \text{(IV.52)}$$

mentioned first by Nye holds only for small rotations. These equations are the conditions that $d\vec{u}$ of eq. (IV.51) is an exact differential. However, this only can hold for small angles of rotation.

## §28 Discussion

In the last two sections we reported applications of non-Riemannian geometry to continuous dislocation distributions. The authors mentioned moreover discussed, some very comprehensively, the relation to known problems of differential geometry--it should be mentioned, for instance, Cartan's holonomic groups and structural equations, or Ricci's rotation coefficients; so they improved understanding of the non-Riemannian concept of dislocations. In the book of Kondo [74] is described the very interesting possibility of treating the same problems by the use of a pure Riemannian geometry in six dimensions, where the holonomic coordinates $x^i$ are no longer sufficient, but we have to use the anholonomic coordinates of Riemann space. In the opinion of this author the related method is

especially useful for treating problems in plasticity. On this basis he developed a new mathematical theory of plasticity. Unfortunately the papers of Kondo and his collaborators came to the knowledge of the author a short time before finishing this report. Hence the last mentioned paper could not be reviewed.

In the opinion of the author the Riemannian and the non-Riemannian geometry will play an important role for large distortions. First these geometries, which were highly developed in another connection, are formulated from the beginning for arbitrarily large distortions. Second, the incompatibility tensor, $\eta_{ij}$, governs the self stresses uniquely only for small distortions, while the Riemann curvature tensor is unique for large distortion also. On the other hand, in these theories we usually have to calculate with tensors of 3rd and 4th rank; however these can be replaced by tensors of lower rank at least in the case of small distortions. [See eq. (IV.9) and IV.27)]. Therefore, it seems that the theory described in the first sections of this book is especially useful in the case of small distortions; moreover, it is more closely connected with the physical procedures of plastic deformation than the other theories.

In the preface of this chapter we pointed out the difference of the points of view of Kondo and BBS. Now we will discuss another difference between the two theories, which exists independently from the first.

For a given reference and final state of the body, the $D^{\alpha}_{a}$ and $E^{a}_{\beta}$ in eq. (IV.34) are uniquely determined,[1] the $A^{j}_{\lambda}$ and $A^{\kappa}_{j}$ in eq. (IV.12), however, are not. Here we have a free choice of the

coordinate system. First we will determine the orientation of the elementary regions in their natural state, which was determined by Kondo, in such a way that they have the same direction everywhere. Furthermore, we imagine that a virtual ideal point lattice most usefully having the same shape as the real atomic lattice in its perfect state, is impressed on the final state, thus, e.g., $\underset{\sim}{i}_j$ are the unit vectors of this lattice, similar to the $i_\alpha$ which are the unit vectors of the reference lattice of BBS. If it would be possible to show this virtual lattice also in the natural state, then it would have suffered exactly the inverse strains and rotations as the real atomic lattice during the transition from the natural state to the final state. The virtual lattice can be completely described in its natural state (in which it is deformed) by a system of unit vectors $\underset{\sim}{e}_\lambda$. If we now choose according to Kondo the basis system $\underset{\sim}{e}_\lambda$, then our considerations lead to the conclusion that eq. (IV.12) has the same meaning for the virtual lattice as eq. (IV.34) for the real lattice. I.e., that $A^{\varkappa}_{J}$, which

(Footnote for preceding page)

[1] With a convention according to footnote 2 pg. 132.

transforms the virtual lattice from the natural to the final state, is of the same magnitude as $D^{\alpha}_{a}$, which transforms the atomic lattice from its natural state or, in other words, from the reference state to the final state. The last holds because of the convention about the orientation in the natural state.

From this it follows that the components of $\Gamma^{i}_{[km]}$ in eq. (IV.21) and $L^{\alpha}_{[\beta\gamma]}$ in eq. (IV.38), which are both related to the Cartesian coordinates of the final state, are not necessarily numerically equal.

Therefore, also dislocation densities $\underset{\sim}{\alpha}$ calculated according to eq. (IV.27) and (IV.45) are generally different. We could call the dislocation density according to Kondo "virtual." It should not be difficult to convert the real and virtual dislocation density.

Finally the difference in the results of the two theories is a matter of convention, which should have no more importance than the sign convention. In the case of small distortions the difference between virtual, local and true dislocation density vanishes, then eq. (IV.21) and (IV.38) become directly the form of (I.17) of the geometric basic equation as will now be shown.

As mentioned above, Kondo's $A^{\mu}_{j}$ and BBS' $D^{\alpha}_{a}$ are numerically equal. Their meaning can be interpreted as a distortion of the lattice from the reference state to the final state. This distortion has the form $\underset{\sim}{I} + \underset{\sim}{\beta}$, where $\underset{\sim}{\beta}$ is identical with our previous distortion tensor ($\underset{\sim}{I} \equiv$ 2nd rank unit tensor). Then for small distortions the reciprocal distortions represented by $A^{i}_{\mu}$ and $E^{a}_{\beta}$ are equal to $\underset{\sim}{I} - \underset{\sim}{\beta}$. If we substitute this into eq. (IV.21) and (IV.38) and we neglect $A^{i}_{\mu}$ and $D^{\alpha}_{a}$ $\underset{\sim}{\beta}$ standing before the parentheses with respect to $\underset{\sim}{I}$, then we get

$$\Gamma^{i}_{[km]} = -\frac{1}{2}\left(\partial_m \beta^{i}_{k} - \partial_k \beta^{i}_{m}\right) \tag{IV.53}$$

and

$$L^{\alpha}_{[\beta\gamma]} = -\frac{1}{2}\left(\frac{\partial\beta^{\alpha}_{\beta}}{\partial x^{\gamma}} - \frac{\partial\beta^{\alpha}_{\gamma}}{\partial x^{\beta}}\right) . \tag{IV.54}$$

These equations are identical (the use of Latin and Greek indices, and the difference of signs is caused by the different conventions used by

Kondo and BBS). Also, we see that eq. (IV.54) is identical with our geometric basic eq. (I.17) in the case of small distortions if we also consider eq. (IV.45). Furthermore, eq. (IV.42) can become a real vector equation if we relate all quantities on the right-hand side to the reference state as in section 10. After comparison with (I.14), we get the geometric basic equation with the interpretation for the large distortions. In the general case of large distortions we need the basic equation related to the coordinate of the final state, since the equilibrium conditions are also related to these coordinates. Then we have to use the basic equations in the form (IV.21) and (IV.38), respectively.

## CHAPTER V

## APPLICATIONS

Problems treated from the point of view of continuum mechanics, as they are no ly formulated in the classic theory of elasticity, have not previ ly been investigated using the continuum theory of dislocations because there had been too little time since its development. The problem previously treated by continuum theoretical methods were mostly of the physical kind and normally dealt with single dislocations and atoms. It is a principal feature of modern plasticity research to understand from first principles basic phenomena from the microscopic point of view. So with this method it is possible in principle to succeed in the investigation of work hardening of metals. In section 29 we will discuss to some extent work hardening, since we assume a certain interest in this by our reader. On the other hand, we will show how mathematical problems appear during such considerations. The phenomena of work hardening is not purely mechanical, but of complex physical nature, and at best we can only give a rough impression of how such problems are solved nowadays. For a representation of the current level of work hardening theory, we refer the reader to the new encyclopedia section of Seeger [135].

According to our opinion the description of point lattice defects (interstitial atoms, vacancies, etc.) as elastic dipoles or centers of polarization has a fundamental significance; therefore we

will discuss four problems in section 31, which will show in a very impressive way which far-reaching problems of such elastic singularities we can solve with the simple formulas of section 19. Surely it would be a very advantageous task for experimental research to measure as much as possible the strength of a dipole and the polarizability for inclusions of atoms B in the solvent A and to put it down in tables, as has been done long ago for electric and magnetic dipoles and the polarizability.

Finally, in section 32 we will show examples of the practical use of the stress function tensor. We think that the complete exploration of this tensor will give some results of practical importance for which we believe that investigation of the three-dimensional and also the rotationally symmetric boundary value problem is very necessary. Now these are problems mainly of mathematical nature, and section 32 should be regarded as a stimulation in this sense for mathematical groups. Beside this, section 32 includes the important results for circular dislocations, which cannot be found in the previous literature.

## §29 The Work Hardening of Face Centered Cubic Metals

One of the most interesting problems, but simultaneously most difficult, of modern solid state physics is the work hardening of metals. Fig. 30 shows the typical work hardening curve of a face centered cubic single crystal as we observe it during a tensile test. It is not possible to deduce the curve from the basic equation of continuum mechanics or any theorems of solid state physics, but it is mostly due to empiricism. We have a certain model of the procedure which occurs in the interior of the body during plastic deformation and we investigate under which conditions this leads to work hardening. Then we carry out the corresponding experiments and examine how far they agree with the theoretical considerations. By this method it was possible to understand the three distinct regions of work hardening I, II and III in Fig. 30.

In the year 1934, Taylor [149] first had the idea that self stress fields are produced by dislocation motion and concentrations during plastic deformation, and these stress fields try to hinder the motion of dislocations. This qualitative concept still holds.

The stress field in the crystal caused by external loads can be decomposed into its components with respect to the slip plane and slip direction. In one of the slip systems the shear stress will be the greatest; we will call it $\tau$. This so-called primary slip system will act first. If the crystal is orientated favorably with respect to the tensile axis, then this slip system remains mostly responsible for the plastic deformation up to large deformations. Fig. 31a,b show how an elongation of the tensile sample occurs by slip on one slip system alone.

For theoretical investigations generally we plot the external shear stress in the primary slip system versus the slip. This quantity is defined to be the ratio of the plastic relative displacement of two lattice planes, separated by the distance d, to d, so that the plastic distortion is $\beta^p_{ij}$ when i is the normal to the slip plane and j the slip direction at the related point of the work hardening curve. Furthermore, we call this slip $\gamma$. Simultaneously $\tau$ has the meaning of a flow stress, for Fig. 30 is a plot of the stress which is necessary to obtain further flow of a crystal which has suffered a strain $\gamma$.

The flow stress of a pure metal can be decomposed according to Seeger [137] into two parts

$$\tau = \tau_S + \tau_G$$

where $\tau_S$ is that part which dislocations in the primary slip plane need to cut dislocations which are in other slip planes and which intersect the primary slip plane (often called the dislocation forest). $\tau_G$ is necessary to overcome the long range stress fields of other dislocations in the primary slip system. In many cases $\tau_S$ is of no importance relative to $\tau_G$, therefore, we will only discuss $\tau_G$.

Frank and Read [51] mentioned a mechanism by which closed dislocation loops can be developed by applying a shear stress on associated slip system. For this we need a sufficiently long piece of dislocation AB (Fig. 32a) which is fastened in some manner at its ends, perhaps so that it makes so-called dislocation nodes, which are often immobile, with other dislocations (this is not shown in the figure). Under an appropriate shear stress, the dislocation first bows out

(Fig. 32b). According to eq. (II.148) the force on the dislocation in the stress field is always perpendicular to the dislocation line; therefore this bows out successively to the form of Fig. 32c,d. The curved pieces at C have the same Burgers vector but opposite line direction, so they are dislocations with opposite sign, therefore, they attract each other according to section 18 till they have annihilated, thus the newly developed loop (e), and the original line AB remain. Afterwards this procedure can happen again. This development of dislocation loops is analogous to the development of soap bubbles, the line tension of the dislocation plays the part of the surface tension of the soap bubble. The necessary initial line AB is also present in undeformed crystals in sufficient numbers, since even during growth of the crystal a "network" of dislocations is developed in the crystal.[1] The number of dislocations cutting an area of 1 $cm^2$

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[1]In theories of the growth of crystals, dislocations play the principal role See Frank [49], Verma [154), Dekeyser and Amelinckx [32].

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is of the order of $10^7$ for many metals and it increases with deformation by several orders of magnitude.

In Stage I of work hardening, usually called the "easy glide" region, the so-called work hardening coefficient, $d\tau/d\gamma$, is relatively small. Dislocations can form and move without great hindrance. From the length of slip lines on the surface of the crystals (which were polished before the deformation) as seen by the electron microscope, we can conclude (Mader [196]) that here the dislocations travel distances similar to the cross-sectional dimension of the crystal (mm).

The much larger work hardening coefficient in the Stage II work hardening is caused by the presence of Lomer-Cottrell dislocations (§24) which greatly restrict the slip distance of dislocations. A dislocation due to a Frank-Read source may meet another dislocation which moves in a second slip system, thus a Lomer-Cottrell dislocation reaction may occur.

All the other dislocations of the source mentioned then cannot pass this obstacle; as a result they develop dislocation walls according to Fig. 24 (however, with variable distance between the dislocations) Seeger, Diehl, Mader and Rebstock [143] investigated theoretically and experimentally the procedure of work hardening starting from this assumption and succeeded in explaining more or less quantitatively the linear increase of the work hardening curve in Stage II.

In Stage III of the work hardening curve, the work hardening coefficient is smaller again. Nowadays it is explained by the assumption that with greater external shear stress the possibility arises that the dislocation can move around the obstacle. For this to occur in every case the dislocation line (§24) dissociated in the slip plane must be constricted over a length of several interatomic distances. This is not possible under the action of stresses which act at the dislocation alone, because this requires an increase of free energy of the order of eV at the place considered (see footnote 1 pg. I. ). Thus the distance between the two partial dislocations is reduced only by the resulting external and internal stress at the dislocation; the rest of the energy which is necessary to constrict the dissociated segment (the so-called activation energy, Q) must be contributed by thermal fluctuations. Evidently Q depends on the stress, and only if the stress is large

enough; i.e., Q is sufficiently small, can the activation energy be contributed by thermal fluctuations.

Accordingly edge dislocations do not have more slip opportunities, since there exists only one slip plane orientation for them, according to §24. However, according to §24, it is possible that screw dislocations lying along ⟨110⟩ directions the intersection of two {111} planes, can dissociate into another {111} plane which becomes the so-called "cross slip plane," after removing the previous dissociation. Fig. 33 shows this procedure. The newly obtained freedom of motion of the screw dislocation leads to the expected decrease of the work hardening rate.

Seeger, Diehl, Mader and Rebstock [143] convincingly proved the truth of these representations by electron microscopic photographs of polished crystal surfaces. For a complete concept of work hardening in Stage III, there is the investigation of a problem which we have waited to discuss until now because its mathematical difficulties have not been sufficiently solved till now.[1]

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[1] This problem is investigated presently.

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Initially we will follow the discussion of Schoeck and Seeger [133] and Seeger [134] (pp. 61p *et seq.*). If the length, $2\ell$, along which the dissociated dislocation is to constrict is very large with respect to the dissociated width, $2\eta$ (Fig. 34), then the dislocation is almost in unstable equilibrium, because the probability that the dislocation will dissociate in a primary slip plane is much larger than that for it to dissociate into a cross slip plane. The shorter the distance

$2\ell$ is, the more the primary slip plane is preferred. However, if a shear stress $\tau_Q$ is present on the cross slip plane, then this plane is preferred because the dislocation bows out into this plane and slips. Thus, the shear stress $\tau_Q$ can do work.[1] Hence, we conclude that an

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[1]Notice: The resulting force on the blocked dislocations in the primary slip plane is approximately zero; otherwise they would move.

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equilibrium length $2\ell$ is related to each such shear stress, and for this length dissociation in the primary and cross slip plane have the same probability. At least along these lengths slip of the blocked dislocations in the primary slip plane must be obstructed if the detour into the cross slip plane is to occur. The activation energy Q for this procedure is that energy which thermal fluctuations have to contribute to produce the configuration of Fig. 34 from two parallel partial dislocations separated by the distance $2\eta$.

From experiments we can find a value for the so-called specific stacking fault energy, the energy necessary to produce a stacking fault which is spread over the whole crystal measured per $cm^2$ of the stacking fault area. The activation energy Q contains the following parts: The (positive) energy $E_{12}$ necessary to bring the partial dislocations into the position of Fig. 34 (in which they are longer than before) and the (negative) stacking fault energy $E_{st}$, which we can recover if we reduce the stacking fault area.

The exact calculation of the activation energy is rather difficult, since the shape of the dislocation line is not known, but has

to be determined by a variational calculation. For this we apparently can use only direct methods. Since there is no physical reason for the symmetry of Fig. 34, we will describe the procedure by the simpler arrangement of Fig. 35. We take for free parameters the cross slip length $2\ell$, the dissociated width $2\eta$ and the tangent to the dislocation line at the nodal points K and K′. The calculation of the part $E_{st}$ is trivial. The part $E_{12}$, which presented great difficulties previously, follows very simply from formula (II.136). The main part of the work is caused by the energy $E_{22}$. The line 2 consists of three parts a,b,c. We write $E_{22} = E_{aa} + E_{bb} + E_{cc} + E_{ab} + E_{ac} + E_{bc}$. $E_{bb}$ can be obtained very easily by eq. (II.144), similarly $E_{ab}$, $E_{bc}$, for then one of the two line integrals of eq. (II.128) can be obtained in an elementary fashion. Since $E_{aa} = E_{bb}$, the calculation of $E_{aa}$ and $E_{ac}$ is the hardest problem. However, the branches a and c are relatively far from each other, thus $E_{ac}$ certainly will give a small value, which is not so important, with respect to $E_{22}$, and we can obtain this quantity by a simplified approximation. Accordingly, the important part is energy $E_{aa}$ used to force the branch a from its straight shape into the curved shape of Fig. 35. We will solve the problem of the self energy of a curved dislocation in the following section, and we will see that we can succeed if the shape is not too complicated.

So nowadays we are in a position to treat successfully with method of continuum theory problems of the activation energy of the kind described above which appear very often in solid state physics, and we can confirm or disprove conclusions drawn from experiments dealing with elementary phenomena in the solid body.

The stacking fault energy is relatively large in aluminum in contrast to copper, therefore the dissociation and the activation energy is very small, and we would expect that Stage III work hardening begins at lower stresses than in copper. This is confirmed very well by experments. It is very satisfying that we can nowadays understand almost quantitatively the differences in work hardening properties of face centered cubic metals, which were very puzzling some years ago. We understand far less the work hardening of body-centered cubic metals.

## §30 An Approximation for the Calculation of the Self Energy of Singular Dislocations

The self energy of a bent dislocation is important for many problems of solid state physics. In contrast to older methods, where we had to calculate at least one line integral of a function given as a surface integral to get the self energy of the dislocation, the reduction of the problem to the double line integral (II.145) is a great advance. However, this also can only be solved exactly in the simple cases. According to Kröner [83], we will succeed in complicated cases with help of the approximation we will describe now.

The starting point is eq. (III.145) in which we assume the cut length $\epsilon$ to be given. We will take $\epsilon$ according to eq. (III.58) for the partial dislocation mentioned in the last section. The integrals in eq. (II.145) will have the form

$$\oint_L \oint_{L'_\epsilon} \frac{dL' \, dL}{|\underset{\sim}{x} - \underset{\sim}{x}'|} \tag{V.1}$$

or forms, which we get if we use the expression $(x_1 - x_1') \cdot (x_3 - x_3')/(\underset{\sim}{x} - \underset{\sim}{x}')^2$ or $(x_3 - x_3')^2/(\underset{\sim}{x} - \underset{\sim}{x}')^2$ as a factor with the integrand. The calculation of the latter integrals does not differ much from that of eq. (V.1), thus we can restrict ourselves to the treatment of (V.1) to get

$$d\underset{\sim}{L} = dx_1\, \underset{\sim}{i}_1 + dx_3\, \underset{\sim}{i}_3, \qquad d\underset{\sim}{L}' = dx_1'\, \underset{\sim}{i}_1 + dx_3'\, \underset{\sim}{i}_3 \tag{V.2}$$

thus we have to calculate integrals of the form

$$|\underset{\sim}{x} - \underset{\sim}{x}'| = \sqrt{(x_1 - x_1')^2 + (x_3 - x_3')^2} \tag{V.3}$$

For this we have to substitute one of the terms $x_1, x_3$ or $x_1', x_3'$, respectively, by the other in the curve equation $x_1 = x_1(x_3)$. If the dislocation is piecewise straight, then the integrals can be determined in an elementary fashion. For second order curves we get elliptic integrals. In other cases the integral can only be determined numerically. In the case of section 29, where the curve equation consists of free parameters, this procedure would be much too complicated. The following method will often lead to results in many cases. Let

$$|\underset{\sim}{x} - \underset{\sim}{x}'| = |x_3 - x_3'|\, \sqrt{1 + s}, \quad s \equiv \left(\frac{x_1 - x_1'}{x_3 - x_3'}\right)^2 \tag{V.5}$$

and we expand the roots $w(s)$ in the interval $0 \le s \le S$ by Legendre polynomials of $s$. If $w(s)$ is a parabola, we can expect a good convergence if we do not take $S$ much larger than 3. The estimation of the error is not very hard because of the simple form of $w(s)$. E.g., if the dislocation never makes an angle $< 30°$ with the $\pm\, x_1$-direction,

then apparently $0 \leq s \leq 3$, and the expansion converges well along the whole dislocation. If, however, the dislocation has nearly vertical pieces in addition to nearly horizontal pieces, we will use in addition to (V.5)

$$|\underset{\sim}{x} - \underset{\sim}{x}'| = |x_1 - x_1'| \sqrt{1 + (1/s)}, \quad s \equiv \left(\frac{x_1 - x_1'}{x_3 - x_3'}\right)^2 \qquad \text{(V.6)}$$

and $w(1/s)$ will be represented by Legendre polynomials. However, it would be worthwhile to investigate whether we can avoid the complications connected with this if we use in addition higher order terms of the expansion of $w(s)$ by which we can enlarge the region $0 \leq s \leq S$, so it may be possible that it is not necessary to use $w(1/s)$. This procedure may be recommended for the calculation of the energy $E_{aa}$.

The amount of calculation is mainly determined by the shape of the dislocation. If $x_1(x_3)$ is a polynomial, then the integrations are elementary. The same holds for hyperbolas $x_1 \pm a = c/(x_3 \pm b)$ and this statement holds also for finite pieces of hyperbolas. Apparently we can describe the line elements a,c in Fig. 35 with such a hyperbola and now we can calculate the energy in an elementary fashion. The amount of the calculation is tolerable.

Evidently we can use the same method to calculate the interaction energy of two bent dislocations according to eq. (II.128). Then we can also determine the activation energy related to the arrangement of Fig. 34.

The component $\tau_S$ of the yield stress is due to the mutual intersection of two dislocations. According to Heidenreich and Shockley [64] the dissociation of the dislocation has to be removed at the point of

intersection. Then according to Figs. 34 and 35 we get a so-called constriction (Stroh [148]) by letting $\ell = 0$. Schöek and Schöek and Seeger [133] were able to calculate the activation energy for the cutting of dislocations in some cases, which agreed sufficiently with experimentally measured activation energies. (See also Seeger [136]).

## §31 Foreign Atoms as Elastic Dipoles and Centers of Polarization

We speak of foreign atoms if there are some atoms of type B in a lattice of atoms of type A. They can be at a regular lattice position of an atom A (substitutional) or they can be in a so-called interstice. This especially happens if atom B is small in comparison to atom A. Even very small numbers of such foreign atoms can influence the "structural sensitive" properties of the medium (Smekal [145]). The changes of the properties of iron in which carbon (C) is dissolved in small concentrations is well known. The C-atoms are in an interstitial position here. According to Cottrell [23] and Cottrell and Bilby [25], the interaction of the C-atom with dislocations in iron governs, e.g., the familiar yield point phenomenon of steel.

Fig. 36a shows how such a C-atom is placed in the body-centered cubic lattice of rion. In order to have enough space it has to push the neighboring atoms. Obviously we get the same state of distortion of the lattice in the surroundings if instead of the C-atom we apply forces of magnitude P which press one atom up and the other down. If a is the interatomic distance in the normal state, then we have here a force dipole with $P_{22} = P$ the only component different from zero.

The model mentioned above is a little bit too simple. For an exact discussion we have to investigate the forces which keep the atoms together. This is determined by the particular distribution of the electrons. We can imagine that the C-atom not only changes the forces between the neighboring atoms, but also that other atoms of the neighborhood will be affected. Then we must realize that C-atoms act not as a single dipole $P_{22}$, but that also other dipole components and higher components will play a part. According to experience the direct interaction between atoms decreases rapidly after one interatomic distance to such an extent that we can describe the C-atom in the position of Fig. 36a to a good approximation by its dipole components $P_{22}$ as well as $P_{11}$ and $P_{33}$.

Fig. 37, an example of a substitutional atom, shows that the foreign atom is replaced by a number of force dipoles which are oriented at an angle of 60° to each other. It can be shown that the displacement field which results from such a dipole arrangement is that of a center of dilatation $P_{22}$.[1] In the continuum a center of dilatation represents a small compressed sphere.

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[1]This statement holds in the case of the face-centered cubic lattice, but not necessarily for the hexagonal lattice, since in the latter the elastic stiffness with respect to dipoles depends on their direction, thus we need different dipole magnitudes to displace oppositely placed atoms the same amount.

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For many problems it is important to know the energy with which a foreign atom is bound to a dislocation. In general, a dislocation

exerts a force on a foreign atom due to its elastic deformation field according to eq. (II.156). If we move a foreign atom from a position with zero strain to a position with strain $\epsilon_{ij}$ in the neighborhood of the dislocation then we may release energy, following from eq. (II.165). The motion of such an atom always becomes possible by thermal fluctuations. If we apply a shear stress then (at least at room temperature) the dislocation may move with a much larger velocity than the interstitial atom which tries to keep up with the dislocation. Therefore there is a tendency for the dislocation and interstitial atom to become separated due to the applied shear stress. However for this to occur an energy of the magnitude of the "binding energy" between the foreign atom and the dislocation must be supplied. According to Cottrell the dislocation in the normal state is surrounded by a whole cloud of C-atoms, thus a rather big energy supply is necessary to separate the dislocation from the cloud and make it able to move. This leads to the familiar yield point effect.[1]

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[1]According to the latest considerations of Seeger [135] this argument should be modified.

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Now we will investigate the interaction of the C-atoms with a screw dislocation, where we will methodically follow the representation of Cochardt, Schöek and Wiedersich [16].[2]

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[2]Cottrell and Bilby described the C-atoms mainly as centers of dilatation, therefore they did not obtain an interaction with a screw dislocation according to eq. (II.118). As this author mentioned and

(Footnote 2 continued)
especially Crussard [27] and Nabarro [107] emphasized, we should obtain an interaction of a C-atom and a screw dislocation because of the tetragonality of the distortion. Cochardt, Schöek and Wiedersich first investigated this quantitatively. However they do not use force dipoles explicitly, but their method is very similar to ours.

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In iron the screw dislocation is directed along the ⟨111⟩ direction. The starting point is the two equations

$$K_k = P_{ij} \nabla_k \epsilon_{ij} \tag{V.7}$$

$$U = - P_{ij} \epsilon_{ij} \tag{V.8}$$

from section 19. In polar coordinates $\rho, \varphi, z$ with the z-direction = [111] and $\varphi = 0$ in the $[\bar{2}11]$ direction, the deformation field follows from the stress field (II.118) of a screw dislocation and has only the component

$$\epsilon_{z\varphi} = \epsilon_{\varphi z} = \frac{b}{4\pi\rho} . \tag{V.9}$$

According to eq. (V.8) a force is only then exerted on a dipole in the field of a screw dislocation if the dipole has a $z\varphi$- or a $\varphi z$-component. The force dipole can be transformed as a tensor into the $\rho, \varphi, z$ system. According to common rules it is

$$P_{z\varphi} = P_{\varphi z} = (\underset{\sim}{i}_\varphi \cdot \underset{\sim}{i}_1)(\underset{\sim}{i}_z \cdot \underset{\sim}{i}_1) P_{11} . \tag{V.10}$$

Let $\underset{\sim}{i}_\varphi = \underset{\sim}{i}_{\varphi_0} \cos\varphi + \underset{\sim}{i}_{\varphi_{90}} \sin\varphi$, then $\underset{\sim}{i}_{\varphi_0}$ is the unit vector perpendicular to the vectors

$$\underset{\sim}{i}_z = [\sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}, \sqrt{\tfrac{1}{3}}] \text{ and } \underset{\sim}{i}_{\varphi_{90}} = [\sqrt{\tfrac{2}{3}}, -\sqrt{\tfrac{1}{6}}, -\sqrt{\tfrac{1}{6}}].$$ [1]

(Footnote for preceding page)

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[1] We see at once that these $\underset{\sim}{i}_z$ and $\underset{\sim}{i}_{\varphi 90}$ are unit vectors in the [111] and $[2\bar{1}\bar{1}]$ direction, respectively.

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$\underset{\sim}{i}_{\varphi_0} = \underset{\sim}{i}_z \times \underset{\sim}{i}_{\varphi 90}$ has no component in the $x_i$-direction, as we may check; thus we finally obtain for eq. (V.10)

$$P_{z\varphi} = P_{\varphi z} = \frac{\sqrt{2}}{3} P_{11} \sin \varphi. \tag{V.11}$$

Since $\epsilon_{\varphi z}$ depends only on $\rho$, according to eq. (V.7) the dipole suffers a force

$$K_\rho = - \frac{b}{3\sqrt{2}\pi} \frac{\sin \varphi}{\rho^2} P_{11}. \tag{V.12}$$

This is an attractive force in the region $0° < \varphi < 180°$, a repulsive force otherwise. If we employ the same procedure for $P_{22}$ and $P_{33}$, we obtain formulas which follow from eq. (V.12) by substituting for $\varphi$ by $\varphi + 120°$ and $\varphi + 240°$, respectively. Apparently this is a result of the fact that the [111] direction makes the same angle with the $x_i$-axis.

From eq. (V.8) with (V.9) and (V.11), we immediately get the energy of the dipole $P_{11}$ in the elastic field of the dislocation to be

$$U = - \frac{bP_{11}}{3\sqrt{2}\pi} \frac{\sin \varphi}{\rho}. \tag{V.13}$$

The application of these formulas is prevented initially for the case of the C-atom, since we do not know its dipole magnitude. Today it is only possible to obtain this experimentally by using a theory which allows us to connect the dipole magnitude with experimentally measured quantities. There is no procedure which we can apply in every case.

Eshelby [39] described the following method for measuring the magnitude of the dipoles composing a center of dilatation: Dissolve a number of atoms of type B in a metal A (e.g., Al in Cu) and measure the change of the lattice parameter which occurs. This depends on the corresponding concentration of B atoms in A and on the strength of the center of dilatation. Eshelby gave the necessary equations for calculating $P_{ii}$ from the change of the lattice parameter.

We will show now that this method can be extended to the case of arbitrary force dipoles. If we distribute a number of force dipoles statistically uniformly in an initially homogeneous body A, then it generally changes shape and volume. If we considered certain physical volume elements, each of which includes many such dipoles, but which are on the other hand small with respect to the external dimensions of the body, then we can state that an average (macroscopic) deformation $\varepsilon^Q_{ij}$ is impressed on each volume element by the dipole distribution (§6). If we assume that the concentration is constant with respect to the volume elements, which is generally possible in experiments, then the connection of the body is not disturbed by this impressed deformation, i.e., there are no (macroscopic) elastic deformations necessary to keep the body compact. Accordingly, the impressed deformation is the observed total deformation $\underset{\sim}{\varepsilon}^T$. The macroscopic stresses vanish.

However it is obvious that we cannot distinguish macroscopically if we have $10^6$ dipoles at the strength $A_{ij}$ or $10^7$ dipoles of the strength $A_{ij}/10$, with other words, we imagine the N dipoles of the strength $P_{ij}$ substituted by a constant dipole density $p_{ij}$, which can be determined by the condition

$$\iiint_V p_{ij}\, dV = p_{ij}\, V = NP_{ij} \,. \tag{V.14}$$

Rieder [123] called such a dipole density, taken negative, an extra stress; the nomenclature "impressed stress" is also meaningful. This is connected with the impressed deformation by the equation [123], [122]

$$p_{ij} = C_{ijk\ell}\, \epsilon^{Q}_{k\ell} \tag{V.15}$$

as we easily see.[1] Since $p_{ij}$ is known, we have simultaneously the

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[1]We imagine that surface forces are applied simultaneously with the dipole density such that initially no deformation occurs. Afterwards we can cut the volume elements and can measure the forces $p_{ij}\, dF_i$ which we have to apply in order that no displacement occurs. The following deformation during the relaxation is connected with the stresses by Hooke's Law. Also we realize that we have to deal with small $p_{ij}$, i.e., small concentrations, for otherwise eq. (V.15) will not hold.

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total deformation of the body $\epsilon^{T}_{ij} = \epsilon^{Q}_{ij}$. Conversely we obtain the dipole strength to be

$$P_{ij} = (V/N)\, C_{ijk\ell}\, \epsilon^{T}_{k\ell} \tag{V.16}$$

where the concentration, N/V, and the total deformation $\epsilon^{T}_{ij}$ is known.

This method holds only for small deformations because we assume Hooke's Law (V.15). Such deformations can be measured easily.[2]

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[2]As the sample must generally be melted to put the dipoles in it, the dimensions with and without dipoles can hardly be compared.

However, more difficult considerations are necessary to assure that the change in the lattice parameter determined by x-ray techniques is really the macroscopic deformation. Microscopically we do have stresses and elastic deformation fields which change cheir sign over distances of the order of the average distance between the dipoles. However, the corresponding investigations of Miller and Russel [101], Huang [66], Tetlow [150] and Eshelby [39] seem to solve this problem.[1]

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[1] See the discussion of Eshelby [39].

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The C-atoms are statistically distributed over the three possible positions in the body centered cubic iron-carbon--we will call them the 1-, 2- and 3-position. In this case we can only notice a continuous dilatation of the lattice and so we obtain only an average statement about $P_{ij}$. However in Martensite the C-atoms are arranged tetragonally (e.g., all in the 1-position) then we expect a strong dipole $P_{11}$ and two weak dipoles $P_{22}$ and $P_{33}$.

Kurdjumov and Kaminski [85] gave an increase of the lattice parameter C in the 1-direction from 2.86 to 2.96 Å for 1 weight % C in Fe (corresponds to $V/N = 2.58 \cdot 10^{-22}$ cm$^3$), while simultaneously the c/a ratio increases from 1. to 1.04. (a = lattice parameter in 2- and 3-direction). This means a strain $\epsilon^T_{11} = 0.035$, $\epsilon^T_{22} = \epsilon^T_{33} = -0.0048$. With $C_{1111} = 2.37 \cdot 10^{12}$ dyne/cm$^2$ $C_{1122} = 1.4. \cdot 10^{12}$ cyne/cm$^2$, it follows[2] from eq. (V.1 6) easily[3]

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[2] $C_{1111} = C_{2222} = C_{3333} \equiv C_{11} \cdot C_{1122} = C_{2211}$, etc. $\equiv C_{12} \cdot C_{11}$ and $C_{12}$ are from Zener [158], page 17.

[3] 1 eV = $1.6 \cdot 10^{-12}$ erg.

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$$P_{11} = 11.2 \text{ eV}, \qquad P_{22} = P_{33} = 4.6 \text{ eV}. \tag{V.17}$$

From eq. (V.13) and the corresponding equations for $P_{22}$ and $P_{33}$, we get the energy of the total dipole represented by the C-atom at a distance b from the screw dislocation and for $\varphi = 90°$ (where $|U|$ has its maximum) to be 0.5 ev.[1]

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[1]The difference between this and the value of 0.75 of Cochardt and collaborators is caused by the fact that these authors used the isotropic Young's modulus. This is an important and useful example of how large the difference can be if we do not consider the elastic anisotropy of the crystals.

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We will briefly discuss the application to the important Snöek effect [146].[2] If we apply a stress $\sigma_{11}$ to the crystal of Fig. 36a

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[2]The Snöek effect is often used to determine the smallest C-concentration in Fe [75]. Zener [158] treated this effect theoretically very sufficiently and in parts we follow his representation. The reduction of the relaxation of the elastic coefficient $s_{ijk\ell}$ due to the force dipole represented by the C-atom is new and very impressive.

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so it is strained in the $x_1$-direction, then the C-atoms prefer to switch over into the 1-position (Fig. 36b) since they have more room there. The total displacement is composed now of the elastic part $\epsilon_{ij} = s_{ijk\ell}\,\sigma_{k\ell}$ and an additional quasilinear deformation

$\epsilon^Q_{ij} = s_{ijk\ell}\, p_{k\ell}$ (eq. (V.15)), where $p_{k\ell}$ is the change of the dipole density due to the motion of n dipoles from the 2- or 3-position to the 1-position. Therefore the total strain is

$$\epsilon^T_{ij} = s_{ijk\ell}(\sigma_{k\ell} + p_{k\ell}). \tag{V.18}$$

In this $p_{k\ell}$ should be expressed in terms of known quantities. $P^1_{ij}$, $P^2_{ij}$, $P^3_{ij}$ are the dipoles in the three different positions. Then because of the similarity of $P^2_{ij}$ and $P^3_{ij}$, it follows from eq. (V.14)

$$P_{k\ell} = \frac{n}{V}(P^1_{k\ell} - P^2_{k\ell}). \tag{V.19}$$

According to Zener [158], we can easily find n from Boltzman statistics to be

$$n = -\frac{2}{9} N \frac{U_1 - U_2}{kT}, \quad \text{for} \quad \frac{|U_1 - U_2|}{kT} \ll 1 \tag{V.20}$$

where $U_1$ is the elastic energy of the dipole in the 1-position, similarly $U_2$. From eq. (V.8) it follows that

$$U_1 - U_2 = -(P^1_{ij} - P^2_{ij})\epsilon_{ij} \tag{V.21}$$

with which we obtain

$$p_{k\ell} = \frac{2}{9}\frac{N}{VkT}(P^1_{k\ell} - P^2_{k\ell})(P^1_{ij} - P^2_{ij})\epsilon_{ij} \tag{V.22}$$

In this we substitute $\epsilon_{ij} = s_{ijmn}\,\sigma_{mn}$ and $p_{k\ell}$ into eq. (V.18) so we get

$$\epsilon^T_{ij} = (s_{ijk\ell} + \Delta s_{ijk\ell})\sigma_{k\ell} \tag{V.23}$$

with

$$\Delta s_{ijk\ell} = \frac{2}{9}\frac{N}{VkT} Q_{ij}\, Q_{k\ell}, \qquad Q_{ij} = s_{ijmn}(P^1_{mn} - P^2_{mn}). \tag{V.24}$$

The quantities $s_{ijk\ell} + \Delta s_{ijk\ell}$ are called the "relaxed" elastic coefficients, and they are measured statically by the ratio of total strain to applied stress. In spite of this we can measure the "unrelaxed" elastic coefficients $s_{ijk\ell}$ with vibrating rods, where the period is so small that a rearranging of the dipoles (the "relaxation"), which always requires a finite time, cannot occur. Zener, whose results are quantitatively the same as ours, furthermore describes the way in which $\Delta s_{ijk\ell}$ is a measure of the magnitude of the damping. For this and the comparison with experimental results, which are very satisfactory, see Zener [158].

Finally, we will discuss briefly the polarizability (§19). We assume the sample does not contain dipoles but rather centers which can be polarized. Important examples are lattice vacancies in many face centered cubic crystals.[1] Analogous to the circumstances in electro-

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[1] Seeger and Bross [142] previously calculated from electron theory that the dipole strength of a lattice vacancy is approximately zero.

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dynamics, we can call such a medium "dielastic" in contrast to a body with dipoles, which is called "parelastic." Parelastic bodies always have a certain amount of dielasticity.

For instance, if we apply a homogeneous stress $\sigma_{ij}$ to a dielastic rod, then dipoles $P_{ij}^{ind}$ of density $p_{ij}^{ind}$ are induced. We obtain

$$\sigma_{ij} + p_{ij}^{ind} = C'_{ijk\ell}\,\epsilon_{k\ell} \tag{V.25}$$

instead of the normal Hooke's Law

$$\sigma_{ij} = C_{ijk\ell}\,\epsilon_{k\ell} \tag{V.26}$$

which we would obtain if no dielasticity were present. The combination of both equations becomes

$$p_{ij}^{ind} = (C'_{ijk\ell} - C_{ijk\ell})\epsilon_{k\ell} \tag{V.27}$$

and

$$r_{ijk\ell} = C'_{ijk\ell} - C_{ijk\ell} \tag{V.28}$$

is the "elastic susceptibility" of the sample. Because $p_{ij}^{ind} = P_{ij}^{ind}(N/V)$, the elastic polarizability of eq. (II.169) is given by

$$R_{ijk\ell} = (V/N)r_{ijk\ell}. \tag{V.29}$$

In many cases $c_{ijk\ell}$ and $C'_{ijk\ell}$ can be accurately measured as elastic moduli of samples with and without centers of polarization; thus the polarizability of defects is relatively easy to obtain.[1] The interaction

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[1]The effective modulus, $C'_{ijk\ell}$, of a sample with centers of polarization appears smaller or larger with respect to the modulus without centers depending on the sign of $p_{ij}^{ind}/\epsilon_{k\ell}$. According to Zener [159], all self stress sources have additional effects which reduce the modulus. This effect is reduced to the determination of the increase of the oscillation entropy of the body by increasing elastic deformation,

(Footnote continued)

so in contrast to our effect it depends very much on temperature. Therefore it should be possible to separate these two effects experimentally.

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of dielastic lattice defects is mainly determined by the polarizability. This is an interaction with a very small region of influence (force is proportional to the -6 power of the distance), while the force of dipoles goes with the -4 power.

## §32 Applications of the Stress Function Tensor $\underset{\sim}{\chi}'$ to Rotationally Symmetric and Three-dimensional Problems

Let cylindrical coordinates $\rho$, $\varphi$, z have $\underset{\sim}{i}_\rho$, $\underset{\sim}{i}_\varphi$, $\underset{\sim}{i}_z$ as the corresponding unit vectors (length $\ell$). The components of the stress function tensor $\underset{\sim}{\chi}'$ may not depend on $\varphi$. Then as can be easily checked we can write the secondary conditions $\nabla_i \chi'_{ij} = 0$ of section 12

$$\frac{\partial(\rho\chi'_{\rho\rho})}{\partial\rho} - \chi'_{\varphi\varphi} + \rho\,\frac{\partial\chi'_{\rho z}}{\partial z} = 0 \qquad \text{(V.30)}$$

$$\frac{1}{\rho^2}\,\frac{\partial(\rho^2\chi'_{\rho\varphi})}{\partial\rho} + \frac{\partial\chi'_{z\varphi}}{\partial z} = 0 \qquad \text{(V.30')}$$

$$\frac{1}{\rho}\,\frac{\partial(\rho\chi'_{\rho z})}{\partial\rho} + \frac{\partial\chi'_{zz}}{\partial z} = 0\,. \qquad \text{(V.30'')}$$

If they are satisfied, then the stresses follow most simply from eq. (II.23)

$$\underset{\sim}{\sigma} = 2G[\Delta\underset{\sim}{\chi}' + \frac{m}{m-1}(\nabla\nabla - \Delta\underset{\sim}{I})\chi'_I]$$

with $\chi'_I \equiv \chi'_{\rho\rho} + \chi'_{\varphi\varphi} + \chi'_{zz}$. It is with $\chi'_{\rho\rho} + \chi'_{\varphi\varphi} \equiv \chi'_+$, $\chi'_{\rho\rho} - \chi'_{\varphi\varphi} = \chi'_-$ that

$$(\sigma_{\rho\rho}+\sigma_{\varphi\varphi})/2G = \Delta\chi'_{+} - \frac{m}{m-1}\left(\Delta + \frac{\partial^2}{\partial z^2}\right)\chi'_{I}$$

$$(\sigma_{\rho\rho}-\sigma_{\varphi\varphi})/2G = \left(\Delta - \frac{4}{\rho^2}\right)\chi'_{-} + \frac{m}{m-1}\left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\chi'_{I}$$

$$\sigma_{zz}/2G = \Delta\chi'_{zz} - \frac{m}{m-1}\left(\Delta - \frac{\partial^2}{\partial z^2}\right)\chi'_{I},$$

$$\sigma_{\rho z}/2G = \left(\Delta - \frac{1}{\rho^2}\right)\chi'_{\rho z} + \frac{m}{m-1}\left(\frac{\partial^2}{\partial\rho\,\partial z}\right)\chi'_{I}$$

$$\sigma_{\varphi z}/2G = \left(\Delta - \frac{1}{\rho^2}\right)\chi'_{\rho z}; \quad \sigma_{\rho\varphi}/2G = \left(\Delta - \frac{4}{\rho^2}\right)\chi'_{\rho\varphi}.$$

The Cartesian components of $\chi'_{ij}$ satisfy the eq. (II.20)

$$\Delta\Delta\,\chi'_{ij} = \eta_{ij}. \tag{V.31}$$

From this immediately follows

$$\Delta\Delta\,\chi'_{+} = \eta_{+}, \quad \Delta\Delta\,\chi'_{zz} = \eta_{zz}. \tag{V.32}$$

More effort is necessary to obtain the other equations

$$\left(\Delta - \frac{4}{\rho^2}\right)\left(\Delta - \frac{4}{\rho^2}\right)\chi'_{-} = \eta_{-}, \qquad \left(\Delta - \frac{4}{\rho^2}\right)\left(\Delta - \frac{4}{\rho^2}\right)\chi'_{\rho\varphi} = \eta_{\rho\varphi}, \tag{V.32'}$$

$$\left(\Delta - \frac{1}{\rho^2}\right)\left(\Delta - \frac{1}{\rho^2}\right)\chi'_{\rho z} = \eta_{\rho z}, \quad \left(\Delta - \frac{1}{\rho^2}\right)\left(\Delta - \frac{1}{\rho^2}\right)\chi'_{\varphi z} = \eta_{\varphi z}. \tag{V.32''}$$

Since $\nabla_i\eta_{ij} = 0$, $\eta_{ij}$ is restricted by the conditions corresponding to (V.30). We notice that the components of $\underset{\sim}{\chi}'$ are initially coupled in the differential equation. however the condition (V.30) always connects $(\chi'_{\rho\rho}, \chi'_{\varphi\varphi}, \chi'_{zz})$, $(\chi'_{\rho\varphi}, \chi'_{\varphi z})$ and $(\chi'_{zz}, \chi'_{\rho z})$. Often it is possible to say that $\sigma_{\varphi z} = \sigma_{\varphi\rho} = 0$, then we can neglect $\chi'_{\varphi z}$ and $\chi'_{\varphi\rho}$. Furthermore,

if $\eta_{\rho z} = \eta_{zz} = 0$, then we only need $\chi'_{\rho\rho}$ and $\chi_{\varphi\varphi}$ to obtain a particular integral. Although we still did not find a rigorous proof, it is very likely that all states of stress in which div $\underset{\sim}{\sigma}$, $\underset{\sim}{\eta}$, $\sigma_{\varphi z}$, $\sigma_{\rho\varphi}$ vanish can be expressed by $\chi'_{\rho\rho}$ and $\chi'_{\varphi\varphi}$ alone. We generally use Love's displacement function for the calculation of these states of stress. In the following we will only consider the case $\chi_{\rho z} = \chi_{\varphi\varphi} = \chi_{\rho z} = \chi_{zz} = 0$.

It is remarkable that $\chi_{\varphi\varphi}$ can be expressed now according to eq. (V.30) by $\chi_{\rho\rho}$. We can also write the condition (V.30) in terms of $\chi_{\rho\rho}$ and $\chi'_+$ and $\chi_-$, respectively. Similarly it holds for $\underset{\sim}{\eta}$ that

$$\chi'_+ = \frac{1}{\rho}\frac{\partial(\rho^2\chi_{\rho\rho})}{\partial\rho}, \quad \chi'_- = -\rho\frac{\partial\chi_{\rho\rho}}{\partial\rho}; \quad \eta_+ = \frac{1}{\rho}\frac{\partial(\rho^2\eta_{\rho\rho})}{\partial\rho}; \quad \eta_- = -\rho\frac{\partial\eta_{\rho\rho}}{\partial\rho} \tag{V.33}$$

Now the stresses can be written

$$\left.\begin{aligned}
(\sigma_{\rho\rho}+\sigma_{\varphi\varphi})\ 2G &= \Delta\chi'_+ - \frac{m}{m-1}\left(\Delta + \frac{\partial^2}{\partial z^2}\right)\chi'_+ \\
(\sigma_{\rho\rho}-\sigma_{\varphi\varphi})\ 2G &= \left(\Delta - \frac{4}{\rho^2}\right)\chi_- + \frac{m}{m-1}\left(\frac{\partial^2}{\partial\rho^2} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\right)\chi'_+ \\
\chi_- &= 2\chi_{\rho\rho} - \chi_+ \\
\sigma_{zz}\ 2G &= -\frac{m}{m-1}\left(\Delta - \frac{\partial^2}{\partial z^2}\right)\chi'_+ \\
\sigma_{\rho z}\ 2G &= \frac{m}{m-1}\frac{\partial^2}{\partial z\partial\rho}\chi_+ .
\end{aligned}\right\} \tag{V.34}$$

By adding eqs. (V.32) and (V.32') related to $\eta'_+$ and $\eta_-$, we obtain with (V.33)

$$\Delta\Delta'\chi_{\rho\rho} = \eta_{\rho\rho}, \qquad \Delta' = \Delta + \frac{2}{\rho}\frac{\partial}{\partial\rho}. \tag{V.35}$$

If now $\chi_+ = 0$, then it follows from eq. (V.33) that $\rho^2\chi'_{\rho\rho}$ can only have the form f(z). If we have $\eta = 0$, then f(z) only can be a third order polynomial (otherwise eq. (V.35) will not be satisfied). The stresses corresponding to f(z) follow from eq. (V.34) very simply $\sigma_{\varphi\varphi} = -\sigma_{zz} = (C_0 + C_1 z)\,\rho^2$; all other components vanish. If this relatively trivial state of stress does not occur in the body considered[1]

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[1]Otherwise we can subtract it anyhow from the total state. Most results of this section for the case $\underset{\sim}{\eta} = 0$ can be found in Marguerre [98].

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(which is the case for every convex body loaded with surface forces) the function $\chi'_+$ (and so $\chi'_-$) has the same value as $\chi'_{\rho\rho}$. Then we can calculate $\chi'_+$, and from this and by use of eq. (V.33) follows the formula

$$\chi'_{\varphi\varphi} = \frac{1}{\rho^2}\int \rho\,\chi'_+\,d\rho\,; \tag{V.36}$$

for calculating $\chi'_{\rho\rho}$, from which we get the stresses by differentiating twice according to eq. (V.34). If we initially calculate $\chi'_{\rho\rho}$, we must differentiate three times to get the stresses. We will not write down eqs. (V.34) in terms of $\chi'_{\varphi\varphi}$.

The presently unsolved problem is how to express most usefully the boundary conditions in $\chi'_{\rho\rho}$ or $\chi'_+$. Every biharmonic function $\chi'_{\rho\rho}$ or $\chi'_+$ represents a possible state of stress, thus the boundary value problem is twice harmonic. From this we may conclude that the actual family of the states of stress which are governed by the Love functions (for div $\underset{\sim}{\sigma} = 0$) can also be represented by $\chi'_{\rho\rho}$ or $\chi'_+$, respectively. Since these functions are more closely related to the stresses, the solution

of the boundary value problem with these functions may be of practical importance.

These considerations showed the variety of the possibilities which the stress function tensor offers, thus we can adjust them very well to actual problems. This is of special importance for using components of the stress function tensor which correspond to curvilinear coordinates. We chose right from the beginning the secondary condition $\nabla_i \chi'_{ij} = 0$, but there are many other possibilities which are presently almost unknown.

Now we will treat an application to circular dislocations. Such a dislocation may lie in the plane $z = 0$ with the center in the origin, and radius R.

The stress function field of this dislocation is mainly given by the integral

$$\oint x \, dL'_i \tag{V.37}$$

according to eq. (II.107) it can easily be shown that (V.37) has the form

$$\underset{\sim}{F}(\rho, z)\underset{\sim}{i}_{\varphi} \tag{V.38}$$

according to Franz and Kröner [53]

$$\underset{\sim}{F} = \frac{s^3}{3k}\,[2k'\underset{\sim}{K} - (2 - k^2)\underset{\sim}{E}] \tag{V.39}$$

where $\underset{\sim}{K} \equiv \underset{\sim}{K}(k)$ and $\underset{\sim}{E} \equiv \underset{\sim}{E}(k)$ are the complete elliptic integrals of the first and second kind. Furthermore,

$$k = \frac{4\rho R}{s^2}, \quad s^2 = z^2 + (R+\rho)^2, \quad k'^2 \equiv 1 - k^2. \tag{V.40}$$

From $\underset{\sim}{F}$ the stress functions of eq. (II.107) follow, as we can easily check:

$$\chi'_{\rho\rho} = \frac{b}{4\pi}\frac{\underset{\sim}{F}}{\rho}, \quad \chi'_{\varphi\varphi} = \frac{b}{4\pi}\frac{\partial \underset{\sim}{F}}{\partial \rho}; \tag{V.41}$$

the other components of $\chi'$ vanish. The equation only holds for the case that the Burgers vector (magnitude b) of the dislocation is directed in the z-direction, because only then is the problem rotationally symmetric.

Starting from (V.41) the author investigated an arrangement of parallel equidistant circular dislocations with the same radius [79]. This arrangement is very similar to the familiar current coil in technical electronics. If we assume that the coil is very long with respect to the radius R, then we can use the same approximations as with the current coil, then we get in the interior of the coil

$$\sigma_{\rho\rho} = \sigma_{\varphi\varphi} = \frac{-G}{m-1}\nu b, \quad \sigma_{zz} = \frac{-2Gm}{m-1}\nu b \tag{V.42}$$

and in the exterior

$$\sigma_{\rho\rho} = -\sigma_{\varphi\varphi} = \frac{-G}{m-1}\nu b\frac{R^2}{\rho^2}, \tag{V.42'}$$

($\nu$ = the number of windings). All the other components vanish with this approximation. The external state is exactly the state described above with $\chi'_+ = 0$. The energy per unit coil volume is found to be

$$e = \frac{mG}{m-1}\nu^2 b^2. \tag{V.43}$$

There is no difficulty in principle to solve this problem exactly by the use of elliptic integrals.[1]

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[1]The stress which occurs during shrinking a hollow cylinder onto a rigid cylinder with a slightly larger radius can be reduced to a

(Footnote Continued)
dislocation arrangement in the boundary surface. But these dislocations are directed in the z-direction according to eq. (I.77) and they have their Burgers vector in the $\varphi$-direction. Our problem corresponds to a welding of two cylinders like the above, however, the interior is elongated elastically with respect to the external one the z-direction.

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Calculations of this nature are of interest for certain problems in metal physics. The important facts we can see in the following problem: During the cold working of the important aluminum-copper alloys (Duraluminum) there are the following two states among others. We have the two-dimensional clustering of Cu-atoms in the $\{100\}$ plane of the Al-lattice, and most likely a one-atom thick layer of copper. These are distributed statistically in one case (called "Guinier-Preston zone I); in another a number of them are arranged in complexes (Guinier-Preston zone II).[1]

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[1]See, e.g., Gerold [59] or Hardy and Heal [63].

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We can describe the rearrangement of the Cu concentration thus: a partial lattice plane of atoms in the Al-lattice is replaced by Cu-atoms. Since the Cu layer is "thinner" there is a missing layer having a thickness equal to the difference between the lattice spacing of Al($d^{Al}$) and Cu($d^{Cu}$). However, the connection is maintained by the atomic cohesion forces and we get elastic reactions. The Cu-layer acts like a dislocation line with the Burgers vector $b = d^{Al} - d^{Cu}$ in the z-direction, if this is taken perpendicular to the layer. We will treat

this "quasidislocation" (it is not a complete edge dislocation since the lattice plane does not terminate) as approximately circular. In experiments it was noticed that (at least in certain temperature regions) they prefer to arrange themselves on top of each other like a "coil." Since the Burgers vectors are parallel we would expect the contrary to be more likely, for similar dislocations with parallel Burgers vectors repel according to section 18.

According to Franz and Kröner [53], the opposite effect can be explained as follows. One of the middle Cu-layers of the complex is bounded by a complete dislocation with Burgers vector opposite to the quasidislocations, i.e., this layer is not continued as an Al-plane externally. Since the Burgers vector of this dislocation is much larger than those of the quasidislocations, it can attract such a large number of quasidislocations the sum of all Burgers vectors of the complex is zero. By this also the long range stress field which requires energy is removed. The following is assumed today for the most probable arrangement of the layers in the Guinier-Preston zone II.

Every fourth lattice plane contains copper.[1] For the number of

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[1]If only every second atom in a layer is a Cu-atom, then the vertical size is about double. Experimentally this has not been proved at present.

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layers in one complex we obtain six [53]. The vertical arrangement of such a zone is 21 lattice planes which corresponds to slightly more than 40Å and which agrees sufficiently with experiments which are very difficult to explain otherwise.

We can treat the phase boundary between two pure metals A and B analogously. For simplicity we assume that the lattice parameter of A and B is only different in the z-direction and that the phase B is a circular cylinder in A with the axis in the z-direction. If the lattice parameter $d^B$ of B is smaller than that of A, then now and then a lattice plane of B must terminate, otherwise we would have the large energy of (V.43). Now we can represent each lattice plane of B as a quasidislocation with Burgers vector $d^A - d^B$. If, e.g., $d^A - d^B = d^B/5$, than after 5 lattice planes of B we must have a complete dislocation. Accordingly, we can describe a phase boundary by an arrangement of dislocations and quasidislocations, as shown in Fig. 38. While the dislocations and the quasidislocations are, considered by themselves, strong self stress sources (§23) together they act mainly as a surface cover of "dislocation dipoles" (or "incompatibility quadrupoles (§23)) so that their elastic effect and therefore its elastic energy is small. If we want to calculate it, then we would have to solve a boundary value problem with respect to the boundary surface and have to consider the different elastic constants in the interior and exterior.

Eqs. (V.43) are no longer sufficient to calculate the surplus energy during the transfer from the Guinier-Preston zone I to zone II, therefore Kröner and Franz [53] calculated exactly the interaction energy of two coaxial circular dislocations with Burgers vectors perpendicular to the dislocation-plane by using elliptic integrals according to eq. (II.128). Pfleiderer [117] obtained the interaction of circular dislocations more generally, also starting from eq. (II.128).

In the following we will summarize the results, some of which are surprisingly simple.

Let

$$\underset{\sim}{F}' = -\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2}\right)\underset{\sim}{F}$$

$$= -\frac{k^2}{2\rho\sqrt{\rho k}}\left\{2(\rho^2+R^2)\underset{\sim}{K} - \left[(\rho+R)^2 + \frac{(R-\rho)^2}{k'^2}\right]\underset{\sim}{E}\right\} \qquad (V.44)$$

$$\underset{\sim}{F}'' \equiv \frac{\partial^2 F}{\partial z^2} = 2\sqrt{(R\rho)}\left\{\left[\frac{2}{k}\left(1+\frac{z^2}{s^2}\right)-k\right]\underset{\sim}{K} - \left[\frac{2}{k}\left(1+\frac{z^2}{s^2}\right) + \frac{kz^2}{k'^2 s^2}\right]\underset{\sim}{E}\right\}. \qquad (V.44')$$

For two dislocations AB in the arrangement of Fig. 39, according to Pfleiderer, it holds that

$$E^{AB} = H_{11} + H_{22} + H_{33}, \qquad (V.45)$$

where

$$H_{11} = \frac{G}{8}\, b_1^A b_1^B \rho\left[-\left(\frac{3m-1}{m-1}\right)\underset{\sim}{F}'' + \underset{\sim}{F}'\right]; \quad H_{33} = \frac{-m}{2(m-1)}\, G b_3^A b_3^B \rho\, \underset{\sim}{F}'.$$

$H_{22}$ follows from $H_{11}$ by substituting $b_1^A b_1^B$ by $b_2^A b_2^B$, $\rho$ indicates the radius of the dislocation B, $b_i^A$ and $b_i^B$ are the Cartesian components of the Burgers vectors of the lines A and B.

The equations simplify greatly if both loops are in the plane $z(= x_3) = 0$ or if both have the same radius. We have then

$$H_{11}\Big|_{\rho=R} = \frac{G b_1^A b_1^B R}{2(m-1)k}\left\{\left[(1-4m)k^2 + 2(3m-1)\right]\underset{\sim}{K} - \left[-mk^2+2(3m-1)\right]\underset{\sim}{E}\right\},$$

$$H_{33}\Big|_{\rho=R} = \frac{mG}{m-1}\, b_3^A b_3^B R k\left[\underset{\sim}{K} - \underset{\sim}{E}\right],$$

$$H_{11}\Big|_{z=0} = \frac{2m-1}{2(m-1)}\, G b_1^A b_1^B (R+\rho)\left[\left(1-\frac{k^2}{2}\right)\underset{\sim}{K} - \underset{\sim}{E}\right],$$

$$H_{33}\Big|_{z=0} = \frac{m}{m-1} Gb_1^A b_1^B (R+\rho)[(1-\frac{k^2}{2})\underset{\sim}{K} - \underset{\sim}{E}]. \tag{V.47}$$

From eqs. (V.45,46) it follows that dislocations with perpendicular Burgers vectors do not interact. We obtained this result earlier in §18. For the problem discussed above, the energy of two dislocations with $\rho = R$ and Burgers vector (magnitude b) in the z-direction is important. We find it to be

$$\frac{m}{m-1} Gb^2 Rk(\underset{\sim}{K} - \underset{\sim}{E}). \tag{V.48}$$

For other uses a case of special interest is when both dislocations have their Burgers vector in the $x_1$-direction and are in the plane $z = 0$ (this is then slip plane, the corresponding equations can be used for calculating the energy of piled up dislocations). We also have then the very simple expression

$$\frac{2m-1}{2(m-1)} Gb^2 (R+\rho)[(1-\frac{k^2}{2})\underset{\sim}{K} - \underset{\sim}{E}]. \tag{V.49}$$

If we choose to make the distance between the dislocations very small, about twice the cut length $\epsilon$ in eq. (II.145), then we obtain the self energy in the approximation described in §18 and §25. The self energy of the dislocation in (V.49) was first obtained by Nabarro [110] in another way (starting with eq. (II.122). In this case $k \approx 1$ (if $\epsilon \ll R$), then we have the approximation $\underset{\sim}{E} \approx 1$, $\underset{\sim}{k} \approx \ln (4/k')$ [69] and we obtain for the self energy of this dislocation

$$\frac{2m-1}{2(m-1)} Gb^2 R(\ln \frac{4R}{\epsilon} - 2) \tag{V.50}$$

in agreement with Nabarro.

All these calculations were carried out with the assumption of an infinite medium. At large distances such a dislocation loop acts

like a force dipole, i.e., the long-range displacement field of the dislocation varies as $1/r^2$ ($r$ = distance to the origin), the stress field with $1/r^3$ and the energy density $\epsilon_{ij}\,\sigma_{ij}/2$ with $1/r^6$. The part of the energy in the infinite medium external to a sphere of radius $r_o$ varies as $1/r_o^3$. If the dimensions of the body are sufficiently large with respect to R, which is true in almost all applications, then the surface need not be considered, i.e., the above equations give the energy to practically the same approximation for the finite body also.

For these calculations the stress function tensor took part only indirectly (eq. (II.128) was derived with its help). Finally, we give equations derived from the stress function field $\underset{\sim}{\chi}'$ for a dislocation with the Burgers vector in the plane of the loop (slip plane) (Keller, [70]). Let $(x,y,z) \equiv (x_1,x_2,x_3)$. Then it holds that (see (V.40))

$$\left.\begin{aligned}
\sigma_{11} &= \alpha\,\frac{m}{m-1}\,(3B - 2C + Dx^2)xz \\
\sigma_{22} &= \alpha[2C + \frac{m}{m-1}\,(B - 2C + Dy^2)]xz \\
\sigma_{33} &= \alpha\,\frac{m}{m-1}\,(C + Fz^2)xz \\
\sigma_{12} &= \alpha[-C + \frac{m}{m-1}\,(B + Dx^2)]yz \\
\sigma_{23} &= \alpha[-B + \frac{m}{m-1}\,(B + Ez^2)]xy \\
\sigma_{31} &= \alpha\{A + By^2 + \frac{m}{m-1}\,[A + Cz^2 + (B + Ez^2)x^2]\}
\end{aligned}\right\} \qquad (V.51)$$

$$\left.\begin{aligned}
&A \equiv \frac{1}{\rho s k^2}\,[2\underset{\sim}{E} - (2-k^2)\underset{\sim}{K}]; \quad B \equiv -\frac{1}{2\rho^2}(a^2 C + 3A) \\
&C \equiv \frac{-3}{\rho s^6 k k'^2}\,\underset{\sim}{F}, \quad D \equiv -\frac{1}{\rho^2}\,(4B + C + Fz^2) \\
&E \equiv -\frac{1}{2\rho^2}\,(a^2 F + 5C), \quad F = -\frac{1}{\rho s^5 k'^4}\,(\rho s k'^2 A - 2k^2 \underset{\sim}{E}). \\
&\alpha = -RbG/\pi, \quad a^2 = z^2 + R^2 - \rho^2.
\end{aligned}\right\} \qquad \text{(V.51')}$$

Notice: The stresses are not rotationally symmetric, therefore the problem solved by Keller with use of the stress function tensor is really three-dimensional.

At the origin the only remaining stress is

$$\sigma_{31} = \frac{Gb}{4}\,\frac{2m-1}{m-1}\,\frac{1}{R}\,. \qquad \text{(V.52)}$$

# APPENDIX

## DECOMPOSITION OF THE 2nd RANK TENSOR FIELD

Partially we will use vector symbols according to Gibbs,[1] but normally we calculate with normal indices, including the Einstein summation convention,[2] i.e.,

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[1]This was recommended previously by the International Union for Pure and Appl. Phys. [67].

[2]This nomenclature is especially emphasized in the books of Duschek and Hochrainer.

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| | | |
|---|---|---|
| $\underset{\sim}{a}\,\underset{\sim}{b}$ or $a_i b_j$ is the dyadic | | products of two |
| $\underset{\sim}{a}\cdot\underset{\sim}{b}$ or $a_i b_i$ is the scalar | | vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ |
| $\underset{\sim}{a}\times\underset{\sim}{b}$ or $\epsilon_{ijk}\, a_j b_k$ is the vectorial | | |

where $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$; $\epsilon_{132} = \epsilon_{321} = \epsilon_{213} = -1$, while all the other components of the totally antisymmetric $\epsilon$-tensor vanish. The following equations will be used very frequently

$$\epsilon_{ijk}\,\epsilon^{\ell mn} = \begin{vmatrix} \delta_i^{\ell} & \delta_i^{m} & \delta_i^{n} \\ \delta_j^{\ell} & \delta_j^{m} & \delta_j^{n} \\ \delta_k^{\ell} & \delta_k^{m} & \delta_k^{n} \end{vmatrix} \tag{A.1}$$

From this it follows for n = k

$$\epsilon_{ijk}\,\epsilon^{imk} = \delta_i^{\ell}\,\delta_j^{m} - \delta_i^{m}\,\delta_j^{\ell} \tag{A.2}$$

and if additionally m = j, we have

$$\epsilon_{ijk}\,\epsilon^{ijk} = 2\delta_i^{\ell} \tag{A.3}$$

Let[1]

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[1]We will write the first letter with a capital to indicate that we are dealing with a tensor field.

---

$$\begin{aligned} &\text{Grad } \underset{\sim}{a} \equiv \nabla \underset{\sim}{a} \equiv (\nabla_i a_j) \\ &\text{Div } \underset{\sim}{\tau} \equiv \nabla\cdot\underset{\sim}{\tau} \equiv (\nabla_i \tau_{ij}) \\ &\text{Curl } \underset{\sim}{\tau} = \nabla \times \underset{\sim}{\tau} \equiv (\epsilon_{ijk}\,\nabla_j\,\tau_{k\ell}) \end{aligned} \tag{A.4}$$

(Read "gradient of $\underset{\sim}{a}$", "divergence of $\underset{\sim}{\tau}$", "curl of $\underset{\sim}{\tau}$".)

In an infinite medium each tensor field $\underset{\sim}{\tau}$ that vanishes at infinity can be uniquely decomposed according to the equation

$$\underset{\sim}{\tau} = \nabla \underset{\sim}{a} + \nabla \times \underset{\sim}{\alpha} \tag{A.5}$$

where $\underset{\sim}{\alpha} \equiv (\alpha_{ij})$.

Also for an arbitrary tensor $\underset{\sim}{\alpha}$ the unique decomposition holds

$$\underset{\sim}{\alpha} = \underset{\sim}{b}\nabla + \underset{\sim}{\beta} \times \nabla \tag{A.6}$$

with $\underset{\sim}{\beta} \equiv (\beta_{ij})$. This substituted into (A.5) with $\nabla \times \underset{\sim}{b} \equiv \underset{\sim}{c}$ (i.e., div $\underset{\sim}{c} = 0$) yields

$$\underset{\sim}{\tau} = \nabla \underset{\sim}{a} + \underset{\sim}{c}\nabla + \nabla \times \underset{\sim}{\beta} \times \nabla \tag{A.7}$$

where

$$(\nabla \times \underset{\sim}{\beta} \times \nabla)_{i\ell} \equiv \epsilon_{ijk}\,\epsilon_{\ell mn}\,\nabla_j\,\nabla_n\,\beta_{km}. \tag{A.8}$$

Symbolically we write for this also[1]

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[1]The name should remind us that Inc $\underset{\sim}{\epsilon} = 0$ are the incompatibility conditions of de St. Venant. These are satisfied if "the incompatibility of $\epsilon$" vanishes.

---

$$(\text{Inc } \beta)_{i\ell} \tag{A.9}$$

(read "incompatibility of $\underset{\sim}{\beta}$"). If $\underset{\sim}{\beta}$ is symmetric, then we can interchange i and $\ell$ in (A.8), thus

$$\nabla \times \underset{\sim}{\beta}^S \times \nabla = (\nabla \times \underset{\sim}{\beta} \times \nabla)^S \tag{A.10}$$

where S indicates that only the symmetric part is considered. If $\underset{\sim}{\beta}$ is antisymmetric, then we can interchange i and $\ell$ in (A.8) by changing the sign, thus

$$\nabla \times \underset{\sim}{\beta}^A \times \nabla = (\nabla \times \underset{\sim}{\beta} \times \nabla)^A \tag{A.11}$$

where A means "the antisymmetric part of."

Accordingly, if we write $\underset{\sim}{a} + \underset{\sim}{c} \equiv \underset{\sim}{g}$, $\underset{\sim}{a} - \underset{\sim}{c} = \underset{\sim}{h}$, the symmetric part of eq. (A.7) reads

$$\underset{\sim}{\tau}^S = \frac{1}{2}(\nabla \underset{\sim}{g} + \underset{\sim}{g}\nabla) + \nabla \times \underset{\sim}{\beta}^S \times \nabla \tag{A.12}$$

and its antisymmetric part

$$\underset{\sim}{\tau}^A = \frac{1}{2}(\nabla \underset{\sim}{h} - \underset{\sim}{h}\nabla) + \nabla \times \underset{\sim}{\beta}^A \times \nabla. \tag{A.13}$$

Symbolically we can also write eq. (A.12)[2]

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[2]For Def read "deformation of". The name should remind us that $\underset{\sim}{\epsilon}$ = Def $\underset{\sim}{s}$ is the relation between the deformation $\underset{\sim}{\epsilon}$ and the displacement field $\underset{\sim}{s}$ ([52], Vol. I, pg. 97).

---

$$\underset{\sim}{\tau}^S = \text{Def}\ \underset{\sim}{g} + \text{Inc}\ \underset{\sim}{\beta}^S. \tag{A.14}$$

Since (A.5) and (A.6) are unique decompositions, then (A.14) is also a unique decomposition of a symmetric tensor field. The following relation can be easily verified

$$\begin{aligned} &\text{Inc Def} \equiv 0 \\ &\text{Div Inc} \equiv 0 \end{aligned} \tag{A.15}$$

This says that a tensor $\underset{\sim}{\tau}^S$ which satisfies the condition div $\underset{\sim}{\tau}^S = 0$, is an incompatibility tensor, while it is a deformator (which can be derived from a vector field) if Inc $\underset{\sim}{\tau}^S = 0$. The importance of the operations Inc and Def for the theory of elasticity is that the state of a body loaded only on the boundary is completely determined elastically by the equations

$$\text{Inc}\ \underset{\sim}{\varepsilon} = 0, \qquad \text{Div}\ \underset{\sim}{\sigma} = 0 \tag{A.16}$$

where we additionally use Hooke's Law and the equation of the elastic energy density.

In eq. (A.13), we can replace $\underset{\sim}{\beta}^A$ (as for each antisymmetric tensor [34]) by the equivalent vector according to

$$\beta^A_{ij} = \varepsilon_{ijk}\ \beta^A_k, \qquad \beta^A_k = \frac{1}{2}\ \varepsilon_{ijk}\ \beta^A_{ij}. \tag{A.17}$$

Accordingly after simple calculations, it follows that

$$\tau^A_{ij} = \varepsilon_{ijk}(\varepsilon_{k\ell m}\ \nabla_\ell\ h_m + \nabla_k\ \lambda), \quad \lambda \equiv -\ \nabla_i\ \beta^A_i, \tag{A.18}$$

or corresponding to eq. (A.17)

$$\tau^A_k = \varepsilon_{k\ell m}\ \nabla_\ell\ h_m + \nabla_k\ \lambda \equiv (\text{Curl}\ \underset{\sim}{h} + \text{Grad}\ \lambda)_k. \tag{A.19}$$

I.e., the decomposition of the antisymmetric tensor field corresponds to the familiar decomposition of the related tensor field into a source and vortex field.

In eq. (A.5) we can add a gradient tensor to $\underset{\sim}{\alpha}$ without changing $\underset{\sim}{\tau}$.[1] Similarly we can add a deformator to $\underset{\sim}{\beta}^S$ in eq. (A.14)

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[1]Evidently the identities Curl Grad $\equiv$ 0, Div Curl $\equiv$ 0 hold.

---

without changing $\underset{\sim}{\tau}^S$. Therefore we can restrict $\underset{\sim}{\alpha}$ in eq. (A.5) and $\underset{\sim}{\beta}^S$ in eq. (A.14), respectively by certain secondary conditions. E.g., Div $\underset{\sim}{\alpha} = 0$ and Div $\underset{\sim}{\beta}^S = 0$ are always allowed conditions; i.e., we can still represent every arbitrary $\underset{\sim}{\tau}$ and $\underset{\sim}{\tau}^S$ by eq. (A.5) or eq. (A.14), respectively, if $\underset{\sim}{\alpha}$ and $\underset{\sim}{\beta}^S$ are governed by the restrictions mentioned [77]. If $\underset{\sim}{\chi}$ and $\underset{\sim}{q}$ are the incompatibilities or sources of $\underset{\sim}{\tau}^S$, respectively, then as we can easily check, we obtain from eq. (A.14) in the case Div $\underset{\sim}{\beta}^S = 0$

$$\text{Inc}\ \underset{\sim}{\tau}^S = \text{Inc Inc}\ \underset{\sim}{\beta}^S = \Delta\Delta\underset{\sim}{\beta}^S = \underset{\sim}{\chi}. \tag{A.20}$$

Hence it follows that $\underset{\sim}{\beta}^S$ is determined by

$$\underset{\sim}{\beta}^S = -\frac{1}{8\pi}\iiint_{\infty} \underset{\sim}{\chi}(\underset{\sim}{x}')\ |\underset{\sim}{x}'-\underset{\sim}{x}'|\,dV' \tag{A.21}$$

uniquely, apart from an unimportant linear function of $\underset{\sim}{x}$.[2]

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[2]We recall that $\underset{\sim}{\tau}^S$ should vanish at infinity. We can easily check that (A.21) satisfies this secondary condition.

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On the other hand, it follows from eq. (A.14)

$$\mathrm{Div}\ \underset{\sim}{\tau}^{S} = (\Delta \underset{\sim}{g} + \nabla\nabla\cdot \underset{\sim}{g})/2 = \underset{\sim}{q}. \tag{A.22}$$

By taking the divergence again, we obtain

$$\Delta\ \mathrm{div}\ \underset{\sim}{g} = \mathrm{div}\ \underset{\sim}{q} \tag{A.23}$$

from which we obtain div $\underset{\sim}{g}$, apart from a constant. Afterwards we can easily get $\underset{\sim}{g}$ from eq. (A.22) up to an arbitrary constant. Thus we have showed how the decomposition (A.14) is carried out in reality in an infinite body.

## ADDITION

We will add some theorems concerning media only containing self stresses.

1. It holds for arbitrary elastic homogeneity and anisotropy that

$$\iiint_V \sigma_{ij}\ dV = 0$$

integrated over the whole volume (in the state of self stress).

2. The total volume change of the medium is

$$\Delta V = t_{ijk\ell} \iiint_V \sigma_{ij}\ \sigma_{k\ell}\ dV + \text{higher order terms}$$

for a nonlinear elasticity law with the material constants

$$t_{ijk\ell} = \frac{1}{2} \frac{\partial^2 \Theta}{\partial\sigma_{ij}\partial\sigma_{k\ell}} \bigg|_{\sigma=0} .$$

Theorem 1 follows from equilibrium considerations [179]. Theorem 2 follows with theorem 1 if we expand the differential volume change $\theta$ in powers of $\sigma_{ij}$. Theorem 2 was found and examined by Zener [178] for elastic isotropy in a slightly different form more adaptable for comparison with existing experiments. Seeger [176] expanded it to cubic crystal systems and applied it to dislocations. The tensor $t_{ijk\ell}$ has the same symmetry and number of components as the elasticity tensor $c_{ijk\ell}$ of the related medium.

The "volume theorem" of Colonnetti mentioned in section 1 follows from theorem 1 by applying Hooke's Law.

## ILLUSTRATIONS

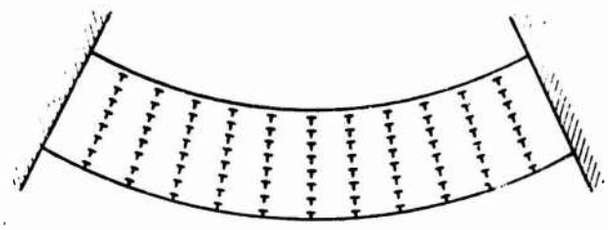

Fig. 1. The symbol ⊤ represents edge dislocation. It appears the first time in §23.

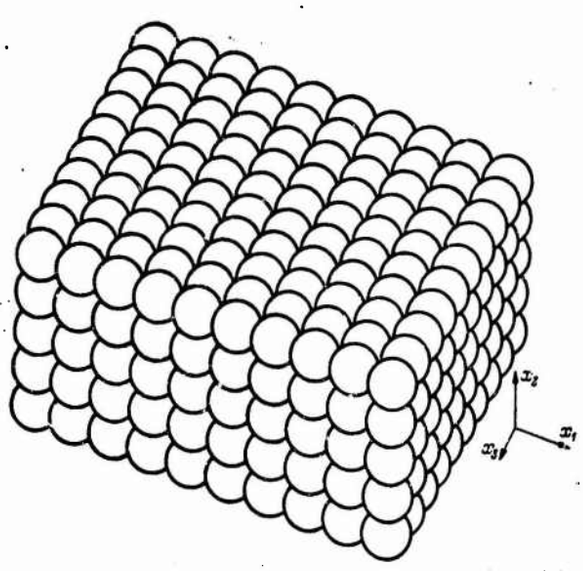


Fig. 2. Ideal crystal, cubic primitive lattice.

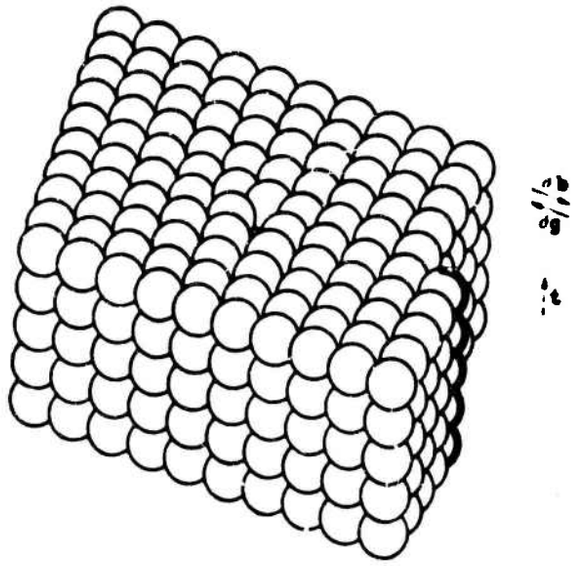

Fig. 3. The crystal in Fig. 2 after an invasion of edge dislocation from the $x_1$-direction.

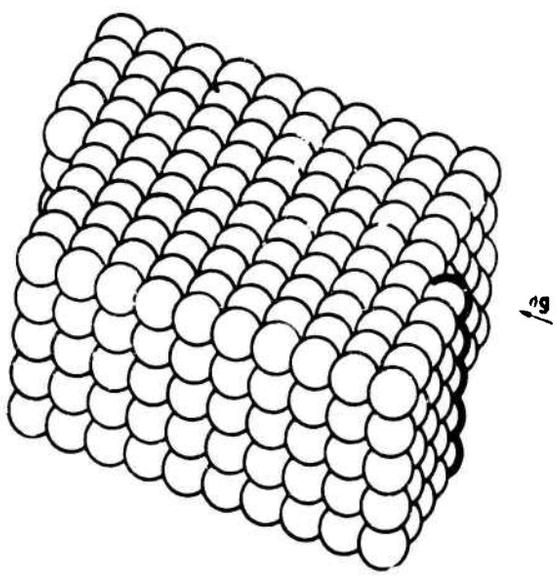

Fig. 4. The edge dislocation in Fig. 3 has moved outside the crystal in the $x_1$-direction.

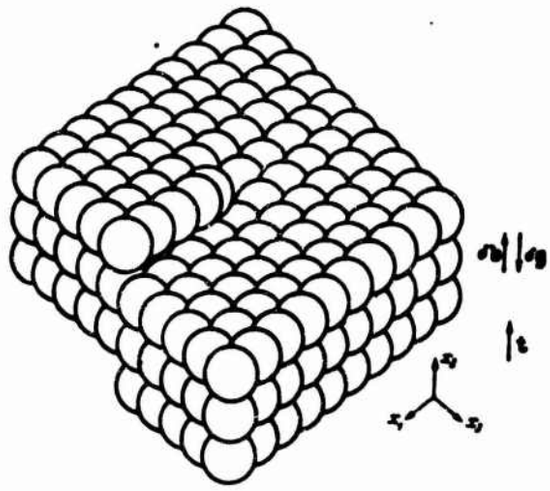

Fig. 5. The top lattice plane of the crystal in Fig. 2 after an invasion of a screw dislocation from the $x_1$-direction.

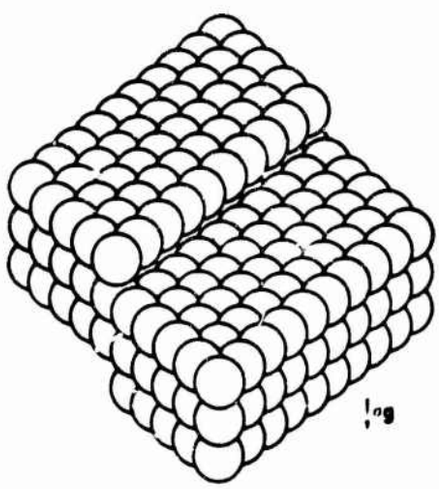

Fig. 6. The screw dislocation in Fig. 5 has moved outside the crystal in the $x_1$-direction.

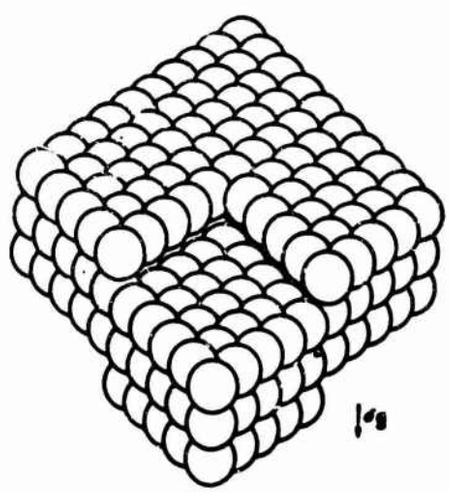

Fig. 7. The screw dislocation in Fig. 5 has moved outside the crystal in the $x_3$-direction.

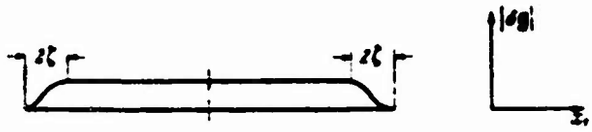

Fig. 8. The formation of a dislocation in a continuum.

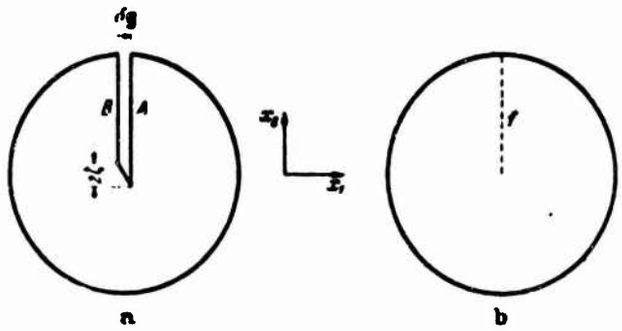


Fig. 9. The formation of a straight edge dislocation in a continuum. One visualizes that the slit in (a) is formed by the removal of material from a complete cylinder.

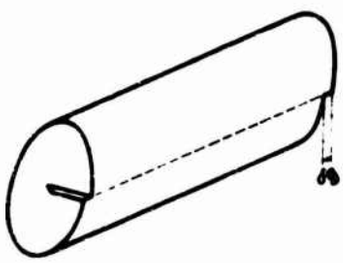

Fig. 10. The formation of a screw dislocation in a continuum.

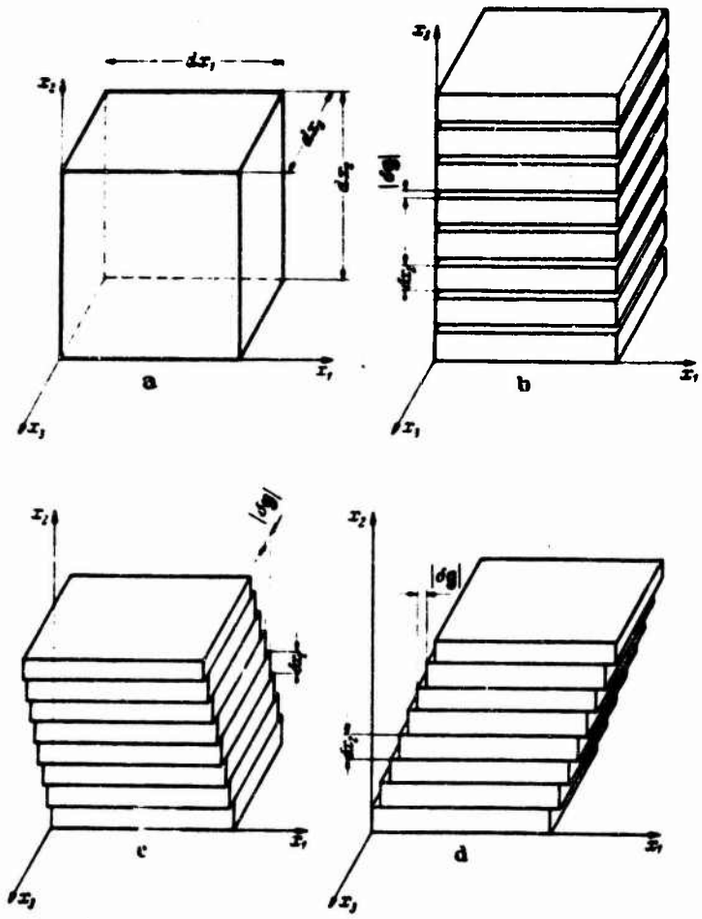


Fig. 11. The definition of macroscopic tensors of plastic distortions.

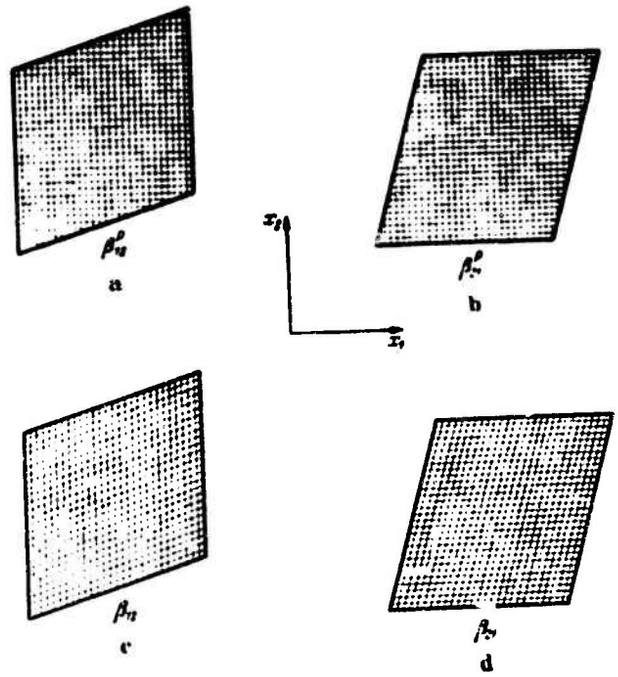


Fig. 12. (a,b) Plastic distortion preserves the original orientation. (c,d) Elastic distortion will in general twist the original orientation.

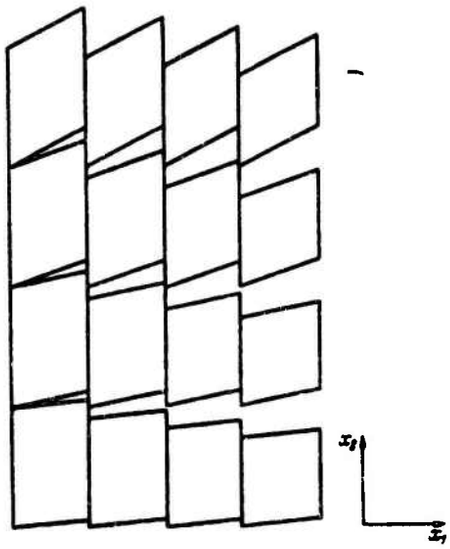

Fig. 13. A plastic distortion, which is accomplished without simultaneous elastic distortion, destroys in general the continuity of the body.

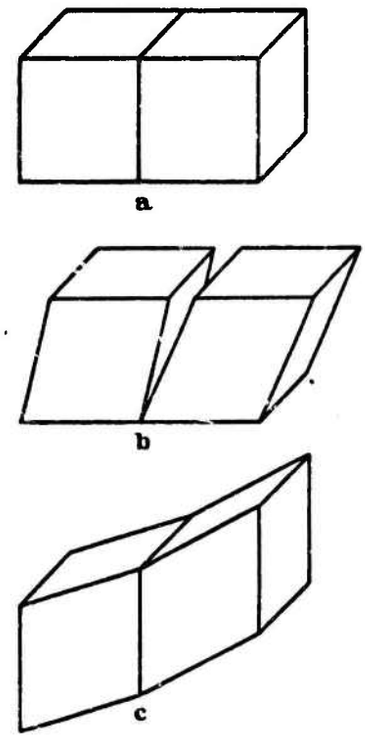


Fig. 14. Coordinate system as in Fig. 13.

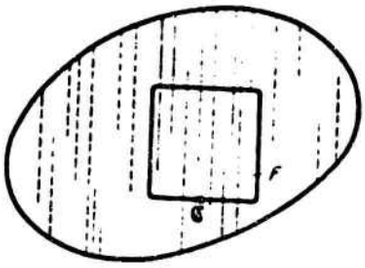


Fig. 15.

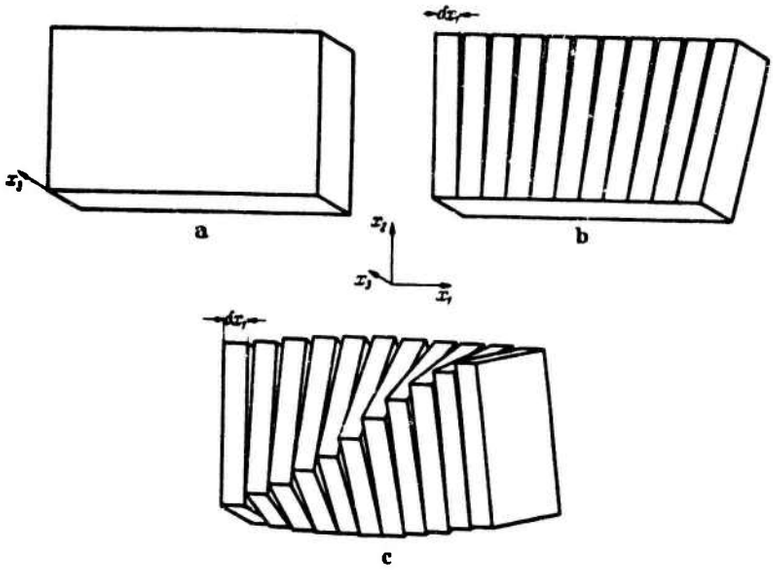


Fig. 16. Between each two layers of $\delta x_1$ in b and c, there is a dislocation wall of constant intensity. The dislocation moves from the right.

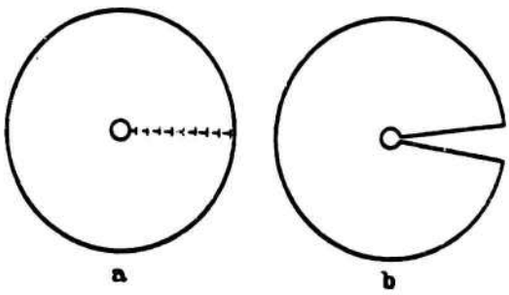


Fig. 17. The formation of a Volterra's distortion of the 2nd kind.

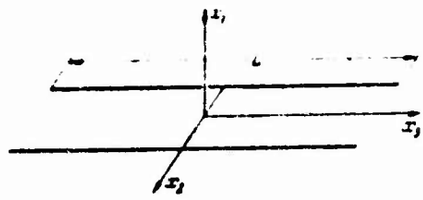


Fig. 18.

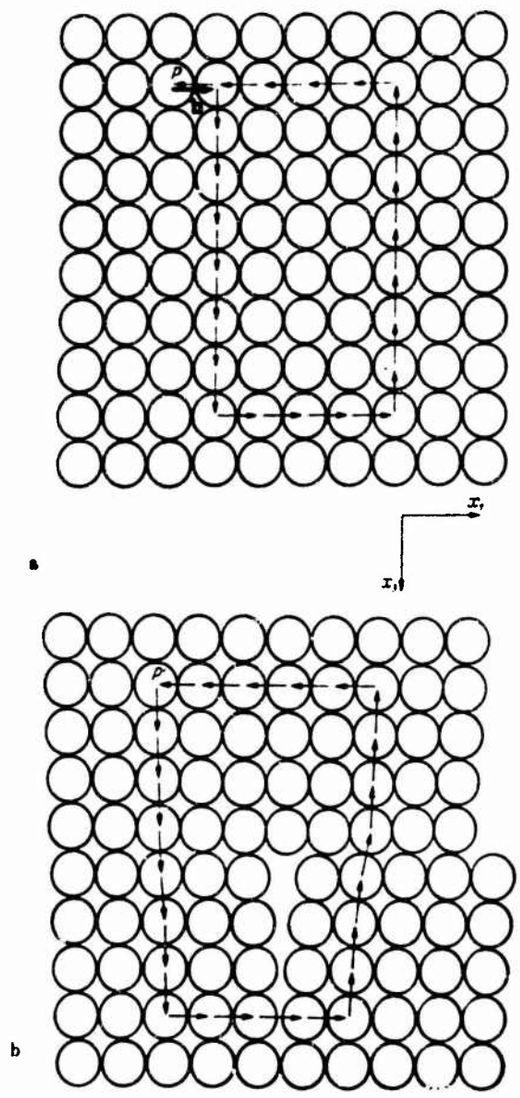


Fig. 19. The illustration of Frank-Burger's circuit. The letter P belongs to the further right atom.

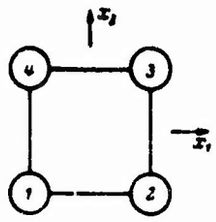


Fig. 20. Plane crystal formed by 4 atoms. The atom pairs 1,3 and 2,4 are not neighboring atoms.

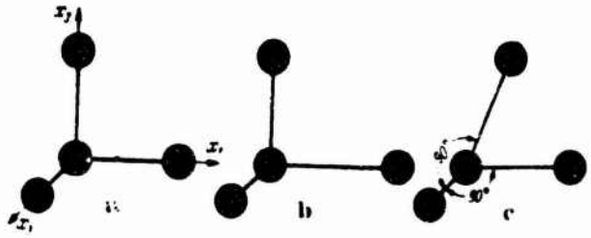


Fig. 21. The definition of microscopic distortion tensors.

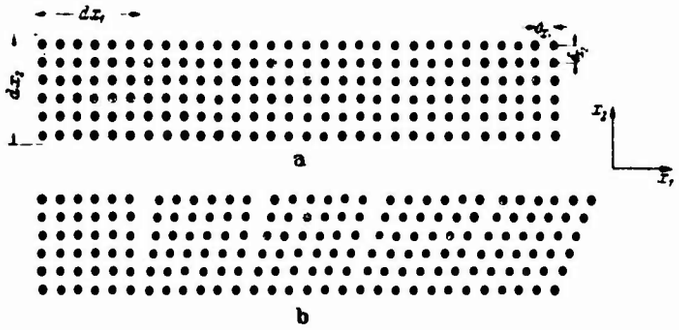


Fig. 22.

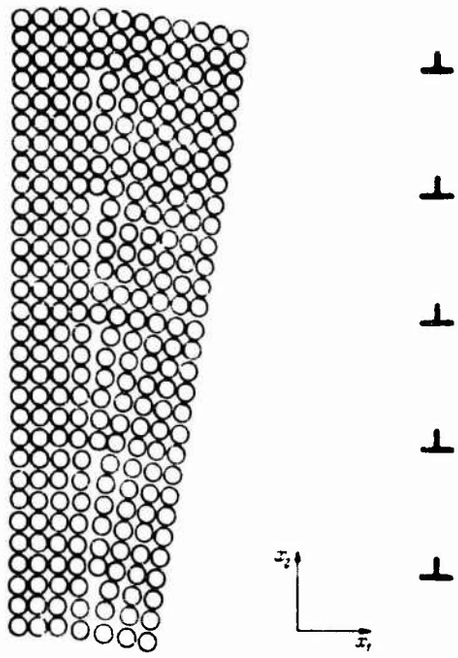


Fig. 23. Grain boundary of the 1st kind. The orientation variation between the adjacent grains is $|\mathbf{g}|/d$, where d is the distance between dislocation lines. This follows from Eq. (III.32).

Fig. 24.

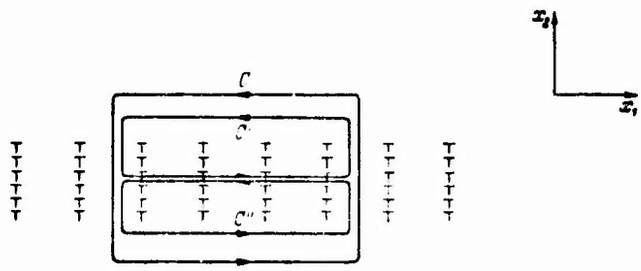


Fig. 25. Dislocation wall as plane-clamped incompatibility dipole.

Fig 26 Face-centered cubic crystal, from Jagodzinski [68].

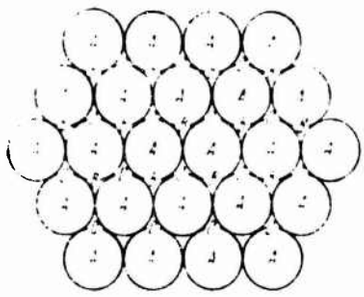

Fig. 27 The most dense lattice plane, from Seeger [134].

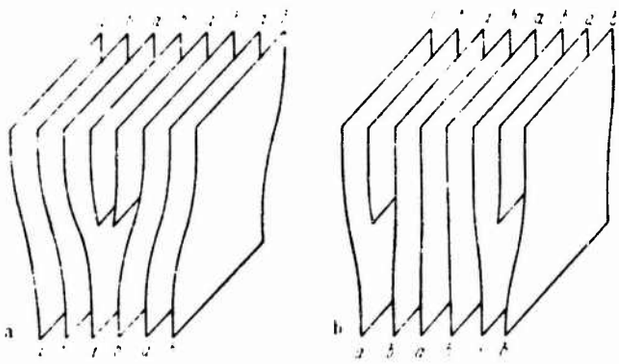

Fig. 28. (a) Schematic representation of an edge dislocation in the face-centered cubic lattice (b) This splits under the formation of a stacking fault into two partial dislocations. With the mark ababab .. it is indicated that the sketched ⟨110⟩ -planes represent a double layer. From Seeger [134].

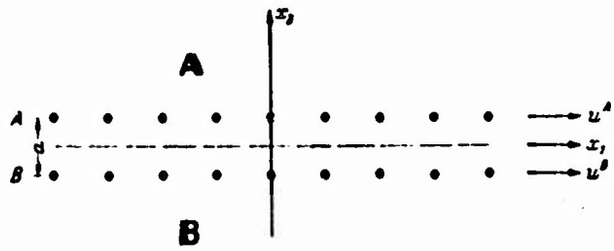


Fig. 29. The illustration of Peierls' model. The lattice planes A and B (perpendicular to the paper) seperate the two half-spaces **A** and **B**.

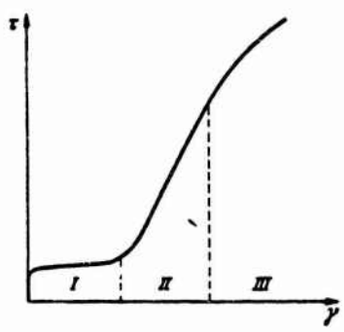


Fig. 30. Typical work hardening of a face-centered cubic metal (e.g.Cu). In the elastic range the curve practically coincides with the t-axis on our scale.

a b

Fig. 31. Model for plastic elongation of a rod. From Schmidt-Boas [129].

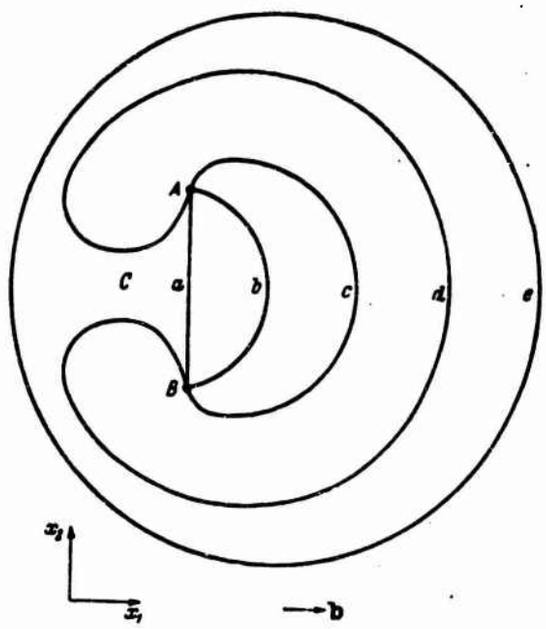


Fig. 32. The construction of a dislocation ring with the help of Frank-Read-Mechanism.

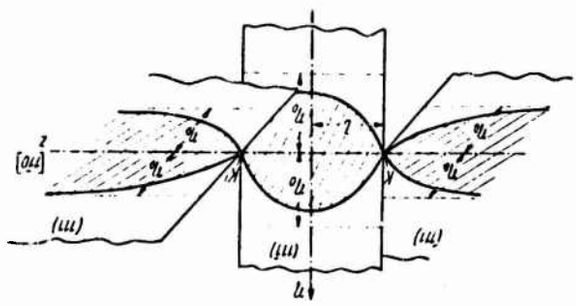

Fig. 33. Cross split of a screw dislocation. Stacking fault is shown by hatched area. Here $z = x_3$. From Seeger [134].

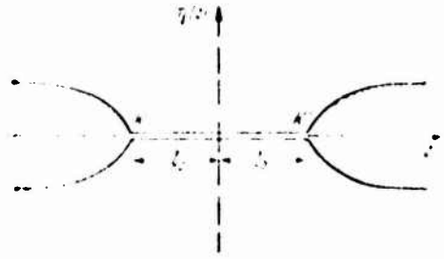

Fig. 34. The calculation of activation energy for the cross split in a screw dislocation. The distance of extension is exactly 2 $\eta_0$. From Seeger [134].

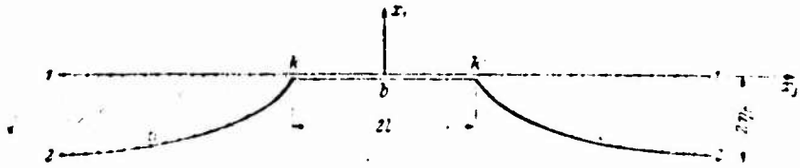

Fig. 35. A simpler model for the calculation of activation energy for the cross split.

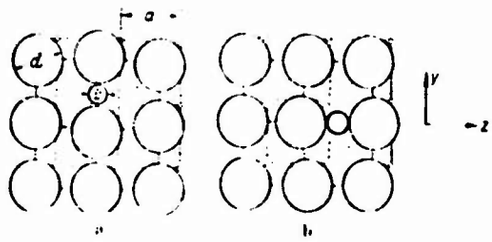

Fig. 36. (a) Interlattice atom in the body-centered cubic crystal, along with model for carbon in iron. Only the front atoms are sketched clearly in whole size. For d = a 3/2 each atom contacts its eight closest neighbors. (b) The same as (a) after changing the position of interlattice atom. Here $y = x_2$ and $z = x_1$.

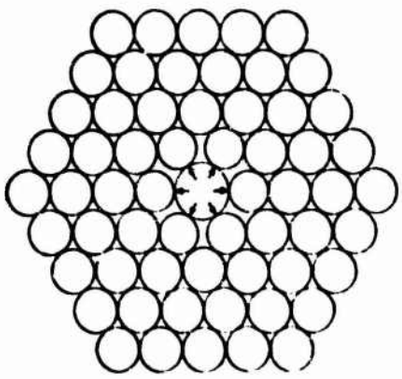

Fig. 37. Substitutional foreign atom in the most dense plane. The atomic array is distorted a little.

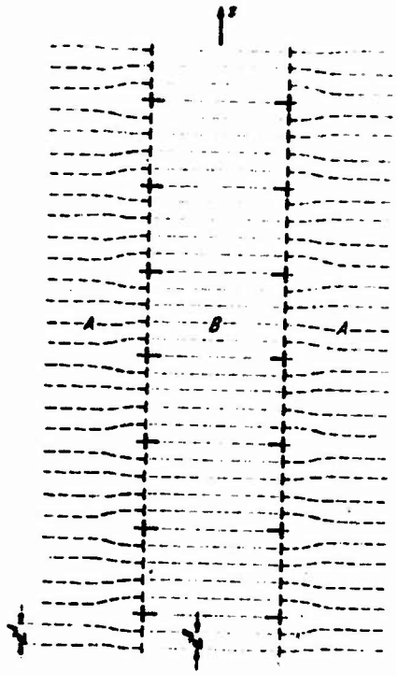


Fig. 38. Phase boundary as plane-clamped dislocation array. Dashed line is lattice plane of longitudinal section.

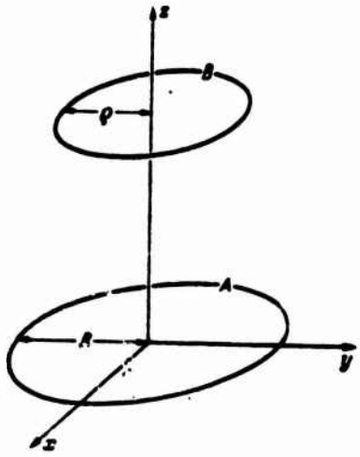


Fig. 39. The calculation of interaction energy between circular dislocations.

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The theory of dislocations in a continuum is developed. Methods are presented for solving boundary value problems for stresses and displacements due to distributions of dislocations. Examples are presented of the application of the theory to crystalline materials.

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