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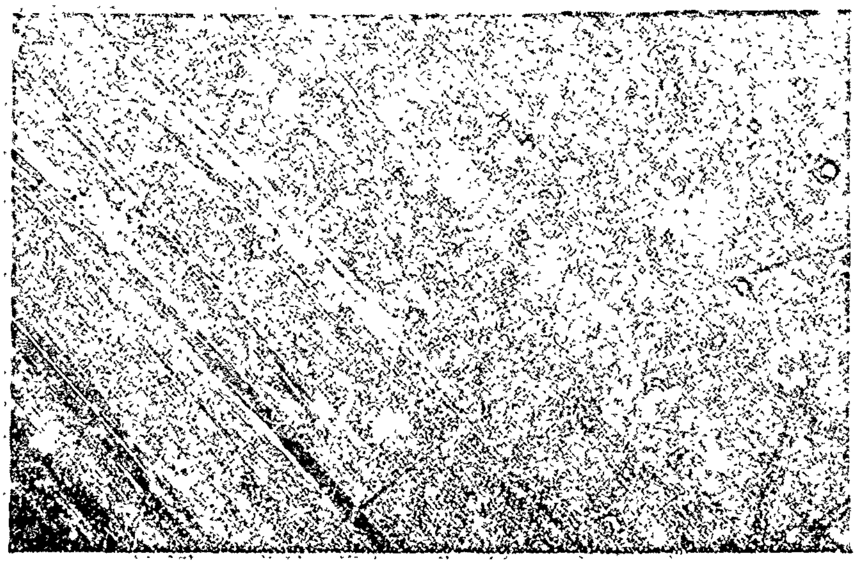

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The Application of Game Theory to ASW Detection Problems

Prepared for
Office of Naval Research

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30 September 1967

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I. INTRODUCTION

This document constitutes MATHEMATICA's Final Report on project work undertaken for the Office of Naval Research under contract N00014-66-C-0215. The central aim of the research undertaken in this project was to investigate the application of mathematical game theory to ASW detection problems.

The major findings, conclusions, and recommendations which have resulted from this project are described in Chapter II, "Game-Theoretic Models for ASW", in non-technical language. Chapter III, "Game-Theoretic Analyses of ASW Problems", consists of a series of technical papers examining various facets of ASW problems and models.

The work on this project was performed by the following members of MATHEMATICA's staff: Norman I. Agin, Michel L. Balinski, Harold W. Kuhn, John P. Mayberry, and Francis M. Sand.

II. Game-Theoretic Models for ASW

II. 1 Construction of Game-Theoretic Models

The basic aim in the research undertaken thus far has been to introduce strategic choices into the problem of submarine-submarine detection and to introduce them in a way that can be analyzed by the mathematics of game theory. This research has consisted of a sequence of models which have incorporated successively more strategic features and more parameters designed to better represent the real situation.

To understand the basic structure of these models, it is useful to recall the underlying theory in which they are set, namely, zero-sum two-person game theory. This theory of conflict deals with situations in which two opposing parties make strategic choices that control their actions throughout a particular contest. The rules of the game define precisely the strategies available to each of the opposing parties, and determine the outcome for each player, when a specific strategy is chosen by each party. This outcome is measured by a numerical payoff to each player as a result of the contest and which is a function of the strategies chosen by the two parties to the conflict. If there are but two parties to the conflict and if what one wins the other loses, then the game is called zero-sum two-person and assumes a particularly simple formal structure. Namely, the game may be described by the sets X and Y of strategies available respectively to the first player and to the second player, and a real-valued function $f(x, y)$ defined for every choice of $x \in X$ and $y \in Y$. (Conventionally, f is taken to be the payoff paid to the first player by the second player.)

In various applications the strategy sets X and Y take on different forms. For instance, they may be chosen to be finite discrete sets -- in which case, the payoff function is a matrix with real entries. In other cases, they may be taken to be all probability mixtures of a finite set of distinct elements -- this is the familiar case of "mixed strategies" for finite zero-sum two-person games. In other instances, the sets X and Y may take a structure dictated by the essential features of the context under study. For example, if the second player is a transiting submarine crossing a rectangular barrier, Y may consist of all possible speeds at which he may travel, if we believe that this is the only relevant parameter in the problem.

The structure of the models studied in our research may now be explained within this framework. First, and very important in its consequences, all of our analyses have been carried out with the payoff function giving the <u>probability of first detection.</u> (Here and throughout the reports the <u>first</u>

player is a submarine patrolling the barrier and the second player is a submarine transiting the barrier.) Whatever the strategy spaces X and Y used for the patroller and the transitor, respectively, the first step in fully defining the game has been to compute the probability of first detection if the patroller chooses $x \in X$ and the transitor chooses $y \in Y$.

A second important feature of the analysis is the choice of a solution concept for the games after they are defined. Two distinct approaches have been adopted. The first is a technique which has been used in situations which are either too complicated to permit a solution with strategic choices on both sides or have a "reasonable" fixed strategy for one of the players. (An example of the latter is provided by a "house" strategy for card games such as Blackjack.) Precisely, if we fix the strategy of the second player to be $\bar{y} \in Y$, the problem of the first player becomes simply: Find $x = \bar{x}$ so as to maximize $f(x, \bar{y})$. If we were certain that the second player would use $\bar{y}$ then the first player surely can do no better than play $\bar{x}$.

The second solution concept which has been used extensively in our analyses has been that of a largest assured payoff or, somewhat more technically, of a maximin strategy. The motivation of this is clear. For each (patrol) strategy

x there is a counter (transit) strategy y that minimizes the probability of first detection. The value of this probability is given by $\min_y f(x, y)$; it is the worst that the first player can gain if he plays x. He then chooses x so as to maximize this probability, giving him a probability of first detection equal to

$$\max_x \ \min_y \ f(x, y) .$$

This probability is a "sure thing" for him; he may do better if the second player does not counter him optimally but on the average he will do at least as well. It is also true that this is the highest probability of first detection that he can assure himself.

II. 2 Analysis of Models: A Summary of Results

We are now in a position to describe how the sequence of models that have been analyzed has been built up. The first models were studied with the first solution concept. Namely, various classes of patrol strategies X were played against an essentially fixed transit strategy $\bar{y}$. This technique is appropriate to the optimization of certain key parameters in the set X . It is well suited to such questions as: What is the optimal angle for a bow-tie patrol against transits in one direction? This technique of simple optimization against a fixed opponent's strategy (which may be a probability mix of a class of strategies) is close in spirit to the kinds of questions posed in gaming and simulation approaches to the submarine-submarine problem.

What kinds of strategy spaces X and Y have been studied? For the patrol submarine, the emphasis has been largely either on the pattern of patrol (varying the parameters of the pattern) or on the speed of the patrol (keeping the pattern essentially fixed). For the transiting submarine the emphasis has been largely placed either on the location of a straight line transit through the barrier or on the speed of the transit. Each choice of a strategy space leads to technical problems peculiar to that choice and hence to assumptions designed to make the analysis more tractable. (For example, if we are comparing patrol patterns, assumptions can be made to make the detection depend only on the closest approach distance of the two submarines.) With these assumptions, the following results were obtained (using maximin probability of detection as the principal solution concept).

(1) A basic proposition which has been widely ignored in ASW analyses is that there can be no sea exercise, no simulation, and no game theoretic analysis without the definite specification of three factors:

(a) the information available to the participants as to the context of the action;

(b) the technological possibilities (such as speeds, maneuverability, and detection capabilities) available to the participants;

(c) the objectives of the participants in terms of quantified measures or payoffs.

In Chapter III of this report, "Game-Theoretic Analyses of ASW Problems," the effects of incorrect specification of these factors are discussed and illustrated by examples. It is shown that, individually, each of these factors, if incorrectly formulated, can lead to solutions which are seriously misleading. Specifically:

(2) The analysis of speed games having an objective of secure detection was compared with the simpler one of

detection, and various approaches were suggested for the solution of these more complex games. Most of the approaches led to the same mathematical model: the "Difference Game", which is partially solved in Chapter III, Section 3 of this report. Optimal strategies require both the patroller and the transitor to use slow speeds.

(3) Extending the previous MATHEMATICA work on matrix games, a new series of analyses has been undertaken in which different range laws are compared. This leads to a more general definition of Secure Sweep Width covering the minimax strategies in a competitive situation. The previous definition was applicable only for "games against nature", i.e., situations in which the transitor chose at random.

(4) Continuous payoff functions representing range laws analogous to those studied in the matrix games were introduced, and it was shown that optimal mixed strategies retained essentially the same character as in the corresponding matrix games. In particular, the optimal strategies required only a finite number of points of entry (for the transitor) and points of defense (for the patroller).

(5) A repeated game with incomplete information on the transitor's side was analyzed. This was an attempt to model the learning aspects of the detection-evasion situation. It led to the result that the game should be played (i.e., the

patrol and transit strategies chosen) as if all payoffs were the average of what the transitor could reasonably expect, given his incomplete information. This type of analysis has the potential of providing an important link between information and strategy in repetitive situations.

II. 3 The Problem of Secure Detection vs Detection

The patroller's mission is in the first place to detect the transitors. Other events may then follow: localization and classification in peacetime (possibly also trailing); finally, approach and kill in wartime. But there is a considerable difference between detection and secure detection in either peace or war. Secure detection means that the patroller is not detected by the transitor in the process of detecting the transitor. The advantages of secure detection are manifold; in wartime it may spell survival, while in peacetime the information on enemy submarine movements is gained without revealing our submarine movements to the enemy.

A game-theoretic model which uses the concept of secure detection in its definition of objective is likely to differ substantially from a model based either on detection or on evasion. Not only is the measure of effectiveness different, but the optimal strategies also turn out to be quite different. In particular, optimal patrol patterns have been found in simple games of secure detection which are quite different compared to the previous one-way street of detection only. Unfortunately the mathematical difficulties of analyzing secure detection probabilities are rather more severe than in the case of detection probabilities. We have considered a compromise model which approximates the probability of secure detection by a weighted combination of the two probabilities: (i) that the patroller detects the transitor, and (ii) that the patroller evades detection by the transitor. In a simple analysis of patrolling strategies using only the speed variable it has been found that for purposes of maximizing detection probabilities the patrolling submarine should be moving at maximal speed for a fraction of the patrol cycle, while for purposes of evasion he should never exceed that speed at which self-noise is known to take a sharp upward turn as a function of speed. What recommendation will the game-theoretic analyst make if the mission emphasizes both detection of transitors and evasion? The answer will in fact depend on the relative importance of the opposing parts of the objective.

By examining a hypothetical case in which it is assumed that there is zero probability of a counterdetection following detection, we show in Chapter III, Section 1, that the compromise objective function is the difference between the probabilities of detection and counterdetection. This model is called "The Difference Game" and a generalization of it allows a utility weight to be applied to evasion of counterdetection. In Chapter III, Section 3, the solution is found for all cases where evasion is rated at least as important as detection. The solution requires both the patroller and the transitor to move at their quiet speeds for optimization with respect to the same weighted difference function. It should be pointed out that the assumptions for the transitor are the same as for the patroller but with signs reversed. The transitor, in other words, is emphasizing detection of the patroller at least as much as evasion of detection. The earlier analyses of speed games with simple detection probability objective functions (see refs [5] and [6])* showed that optimal transit speed according to the model could be faster than quiet speed, but ignored the possibility that the resulting secure sweep width for the transitor might be less than for the patroller. In some cases (depending on ratio of noise function slopes) this actually did occur, invalidating the solution on logical grounds. The present analyses, although they do not solve the cases where the patroller rates detection more important than evasion, correct the error for an important class of game-theoretic models including the "equal importance" case which is represented by an unweighted difference of detection probabilities.

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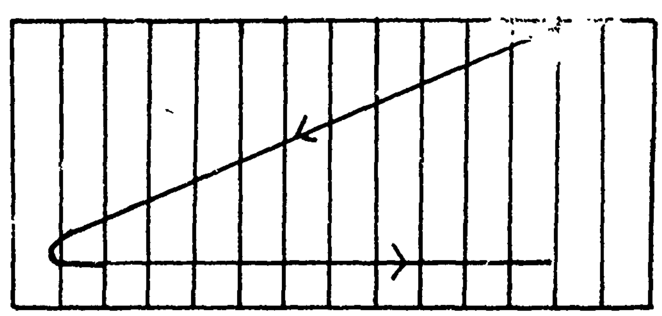
* References are to the Bibliography on p. 109.

II. 4 The Analysis of Position of the Patroller within a Zone (Matrix Games)

By keeping speed, other strategic variables and exogenous parameters constant, it is possible to analyse the relationship between detection range laws and the choice of patrol positions (and transit lanes). The zone may be divided into an indefinitely large number of lanes; by occupying one or more of these lanes the patroller is said to take a position within the barrier zone.



n transit lanes

Figure II. 1

No question of patrol path configuration in two dimensions is involved. Each patrol path is evaluated only in terms of the fraction of patrol cycle time spent in each position. In ref. [6], MATHEMATICA's previous analysis of the position strategies, two types of strategy were distinguished.

(i) The on-station strategy. The patroller occupies a lane and remains close to its center.

(ii) Continuous-motion back-and-forth strategy. The patroller moves at constant speed back and forth from end to end thus dividing up a patrol cycle uniformly among the lanes.

In all cases the transitor was assumed to come straight through the zone at constant speed. Mixed strategies could be interpreted in terms of a unique random choice of a pure strategy, or as a time mixture assigning appropriate fractions to different lanes. The range laws considered were of the cookie-cutter type except for the 3-lane analysis of Appendix IV which included a range law of the form (0, p, 1, p, 0) where the probability is 1 only for the lane actually occupied. The continuous-motion patrol assumed that the probability of detection was 2q (not necessarily the same as p) at the center and q at the edges.

In this report the position-strategy analysis is extended in two ways.

(i) A more realistic range law is assumed and the matrix games are solved in Chapter III, Section 4. The range law incorporates a linear decrease in the probability of detection symmetrically outward from the center, dropping gradually off to zero at a finite range.

(ii) The division of the zone into a finite number of lanes and positions is replaced by the more realistic assumption of a continuum of lanes and positions in Chapter III, Section 5. The range law described above is replaced by its continuous analog, i. e., a triangular-shaped function. Again the extreme range is finite, but there is a continuous variation in probability of detection unlike the discontinuous cookie-cutter range law.

While the results for these new game-theoretic models are not easily described in a few words, three salient features may be noted:

First, there are a finite number of favored lanes and positions, and in the case of the matrix games this number is generally less than the total number of lanes. Even when the transitor is permitted to enter the zone at any point across the entire width, optimal minimax strategy calls for the use of only a small number of lanes. Similarly and symmetrically, the patroller's strategy requires him to ignore the majority of positions in favor of the few. This result is naturally related to the assumed form of range law, but ref [4] shows that it is qualitatively similar to the results found in the analysis of continuous convex games with bell-shaped kernels. For these games also there are only a finite number of pure strategies which are used to define the optimal mixed strategies.

Second, a salient feature of the results described in III. 4 and III. 5 is the general behavior of the value of the game, as the parameters change. Consider a zone of width D and the triangular detection law extending from $x - d$ to $x + d$ when position x is occupied by the patroller. Then $\frac{d}{D + d}$ is a good approximation to the value of the game in most cases*; thus a secure sweep width** (SSW) of $\frac{dD}{D + d}$ is assured by the use of the minimax strategies by both submarines. For example, if d is very much smaller than D, a SSW of approximately d is achieved: this is only half of the corresponding figure for a cookie-cutter range law. Now consider a smaller zone (or, equivalently, larger range of probable detection) and let $D = 2d$. Then the SSW is just $(2/3)d$.

Finally, we have also extended the concept of secure sweep width to a strategic confrontation. The "secure sweep width" concept, introduced in ref [8], is intended to be a numerical measure of the patroller's ability to detect the transitor without being previously counterdetected. It is defined as the width of frontage over which target crossings are equally likely at all points, times the fraction of targets on which the patroller makes secure detection. In other words, it is the product of the zone width D with the proba-

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* Except when d is nearly as large as D.

** The concept is extended in the natural way for non uniform transits; see next paragraph.

bility of the patroller detecting the transitor, given that the transitor is assumed to use all transit lanes with equal probability. This last assumption makes of this definition a non-strategic measure since the transitor is not endowed with the possibility of making strategic choices to counter the patroller's strategy. If we change and extend the definition of SSW by allowing the transitor to adopt strategic choices and assign both participants minimax strategy choices then the models studied give SSW's under varying range laws. Thus, if a patroller is assumed to make secure detection of any transitor within d distance an SSW of approximately $2d$ is found. On the other hand, if a patroller is assumed to detect a transitor with probability $1 - x/d$, if $x \leq d$ is the distance between the subs, and not detect otherwise, then the SSW is approximately $dD/(d + D)$.

It appears that the SSW concept, which is intended to give a rough measure of ability to detect, can in fact be a truly strategic measure. To be precise this suggests the definition: secure sweep width is the width of frontage over which the transitor may attempt crossings times the probability of the patroller detecting the transitor when both parties use minimax strategies. This definition will remain useful even when the transitor is known to employ a different strategy so long as the patroller is using the minimax strategy.

Analogous definitions of SSW can be constructed for other strategy concepts, and in the simplest case of a uniform patrol the new definition proposed is equivalent to the previous definition.

II.5 Conclusions

Considerable effort has been devoted to the study of problems concerning submarine-submarine detection by mathematical analysis, gaming and computer simulation. MATHEMATICA has, from May 1965 to the present,been reviewing for ONR* the question: "How can the theory of games be fruitfully applied to the study of ASW detection and evasion strategies?" We initially considered patrols by an individual submarine in a known rectangular zone; subsequently, while the results for a single zone have been extended and improved, attention has also been devoted to the construction of barrier models in a more general setting. For reasons which are detailed below, MATHEMATICA's investigators have concluded that the mathematical theory of games cannot, at present, provide a suitable methodology for <u>completely</u> analyzing the strategic alternatives which are available to patrolling submarines -- even in the restricted case of a single submarine patrolling a fixed zone. (By analogy with economics, we might express this viewpoint differently by the statement that the <u>micro-</u> strategies are inadequately represented in any mathematical models which can be formulated at present.) We anticipate that game-theoretic methods, in combination with other analytical ap-

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*Under contracts Nonr 4937(00) and N00014-66-C-0215.

proaches, will provide useful and practical insights for the macro-strategies of large barrier situations, rather than complete formal solutions.

These conclusions should not be regarded as mainly negative, because only the most elementary of practical problems in other fields possess complete mathematical solutions. There are two primary difficulties at present:

First, the information which MATHEMATICA has obtained on the determinants of patrol and transit strategies -- such as the range laws of sonar detection, the effects of sea state and speeds of both patrolling and transiting submarines on sonar detection and counterdetection, the effects of depth of the submarine, convergence zones, thermal layer and so forth -- seems to be at present a large, complex, and poorly structured body of data, and is therefore not easily provided as an input to the conventional game-theoretic models of strategic conflict. (Such models might be used either to generate hypotheses, or to test and verify hypotheses; the data requirements would be somewhat different.)

Second, the variables, which are known to determine the probability of a detection or counterdetection, are numerous and interdependent in complex ways. We have firm information about so few of these potential interactions that a useful mathematical theory is almost inaccessible at present. Such a theory apparently would require the analysis of strategies as functions of several control variables; the resulting mathematical problems are difficult even to formulate, and still more difficult to use as a basis for practical results of importance for application to the real ASW situation.

Both in on-going research and in the sea exercises that are undertaken to test the various recommended modes of patrolling and barrier design, a large body of information is being built up about ASW and its interface with oceanography. MATHEMATICA's analysts, on the basis of their familiarity with that body of information, have found that the current level of sophistication in understanding and structuring that data does not appear to permit a successful mathematical analysis of the patrol and transit strategies at the zonal level. Several game-theoretic results have been obtained by MATHEMATICA which can serve as a beginning for a mathematical theory of barrier detection in the large: for example, minimax speeds for certain detection-and-evasion conflict models have been derived (with other parameters held constant). These show that, the greater the emphasis on evasion (by the patroller) the more plausible is the recommendation of slow speed. Another example is the investigation of the relationship between probability of detection and range law. The investigation has led to an extension of the concept of Secure Sweep Width to the strategic situation. As a third example, we have studied a repeated game, the individual steps of which are opportunities for detection and counter-detection. It is assumed that the patroller knows more about his own range of detection under the prevailing environmental conditions than does the transitor, and the patroller may choose whether to make use of this information. The transitor may be able to infer (from observing the patroller's strategic moves in the steps, which are called "stage games") something about the unknown range law of the patroller. This analysis shows, in one particular case of interest:

(a) that the patroller cannot profit by using his information in a strategic mode, and

(b) that the transitor, to optimize, must play the average game expected under his prior beliefs about the patroller's detection capability.

Such results are obtained from zonal analysis, but because of the previously described complexities they are really of limited value for the improvement of ASW patrols, within the zone. We feel that the generalization of the task to the overall discussion of barrier strategy would permit the use of the results in an interesting and potentially valuable theory. For instance, if the relative positioning of a number of submarines of differing detection capabilities were analyzed, the methodology of MATHEMATICA's described work on Matrix Games could be brought to bear.

Game theory requires precise detailed infromation on the three factors mentioned above, (and discussed in greater detail in the next Chapter) for its successful application to ASW. Whenever some of this information is lacking, the partial analysis will yield only qualitative results. Whereas such results may provide meaningful insights into the principles of a rather large systems problem such as barrier design, they do not at present appear to be very practical at the level of components such as a patrol zone. For a successful strategic analysis of the zonal problem the requirements are at least the following:

(a) The functional relation between detection range and its determinants -- patrol speed, oceanographic conditions, target speed, depth,etc. -- should be known and parametrized.

(b) The forms and limits of strategic behavior allowed to the patrolling and transiting submarines must be clearly spelled out. Excessive complexity and variety here creates difficulties for the analyst.

(c) The objectives must be formally stated, inasmuch as the solutions will depend critically on them. Several alternative objectives may be used, requiring separate analyses and leading of course to distinct solutions. In view of the variety of missions to which a barrier submarine is assigned in peacetime and war, there is clearly a necessity for several of these alternative analyses.

II.6 Recommendations

As a result of the conclusions presented above, MATHEMATICA recommends:

(i) that the attempt to model the single patrol zone ASW problem as a game be discontinued for the present;

(ii) that further research be conducted on the use of mathematical models of ASW barriers as a whole at a higher level of aggregation than the zonal unit;

(iii) that the work which MATHEMATICA and other investigators have begun, on the effects of range law detection on optimal strategies, be continued; and

(iv) that the above work should make use of recent advances in the theory of Repeated Games of Incomplete Information.

III. Game-Theoretic Analyses of ASW Problems

III. 1 Games with Different Objective Functions

Introduction

The purpose of this section is to discuss several basic questions which underlie any analysis of the conflict situation involving one or more submarines patrolling a barrier and one or more submarines attempting a transit across the barrier. These questions arise whether the analysis proceeds via sea exercises, simulation, gaming, or game theoretical models. They have been largely ignored in previous ASW analyses and our purpose is to show, by simple examples, how this neglect has influenced the results obtained and restricted their practical usefulness.

A first step in any analysis of the patrol-transitor conflict is the definition of the actions open to the parties involved and their objectives in the conflict. This is true whatever the nature of the analysis. If we are designing a sea exercise, we must give the "rules of the game" to the two sides and, in general, these will circumscribe their range of free action rather sharply. The rules, as given to the participants, will consist of three parts: (1) the types of information permitted to them either from their own equipment or observations or from outside sources; (2) the range of actions that are permitted to them at various points of the situation as a function of the information available to them at those points; (3) the object of each antagonist in the exercise, preferably in terms of some index of merit which could be computed at the end of a contest. (It may happen that the index of merit could only be evaluated by an umpire who will have more information than any of the individual participants.)

To be more specific about these three ingredients in a sea exercise, we may consider a simple patrol-transitor exercise run on a rectangular area of the ocean, assumed to be oriented so that the sides run North-South and East-West. For the patrol submarine, under (1) we may specify the exact location of the patrol rectangle, the length of time of the exercise, the fact that one submarine of a given type will be attempting a single transit through the area from East to West at given depth sometime during the exercise, and that the patrol will start the exercise at a given point in the area. Furthermore, we may specify the types of detection equipment that may be used by the patrol submarine and thus the kinds of information about its own location and speed and about the transitor's location and speed. All of these factors constitute a part of the patrol's knowledge of the extensive form of the exercise; indeed, that part which is exclusive of the alternatives open to him and the transitor or based on that information. As for similar information to be provided to the transit submarine in the exercise, we may tell him the location of the rectangle, the length of time of the exercise, the fact that one submarine of given type will be patrolling somewhere in the area at a given depth. Furthermore, we may specify the types of detection equipment available to the transitor and thus the kinds of information about its own location and speed and about the patrol's location and speed. Again these constitute the transitor's knowledge of the extensive form of the exercise, exclusive of the alternatives open to him or based on that information.

As for (2), the range of actions open to the two submarines, we may specify them in great detail, such as only allowing the transitor straight-line crossings of the rectangle at constant speed, or we may allow considerable choice, such as zig-zag paths at varying speeds depending on information reaching the transitor through its detection equipment. Similar comments apply to the patrolling submarines.

The all-important result of the combination of (1) and (2) is the concept of a <u>strategy</u> for a submarine in an exercise. This concept reflects the <u>planned interaction of information with the freedom of action</u> allowed to the submarine. Without the specification of, or explicit assumptions concerning, (1) the information about the extensive form of the conflict and

(2) the restriction of the range of action open to the participants, there can be no exercise, no simulation, or no game-theory model.

These ingredients are by no means the whole story, for they only speak to what the submarines know and can do. They say nothing about why they are doing it, what their objectives are or, in game-theoretic terms, what the payoff is. If we return to the question of an operational exercise, we are concerned with a measure of the performance of the participating submarines. A typical specification for the patrol submarine would be to say that he wins if he detects the transit without being detected himself, that the transitor wins if he crosses the rectangular area without being detected and that the exercise is a draw if both submarines correctly detect each others' presence. If we are dealing with a larger number of transits than one, we may use the ratio of first detections by the patrol to the number of transits as a figure of merit to measure the performance of the patrol side of the exercise. Both the simplicity of these suggestions and the dependence on the structure of the underlying actions and information possibly available to the sides in the exercise make [illegible] act crystal clear: the optimal behavior of the participants may vary significantly if we change the objective function. Fundamentally it is this phenomenon that we wish to illustrate in this chapter.

To recapitulate, there can be no exercise, simulation, or game without the specification of three factors:

(1) the information available to the participants as to the rules of the game;

(2) the technological possibilities (that is, the speeds, maneuverability, and detection capabilities) available to the participants;

(3) the objectives of the participants, preferably in the form of numerical measures that could be computed by an observer of the complete action.

Incorrect specification of any one of these can lead to seriously incorrect calculation of optimal behavior as we shall now show by means of examples.

Misinformation about the Extensive Form

The difficulty described by the title of this subsection may be encountered in a number of forms and may differ in importance according to the situation. That it is of practical relevance is easily seen: surely it is unrealistic to assume that the transiting submarine has an accurate map of the barrier with its complete geometry and the number and type of patrol submarines present in it in a given period of time, and equally unreasonable to assume that the patrolling submarines will know the number and type of the submarines attempting to transit through the barrier. This second possibility is illustrated by the following example, patterned on a game which we have discussed previously:*

First Extensive Form: Let the payoff matrix

be

	T_1	T_2	T_3
P_1	1	0.25	0
P_2	0.25	1	0.25
P_3	0	0.25	1
P_4	0.25	0.5	0.25

The patroller's strategies are:

P_1 = patrol in Northern boundary cell

P_2 = patrol in central cell

P_3 = patrol in Southern cell

P4 = continuous motion patrol

The transitor's strategies are:

T_1 = transit through Northern cell

T_2 = transit through central cell

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* Pp. 44-51 of ref [6].

T3 = transit through Southern cell .

The underlying assumption is that one transit will be attempted and the payoff is the probability that all transits will be detected.

This example is easily solved. The optimal strategies are

$$P^* = 3/8\, P_1 + 1/4\, P_2 + 3/8\, P_3$$

$$T^* = 3/8\, T_1 + 1/4\, T_2 + 3/8\, T_3$$

and the minimax value is 7/16 .

Now let us alter the game by having the transitor attempt to pass at least one of two submarines through the barrier. If we denote the resulting pure strategies for the transitor by (T_i, T_j) for $1 \leq i \leq j \leq 3$, then the payoff matrix becomes:

	(T_1, T_1)	(T_1, T_2)	(T_1, T_3)	(T_2, T_2)	(T_2, T_3)	(T_3, T_3)
P_1	1	0.25	0	0.0625	0	0
P_2	0.0625	0.25	0.0625*	1	0.25	0.0625
P_3	0	0	0	0.0625	0.25	1
P_4	0.0625	0.125	0.0625	0.25	0.125	0.0625

Here, as before, the payoff is interpreted as the probability that all transits are detected.

This matrix has a saddlepoint solution (indicated by the asterisk) consisting of

$$\overline{P} = P_2$$

$$\overline{T} = (T_1, T_3)$$

with minimax value $\frac{1}{16}$. This has the obvious interpretation that the patrol should sit on the center station while the transitor's submarines should run the sides of the barrier. If the patrol submarine plays the optimal strategy for the game with but one transit submarine, then the probability that he detects all of the transits is reduced to $\frac{1}{64}$!

We have not solved the problem of constructing reasonable patrol strategies when the number of submarines attempting transit is unknown. The problem is clearly but one aspect of a central theme of this report, namely, that the well-defined area games may mislead if unrelated to a larger context.

Misspecification of the Technological Possibilities

In our definition of the context of the analysis of a transit-patrol conflict, we included under the category of technological possibilities open to the participants such factors as speed, detection capability, and patrol or transit pattern. Naturally, in any theoretical modeling of the situation we will use the best possible estimates of speed and will attempt completeness (within the context) in listing the geometric patterns allowed to the patrolling and transiting submarines. Even if we assume that these parameters are known accurately, the remaining variability in the detection capabilities are such to render the results suspect. If we deal with sea exercises, the variability of sea state, equipment performance, and the small samples obtained render most results statistically insignificant. If we consider simulation or gaming experiments, the lack of reliable hypotheses to program for detection renders the results unreliable when interpreted as guides to practice. If we are attempting game-theoretical analyses, the assumptions about detection capability can alter the conclusions radically (even when all other parameters are held constant.)

This phenomenon can be illustrated by a simple example of the same type used in the previous subsection. Consider the general game matrix:

	T_1	T_2	T_3
P_1	1	p	0
P_2	p	1	p
P_3	0	p	1
P_4	q	2q	q

where the strategies have the same interpretation as in the previous subsection. If we consider the case of detection probabilities $p = q = 1/4$ as before, the optimal strategies are

$$P^* = 3/8P_1 + 1/4P_2 + 3/8P_3$$

$$T^* = 3/8T_1 + 1/4T_2 + 3/8T_3$$

with minimax probability of detection equal to $\frac{7}{16}$. If we vary the parameters of detection through the interval $\frac{1}{4} \leq p = q \leq \frac{1}{2}$, the optimal strategy for the patroller varies in the following manner:

$$P = \frac{1 - p}{3 - 4p} P_1 + \frac{1 - 2p}{3 - 4p} P_2 + \frac{1 - p}{3 - 4p} P_3 \ .$$

However, the endpoint $p = q = \frac{1}{2}$ this formula yields

$P = \frac{1}{2}P_1 + \frac{1}{2}P_3$, whereas the strategies P_2 and P_4 dominate this weakly. If this problem were presented at the endpoint $p = q = \frac{1}{2}$ the payoff matrix is

	T_1	T_2	T_3
P_1	1	$\frac{1}{2}$	0
P_2	$\frac{1}{2}$	1	$\frac{1}{2}$
P_3	0	$\frac{1}{2}$	1
P_4	$\frac{1}{2}$	1	$\frac{1}{2}$

Common sense dictates the choice of P_2 or P_4 (the center station or the continuous patrol). Nevertheless with $p = q = \frac{1}{2} - \epsilon$, these strategies are hardly used at all:

$$P = \frac{\frac{1}{2} + \epsilon}{1 + 4\epsilon}P_1 + \frac{2\epsilon}{1 + 4\epsilon}P_2 + \frac{\frac{1}{2} + \epsilon}{1 + 4\epsilon}P_3 \ .$$

<u>Thus a slight change in the detection specification can cause a radical change in the strategy proposed as optimal.</u>

Misspecification of the Objective Function

This is perhaps the most important part of formulating a theoretical model of any process and yet many previous efforts in ASW seem, in retrospect, to have been extremely oversimplified. Consider some of the possible objectives of a barrier:

(1) Make a statistical census of the transiting submarines passing through. This in turn could have two possible variations:

> (1a) The purpose of the survey could be an absolute census of the transiting submarines, in order to keep a rough count of the opposing force in various areas.
>
> (1b) The purpose of the survey could be a check on large changes in activity. Thus, if the average has been 1 transit per week a change to 2 per week might not be significant, while 10 in one week would constitute an important change in the situation.

(2) Make a complete census of the number and type of submarines crossing the barrier. (The barrier might or might not be interested in whether the patrol submarines are in turn detected by the transitors.)

(3) Prevent all transits. (The patrolling submarine will necessarily follow a detection by other, more active, phases of identification, pursuit, and attack.)

Even this simple enumeration of possible objectives for the patrolling submarine should make clear the inadequacy of the "probability of first detection" as an objective function. However, the difficulty is even more serious than this. Namely, it is not clear that the essence of the real situation can be reflected by any zero-sum model. For example, if a mission is assigned to one side, the other side may take "prevention of that mission" as its criterion; so far, the game is zero-sum. But if one side risks loss of a submarine, the other side is likely to be *prima facie* indifferent to whether the sub is lost; this is a non-zero-sum aspect. Under these circumstances, any analytical approach must acknowledge that the situation is one of partial competition, and must somehow deal with the inherent conceptual difficulties.

The unavoidable conclusion of this discussion is that, although the available methods may analyze local situations and produce optimal solutions, these are likely to be misleading if not erroneous without more realistic formulations of the global contexts of which they are a part.

III. 2 First-Detection Games

An important step in the further development of submarine-versus-submarine detection games is the determination of improved objective functions. In a previous MATHEMATICA study [6], consideration was given to a detection problem in which a patroller attempts to maximize his probability of detecting a transitor, while the transitor attempts to minimize that probability. This objective function was also used by Wagner and Associates in [5]; optimal strategies were found for that detection problem. Unfortunately, those strategies, although they maximized the patroller's probability of detecting the transitor, did not assure that it was a first detection; realistically, if the transitor detected the patroller first, the transitor could probably avoid detection by the patroller. The previous objective function would therefore only be applicable if (for some reason) the transitor had fixed his route and speed in advance, and could not alter them even if he detected the existence and location of a patroller.

In this section several games will be examined which incorporate an improved objective function, namely, the probability that the patroller detects the transitor before he is himself detected. These games make the essential assumption that the range at which detection first occurs is a random variable; otherwise we could not represent the random events "transitor detects patroller first" and "patroller detects transitor first."

In order to facilitate the development we will first set down a sequence of definitions which will also apply to the following section, III. 3.

1. Events

Attempted Transit (AT) An occasion for possible detection of an enemy submarine attempting to transit the barrier, which is patrolled.

Detection The reception of signals (on sonar or other equipment) which have in fact originated from an enemy submarine, although they may not have been identified as such with certainty.

2. Range Variables

Closest point of approach (CPA) We shall use D for the random variable which corresponds to the distance between the submarines at the CPA, when the AT occurs at a random point of time. D is also dependent on the two speeds, the patrol path and the transit path. We will explicitly treat only the speeds here.

Maximum range for detection Two random variables are defined here, S and T. They are symmetrical with respect to patroller and transitor submarines.

$S \equiv$ maximum range at which the patroller is capable of detecting the transitor under the conditions which prevail on a specific AT.

$T \equiv$ maximum range at which the transitor is capable of detecting the patroller under the conditions which prevail on a specific AT.

$R \equiv$ the maximum of S and T.

These are random variables because the definition allows them to vary between AT's. They also depend on the speeds of the two submarines in a manner which we shall make explicit below.

3. The Objectives

To maximize the probability of detection. For the patroller, a detection is feasible, but not certain, if $D \leq S$. He may wish to maximize $P(D \leq S)$. For the transitor, a detection is feasible if $D \leq T$. He may wish to maximize $P(D \leq T)$.

To minimize the probability of counterdetection. Against the patroller, a counterdetection is feasible under the same conditions which allow detection for the transitor and vice versa. Thus the patroller may wish to minimize $P(D \leq T)$ and transitor may wish to minimize $P(D \leq S)$. Notice that these are logically the negations of the first pair of objectives.

To maximize the probability of secure detection. For the patroller, if $D \leq S$ and $T \leq S$, secure detection is feasible but not certain. The patroller may wish to maximize the probability of a secure detection, $P(D \leq S, T \leq S)$.

Similarly the transitor may wish to maximize the pr bability of a secure detection of the patroller, $P(D \leq T, S \leq T)$. Notice these two objectives are not complementary since $P(D > S \text{ or } T > S) = 1 - P(D \leq T \text{ and } S \leq T)$.

<u>To minimize the probability of</u> <u>re counterde-tection.</u> We merely remark that the trans y wish to minimize the probability $P(D \leq S, T$ hich is the complementary objective to the patroller's he previous paragraph. It seems unlikely that the patroller would concern himself with minimizing $P(D \leq T, S \leq T)$ explicitly, although that would partly be a result of his previous objective.

4. The Games

We consider in this section and the following, a class of detection games with one of the above objectives, or some objective derived from them by compounding. The strategy spaces are simple: the normalized speeds, u and v which vary over the intervals $[u_o, u_m]$, $[v_o, v_m]$. It is assumed throughout that both submarines follow straightline courses at right angles to each other in the present simple model.

It is important to note that the game stops when <u>either</u> an undetected transit is completed, <u>or</u> a detection (counterdetection) occurs. For the subsequent events in the latter case a new game model is started, which will have different features. Strictly speaking, the game may stop even earlier in the transit sequence: after the transitor has passed the CPA undetected. However, this distinction makes no difference to the analysis, so we may ignore it.

As previously developed*, the probability of the distance between the patroller and the transitor ever falling below r during an attempted transit is:

$$P(D \leq r) = \min\left(\frac{Mr}{w}, 1\right),$$

where

$$M = \sqrt{1 + \frac{u^2}{v^2}}$$

and where u and v are normalized speeds for the patroller and the transitor respectively while w is the half-width of the barrier. Let

$$F_D(r) = P(D \leq r) \quad \text{and} \quad f_D(r) = \frac{dF_D(r)}{dr}$$

$$F_S(r) = P(S \leq r) \quad \text{and} \quad f_S(r) = \frac{dF_S(r)}{dr}$$

$$F_T(r) = P(T \leq r) \quad \text{and} \quad f_T(r) = \frac{dF_T(r)}{dr}$$

---

* See page 31 of Reference [6].

We define $p(u, v)$ as the probability of a secure detection by the patroller when the respective speeds are u and v. It is this objective which we shall examine in more detail now. As mentioned above, the patroller attempts to maximize and the transitor to minimize:

$$p(u, v) = P(D \leq S, \ T \leq S) \ .$$

It we assume that D and S are independent we can write

$$(2) \qquad p(u, v) = \int_0^\infty F_D(r)F_T(r)f_s(r)dr$$

$$= \int_0^{w/M} (Mr/w)F_T(r)f_s(r)dr + \int_{w/M}^\infty F_T(r)f_s(r)dr \ .$$

If w/M is large, then an approximation for equ. (2) will be:

$$(3) \qquad p(u, v) = \frac{M}{w} \int_0^\infty rF_T(r)f_s(r)dr \ ,$$

which is of the form:

$$(4) \qquad p(u, v) = \left(\frac{M}{w}\right) E(S \mid T \leq S)P(T \leq S) \ .$$

If, furthermore, T and S have zero variances we find $T \leq S$ if and if only $cv \leq u$, where c is the ratio of the transitor and the patroller noise slopes. In this case: $E(S|T \leq S) = Ke^{cv-u}$, and:

$$(5) \qquad p(u,v) = \begin{cases} \left(\dfrac{(1+u^2/v^2)^{1/2}}{w}\right) Ke^{cv-u} & \text{if } cv \leq u \\ 0 & \text{if } cv > u \end{cases}$$

which is equivalent to the results obtained previously in [5].

Exponential Example

As an example, assume that S and T are independently distributed according to the negative exponential probability law, i.e., that for $r \geq 0$:

$$E(S) = a_1^{-1}$$

$$E(T) = a_2^{-1}$$

$$P(S \leq r) = 1 - e^{-a_1 r}, \text{ and}$$

$$P(T \leq r) = 1 - e^{-a_2 r} .$$

Then:

$$(6) \qquad p(u,v) = \frac{M}{w}\int_0^\infty r(1 - e^{-a_2 r})a_1 e^{-a_1 r}\,dr$$

$$= \frac{a_1 M}{w}\left[\int_0^\infty re^{-a_1 r}\,dr - \int_0^\infty re^{-(a_1+a_2)r}\,dr\right]$$

$$= \frac{M}{w} a_1 \left(\frac{1}{a_1^2} - \frac{1}{(a_1+a_2)^2} \right) = \frac{M}{w} \frac{a_2}{a_1} \cdot \frac{2a_1+a_2}{(a_1+a_2)^2}$$

$$= \frac{M}{w} \cdot \frac{1}{a_1} \cdot \frac{(2+b)b}{(1+b)^2}$$

where $b = \frac{a_2}{a_1}$.

A particular version of this example is obtained by also assuming that the S and T distributions have means equal to the thresholds for detection and counterdetection:

$$E(S) = \frac{1}{a_1} = Ke^{cv-u}, \text{ and } E(T) = \frac{1}{a_2} = Ke^{u/c-v} .$$

The payoff probability reads:

$$(7) \qquad p(u,v) = \frac{M}{w} Ke^{cv-u} \left(\frac{(2+b)b}{(1+b)^2} \right) \quad \text{where } b = e^{(c+1)(v-u/c)}$$

If $p(u,v)$ were shown to be convex in either u or v, optimal (minimax) strategies would be easily obtained. The function in (7) has been numerically evaluated for $c=0.1,\ 0.2,\ 0.5,\ 0.75,\ 1.0,\ 1.5,\ 2,\ 3,\ 4,\ 10$ and $u, v = 0.1(0.1)\ 1.5$. Table III. 2.1 shows the values for the case $c = 1.0$. In every case, $u = 0.1$, $v = 0.1$ is found to be a saddlepoint of the matrix. If u_o and v_o were different from 0.1, the saddlepoint effect would still be preserved since the function $p(u,v)$ is monotonic decreasing in u and increasing in v throughout the range considered.

TABLE III. 2. 1

Secure Detection Probabilities When Range Variables are Independent Negative Exponentials* (c = 1)

v \ u	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	1.1	1.2	1.3	1.4	1.5
0.1	0.753	0.643	0.548	0.465	0.392	0.329	0.274	0.226	0.186	0.151	0.122	0.098	0.078	0.062	0.049
0.2	0.885	0.764	0.659	0.565	0.482	0.409	0.344	0.287	0.238	0.195	0.159	0.128	0.103	0.082	0.065
0.3	1.029	0.898	0.783	0.679	0.587	0.503	0.428	0.361	0.302	0.250	0.205	0.167	0.135	0.108	0.086
0.4	1.186	1.044	0.920	0.807	0.705	0.612	0.527	0.449	0.379	0.317	0.263	0.215	0.175	0.141	0.113
0.5	1.355	1.203	1.069	0.949	0.838	0.736	0.641	0.553	0.472	0.399	0.333	0.276	0.226	0.184	0.148
0.6	1.537	1.375	1.232	1.103	0.985	0.874	0.770	0.672	0.581	0.496	0.419	0.350	0.290	0.237	0.193
0.7	1.733	1.559	1.407	1.271	1.145	1.027	0.915	0.808	0.706	0.610	0.52[illegible]	0.441	0.368	0.304	0.249
0.8	1.944	1.753	1.596	1.452	1.319	1.194	1.075	0.960	0.849	0.742	0.642	0.548	0.463	0.386	0.319
0.9	2.173	1.973	1.800	1.647	1.567	1.376	1.250	1.128	1.009	0.892	0.780	0.674	0.576	0.485	0.405
1.0	2.422	2.205	2.020	1.857	1.709	1.572	1.440	1.312	1.185	1.060	0.938	0.820	0.708	0.604	0.509
1.1	2.693	2.457	2.257	2.084	1.928	1.783	1.645	1.511	1.378	1.246	1.114	0.986	0.861	0.743	0.633
1.2	2.989	2.732	2.516	2.329	2.163	2.011	1.866	1.726	1.588	1.449	1.310	1.171	1.035	0.903	0.778
1.3	3.313	3.033	2.797	2.595	2.418	2.256	2.105	1.958	1.814	1.669	1.523	1.376	1.230	1.086	0.946
1.4	3.669	3.362	3.105	2.886	2.694	2.522	2.361	2.208	2.057	1.906	1.754	1.600	1.445	1.290	1.138
1.5	4.062	3.724	3.442	3.203	2.995	2.810	2.639	2.477	2.320	2.162	2.004	1.843	1.680	1.516	1.352

* The entries in this table are proportional to secure detection probabilities. They must all be scaled down by dividing them by the half-width of the barrier zone in nautical miles.

III. 3 A Difference Game

The objective function used for submarine-versus-submarine detection games in the previous MATHEMATICA report [6] and also in Wagner [5] was the probability of detection of the transitor by the patroller, as a function of their speeds. Parameters representing the range law and oceanographic conditions were held constant. Using the notation of [5]; if u and v are normalized speeds, then the model, which we shall call the "Patroller's Speed Game" G^P, is de ribed as follows:

$$\max_{u_o \leq u \leq u_m} \min_{v_o \leq v \leq v_m} \left[F^P(u,v) = e^{cv - u\sqrt{1+u^2/v^2}} \right],$$

where c is a positive constant. It is obvious that there is a symmetrical position for the transitor to take in this game: this results in a model, which we shall call the "Transitor's Speed Game", G^T which is described as follows:

$$\min_{u_o \leq u \leq u_m} \max_{v_o \leq v \leq v_m} \left[F^T(u,v) = e^{u/c - v\sqrt{1+u^2/v^2}} \right].$$

Note that the min and max variables are reversed. In both games, $c = a/b$ is a positive constant and a and b are the slopes of the noise functions for the transitor and the patroller in terms of their speeds:

$$N_P(u) = b(u-u_o) + k_o \quad \text{if} \quad u_o \leq u \leq u_m \quad ,$$

$$N_T(v) = a(v-v_o) + \ell_o \quad \text{if} \quad v_o \leq v \leq v_m \quad .$$

At speeds below u_o, the transitor's noise output $N_P(u)$ is (roughly) constant and similarly for $N_T(v)$ below v_o. The maximum achievable speeds for the patroller and the transitor are assumed to be u_m and v_m respectively.

The analysis of a patrol speed strategy based on G^P is faulty if it yields solutions which in fact result in a situation where the transitor is more likely to secure first detection. An example of this error occurs in the game described in Figure 6 of ref. [5], which assumes that $a = b$; the analysis leads to "optimal" normalized speeds for the patroller of 0.7500 and for the transitor of 0.6054. The value for the patroller's game G^P is 1.38; but the transitor's game G^T has a value of 1.74. This means that the transitor can assure himself of a larger "secure sweep width" and accordingly also of first detection. Given the broad situation, there is no reason for the patroller to adopt the optimal strategy of G^P.

Our formulation of a Secure Detection Speed Game in the first place will involve objectives for the patroller and the transitor which are not complementary; hence, a nonconstant-sum game. In the subsequent analysis we will reformulate the game in such a way that it becomes a constant-sum game. Then various simplifying assumptions will be introduced which lead to mathematically tractable payoff functions.

Consider first the game:

(SD_o) patroller to maximize $P(T < D \leq S)$;

transitor to maximize $P(S < D \leq T)$.

This game assigns symmetrical objectives to the patroller and the transitor; both are to choose strategies which maximize the probability of secure detection of the enemy. The solution of this game is conceptually difficult because of its noncon-stant-sum nature.

A constant-sum relative of (SD_o) is derived next. Let μ_i and $'_i$ $(i = 1, 2)$ be weights reflecting the relative importance of detection and evasion to the patroller $(i \quad 1)$ and the transitor $(i = 2)$. Then it is reasonable to represent the payoffs to the patroller and the transitor in the following way:

$$\text{patroller to maximize } \mu_1 P(T < D \leq S) + \lambda_1 [1 - P(S < D \leq T]$$
$$\text{transitor to maximize } \lambda_2 P(S < D \leq T) + \mu_2 [1 - P(T < D \leq S]$$

which, in case $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$, is equivalent to the zero-sum game:

$$(SD_1) \quad \min_{\Sigma_T} \max_{\Sigma_P} [\mu P(T - D \leq S) - \lambda P(S < D \leq T)]$$

where Σ_T and Σ_P are respectively the patroller's and the transitor's strategy spaces.

It is interesting and helpful to note the effect of assuming <u>a priori</u>, that one of the two probabilities in the objective function of (SD_1) must be zero. This assumption would be natural if, for instance, the inherent differences between the patroller's and the transitor's equipment made it virtually impossible to achieve either $S > T$ or $T > S$ at the same time as fulfilling other requirements of the mission. Suppose then that $P(T < D \leq S) = 0$ <u>a priori</u>. This implies, since

$$P(D \leq S) = P(D \leq S, D \leq T) + P(T < D \leq S) \text{ and}$$
$$P(D \leq T) = P(D \leq S, D \leq T) + P(S < D \leq T),$$

that

$$P(D \leq T) - P(D \leq S) = P(S < D \leq T),$$

and therefore that the game (SD_1) is equivalent to:

$$(SD_2) \quad \min_{\Sigma_T} \; \max_{\Sigma_P} \; [P(D \le S) - P(D \le T)] \; .$$

The same conclusion is reached if we start with the *a priori* assumption that $P(S < D \le T) = 0$.

The game (SD_1) will be referred to herein as the "Difference Game". [illegible] as the assumptions on which (SD_2) is based are unnecessarily restrictive, and the mathematical formulation of (SD_1) is analytically too complex, we shall analyze a slightly different specification of (SD_1):

$$(SD_3) \quad \min_{\Sigma_T} \; \max_{\Sigma_P} \; [\mu P(D \le S) - \lambda P(D \le T)] \; .$$

Substitution of parametric forms of the probabilities in (SD_3) in terms of the strategy variables u and v (the normalized speeds) results in the payoff function:

$$G(u,v) = \mu F^P(u,v) + \lambda(1 - F^T(u,v)) \; ; \mu > 0 \; , \; \lambda > 0 \; .$$

The patroller is the u-player and maximizer; the transitor is the v-player and minimizer. The analysis can be conducted without specifying the values of the utility "weights", μ and λ.

Without loss of generality, since $\mu > 0$ and both μ and λ are held constant, we can divide through by μ and represent the game as:

$$G^s(u,v) = F^P(u,v) - sF^T(u,v) = \left(e^{cv-u} - se^{u/c-v}\right)\sqrt{1+u^2/v^2} \ ,$$

where $s = \frac{\lambda}{\mu} > 0$ but may be indefinitely large. In this game, the patroller is maximizing and the transitor is minimizing a weighted combination of the probability of the patroller detecting the transitor and the probability of the patroller avoiding counter-detection by the transitor.

Both $F^P(u,v)$ and $F^T(u,v)$ are convex in the minimizing player's variable: v for the former, u for the latter. The proof is easily found by examination of the first or second derivatives of the functions. Unfortunately, the convexity of G^s is not so easily explored. In a related secure-detection game, examination of the numerical evaluation* of the payoff function (Table III.2.1 in this report) shows that optimal pure strategies exist: they are the slow speeds u_o and v_o for the patroller and the transitor respectively. Under a mild restriction on three parameters of the game (u_o, c and s), we will prove that the same strategies are minimax for (SD_3).

<u>Lemma III.3.1</u> If $v = v^*$ is any fixed number in $[v_o, v_m]$, then $\max\limits_{u_o \le u \le u_m} G^s(u,v^*) = G^s(u_o,v^*)$ when $s \ge 1$.

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* The numerical calculations were performed for detection distributions which are exponential with range, and statistically independent.

Proof: $\quad G_1^s(u, v^*) \geq 0 \Leftrightarrow 1 + sbe^{v^*} \geq \frac{b+1}{1+2v^*}$

$$s \geq 1 \Rightarrow 1 + sbe^{v^*} \geq 1 + sb \geq \frac{b+1}{1+0} \geq \frac{b+1}{1+2v^*}$$

$$s \geq 1 \Rightarrow G^s(u_o, v^*) \geq G^s(u, v^*) \text{ for all } u \geq u_o \, .$$

<u>Lemma III. 3. 2</u> $\qquad$ If $\frac{u_o}{c} > \frac{-\ln(cs)}{1+c}$, then

$\min\limits_{v_o \leq v \leq v_m} G^s(u^*, v) = G^s(u^*, v_o)$ for any fixed u^* such that $u_o \leq u^* \leq u_m$

Proof:

The minimum over all $v \in [v_o, v_m]$ of the function $G^s(u_o, v)$ occurs either when $v = v_o$ or when $v = v_m$ or at a root of the equation $G_2^s(u_o, v) = 0$. Rearranging terms and cancelling non-zero common factors, the latter equation can be represented as

$$1 + v + v^3/u_o^2 = \frac{1 + c}{c\left[1 + se^{(c+1)(u_o/c - v)}\right]} \, .$$

Examination of the two functions of v on the left-hand side and the right-hand side of the last equation shows that it either has exactly one real root or none in the interval (v_o, v_m), and that the condition of the Lemma guarantees there will be none. Furthermore, under the condition, $G_2^s(u_o, v) > 0$ for all

v in the permitted range, so that $G^s(u_o, v)$ takes its minimum at the lower end point of the range: $v_{min} = v_o$.

<u>[illegible] III.3.3</u> The conditions $\frac{[illegible]}{c} > \frac{-\ln(cs)}{1+c}$ and $s \geq 1$ are sufficient for (u_o, v_o) to be a saddle point of $G^s(u, v)$. For all (u, v) such that $u_o \leq u \leq u_m$, $v_o \leq v \leq v_m$, $G(u, v_o) \leq (G(u_o, v_o) \leq G(u_o, v)$. When $c \geq 1$, the first condition is unnecessary, as it is implied by the second.

Proof: This is a direct consequence of Lemmas III.3.1 and III.3.2. We have now proved:

<u>Theorem</u> The pure strategies $u = u_o$ and $v = v_o$ are optimal for the game G^s, when s is sufficiently large (and in any case not less than 1).

As an interesting special case, suppose that the patroller and the transitor, being matched in equipment, have the same noise functions so that $c = 1$. Then the theorem's conclusion provides a recommendation that if $s = 1$ both submarines should travel at their minimal speeds* (u_o, v_o) . Now suppose that the transitor has equipment that is twice as quiet as the patroller's: $c = 0.5$. Then the theorem is only operative when $s \geq 2$; but the pure strategy solution (u_o, v_o) may

* (in terms of the noise functions this is the speed such that any slower speed is likely to produce essentially the same noise output).

be true for $s <$ [illegible] as well. The transitor's advantage of [illegible] implies that he has a greater chance to detect the patroller before being detected. As a result, we can only recommend the strategy (u_o, v_o) if the patroller values avoiding counter-detection twice as highly as a detection of a transitor, or believes it twice as likely that the transitor will play the Transitor's Speed Game G^T as that he will play the Patroller's Speed Game G^P.

The pure strategy solution (u_o, v_o) of the Difference Games G^s is not in agreement with most of the solutions found for the examples in [5] for the Patroller's Speed Game. We show in the table below a comparison of the solutions in the case of eleven examples presented in [5]. The last column of the table indicates the probability that the solutions offered in [5] violate the non-probabilistic condition $(cv > u)$ for a secure detection by the patroller.

These examples make it clear that, while the simplified speed model for secure detection of the present section cannot guarantee that detection will be secure in cases where the patroller must use a higher speed than the transitor (examples 1, 2, 6, 7, 8, 9), it can provide a better guide to strategy in other cases (4 and 6). For the remaining examples, the comparison shows interesting differences (except 10) which are not yet fully explained.

T[illegible]

Example in [5]		MATHEMATICA Solution	Solution in [5]	Violation
1.	c=2	(.50, .05)	(.8, .4)	50%
2.	c=1	(.75, .25)	(.7500, .6054)	100%
3.	c=2	(.00, .02)	(.2000, .2229)	0
4.	c=1	(.15 .25)	$\left(\begin{cases} .1500 \text{ wp } .3972 \\ .6981 \text{ wp } .6028 \end{cases}, .4591\right)$*	100%
5.	c=1	(.00, .02)	$\left(\begin{cases} .0000 \text{ wp } .1350 \\ .2000 \text{ wp } .8650 \end{cases}, .2852\right)$	0
6.	c=1	(.10, .05)	(.90, .30)	100%
7.	c=1	(.70, .30)	(.70, .55)	100%
8.	c=1	(.05, .02)	(.65, .30)	100%
9.	c=2	(.50, .45)	(.7179, .4500)	0
10.	c=1	(.50, .55)	(.50, .55)	0
11.	c=2	(.40, .40)	(.70, .40)	0

* wp = with probability

III. 4 Large Structured Matrix Games

Introduction

We consider classes of large matrix games in studying either game-theoretic aspects of barriers, i.e., an array of specified zones in which patrolling submarines attempt to detect transitors, or aspects of the patrol strategies used by a patroller within a zone. In these matrices, entries correspond to (are proportional to) the probability of the patroller detecting the transitor, columns representing a choice of transit lane by the transitor, and rows the choice of a "pure patrol strategy" by the patroller. The motivation for considering matrix games should be clear; computational methods for solving them and theorems for analyzing them abound. However, in order to obtain any results of interest for the problems at hand large games need to be considered. Analysis indicates that if large matrix games are to be solved, either they must possess a strikingly symmetric structure which permits explicit analytic solution, or realistic-looking matrices must be generated from data obtained in fleet exercises and solved by techniques from linear programming. Both of these approaches are discussed below.

Toeplitz Matrix Games

We consider square matrix games $G = (g_{ij})$ where $g_{ij} = f(|i-j|)$, that is, the entry g_{ij} is a function of the distance of the entry (i, j) from the main diagonal. It seems particularly appropriate to consider functions f which are monotonic non-increasing if we interpret pure row strategy i as that of hovering at station i $(i = 1, \ldots, n)$. Implementation of successive row strategies i and $i + 1$ can be made in a straight-line patrol between stations i and $i + 1$ within a zone. A mixed strategy may then be interpreted as the proportion of time spent hovering in the vicinity of i in a back-and-forth patrol.

In previous work MATHEMATICA considered an $m(2k - 1)$-station problem G with $g_{ij} = f(|i-j|) = 1$ for $|i-j| \leq k - 1$ and $g_{ij} = f(|i-j|) = 0$ for $|i-j| > k - 1$ where $k \geq 1$. This corresponds to the assumption that the transitor is detected if he traverses at any one of the $k - 1$ adjacent stations on either side of the patroller's station. Call such games, which are defined by specification of m and k, $H(m, k)$. For example, the game $H(2, 2)$ is defined by the 6×6 matrix

$$\left[\begin{array}{ccc|ccc} 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ & 1 & 1 & 1 & & \\ \hline & & 1 & 1 & 1 & \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \end{array}\right] = H(2,2)$$

It is convenient to think of m as the number of identical blocks or square submatrices of dimension $2k - 1$ which lie along the main diagonal of $H(m,k)$. Using these concepts it is easy to show that the value of the game $H(m,k)$ is $1/m$; an optimal strategy for the patroller or row player is to choose row i with probability $1/m$ if $i \equiv k \pmod{k+1}$ and with probability zero if $i \not\equiv k \pmod{k+1}$; and an optimal strategy for the transitor or column player is to choose some column from each block and play this column with probability $1/m$. Thus, for example, the value of $H(2,2)$ is $1/2$; the patroller's optimal strategy is $(0, \frac{1}{2}, 0, 0, \frac{1}{2}, 0)$; and the transitor's optimal strategy is $(\frac{1}{2}, 0, 0, 0, 0, \frac{1}{2})$.

The unsatisfactory nature of games $H(m,k)$ is that they make a "cookie cutter" assumption for the range law: any transitor within $k - 1$ stations of the patroller is surely detected, any transitor further away is never detected.

Graphically,

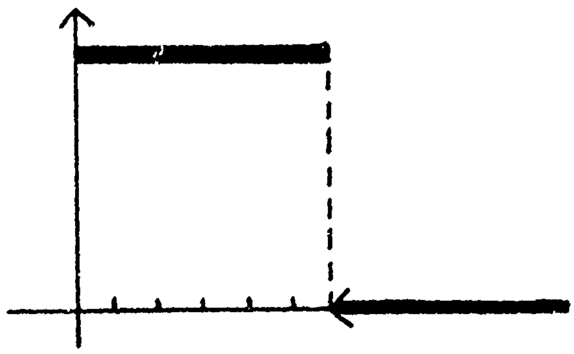

Figure 3.3

Consider, now, an mk-station problem G with $g_{ij} = f(|i-j|) = \frac{k - |i-j|}{k}$ for $|i-j| \leq k-1$ and $g_{ij} = f(|i-j|) = 0$ for $|i-j| \geq k-1$ where $k \geq 1$. This corresponds to the assumption that the transitor is detected with probability $1 - \ell/k$ if he traverses in a lane ℓ-distant from the patroller's station. Thus the cookie-cutter range law is discarded and, instead, the range law is that probability of detection decreases as distance increases. Specifically,

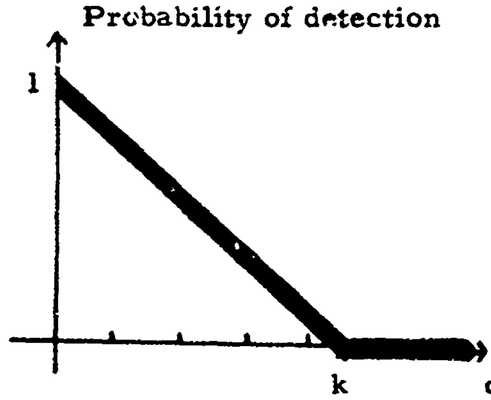

Figure 3.4

We call such games, which are again defined by specification of m and k, $G(m,k)$. Again, it is convenient to think of m as the number of identical blocks or square submatrices of dimension k which are of the form

$$\frac{1}{k}\begin{bmatrix} k & k-1 & \cdot & \cdot & \cdot & 1 \\ k-1 & k & k-1 & \cdot & \cdot & 2 \\ \cdot & \cdot & & & & \cdot \\ \cdot & \cdot & & & & \cdot \\ 1 & 2 & \cdot & \cdot & \cdot & k \end{bmatrix}$$

and lie along the main diagonal of $G(m,k)$. For example, $G(2,3)$ is the 6×6 matrix

$$\frac{1}{3}\left[\begin{array}{ccc:ccc} 3 & 2 & 1 & & & \\ 2 & 3 & 2 & 1 & & \\ 1 & 2 & 3 & 1 & 2 & \\ \hdashline & 1 & 2 & 3 & 2 & 1 \\ & & 1 & 2 & 3 & 2 \\ & & & 1 & 2 & 3 \end{array}\right] = G(2,3)$$

We have found that the value and optimal strategies for any game $G(m,k)$ can be specified.

The value of the game $G(m,k)$ is $\frac{mk+1}{km(m+1)}$. An optimal strategy for the patroller or row player is to choose row $ik+1$ $(i = 0, \ldots, m-1)$ with probability $(m-i)/m(m+1)$; row ik $(i = 1, \ldots, m)$ with probability $i/m(m+1)$; and all other rows with probability zero. Symmetrically, the transitor or column player is to choose columns i with the same probability with which the row player chooses rows i . Thus, for example, the value of the game $G(2,3)$ is $7/18$ and optimal strategies for both players are $1/6[2,0,1,1,0,2]$.

To prove this statement it suffices to show that the expected gains or winnings of the row player against any pure strategy of the column player is precisely the value of the game. Then, since the game matrix is symmetric, the identical strategy for the column player assures him of losing the value as well. So, consider any column, say column $\ell k+j$. Its non-zero entries lie within the rows $\ell k+j-k$ through

$\ell k + j + k$. Within these, the value of the entry in row ℓk is $k - j$; in row $\ell k + 1$ is $k - j + 1$; in row $(\ell+1)k$ is j; and in row $(\ell+1)k + 1$ is $j - 1$. Thus, since the mixed strategy above plays only these rows with non-zero probabilities, the expected winnings against column $\ell k + j$ is

$$\frac{1}{m(m+1)} \cdot \frac{1}{k}\left\{\ell(k-j) + (m-\ell)(k-j+1) + (\ell+1)j + (m-\ell-1)(j-1)\right\}$$

$$= \frac{mk+1}{km(m+1)}$$

thus proving the assertions.

Notice that as, k becomes large, the value of the game approaches $(m + 1)^{-1}$. Further, the parameter k enters only for purposes of determining the cycle of non zero choices of pure strategies. Thus, as an average figure, we obtain the approximate result that a patroller able to detect a transitor with probability $1 - \frac{x}{d}$, if $x \leq d$ where x is the distance between them, d the maximum distance at which detection is possible, has probability $(m + 1)^{-1}$ of detecting a transitor, where D is the width of the patrol zone. In $G(m, k)$ we interpret $d = k$ and $D = mk$ so that $\frac{d}{D+d}$ is $(m + 1)^{-1}$, agreeing with the result above as k becomes large (which is essentially a change of scale). Thus it would seem that the "measure of effectiveness" $\frac{d}{D+d}$ should have a bearing on the design of barriers, and, more particularly, on the definition of the width D of a patrol zone, in relation to the secure sweep width, which is proportional to d.

Assume that d is known and that D may be chosen in such a way as to make $D/d = m$ integer. The implications of the assumed range law, with its maximum range d for a detection to occur with non-zero probability, are then as follows: When m is 1 and the range of detection is the whole width of the barrier zone, the patroller's mixed strategy assigns equal weight to each end of the zone and zero and zero weight to the "center". When $m = 2$ and the range of detection is half the zone width, the mixed strategy assigns essentially equal weight to the "center" of the zone and each end of the zone. At the other extreme, when m is very large (range of detection is a very small fraction of zone width) the mixed strategy for patrol assigns equal non-zero weight to pairs of adjacent interior stations spaced out d units apart, and double weight to each end station. For intermediate values of m the results are qualitatively the same, except that the optimality is more sensitive to the correct positioning of the individual stations. The main point to realize about the nature of these mixed strategies is that they decisively do <u>not</u> recommend a uniform patrol back and forth across the zone. Instead, the designation of a particular set of $m+1$ rows out of the mk rows in the game matrix for use in the patrol strategy is equivalent to a recommendation of a number of favored positions for optimal patrol in the barrier. Of course the recommendation is contingent upon the particular form of the range law assumed. Other range laws will produce different mixed strategies; in general it is reasonable to expect that they will also show a pattern which favors certain distinguished points of the zonal width.

Large Matrix Games

A difficulty with many of the analytic models discussed previously is that the strategy spaces of both the patroller and the transitor are considerably too restrictive to adequately represent actual strategy alternatives. A methodological approach which would allow at least some analytic probing of tradeoffs between such factors as speed, use of active sonar, patrol patterns, etc., is described below. Allow the transitors pure strategies which depend upon the choice of transit lane and choice of speed; allow the patrollers pure strategies which depend upon varying speeds, type of patrol (e.g., bow-tie, back and forth); but in all cases assume that the set of choices within each category is finite. For every choice, which can be represented by a t-tuple of integers for the transitor and a p-tuple of integers for the patroller, find, according to given functions or from data produced by fleet exercises, probabilities of detection, or of first detection, or of no detection, etc., by the patroller and the transitor. On the basis of these, assign subjective values to the patroller for mutually exclusive outcomes. This results in a matrix having as many rows as the product of the number of speeds, patrols, etc., available to the patroller, and as many columns as the product of the number of transit lanes, speeds, etc., available to the transitor. This matrix game could then be solved by means of linear programming to determine optimal mixed strategy patrols for the patroller and transits for the transitor. The difficulties with this approach lie in two areas: availability of data concerning probabilities of detection as a function of speeds, patrol patterns, etc., and the size of the resulting matrix. Nevertheless, extremely useful insights could be obtained.

The size of the resulting problem may appear to be too large. Current computer limitations would allow at most some 1000 pure strategies for one player, though a practically unlimited number for the other. The hope, of course, is that some form of decomposition approach might lend itself to permit efficient computation of large problems. Unfortunately it seems that the structure of most matrix games does not allow such an approach. The basic reason for this can be summarized by saying that the value of the game is a complicated rational function of the values of the matrix entries. The simplest applicable type of decomposition is for matrices which are the tensor product of other matrices. This type of approach was investigated in ref [7] through rather involved and lengthy arguments. We present those ideas in a direct and simple manner here.

Let A be a matrix game, x a mixed strategy on rows, y a mixed strategy on columns. Then, by definition, x^* and y^* are optimal and λ is the value of the game A if and only if

$$x^*A \geq \lambda e \, , \qquad \text{and} \qquad Ay^* \leq \lambda e \, .$$

where e is a vector of 1's (one's) of appropriate dimension. We have the following result:

Let x and y be optimal for A with value $\lambda \geq 0$, and $\overline{x}$, $\overline{y}$ be optimal for $\overline{A}$ with value $\overline{\lambda} \geq 0$. Assume $Ay \geq 0$ and $\overline{A}\overline{y} \geq 0$ (it is sufficient to have $A, \overline{A} \geq 0$ for this assumption to hold). Then $x \otimes \overline{x}$ and $y \otimes \overline{y}$ are optimal for the game $A \otimes \overline{A}$, the tensor product of A and $\overline{A}$, with value $\lambda\overline{\lambda}$.

The proof of this is trivial, for

$$(x \otimes \overline{x})(A \otimes \overline{A}) = xA \otimes \overline{x}\overline{A} \geq \lambda\overline{\lambda}\, e \quad \text{and}$$

$$(A \otimes \overline{A})(y \otimes \overline{y}) = Ay \otimes \overline{A}\,\overline{y} \leq \lambda\overline{\lambda}\, e\ .$$

the last inequality holding by the explicit assumption made in the statement above.

As an example consider

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}. \qquad \overline{A} = \begin{pmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \\ 1 & 2 & 1 \end{pmatrix}.$$

Then $x = (3/5, 2/5)$, $y = (3/5, 2/5)$, $\lambda = 1/5$; and $\overline{x} = (1/16, 4/16, 11/16)$, $\overline{y} = (6/16, 2/16, 8/16)$,

$\bar{\lambda} = 18/16$; and the assumption $Ay \geq 0$, $\overline{Ay} \geq 0$ is satisfied. Therefore, the optimal solution to

$$A \otimes \bar{A} = \left(\begin{array}{rrr|rrr} -1 & 0 & 3 & 1 & 0 & -3 \\ 2 & -1 & 1 & -2 & 1 & -1 \\ 1 & 2 & 1 & -1 & -2 & -1 \\ \hline 1 & 0 & 1 & -2 & 0 & -1 \\ -2 & 1 & -1 & 4 & -2 & 2 \\ -1 & -2 & -1 & 2 & 4 & 2 \end{array} \right)$$

is

$$x \otimes \bar{x} = \frac{1}{80} (3, 12, 33, 2, 8, 22),$$

$$y \otimes \bar{y} = \frac{1}{80} (18, 6, 24, 12, 4, 16),$$

with value

$$\lambda\bar{\lambda} = \frac{18}{80} \quad .$$

This result is applicable to situations where the effects of two parameters are multiplicative. For example, if the effect of patrol pattern on detection-probability was independent of the effect of speed, the payoff-matrix G_{ps} obtained when both parameters were taken as strategic variables would be proportional to the tensor product $G_p \otimes G_s$ of the payoff matrix G_p of the "pattern game" with the payoff matrix G_s of the "speed game".

True independence of such a pair of parameters, of course, would be a rare circumstance -- but if the factors were nearly independent, the above tensor-product theorem would provide a useful first approximation to a solution.

If the strategic choices for each player involved more than two parameters, we could employ the obvious generalization of the above theorem to a tensor product of $\underline{n}$ matrices.

Theorem: For each $\alpha = 1, \ldots, n$, suppose A^α is the matrix of a game which has optimal strategies x^α and y^α, and value $\lambda^\alpha \geq 0$. If also $A^\alpha y^\alpha \geq 0$ for each α, then the game whose matrix is the n-fold tensor product $A = A^1 \otimes A^2 \otimes \ldots \otimes A^n$ has value $\lambda = \lambda^1 . \lambda^2 \ldots \lambda^n$ and optimal strategies $x = x^1 \otimes x^2 \otimes \ldots \otimes x^n$ and $y = y^1 \otimes y^2 \otimes \ldots \otimes y^n$.

III. 5 Continuous Analogs of Toeplitz Matrix Games

In Part III. 4 "Large Structured Matrix Games," a class of large matrix games $G(m, k)$ was introduced in which the transitor has as pure strategies the choice of one of a finite number of transit lanes (choice of a column in $G(m, k)$) and the patroller has as pure strategies the choice of one of a finite number of stations at which he can hover. Then, the probability that the patroller detects the transitor is $(k - d)/k$ if $d \leq k$ and 0 if $d > k$, where d is the distance between the patroller and the transitor.

It is natural to consider the continuous analog of this class of matrix games. This game is defined as follows: Let

$$\phi(x, y) = 1 - \frac{|x - y|}{d} \quad \text{if} \quad |x - y| \leq d$$

$$= 0 \quad \text{if} \quad |x - y| > d$$

where $0 \leq x, y \leq D$, be the payoff of player II, the y-player, to player I, the x-player. The interpretation for this game is: the patroller has as pure strategies the choice of a point x on the line $[0, D]$, the transitor has as pure strategies the choice of a point y on the line $[0, D]$; the probability of the patroller detecting the transitor is given by $\phi(x, y)$. Thus d is the maximum range in which detection is possible; D is the width of the patrol zone. For fixed x, say $x = \bar{x}$, $\phi(\bar{x}, y)$ describes the range law:

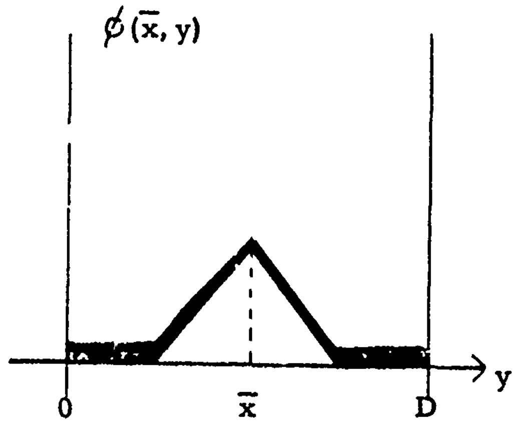

Figure 3.5

It is most surprising to find that both players have optimal mixed strategies which each use only a finite number of pure strategies. A mixed strategy is a probability distribution f on the choice point x for player I, and a probability distribution g on the choice of point y for player II.

The value of the game defined above, for D/d not integer-valued, is

$$\frac{2n - D/d}{n(n+1)}$$

where $n = [D/d] + 1$, with $[.]$ denoting "integer part of." Note that $n \approx D/d$ so that the value of the game is about $\frac{d}{D+d}$, as was indicated in an approximate manner in Part III. 4 . An optimal strategy for player I is to choose a distribution $f^*(x)$ defined as follows:

$$f^*(x) = f_L^*(x) + f_R^*(x)$$

where

$$f_L^*(x) = f_R^*(x) = 0 \text{ if } x \neq kd \text{ ; } x \neq D - kd \text{ ,}$$

$$k \text{ integer ,}$$

$$f_L^*(kd) = \frac{n-k}{n(n+1)} = f_R^*(D-kd) \text{ ,}$$

where, again, $n = [D/d] + 1$. An optimal strategy for player II is to choose a distribution $g^*(y)$ where $g^*(y) = g_L^*(y) + g_R^*(y)$, with $g_L^*(y) = f_L^*(y)$, $g_R^*(y) = f_R^*(y)$.

For example, if $d = 5$, $D = 16$ we have $n = [16/5] + 1 = 4$; the value of the game is 24/100 ; and each player uses 8 pure strategies

$$f_L^*(0) = \frac{4}{20} = f_R^*(16) \text{ ; } f_L^*(5) \qquad = f_R^*(11)$$

$$f_L^*(10) = \frac{2}{20} = f_R^*(6) \quad ; \quad f_L^*(15) = \frac{1}{20} = f_R^*(1) \ .$$

g^* is the same as f^*. Graphically this means that the patroller should circle at one of 8 stations as indicated in Figure 3.6 below.

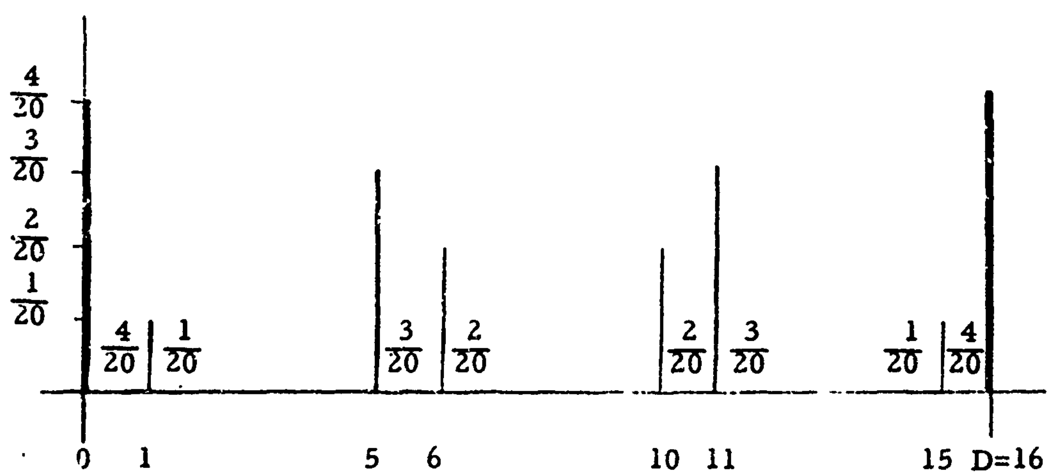

Figure 3.6

Let $E(\phi; f, g)$ denote the expected winnings of player I given that I uses mixed strategy f and II uses mixed strategy g. To prove the contention above it suffices to show

$$E(\phi; f, g^*) \leq E(\phi; f^*, g^*) \leq E(\phi; f^*, g) \ ,$$

that is, f^* is optimal against g^*, and g^* is optimal against f^*.

We first compute $E(\phi; f, g^*)$.

$$E(\phi; f, g^*) = \frac{1}{n(n+1)} \sum_{j=0}^{n} (n-j) \left\{ \frac{1}{d} \int_{jd}^{(j+1)d} (d - x + jd)\, df + \frac{1}{d} \int_{jd-d}^{jd} (x + d - jd)\, df \right\}$$

$$+ \frac{1}{n(n+1)} \sum_{k=0}^{n} (n-k) \left\{ \frac{1}{d} \int_{D-kd-d}^{D-kd} (d - D + kd + x)\, df + \frac{1}{d} \int_{D-kd}^{D-kd+d} (d - kd + D - x)\, df \right\}$$

since $\phi(x,y) = 0$ for $|x-y| > d$, where we have "broken up" the interval $[0, D]$ into $[0, D-nd+d, d]$, $[D-nd+d, d]$, $[d, D-nd+2d]$, etc.; or, in general into $[hd-d, D-nd+hd]$, $[D-nd+hd, hd]$. Consider, first, the contribution to $E(\phi; f, g^*)$ due to the interval $[hd-d, D-nd+hd]$. From the first sum, with $j = h-1$ and $j = h$ we get

$$\frac{n-h+1}{n(n+1)d} \int (d-x+hd-d)df \quad \text{and} \quad \frac{n-h}{n(n+1)d} \int (x+d-hd)df$$

while from the second sum with $k = n-h$ and $k = n-h+1$

we get

$$\frac{(n - n + h)}{n(n+1)d} \int (d - D + (n - h)\, d - x)\, df$$

and

$$\frac{(n - n + h - 1)}{n(n+1)d} \int (d - (n - h + 1)d + D - x)\, df \; .$$

Combining these results we obtain

$$\int \frac{(2n - D/d)}{n(n+1)}\, df \; .$$

Precisely the same result obtains when integration is performed over $[D - nd + hd,\ hd]$. Thus we find

$$E(\phi; f, g^*) = \frac{(2n - D/d)}{n(n+1)} \int_0^D df = \frac{2n - D/d}{n(n+1)} \; .$$

This says that against the strategy g^* player I can choose any mixed strategy and obtain the value $(2n - D/d)/n(n+1)$. Thus, in particular, $f = f^*$ maximizes $E(\phi; f, g^*)$. Similarly, $g = g^*$ minimizes $E(\phi; f^*, g)$. Therefore, f^*, g^* are optimal strategies and the value of the game is $(2n - D/d)/n(n+1)$.

For the D/d integer valued it is easy to verify that the value of the game is $(n+1)^{-1}$ where $n = D/d$, and both players play points kd $(k = 0, \ldots, n)$ with probability $(n+1)^{-1}$.

It seems clear, as is the case for the optimal strategies for the games $G(m, k)$, that f^* and g^* are unique. So we obtain similar results, about the same "measure of effectiveness" $\frac{\alpha}{D+d}$, and, again, a recommendation that the patrollers should use a finite number of favored stations or positions for specified proportions of time. Of course, the result depends upon the particular range law $\phi(x, y)$ which is chosen. Nonetheless, reasonable range laws would seem to lead to the same type of qualitative result; certain distinguished stations should be used.

The game $\phi(x, y)$ which has been analyzed above is the continuous analog of the matrix games $G(m, k)$. For purposes of comparison it is interesting to consider the continuous analog of the matrix games $H(m, k)$ (see Part III. 4). It is natural to formulate this game as follows. Let

$$\begin{aligned} \Psi(x, y) &= 1 \quad \text{if} \quad |x - y| \le d \quad , \\ &= 0 \quad \text{if} \quad |x - y| > d \quad , \end{aligned}$$

where $0 \le x, y \le D$, be the "payoff" of player II, the y-player, to player I, the x-player. Here, again, D is the width of the patrol zone, and d is the maximum range in which sure detection is possible with probability $\Psi = 1$. Thus, we have the cookie-cutter range law: for fixed x, say $x = \bar{x}$, $\Psi(\bar{x}, y)$ describes the range law

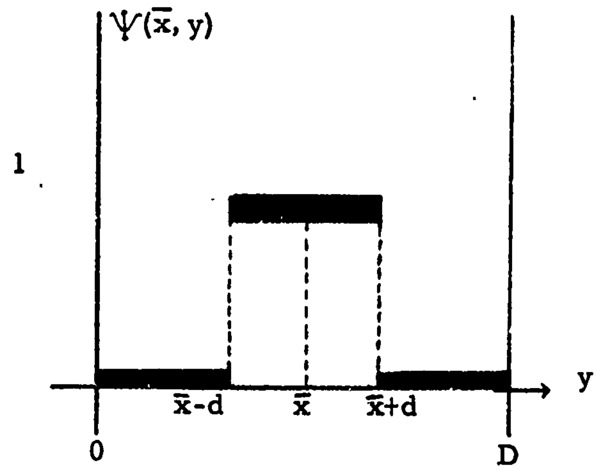

Figure 3.7

The value of the game Ψ, for D/2d not integer valued, is

$$\frac{1}{m}$$

where $m = [D/2d] + 1$. Note that $m \approx D/2d$ so that the "measure of effectiveness" is about d/D. An optimal strategy for player I is to choose a distribution $f^{*}(x)$ for which

$$f^*((2k+1)\,d) = \frac{1}{m}, \quad k = 0, \ldots, m-2 \quad ,(k = 0, m-2),$$

$$f^*(x^*) = \frac{1}{m} \quad \text{where} \quad x^* = \min\left\{(2m-1)\,d, D\right\},$$

$$f^*(x) = 0 \quad \text{otherwise;}$$

and an optimal strategy for player II is to choose a distribution $g^*(y)$

$$g^*(k(2d+\varepsilon)) = \frac{1}{m}, \quad k = 0, \ldots, \quad -1 \; (k = 0 \ldots, m-1)$$

$$g^*(y) = 0 \quad \text{otherwise}.$$

Here $0 < \varepsilon \leq (D/2d - m + 1)2d/(m-1)$. The basic point of the transitor's strategy is that m points y should be used, but that each pair of such points should be distant <u>more than</u> $2d$.

For example, if $d = 5$, $D = 16$, then $m = 2$; the value of the game is $\frac{1}{2}$ and each player uses 2 pure strategies; ε may be between 0 and 6 (we choose 0.5):

$$f^*(5) = f^*(15) = \frac{1}{2}, \quad g^*(0) = g^*(10.5) = \frac{1}{2}.$$

The proof of this statement is straightforward.

$$E(\Psi; f, g^*) = \frac{1}{m}\int_0^d df + \frac{1}{m}\sum_{k=1}^{m=1} \int_{(2k-1)d+k\varepsilon}^{(2k+1)d+k\varepsilon} df .$$

Thus, an f which maximizes is any distribution f which has all its weight concentrated in the intervals specified in the above definite integrals. $f = f^*$ is such a distribution. So, $E(\Psi; f, g^*) \leq E(\Psi; f^*, g^*)$. The inequality $E(\Psi; f^*, g^*) \leq E(\Psi; f^*, g)$ is immediate.

If, on the other hand, $D/2d$ is integer valued it is easy to see that the value of the game is $1/m$, with $m = D/2d$, player I plays points $(2k + 1)d$, $k = 0, \ldots, m - 1$, with probability $\frac{1}{m}$, player II plays points $k(2d + \varepsilon)$, $K = 0, \ldots, m - 1$, with probability $\frac{1}{m}$, where $0 < \varepsilon \leq 2d/m - 1$.

This singles out an interesting observation. Namely, if one believes that the range law is of cookie-cutter type, then the width of patrol zone D should be chosen so that $D/2d$ is integer valued. For if it is not so chosen then the patrol width might just as well be increased: this can lead to no decrease in the probability of detection. This seemingly peculiar result is a consequence of the discontinuous nature of $\Psi(x, y)$. Another way of pointing to this peculiarity is to note that if $D/2d = m$ is integer-valued then the value of the game is $\frac{1}{m}$, while if the width is increased to $D + \delta$, with $\delta > 0$ but arbitrarily small, then the value of the game jumps down to $1/(m + 1)$.

Finally, notice that the measure of effectiveness for Ψ with defining parameters d, D is about 2d/D while that for ϕ with defining parameters d, D is about $\frac{d}{D+d}$. In terms of the extended definition of SSW given in Part II this means that the cookie-cutter range law yields an SSW of 2d while that of the "linearly decreasing" range law yields an SSW of $\frac{dD}{D+d}$.

III. 6 Repeated Games

Recent research in the mathematical theory of games by J. Harsanyi [3]*, and R. Aumann and M. Maschler [1], [2] has resulted in a theory which provides strategic analysis for a sequence of repeated conflict situations in which the players -- the patroller and the transitor, for instance -- have incomplete information about the payoffs. This section represents an attempt to apply some of their results to ASW, and to interpret them in the context of a barrier situation.

Repeated Zero-sum Games of Incomplete Information are formal models of a sequence of closely related strategic situations. Two players are to make choices of strategy at each stage of an indefinitely long sequence of matrix games. One or both of the players may lack certain information which would specify the true payoffs at each stage. The alternative matrix games which determine the actual sequence being played are known to both: call them $G_1, G_2, \ldots, G_\alpha$, where G_i is an $m \times n$ matrix game for each i. In addition, a prior probability distribution over the alternatives, $(q_1, q_2, \ldots, q_\alpha)$ representing the state of information of the uninformed player** is known to both players.

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* See Bibliography on page 109 of this report.

** Assuming for the purposes of discussion that only one player has incomplete information.

Naturally the sum of the q_i's is 1; in the case where one of the q_i's is exactly equal to 1 the inforn ation of both players is complete. For all other cases there is said to be incomplete information.

In the sequence of repeated games, each player attempts to maximize his gain (minimize his loss) in the sense of the long-run average payoff per stage. The players are generally not able to learn exactly what the payoffs are at each stage, but if one player has complete information he, of course, can deduce them. The uninformed player must try to discover the payoffs by observing the strategy choices of the opponent, and he may in some cases approach a state of complete information. There are also examples where no information is revealed to him by an intelligent opponent's choices. In the latter cases the players are simply playing the game (one might call it the expected or averaged game) whose matrix is given by

$$\overline{G} = \Sigma_{i=1}^{\alpha} q_i G_i \quad ,$$

and the optimal strategies for both players are determined by the usual minimax solution to the matrix game defined by $\overline{G}$.

The analysis of the repeated games with incomplete information proceeds as if the first move were a random choice (by nature) of one of the matrix games $G_1, G_2, \ldots,$ G_α. The informed player is, in effect, told the outcome of the "choice of chance" while the uninformed player is not. One of the questions which the theory has sought to answer is: to what extent and by which type of strategy should the informed player make use of his extra information? In a single stage of the sequence, the information, if it is advantageous, can clearly be exploited. However, it is not clear that the informed player can continue to gain advantage from his information in the long run. One of the surprising results of the work of Aumann and Maschler is that, in some situations, the informed player must act as if he were uninformed if he wishes to maximize his long-run average gains. We turn now to the consideration of some rather simple examples in which these considerations may play a rôle for the patroller and the transitor.

The notion of games of incomplete information may cast new light on situations which can arise in an ASW barrier context, where the patroller (Player I) has certain information which the transitor does not have -- such as the existence of abnormal acoustical situations, or the likely sonar range under various conditions. The possibility exists that the transitor (Player II) could infer information, which would be useful to him, by observing the strategy of the patroller -- i.e., by observing how the patroller used his additional information.

We have therefore formulated several games of this type.

Example 1. The patroller knows detection-range, the transitor does not.

To be specific, we may assume that the patroller (Player I) knows the sonar detection-range, that the transitor (Player II) knows only a probability-distribution over the sonar detection-ranges, and that the transitor discovers I's patrol strategy with some time-lag. For example, long-term predictions of detection-range will be known to both, actual ranges may change only slightly from day to day, and the previous days' patrolling orders may be assumed deciphered* and available to the would-be transitor.

Furthermore, we assume that, when the sonar range is known, the detection game will of the form called G(m, k) in Chapter III. 4 of this report. G(m, k) distinguishes among mk possible positions for the patroller, and among mk corresponding lanes for the transitor. The optimal strategy for Player I in the game G(m, k) is given in Chapter III. 4 of this report; the optimal strategy for Player II is identical; the value of the game is $\frac{mk+1}{(m+1)mk}$.

*It is usual to assume that security and cryptological measures can only delay, and not prevent, the enemy's interpretation of a message. For that reason we assume that the transitor will eventually discover information which the patroller continues to use.

Now consider two such games, defined as $G(m_1, k_1)$ and $G(m_2, k_2)$, such that $m_1 k_1 = m_2 k_2$. The matrices of these games are of the same size -- viz., $(m_1 k_1) \times (m_1 k_1)$ -- and therefore a game of incomplete information can be defined with a prior probability q_1 of playing the game $G(m_1, k_1)$ and a prior probability q_2 of playing the game $G(m_2, k_2)$. An example of the appearance of the matrices with $m_1 = k_2 = 3$, $m_2 = k_1 = 2$, could look like this:

Probability q_1 :

$$G_1 = G(3,2) = \frac{1}{2}\begin{pmatrix} 2 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 2 & 1 & & \\ & & 1 & 2 & 1 & \\ & & & 1 & 2 & 1 \\ & & & & 1 & 2 \end{pmatrix} \qquad v = \frac{7}{24}$$

Probability $(1 - q_1) = q_2$:

$$G_2 = G(2,3) = \frac{1}{3}\begin{pmatrix} 3 & 2 & 1 & & & & \\ 2 & 3 & 2 & 1 & & & \\ 1 & 2 & 3 & 2 & 1 & & \\ & 1 & 2 & 3 & 2 & 1 & \\ & & 1 & 2 & 3 & 2 & 1 \\ & & & 1 & 2 & 3 & 2 \\ & & & & 1 & 2 & 3 \end{pmatrix} \qquad v = \frac{7}{18}$$

The average matrix, $\overline{G}$, is then given by

$$\overline{G} = G_1 + q_2 \cdot (G_2 - G_1) \ .$$

To simplify the appearance of subsequent formulae we will take $x = q_2$, so that $q_1 = 1 - x$, and the matrix $\overline{G}$, which we shall denote by $\overline{G}(x)$ to recall its dependence on x, becomes

$$\overline{G}(x) = \frac{1}{6}\begin{pmatrix} 6 & 3+x & 2x & & & \\ 3+x & 6 & 3+x & 2x & & \\ 2x & 3+x & 6 & 3+x & 2x & \\ & 2x & 3+x & 6 & 3+x & 2x \\ & & 2x & 3+x & 6 & 3+x \\ & & & 2x & 3+x & 6 \end{pmatrix}$$

The value of $\overline{G}(0)$ is then $v(G(3,2))$, which is known to be $\frac{7}{24} = 0.2917$; the value of $\overline{G}(1)$ is then $v(G(2,3))$, which is known to be $\frac{7}{18} = 0.3889$. Calculation* gives for $0 \leq x \leq 1$ the value

$$v(\overline{G}(x)) = \frac{189 - 99x - 45x^2 + 11x^3}{72\,(9 - 8x + x^2)}$$

which is tabulated below as a function of x, and is shown in Figure 3.8.

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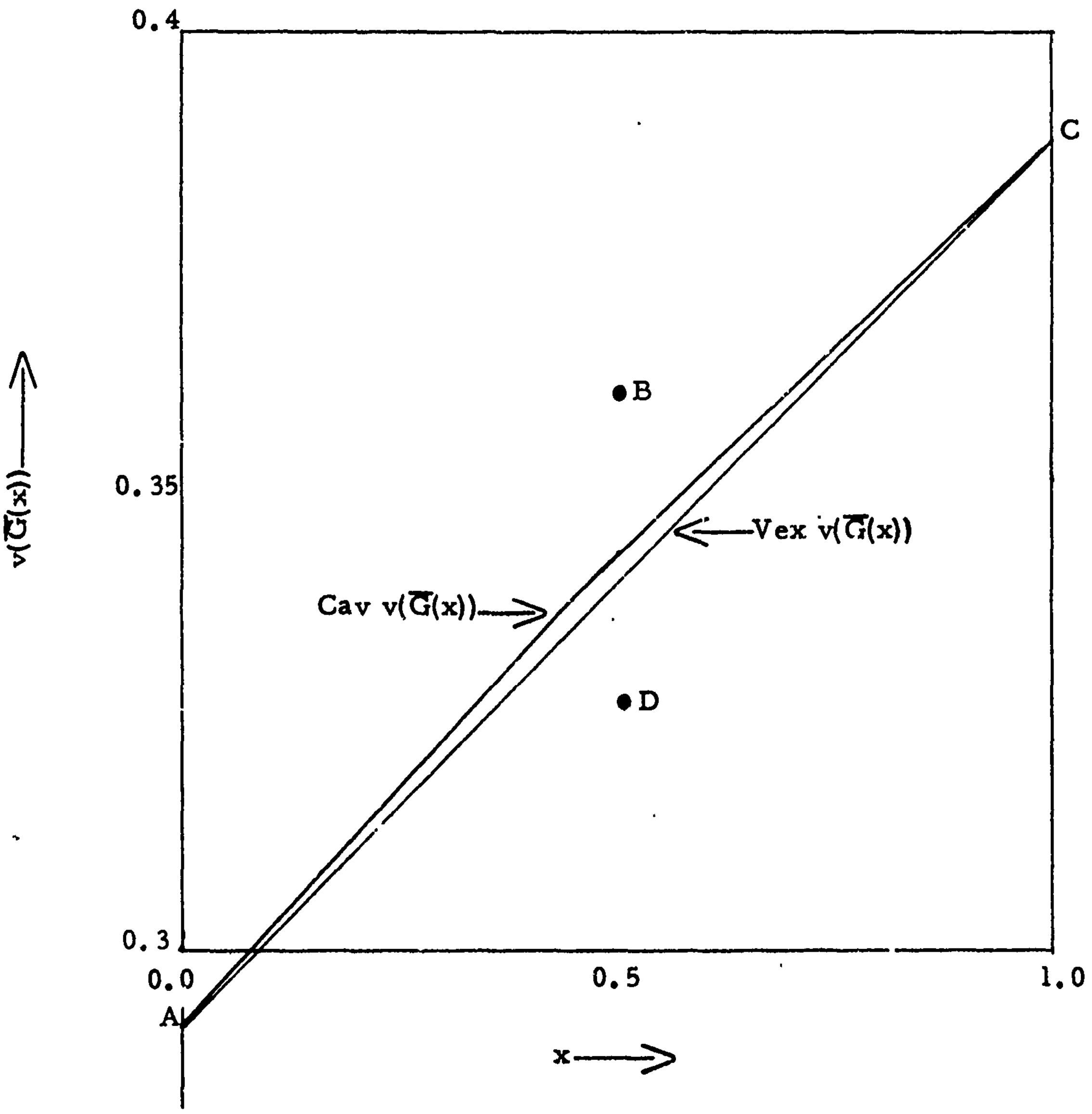
* too tedious to be reproduced



A, C are intersection points of Cav $v(\overline{G}(x))$ and Vex $v(\overline{G}(x))$

B = Γ_1, one-shot game with I informed

D = Γ_2, one-shot game with II informed

Figure 3.8

Value of Games of Example 1

x	v $(\overline{G}(x))$	
0.0	7/24	= 0.2917
0.1	178661/591120	= 0.3022
0.2	2617/8370	= 0.3127
0.3	5716/17840	= 0.3229
0.4	17863/53640	= 0.3330
0.5	1037/3024	= 0.3429
0.6	402/1115	= 0.3526
0.7	101423/280080	= 0.3621
0.8	10829/29160	= 0.3714
0.9	2647/6960	= 0.3803
1.0	7/18	= 0.3889

When we examine the convexity of this function of x, we find that it is a concave function; from this fact, and from the general theorems referred to above, we conclude that Player I (the patroller) should not make use of his additional information to improve his strategy when the transitor could observe the strategy -- because the transitor could then use those observations to deduce which stage game had actually been chosen by chance (i.e., what the patroller had discovered about the sonar detection-range).

This result is, to put it mildly, astonishing. We shall describe the optimal strategies in the games $G(0)$, $G(\frac{1}{2})$, $G(1)$, and attempt to clarify the situation.

Optimal strategy in $G(0) = G(3,2)$ is: $\frac{1}{12}(3,1,2,2,1,3)$;

Optimal strategy in $G(1) = G(2,3)$ is: $\frac{1}{6}(2,0,1,1,0,2)$;

Optimal strategy in $G(\frac{1}{2})$ is: $\frac{1}{252}(71,15,40,40,15,71)$.

(In each case, the strategy cited is optimal for both players). Now if the game is, with equal probability, $G(2,3)$ or $G(3,2)$, but neither player knows which, they must play against the expected outcome $G(\frac{1}{2})$ and the value of the game will be <u>0.3429</u> . If Player I(the maximizer) knows which it is, he may profit by adjusting his strategy appropriately.* However, if he does so adjust his strategy, and if Player II (the minimizer) can observe his play and deduce which game is really being played, they will find themselves in the long run playing either $G(3,2)$ or $G(2,3)$ -- with values 0.2917 and 0.3889 respectively, and with probability $\frac{1}{2}$ of each. The expected value of that game to Player I (the patroller)

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* We specify below the optimal strategies for that one-shot game.

is then $\frac{1}{2}(0.2917 + 0.3889) = \underline{0.3403.}$

It therefore profits the patroller to <u>ignore</u> the information he gets on the detection-range, under these circumstances. If the patroller is not concerned with his strategy being revealed -- (see Fig. 3.9) -- to the transitor, the fact that the game might be repeated is of no importance. Exact determination of the optimal strategy, and the value, for this one-shot game which we call Γ_1 will tell us how much the patroller <u>could</u> benefit from knowing the true detection-range when the transitor does not know it.

Symmetry tells us that optimal strategies will involve location in position 1 as often as in 6, 2 as often as 5, and 3 as often as 4; we may therefore draw up the matrix of strategies for the "one-shot" game, wherein neither revelation of intelligence nor repetition of the same conflict are significant factors. A strategy for Player I consists of a pair (r_1, r_2) where $r_i = 1, 2,$ or 3; $r_i = 1$ denotes that the patroller patrols in positions 1 and 6, $r_i = 2$ denotes positions 2 and 5, and $r_i = 3$ denotes positions 3 and 4; the first number, r_1, tells how player I patrols if the game is really G_1, and the second number, r_2, tells how he patrols if the game is really G_2. The strategies for Player II are 1, 2, or 3, corresponding respectively to transits in lanes 1 and 6, lanes 2 and 5, or lanes 3 and 4.

The payoff matrix for the game Γ_1 is then as shown

Chance Move

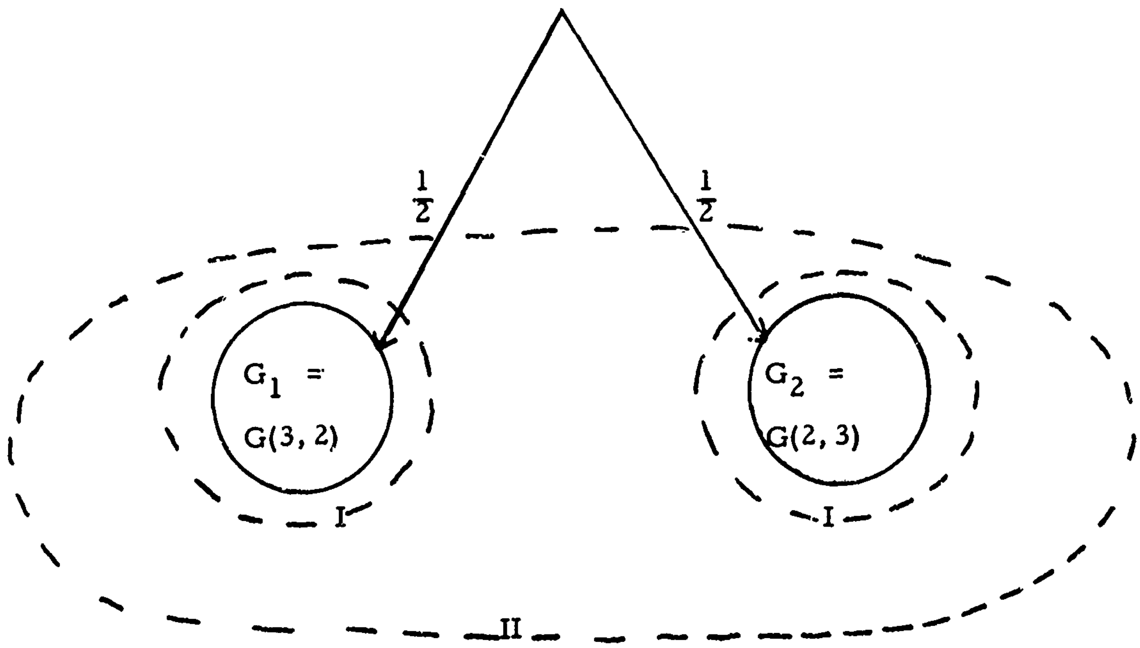

Figure 3.9: The Game Γ_1

Player I is told the result of the first chance move, and therefore knows whether G_1 or G_2 is being played; Player II is not given that information.

(we actually show $24\ \Gamma_1$ to avoid fractions):

$24\Gamma_1$

		Pure Strategy for II		
		1	2	3
Pure Strategy for I	(1, 1)	12	7	2
	(1, 2)	10	9	6
	(1, 3)	9	9	10
	(2, 1)	9	10	5
	(2, 2)	7	12	9
	(2, 3)	5	12	13
	(3, 1)	6	7	11
	(3, 2)	4	9	15
	(3, 3)	2	9	19

It turns out that the optimal strategies for this one-shot game Γ_1 are:

(A)

$$\text{for Player I:} \left\{ \begin{array}{l} \frac{1}{3}(1,2) + \frac{2}{3}(1,3) \ , \\ \text{or } \frac{1}{6}(1,1) + \frac{5}{6}(1,3) \ . \\ \\ \text{or any linear combination of those;} \end{array} \right\}$$

for Player II: $\frac{2}{3}(1) + \frac{1}{3}(3)$.

The value of this game is then $\frac{13}{36} = 0.3611$.

Although the numerical values obtained in this one particular case are not in themselves of great importance, they serve to illustrate several important points:

(1) if the actual situation is either G_1 or G_2, with equal probability, but <u>neither</u> player knows which, the value is simply $v(G(\frac{1}{2}))$ because the expected payoff matrix is simply $\frac{1}{2}G_1 + \frac{1}{2}G_2 = G(\frac{1}{2})$. Value = <u>0.3429</u> ;

(2) if the actual situation is either G_1 or G_2, with equal probability, and the patroller knows which, but the transitor does not, then the patroller can patrol the end lanes (#1 and #6) when the game is G_1 and some mixed behavior-strategy satisfying (A) above when the game is G_2, then the value is increased due to his additional knowledge; value = <u>0.3611</u> ;

(3) if the transitor is unable to observe the detection-range (i.e., the random choice of G_1 or G_2) directly but is able to observe strategies of the patroller over a long period, he could infer from the strategies as described by (A) above whether the game was really G_1 or G_2; then, in the long run, he would be playing either G_1 or G_2 (whichever happened to be chosen by chance) -- and they would be equally likely, so that the value would be $v(G_1)$ half the time and $v(G_2)$ the other half. The expected value would then be $\frac{1}{2}(v(G_1) + v(G_2))$; value = <u>0.3403.</u>

From the above numerical result we can see that, in this case, the benefit which Player I could receive due to an advantage in intelligence is substantial -- but that benefit would be essentially nullified if the game were played repeatedly and the transitor could discover the patroller's past strategies. In fact, under those circumstances the patroller should refrain from using the information which he possesses.

Example 2. The second example, which is similar in principle, results in answers which illustrate the opposite possibility. The example is identical with the preceding example, but we imagine that Player II (the minimizer, who is the transitor) can get the additional information about the detection-range (i.e., about whether G_1 or G_2 was chosen in the first random move.) To Player II, the average game $\overline{G}(\frac{1}{2})$ with expected payment of 0.3429, is less desirable than the opportunity of playing G_1 half the time and G_2 the other half (a situation which has expected payment of 0.3403). Therefore Player II would benefit by using the information, even though he would reveal it in using it. If, as the opposite extreme case, Player II could use the additional information without revealing anything, the asymmetrical one-shot game Γ_2 would be played: the matrix of the game Γ_2 is simply the transpose of matrix Γ_1 shown above; the optimal strategy for the well-informed transitor is then

$$\frac{1}{6}(1,1) + \frac{5}{18}(1,2) + \frac{5}{9}(1,3)$$

and the optimal strategy for the uninformed patroller is

$$\frac{1}{2}(1) + \frac{1}{6}(2) + \frac{1}{3}(3) \ .$$

The value is then $\frac{47}{144} = \underline{0.3263}$.

The results of this example are also shown in Figure 3.8, p. 91.

The above examples used a special form of the detection range law -- viz.,

$$\begin{cases} \frac{P_o}{k}(k - |i-j|) & \text{for } i - j \leq k \ , \\ 0 & \text{if } i - j \geq k \ . \end{cases}$$

This "triangular" detection range law results in a discrete optimal strategy for each player, as described in Chapter III, Section 4 of this report. It also guarantees that the optimal defense will be equal-gamma -- i.e., that an optimal patroller's strategy would cause the probability of detection to be independent of which transit lane was chosen by the transitor.

We have already considered the unilateral incomplete information game, in which the patroller knows detection-range but the transitor does not, and the transitor will discover the patroller's strategy in the long run. Because the transitor is always facing an equal level of defense in each lane for either of the stage games, it is not surprising that (i) he also faces an equal level of defense in each lane for the games of incomplete information obtained from combination of those stage games, and (ii) the value of those combination games is close to the value which would be expected if the stage game (in the above examples, the detection-range) were revealed immediately to both players as soon as the random choice was made.

We will now proceed with examples where the stage games are not equal-gamma games, where intuition is less useful in predicting good strategies for the unilateral game of incomplete information and where the better-informed player might be able to reap a substantial profit from his intelligence advantage.

In all these examples we continue to suppose that Player II must select a lane through which he attempts to transit undetected, and that Player I must select a position i in which to patrol. The probability of detection if I chooses i and II chooses j is g_{ij} and the array (g_{ij}) makes up the matrix G. The several alternative stage-game matrices are called G_1, G_2, ..., G_α.

Example 3. If there is a "blind" lane with smaller detection-probabilities than elsewhere, which is *a priori* equally likely to be any one of the lanes; and if the expected detection-probabilities in each lane are such that the "expected game" $\overline{G}$ has an "equal-gamma" solution, then it seems intuitively obvious that the transitor could profit by any hint (however small) as to which lane was actually the blind one -- consequently we would expect the payoff of such a $\overline{G}(q)$, as a function of q to be a concave function. We can illustrate this with a small example, and then prove it for the simple special case in which a transitor can only be detected by a patroller in the same lane -- so that all the matrices $G_1, G_2, \ldots, G_\alpha$ are diagonal.

Take

$$G_1 = \frac{1}{4} \begin{pmatrix} 4 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix} .$$

$$G_2 = \frac{1}{4} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix} ,$$

and $\overline{G}(x) = (1 - x) G_1 + x G_2$.

Then, explicitly:

$$\overline{G}(x) = \frac{1}{4} \begin{pmatrix} 4-2x & 1 & 0 \\ 1 & 2+2x & 1 \\ 0 & 1 & 4 \end{pmatrix} ;$$

strategy for Player I in $\overline{G}(x)$ is optimal

$$\frac{1}{(8 + 4x - 2x^2)} \cdot (2 + 4x,\ 4 - 3x,\ 2 + 3x - 2x^2) ;$$

optimal strategy for II is the same; and the value is

$$\left(\frac{12 + 9x - 8x^2}{32 + 16x - 8x^2} \right) .$$

This is tabulated below and is shown in Figure 3.10 on p. 103.

Note that

$$v(\overline{G}(0)) = v(G_1) = \frac{3}{8} = 0.3750 ,$$

$$v(\overline{G}(1)) = v(G_2) = \frac{13}{40} = 0.3250$$

$$v(\overline{G}(\tfrac{1}{2})) = \frac{29}{76} = 0.3816 .$$

(Incidentally, $v(\overline{G}(\frac{1}{2}))$ is not only greater than the average between $v(G_1)$ and $v(G_2)$, but is in this case actually greater than either of them.)

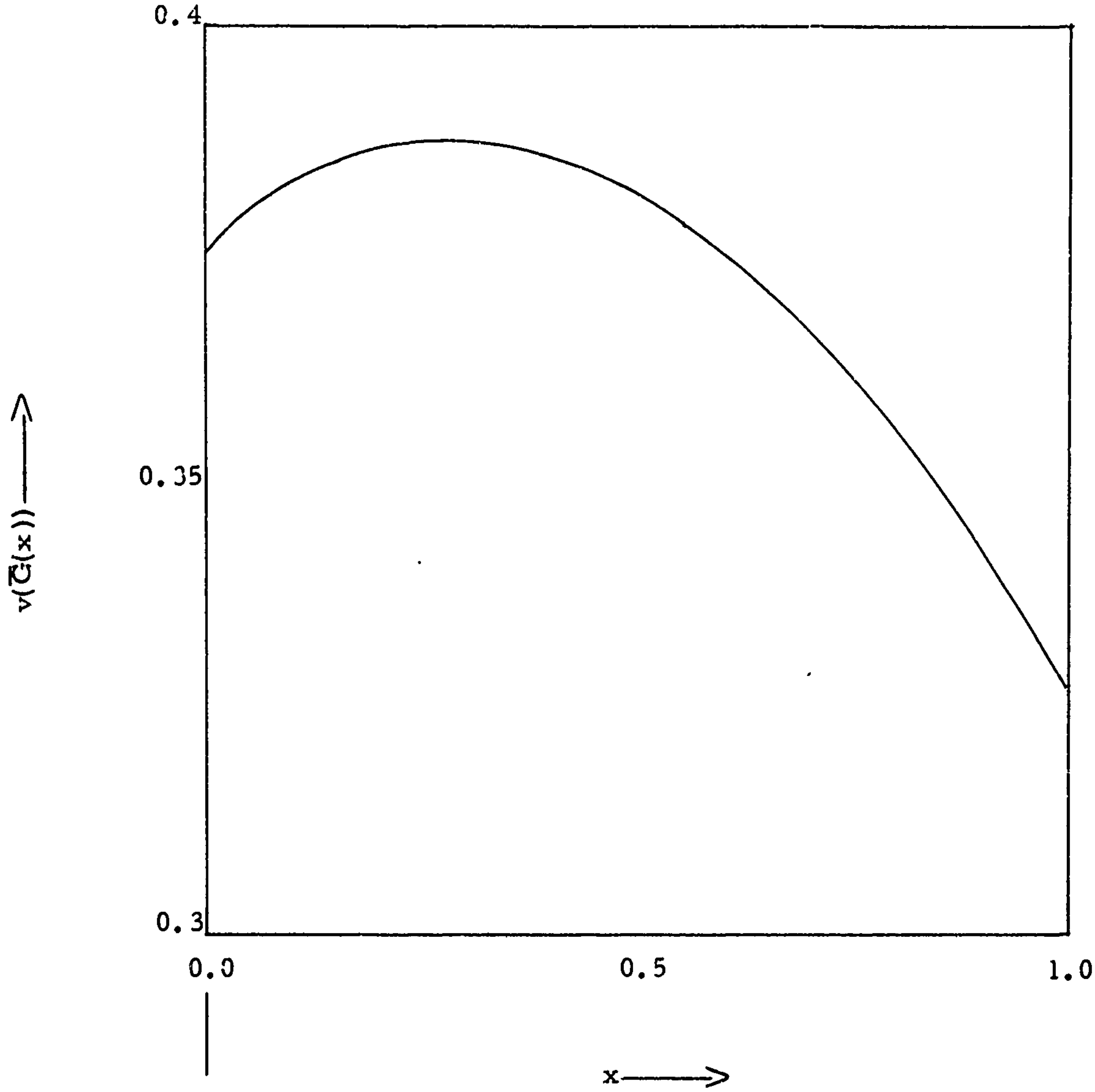

Figure 3.10

Values of Games of Example 3

x	$v(\overline{G}(x))$
0.0	.3750
0.1	.3825
0.2	.3865
0.3	.3875
0.4	.3858
0.5	.3816
0.6	.3750
0.7	.3661
0.8	.3548
0.9	.3412
1.0	.3250

Value of the Games of Example 3

Now the value of a diagonal-matrix game of positive elements $g_{11}, \ldots, g_{nn}$ is simply

$$v(\overline{G}) = (\Sigma_i g_{ii}^{-1})^{-1} .$$

By appealing to continuity of the elementary functions, it suffices to prove our assertion for cases where only one entry (say g_{11}) changes.

Because the second derivative of $(g_{11}^{-1} + \Sigma_2^n g_{ii}^{-1})^{-1}$

with respect to g_{11} is negative,* the desired result follows.

Example 4. In this example, we take a pair of stage matrices which represent different detection-ranges, but which do not have the "triangular" property of the matrices in Examples 1 and 2. Specifically,

$$G_1 = \frac{1}{3}\begin{pmatrix} 3 & & & & & \\ & 3 & & & & \\ & & 3 & & & \\ & & & 3 & & \\ & & & & 3 & \\ & & & & & 3 \end{pmatrix} ;$$

---

* It is easy to verify that

$$\frac{d^2}{dx^2}\left((x^{-1} + y)^{-1}\right)$$

$$= \frac{d}{dx}\left(\frac{(-1)\,.\,(-1)}{(x^{-1} + y)^2\,.\,x^2}\right)$$

$$= \frac{d}{dx}\left(\frac{1}{(1 + xy)^2}\right)$$

$$= \left(\frac{-2}{(1 + xy)^3}\right) . \; y \;, \text{ which is}$$

always negative for positive x and y .

$$G_2 = \frac{1}{3}\begin{pmatrix} 3 & 1 & & & & \\ 1 & 3 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 3 & 1 & \\ & & & 1 & 3 & 1 \\ & & & & 1 & 3 \end{pmatrix} .$$

and $\overline{G}(x) = (1 - x)\ G_1 + x\,G_2$.

Explicitly,

$$\overline{G}(x) = \frac{1}{3}\begin{pmatrix} 3 & x & & & & \\ x & 3 & x & & & \\ & x & 3 & x & & \\ & & x & 3 & x & \\ & & & x & 3 & x \\ & & & & x & 3 \end{pmatrix} .$$

Since we are again dealing with a symmetric matrix, we can describe the strategies for each player by a triple of numbers: the first is the probability of playing $(\frac{1}{2}\text{ row } 1 + \frac{1}{2}\text{ row } 6)$; the second is the probability of playing $(\frac{1}{2}\text{ row } 2 + \frac{1}{2}\text{ row } 5)$; and the third is the probability of playing $(\frac{1}{2}\text{ row } 3 + \frac{1}{2}\text{ row } 4)$.

The optimal strategy for Player I in the game $\overline{G}(x)$ is

$$\left(\frac{1}{27 - 6x - 2x^2}\right) \cdot (9 - x^2,\ 9 - 3x - x^2,\ 9 - 3x) \quad ;$$

and the value is

$$v(\overline{G}(x)) = \left(\frac{27 + 9x - 6x^2 - x^3}{6(27 - 6x - 2x^2)}\right) .$$

Note that

$$v(\overline{G}(0)) = v(G_1) = \frac{1}{6} = 0.1667 \ ,$$

$$v(\overline{G}(1)) = v(G_2) = \frac{29}{114} = 0.2544 \ ,$$

$$v(\overline{G}(\tfrac{1}{2})) = \frac{239}{1128} = 0.2119 \ .$$

Values of $v(\overline{G}(x))$ for Example 4 are tabulated below and graphed in Fig. 3..11. Note that the function $v(\overline{G}(x))$ is again concave, so that again it would be prudent for the patroller to ignore information he might obtain as to which of these would be the true detection-function.

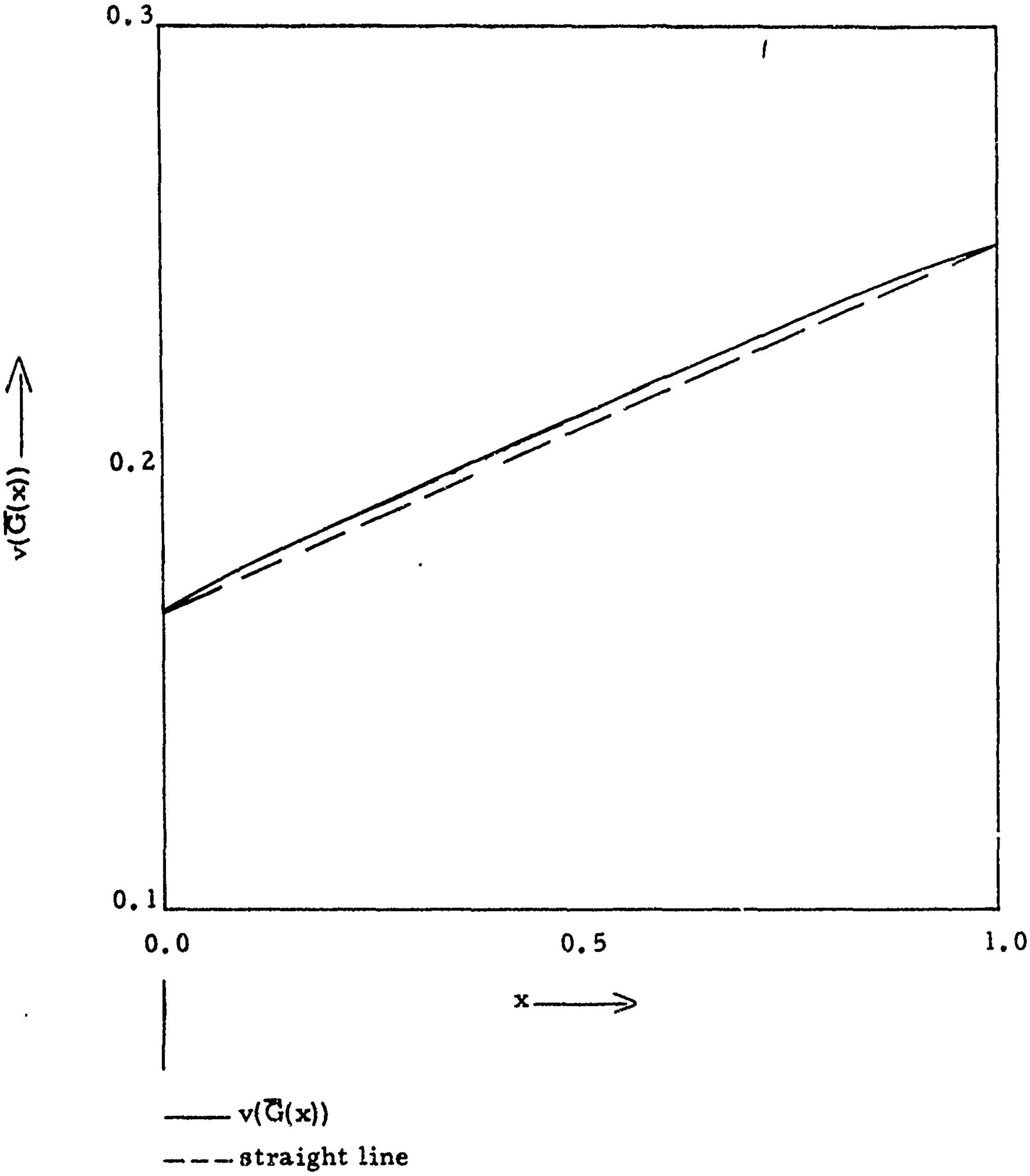

Figure 3.11

Values of Games of Example 4

x	$v(\overline{G}(x))$
0.0	.1667
0.1	.1759
0.2	.1850
0.3	.1941
0.4	.2030
0.5	.2119
0.6	.2206
0.7	.2293
0.8	.2378
0.9	.2462
1.0	.2544

Values for the Games of Example 4

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13. ABSTRACT

The use of the mathematical theory of games for strategic analysis of ASW detection problems is surveyed, emphasizing the specification of strategic and environmental variables, the payoffs and the limitations on patrol and transit submarine movements, detection capabilities and evasive tactics. Games within a patrol zone are compared with games involving the whole barrier. "Secure-detection" objectives are compared with "detection" objectives.

Continuous games involving only the choice of speeds for straight-line patrols and transits are analyzed, using secure-detection probability as a payoff function. In the analysis of patrol strategies, a non-uniform range law for detection by the patroller is assumed, and large matrix games are solved to discover the optimal patrol positions. A continuous version of these matrix games is shown to lead to similar results. From these analyses, a new definition of Secure Sweep Width is derived, which applies to strategic (minimax) transits and patrols.

Repeated games with incomplete information are applied to an ASW patrol-transit confrontation.

The report concl[illegible] that game-theoretic models for ASW should be applied to barrier situations as a [illegible]e rather than to individual patrol zones. Further analysis of the zonal pro[illegible]m will become feasible when we have precisely formulated the determinants of secure detection-- such as range law, effects of thermal layer, convergence zones, varying depth and speed.

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Toeplitz Matrix						
Continuous Games						
Optimal Speeds						
Mixed Strategy						

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