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ON RESOLVING ANGLE AMBIGUITIES OF n-CHANNEL INTERFEROMETER SYSTEMS FOR ARBITRARY ANTENNA ARRANGEMENTS IN A PLANE

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13. ABSTRACT

The concept of array ambiguity plot is introduced, and it is shown that there are exactly five distinct topological cases for antennas arranged in a plane. Some examples are given, including all regular polygon arrangements.

The remainder of the paper is concentrated on those antenna arrays having a two-dimensional lattice point topology for their array ambiguity plots, as they appear to have the most forgiving tolerance requirements. Angle processing implementation and measurement tolerances for ambiguity resolution are discussed. It is shown that, for a given tolerance and region within which antennas are to be located, the ambiguities of the plot move away exponentially with number of antennas if they are placed judiciously. It is therefore possible to obtain the advantages of wide baseline systems which are ambiguity-free with just a few antennas.

The one-dimensional antenna array situation is treated, it bein a special case of the theory.

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ON RESOLVING ANGLE AMBIGUITIES OF n-CHANNEL INTERFEROMETER SYSTEMS FOR ARBITRARY ANTENNA ARRANGEMENTS IN A PLANE

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1. INTRODUCTION AND SUMMARY

One important problem in interferometer antenna system design is the conflict between the need for wide baselines to measure angles accurately and the need for no angle ambiguities. In this document, it is shown that the conflict can be resolved by judicious placement of a few antennas in a plane. A general mathematical theory is developed which defines geometrical constraints dictated by measurement error tolerances.

Section 2 develops the mathematical solution of ambiguity locations for arbitrary antenna arrangements. After introducing the concept of array ambiguity plot, it is shown that there are exactly five distinct plot topologies: a grid of equally spaced parallel lines, a single line, a two-dimensional lattice of points, a one-dimensional lattice of points, and no ambiguities.

In Section 3, the effects on plot topologies of constraining the antennas to lie on a circle are examined. It is shown that the single line topology is not achievable and the other four are. The general regular polygon case is solved.

Section 4 is devoted entirely to the two-dimensional lattice topology, it being conjectured by the author that such antenna arrays are generally superior when measurement error tolerances are taken into account.

Two important results are proved in Section 4. The first is that if two triangular arrays have the same ambiguity plot, then they enclose the same area. This area is denoted by T(P), and is therefore an invariant of the plot P.

The second result of Section 4 has been termed the fundamental area theorem. It states that a necessary and sufficient condition that an antenna array in a plane have a two-dimensional lattice topology for its array ambiguity plot is that the ratio of the areas of all pairs of triangles formed by antenna elements is a rational number. In this case, T(P) is the largest number which divides into the areas of all triangles of the array with integer quotients.

Section 5 extends Section 4 in several directions with emphasis on the two-dimensional lattice topology. The questions of antenna placement, angle processing, and measurement error tolerance are addressed. Several examples are given as illustrations of the general theory. The theory for one-dimensional arrays having a line grid topology is also given, as it turns out to be a special case of the two-dimensional theory.

The most important conclusion is that for a given measurement tolerance and region within which one must place antennas, the ambiguities can be made to move away exponentially as the number of antennas increases, provided the antennas are placed judiciously. A summary of the key results needed in reaching this conclusion follows.

For arrays having a two-dimensional lattice ambiguity plot, if source directions in the forward hemisphere within a cone having its axis perpendicular to the antenna plane can be found unambiguously, then

$$T(P) \leq \frac{\sqrt{3}\,\lambda^2}{12 \sin^2 \rho}$$

where ρ is the cone half-angle and λ is the wavelength.

A quantity T_φ (units of cycles of phase) is introduced, and is called the sufficient phase difference tolerance. Specifically, if the errors in phase at each antenna are all less than $\frac{1}{2} T_\varphi$ in absolute value, thereby implying that phase difference errors are all less than T_φ in absolute value, then all phase differences can be resolved unambiguously except for ambiguities of the entire array.

Another quantity, denoted by E_φ, is termed the tolerance efficiency of the array. For more than three antennas, it is defined by

$$E_\varphi = T_\varphi \left[\sqrt[n-3]{\frac{(n-2) \times (\text{area enclosed by antenna array})}{T(P)}} \right]$$

where n is the number of antennas.

The major result of Section 5 is that $E_\varphi \leq 1$. This is proved for n = 4, 5, 6, and conjectured to be true for n > 6. The significance of this inequality is that it puts a bound on how small T(P) can be. Together with the first inequality, which can be used to determine how small T(P) must be, one can then estimate how many antennas will be needed for a given area and tolerance.

Although the determination of antenna locations is presently somewhat of an art, the inequalities serve at least two purposes. In some cases they can tell one that it is a waste of time to consider fewer than a certain number of antennas, and in other cases they can be used as measures of goodness for any given array arrangement with a two-dimensional lattice ambiguity plot.

2. ARRAY AMBIGUITY PLOTS AND THEIR TOPOLOGIES

GEOMETRY NOTATION

Let $\vec{i}$, $\vec{j}$, $\vec{k}$ denote a right-handed triad of unit vectors fixed in the array. $\vec{i}$ and $\vec{j}$ lie in the plane of the antennas, and $\vec{k}$ is perpendicular to this plane. A unit vector in the direction of a point radiating scurce is denoted by $\vec{s}$, and it is assumed that the source is sufficiently into the far field of the array so that the lines joining each antenna with the source may be considered parallel. The direction cosines of the source direction are denoted by x, y, z so that

$$\vec{s} = x\vec{i} + y\vec{j} + z\vec{k}. \tag{1}$$

It is assumed that the source lies in the forward hemisphere and within a cone having half-angle ρ, so that

$$0 \leq \cos\rho \leq z \leq 1. \tag{2}$$

Since $x^2 + y^2 + z^2 = 1$, one also has

$$0 \leq \sqrt{x^2 + y^2} \leq \sin\rho. \tag{3}$$

A coordinate system has now been established in the boresight plane, see Fig. 1, where there is a one-to-one correspondence between source direction and points (x,y) lying within a circle having radius sin ρ.

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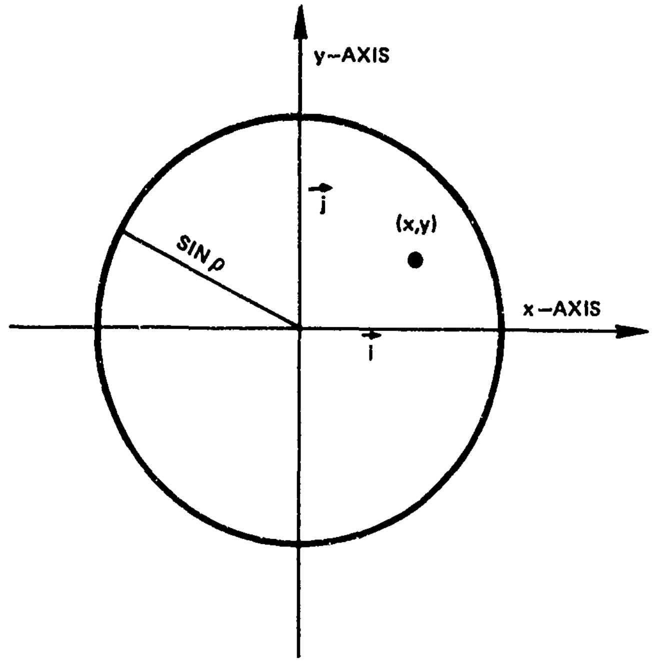

Fig. 1 COORDINATE SYSTEM

MEASUREMENTS FROM ONE ANTENNA PAIR

For the present analysis, each antenna is assumed to be located at a point in the antenna plane, and all antennas receive the signal from the source with the same amplitude. The only difference between signals from antennas lies in the phase difference arising from different path lengths from the source to the antennas.

Consider two antennas 1 and 2 in the antenna plane, see Fig. 2, and let $\vec{d}$ denote the vector with tail at 1 and head at 2.

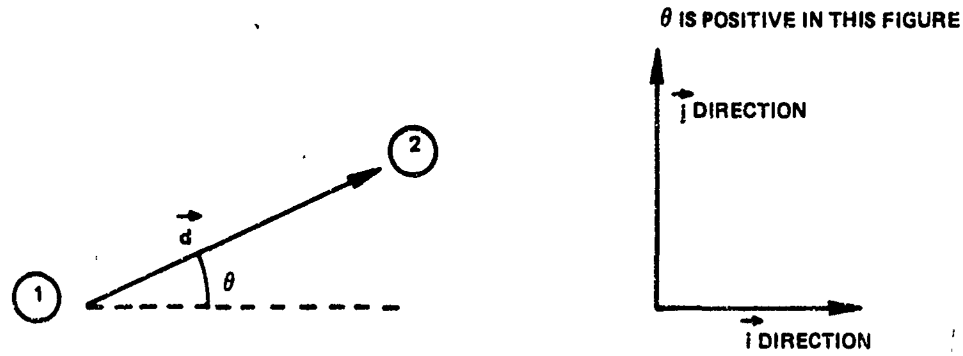

Fig. 2 GEOMETRY FOR ANTENNA PAIR

If $\vec{d}$ is inclined at an angle θ relative to $\vec{i}$, then

$$\vec{d} = (d \cos \theta)\vec{i} + (d \sin \theta)\vec{j} \tag{4}$$

where d is the (scalar) distance between the antennas.

The two antennas and the direction to the source define a plane, depicted in Fig. 3.

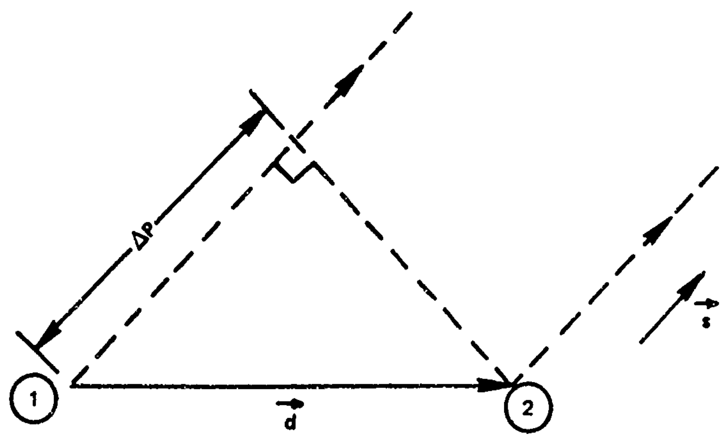

Fig. 3 PLANE FORMED BY TWO ANTENNAS AND SOURCE DIRECTION

The path length difference of the two signal routes is ΔP. Clearly,

$$\Delta P = \vec{d} \cdot \vec{s}, \tag{5}$$

where ΔP is negative if the distance from the source to antenna ② is greater than the corresponding distance to antenna ①.

From Eqs. (1) and (4), then

$$\Delta P = d(x \cos \theta + y \sin \theta). \tag{6}$$

If the signal in antenna ① is denoted by

$$S_{①} = V \cos (\omega t + \mu), \tag{7}$$

then the signal in antenna ② will be

$$S_{②} = V \cos \left[\omega\left(t + \frac{\Delta P}{c}\right) + \mu\right] \tag{8}$$

where c is the speed of light.

If f and λ are the frequency and wavelength of the signals, respectively, then

$$\omega \frac{\Delta P}{c} = \frac{2\pi f}{c} \Delta P = \frac{2\pi}{\lambda} \Delta P. \tag{9}$$

If φ' denotes the phase difference between the two antennas in units of cycles, then

$$\varphi' = \frac{\Delta P}{\lambda} = \frac{d}{\lambda} (x \cos \theta + y \sin \theta). \tag{10}$$

The only information about source direction in these two signals is embodied in sin $(2\pi\varphi')$ and cos $(2\pi\varphi')$, which can be extracted from $S_{①}$ and $S_{②}$. This means that if two distinct source directions, whose maps onto the coordinate system of Fig. 1 both lie within the circle, yield the same values of sin $(2\pi\varphi')$ and cos $(2\pi\varphi')$, then there is no way of telling them apart without additional information. No signal processing tricks based on these two signals alone can possibly alter this conclusion.

Entirely equivalent to the knowledge of sin $(2\pi\varphi')$ and cos $(2\pi\varphi')$ is the knowledge of φ except for some additive integer.

To be more precise, if (x,y) denotes the true source projection onto the coordinate system of Fig. 1, and if (x',y') is some other point within the circle such that

$$x' \cos \theta + y' \sin \theta = n \frac{\lambda}{d} + x \cos \theta + y \sin \theta \tag{11}$$

where n is an integer, then one cannot tell whether the source coordinates are (x',y') or (x,y) or maybe something else.

AMBIGUITY PLOTS

It is instructive to examine the geometrical significance of Eq. (11) by plotting the locus of points (x',y') for fixed (x,y). Equation (11) is the equation of parallel lines, all perpendicular to $\vec{d}$, each separated from another by a distance of a multiple of $\frac{\lambda}{d}$, and one of them passes through (x,y). Only those portions of the lines within the circle contribute to the problem. This is depicted in Fig. 4.

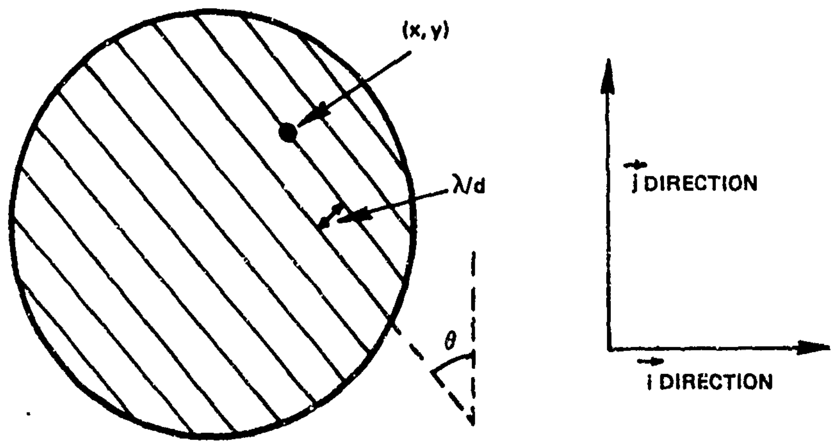

Fig. 4 AMBIGUITY PLOT FOR ONE ANTENNA PAIR

Several things are of interest about ambiguity plots. First, one can see from Fig. 4 that ambiguities are unavoidable with two antennas. Second, ambiguities can be confined to a single line for all sources within the cone of interest if and only if $\frac{\lambda}{d}$ is greater than the diameter of the circle, that is, if

$$\frac{\lambda}{d} > 2 \sin \rho. \tag{12}$$

Third, one can observe that, except for the circle, the ambiguities translate with (x,y). This property holds for ambiguity plots in general, so that one can plot the ambiguities as if the source coordinates were (0,0) on one piece of graph paper, draw a circle on another, put one piece of paper atop the other, and slide it around to produce the equivalent of Fig. 4.

AMBIGUITY PLOT FOR THREE ANTENNAS ARRANGED IN AN EQUILATERAL TRIANGLE

This last comment will be illustrated for the equilateral triangle case. Figure 5 shows the antenna geometry.

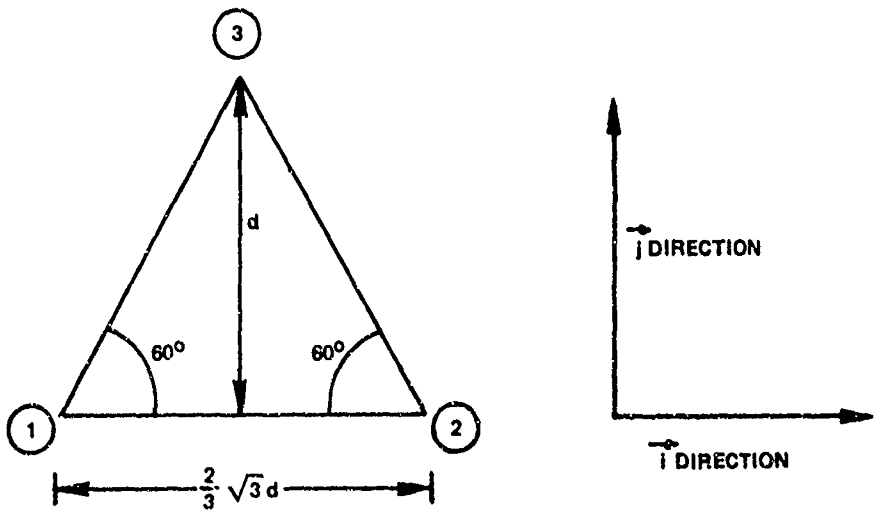

Fig. 5 EQUILATERAL TRIANGLE CASE

Let φ_1' denote the phase difference between antennas (1) and (2), and φ_2' the phase difference between antennas (2) and (3). If two distinct source directions produce the same values of φ_1' and φ_2' except for additive integers in each case, then (except for the circle restriction) they are relatively ambiguous. It does not help to consider the phase difference between antennas (3) and (1) as no new information has been added. This is so because the sum of the phase differences around the clock is zero, or what is the same thing, the final phase difference between (3) and (1) is the negative of the sum of the other two. This implies that if two source directions yield the same values of φ_1' and φ_2' except for additive integers, then they will yield the same value of the phase difference between antennas (3) and (1) except for an additive integer.

The ambiguity plot for the equilateral triangle case is determined as follows: assume the source coordinates are (0,0) in the x,y plane. Draw the grid lines for ambiguities associated with antennas (1) and (2) as discussed previously. Next draw the grid lines for ambiguities associated with antennas (2) and (3). The lattice points determined by the intersections of these two families of lines make up the ambiguity plot for the equilateral triangle case. The lattice points form a hexagonal pattern, as shown in Fig. 6.

By sliding a circle around on top of Fig. 6, it can be seen by inspection that there will be no ambiguities for any source within the cone of interest if and only if the diameter of the circle is less than $\frac{\lambda}{d}$, that is, if and only if Eq. (12) holds.

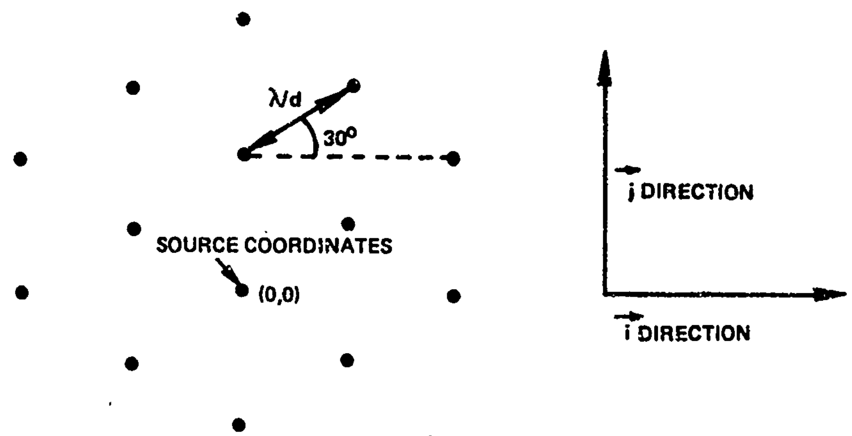

Fig. 6 HEXAGONAL AMBIGUITY PATTERN FOR EQUILATERAL TRIANGLE CASE

AMBIGUITY PLOTS FOR THE GENERAL CASE

Suppose there are m+1 antennas ($m \geq 1$) arranged in an arbitrary fashion on the antenna plane. The first step in determining ambiguities is to order them in some arbitrary but fixed way, so that one can refer to the antennas by number: 1, 2, ..., m+1. One can then let $\varphi_1', \ldots, \varphi_m'$ denote the unambiguous phase differences between 1 and 2, ..., m and m+1, respectively.

The second step is to draw the m families of grid lines associated with antenna pairs 1 and 2, ..., m and m+1 respectively, assuming the target coordinates to be (0,0). A point other than (0,0) which lies on one of the lines for each of the m families will be called an array ambiguity, and the plot of all such points will be called the array ambiguity plot.

Except for the geometrical restrictions having to do with the cone of directions of interest, which can be handled by sliding a circle around on the plot (as discussed previously), the array ambiguity plot has the remarkable property that it is independent of the ordering of the antennas. This is easily seen as the signal from a source in any of these directions arrives at all m+1 antennas in phase. Obviously, there is no way of distinguishing between two such directions without more information. No signal processing tricks will help. The general case where the target is not at (0,0) has an array ambiguity plot which is a translation of the plot for the (0,0) case, as discussed previously, so the conclusion that the ordering of the antenna elements is immaterial is still valid.

ARTIFICIAL AMBIGUITIES AND THE DIAMOND ARRANGEMENT

By not using all of the information present in a given antenna array, it is possible to introduce ambiguities which are not array ambiguities, herein termed artificial ambiguities. The converse is of course not possible; that is, there is no way to get rid of an array ambiguity without additional information.

A classic example of this is the commonly used diamond antenna array depicted in Fig. 7.

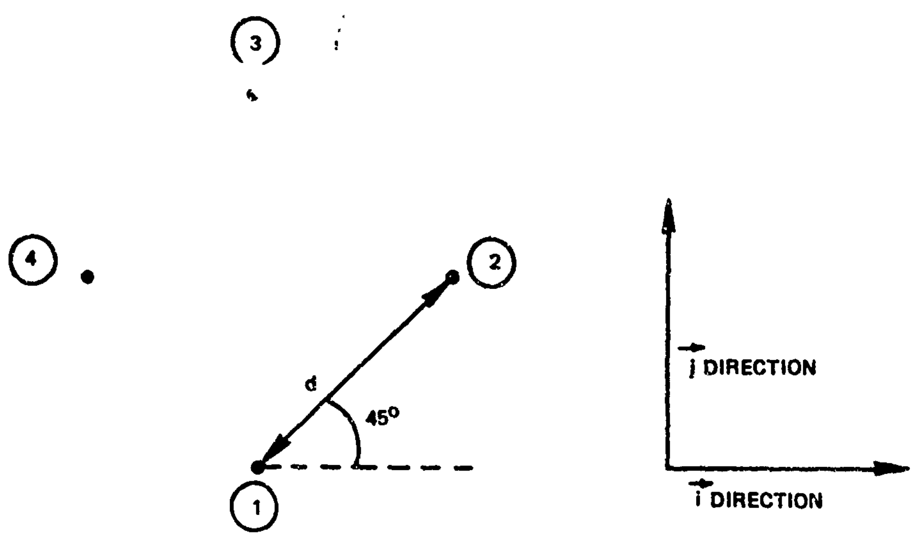

Fig. 7 DIAMOND ANTENNA ARRAY

The array ambiguity plot for this case is the set of circled points in Fig. 8.

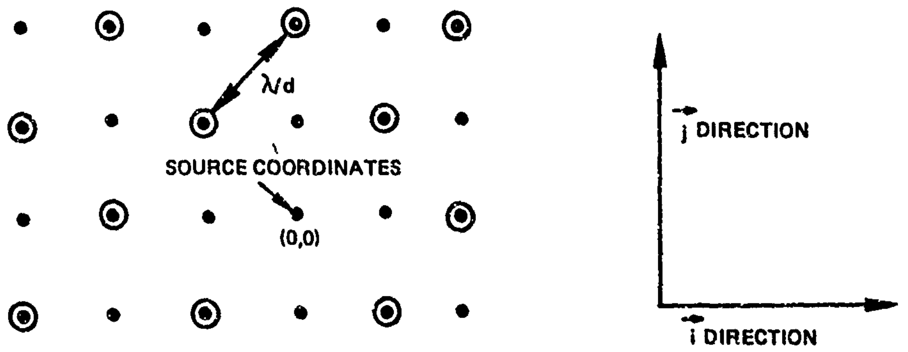

Fig. 8 ARRAY AMBIGUITY PLOT FOR DIAMOND CASE

In some systems, the phase difference between (2) and (4) is used for the x-coordinate and the phase difference between (1) and (3) is used for the y-coordinate. No other antenna pairs are used. In this case the ambiguity plot includes the uncircled points as well as the circled points. The uncircled points (except for (0,0)) are the artificial ambiguities for this processing using a diamond antenna array. One can see by inspection that the nearest ambiguity can be pushed out a factor of $\sqrt{2}$ by using all of the information present in the array.

MATHEMATICAL SOLUTION FOR ARRAY AMBIGUITIES WHEN ANTENNAS LIE ON A LINE

As before, suppose there are m+1 antennas ($m \geq 1$), ordered in an arbitrary way. Let $\varphi_1', \ldots, \varphi_m'$ be as before, and in addition, let d_k and θ_k denote the distance between antennas (k) and (k+1) and the angle that the vector from antenna (k) to antenna (k+1) makes with the positive $\vec{i}$ axis, respectively, for $k = 1, \ldots, m$. As before, the true source coordinates are taken to be (0,0).

In general, it follows from Eq. (10) and the discussion following it that a point (x,y) other than (0,0) is an array ambiguity if and only if the system of m equations

$$(x \cos \theta_k + y \sin \theta_k) = \frac{\lambda n_k}{d_k}, \qquad k = 1, 2, \ldots, m \qquad (13)$$

has a solution in integers for n_k, $k = 1, 2, \ldots, m$.

In the present case, all of the θ_k's differ from each other by 0° or 180°, so that solving the system of equations is equivalent to finding integer solutions* for n_k

*The n's in Eq. (14) may differ in sign from the n's in Eq. (13).

in the set of equations

$$(x \cos \theta_1 + y \sin \theta_1) = \frac{\lambda n_1}{d_1} = \frac{\lambda n_2}{d_2} = \cdots = \frac{\lambda n_m}{d_m}. \qquad (14)$$

It is clear that if the ratio of any two of the d's is an irrational number, then the only solutions for (x,y) are given by

$$x \cos \theta_1 + y \sin \theta_1 = 0 \qquad (15)$$

which is the equation of a single line passing through (0,0) and perpendicular to the line containing the antennas.

If the ratios of all possible pairs of d's are rational numbers, then let

$$\frac{d_1}{d_k} = \frac{a_k}{b_k}, \qquad k = 1, 2, \ldots, m \qquad (16)$$

where a_k and b_k are integers, and $\frac{a_k}{b_k}$ is in lowest terms. (Obviously, $a_1 = b_1 = 1$.)

Also, let L = the least common multiple of a_1, a_2, ..., a_m. Then the solution to Eq. (14) is given by

$$n_k = nb_k \frac{L}{a_k}, \qquad k = 1, 2, \ldots, m \qquad (17)$$

and

$$x \cos \theta_1 + y \sin \theta_1 = \frac{n\lambda L}{d_1} \qquad (18)$$

where n is any integer.

The array ambiguity plot is therefore identical to that for a single pair of antennas separated by a distance

d', where $d' = \frac{d_1}{L}$, and the grid of lines is therefore determined by

$$x \cos \theta_1 + y \sin \theta_1 = \frac{\lambda n}{d'}, \tag{19}$$

where n is any integer.

The irrational fraction case whose solution is given by Eq. (15) can be included in this case by the artifice of setting $d' = \infty$ in Eq. (19).

MATHEMATICAL SOLUTION FOR ARRAY AMBIGUITIES, GENERAL CASE

Now consider a general planar array of m+1 antennas, and without loss of generality assume the true source coordinates to be (0,0). Assume some ordering of the antennas, and let φ_k', d_k, θ_k be defined as before, k = 1, 2, ..., m.

The array ambiguity plot is then determined by solving the system of Eqs. (13). Next, split the equations into groups of equations, where two equations belong to the same group if and only if the two θ_k's associated with the two equations are the same or differ by 180°.

The analysis of the previous section then shows that each group may be replaced by a single equation.

The system of equations to be solved is therefore the system

$$x \cos \theta_k' + y \sin \theta_k' = \frac{\lambda n_k'}{d_k'}, \qquad k = 1, 2, \ldots, p, \tag{20}$$

where no two θ_k''s are equal or differ by 180°, $p \leq m$, and some of the d_k''s may be ∞.

In order to ease the typist's task in the following development, the primes will be removed from Eq. (20), so the reader should bear in mind that the following unprimed variables are not necessarily the same as in the previous definitions.

The system of equations to be solved is thus

$$x \cos \theta_k + y \sin \theta_k = \frac{\lambda n_k}{d_k}, \qquad k = 1, 2, \ldots, p, \tag{21}$$

where no two θ_k's are equal or differ by 180°, $p \le m$, and some of the d_k's may be ∞.

First, if there is just one equation, the antennas all lie on a line, and this case has been solved already.

Second, suppose there are just two equations. Since θ_1 and θ_2 are not equal nor differ by 180°, one can solve the equations for x and y, giving

$$x = \lambda \csc (\theta_2 - \theta_1) \left[\frac{n_1}{d_1} \sin \theta_2 - \frac{n_2}{d_2} \sin \theta_1 \right] \tag{22}$$

$$y = \lambda \csc (\theta_2 - \theta_1) \left[-\frac{n_1}{d_1} \cos \theta_2 + \frac{n_2}{d_2} \cos \theta_1 \right]. \tag{23}$$

If both d_1 and d_2 are ∞, then $x = y = 0$ is the only solution, and there are no array ambiguities. If just one is ∞, say $d_1 = \infty$, then

$$x = \lambda \csc (\theta_2 - \theta_1) \left[-\frac{n_2}{d_2} \sin \theta_1 \right] \tag{24}$$

$$y = \lambda \csc (\theta_2 - \theta_1) \left[\frac{n_2}{d_2} \cos \theta_1 \right] \tag{25}$$

which represents a set of lattice points all of which lie on the line $x \cos \theta_1 + y \sin \theta_1 = 0$. The case where d_1 and d_2 are both ∞ can be achieved with five antennas, and the case where just $d_1 = \infty$ can be achieved with four antennas.

The cases when neither d_1 nor d_2 are infinity include all triangular array cases. From Eqs. (22) and (23), it can be seen that there always exists a lattice of array ambiguities which covers the whole (x,y) plane as n_1 and n_2 range through all possible integers. This was to be expected from the geometrical nature of the array ambiguity plot, since the intersection of two sets of non-parallel grid lines must intersect in a lattice of points.

Next, suppose there are at least three equations. First, consider the case where none of the d_k's are infinity.

The first two equations are linearly equivalent to Eq. (22) and Eq. (23). If one multiples the first equation of Eq. (21) by $\sin(\theta_3-\theta_2)$, the second by $\sin(\theta_1-\theta_3)$, the third by $\sin(\theta_2-\theta_1)$, and adds, one obtains the equation (after dividing through by λ)

$$\frac{n_1}{d_1}\sin(\theta_3-\theta_2)+\frac{n_2}{d_2}\sin(\theta_1-\theta_3)+\frac{n_3}{d_3}\sin(\theta_2-\theta_1)=0. \quad (26)$$

Note that x and y do not appear in Eq. (26). Since the coefficient of n_3 is not zero, Eqs. (22), (23), (26) give x, y, n_3 as functions of n_1 and n_2. Similarly, using the second, third, and fourth equations, one can solve for n_4 in terms of n_3 and n_2 (assuming there are at least four equations). Therefore, a linearly equivalent set of equations to Eq. (21) is given by Eqs. (22), (23), and the system of equations

$$\frac{n_k}{d_k}\sin(\theta_{k+2}-\theta_{k+1})+\frac{n_{k+1}}{d_{k+1}}\sin(\theta_k-\theta_{k+2})$$
$$+\frac{n_{k+2}}{d_{k+2}}\sin(\theta_{k+1}-\theta_k)=0, \qquad k=1,2,\ldots p-2. \quad (27)$$

In the general case where no d_k is ∞ and there are at least three equations, if one first finds all integer solutions for the n_k's of the system of Eqs. (27), and then substitutes all possible values of n_1 and n_2 resulting from that solution into Eqs. (22) and (23), one has the coordinates of all points (except for (0,0) which is always a solution) of the array ambiguity plot. The above procedure utilizing Eqs. (22), (23), (27) also produces the array ambiguity plot if some of the d_k's are ∞, as can be seen by observing that Eqs. (22), (23), (27) are linearly equivalent to Eq. (21) if one takes the variables in question to be $x, y, \frac{n_1}{d_1}, \ldots, \frac{n_p}{d_p}$ instead of $x, y, n_1, \ldots, n_p$. By appropriate linear combinations of Eqs. (22), (23), and the equations of Eq. (27), one can derive Eq. (21) regardless of whether or not some of the $\frac{n_k}{d_k}$'s are zero. Of course, if more than one d_k were ∞, we know from the two equation case that there are no array ambiguities, and one need proceed no further. If only one d_k is ∞, then we know that all array ambiguities are confined to a line.

The general array ambiguity plot problem has now been completely solved mathematically. Unfortunately, most general mathematical solutions do not convey much insight into interesting cases. Before concluding Section 2, some special situations will therefore be examined, and the general topological nature of all possible plots will be derived.

First, it has been shown that the array ambiguity plot for a given antenna array, determined by intersections of grid lines of successive pairs of antennas, does not depend on the ordering of the antennas. The number of equations in Eq. (21), however, may depend on the ordering, and so may the existence of infinity values for the d_k's. It is of course assured that whatever set of equations one uses, the resulting solutions for x and y must be identical.

To illustrate the above phenomena, consider the arrangement of antennas in Fig. 9.

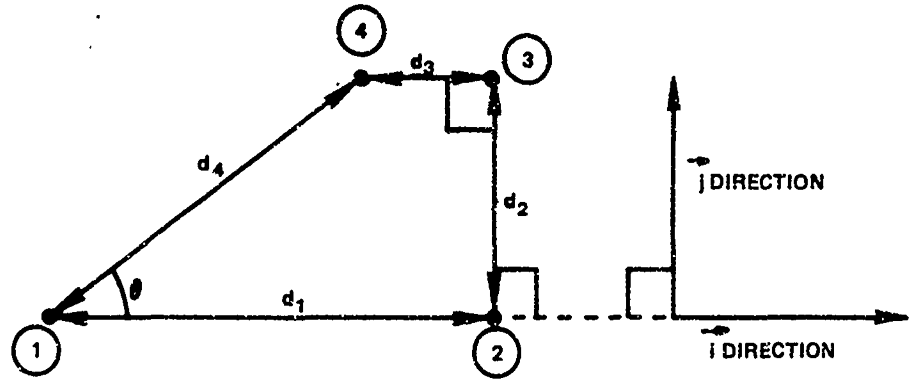

Fig. 9 AN ARRAY

In this figure, suppose that the ratio of d_1 to d_3 is irrational. In the ordering given in the figure, there are just two equations in Eq. (21), namely $x = 0$ and $y = n\,\frac{\lambda}{d_2}$, so that the ambiguity plot obviously consists of lattice points confined to the y-axis and separated by multiples of $\frac{\lambda}{d_2}$.

On the other hand, if the antennas were ordered ①, ④, ③, ②, there are three equations, namely

$$x \cos\theta + y \sin\theta = \frac{\lambda n_1}{d_4} \tag{28}$$

$$x = \frac{\lambda n_2}{d_3} \tag{29}$$

$$-y = \frac{\lambda n_3}{d_2}. \tag{30}$$

Since there are three equations in this case, the equivalent set of Eqs. (22), (23), (27) is

$$x = \frac{\lambda n_2}{d_3} \tag{31}$$

$$y = \lambda \csc \theta \left(\frac{n_1}{d_4} - \frac{n_2}{d_3} \cos \theta \right) \tag{32}$$

$$-\frac{n_1}{d_4} + \cos \theta \frac{n_2}{d_3} - \sin \theta \frac{n_3}{d_2} = 0. \tag{33}$$

From the law of sines,

$$\frac{\sin \theta}{d_2} = \frac{1}{d_4} = \frac{\cos \theta}{d_1 - d_3} \tag{34}$$

so that sin θ and cos θ may be removed from Eq. (33), and Eq. (33) is replaced by

$$-\frac{n_1}{d_4} + \frac{n_2}{d_4}\left(\frac{d_1}{d_3} - 1 \right) - \frac{n_3}{d_4} = 0. \tag{35}$$

Multiplying by d_4, there results

$$n_1 + n_2 + n_3 = n_2 \frac{d_1}{d_3}. \tag{36}$$

Since $\frac{d_1}{d_3}$ is irrational, it follows that $n_2 = 0$ and n_1 can be any integer (taking $n_3 = -n_1$).

Substituting $n_2 = 0$ in Eqs. (31) and (32), one has $x = 0$ and $y = \lambda \csc \theta \frac{n_1}{d_4}$, and from Eq. (34), $y = n_1 \frac{\lambda}{d_2}$, the same result as before.

In addition to illustrating that the equations depend on the antenna ordering, this example also shows that some orderings simplify the analysis compared to others.

We will conclude Section 2 by summarizing some observations about array ambiguity plots which follow from the preceding discussions when the number of antennas is 2 or 3; or more if they all lie on a line. The $n = 4$ and $n > 4$ case will then be treated.

TWO ANTENNA SUMMARY

For two antennas, the array ambiguity plot is always a grid of parallel lines, each separated from its neighbor by a distance $\frac{\lambda}{d}$.

THREE ANTENNA SUMMARY

If the antennas do not lie on a line, then the array ambiguity plot consists of lattice points extending throughout the (x,y) plane, and are formed by the intersection of two sets of grid lines, each set consisting of equally spaced parallel lines, but no line of the first set is parallel to any line of the second set.

If the antennas lie on a line, then the ambiguity plot consists of either a single straight line or a grid of equally spaced parallel lines. In Fig. 10, if $\frac{d_2}{d_1}$ is an irrational number, the former case results, otherwise the latter. The orientation of this array with respect to the coordinate axes has no bearing on this conclusion.

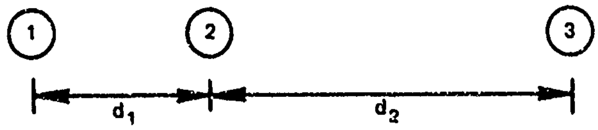

Fig. 10 THREE ANTENNAS ON A LINE

n ANTENNAS ON A LINE SUMMARY, $n \geq 3$

The conclusion here is the same as for the $n = 3$ case in that the array ambiguity plot consists of either a single straight line or a grid of equally spaced parallel lines. If one lists the distances between all possible pairs of antennas, and can find two such distances whose ratio is an irrational number, the former case results, otherwise the latter.

FOUR ANTENNAS

If all four antennas lie on a line, the result is given by the preceding paragraph.

If not, then there always exists an ordering of the antennas such that no pair of θ's selected from θ_1, θ_2, θ_3 are equal or differ by 180°. To see this, suppose first that only three antennas lie on a line, and suppose that ①, ③, ④ are these antennas. Then the ordering ①, ②, ③, ④ will have the desired property. If no three antennas lie on a line, then label the antennas ①, ②, ③, ④ arbitrarily. Neither ③ nor ④ lie on the line joining ① and ②. If the line joining ③ and ④ is parallel to the line joining ① and ②, then at least one of the orderings ③, ①, ②, ④ or ④, ①, ②, ③ will have the desired property. If the line joining ③ and ④ is not parallel to the line joining ① and ②, then the ordering ①, ②, ③, ④ has the desired property. This exhausts all cases.

So suppose there are four antennas not all on a line, and let them be ordered so that no pair of θ's selected from θ_1, θ_2, θ_3 are equal or differ by 180°. For this ordering, the array ambiguity equations are given by Eqs. (22), (23) and (26), where d_1, d_2, d_3 are all finite. The nature of the integer solutions to Eq. (26) will obviously depend on the rationality of the ratios of the coefficients of n_1, n_2, n_3 in Eq. (26). So let

$$c_1 = \frac{d_3 \sin(\theta_1-\theta_3)}{d_2 \sin(\theta_2-\theta_1)} \tag{37}$$

$$c_2 = \frac{d_3 \sin(\theta_3-\theta_2)}{d_1 \sin(\theta_2-\theta_1)} . \tag{38}$$

Case 1: There is no linear combination of c_1 and c_2 which is rational when the coefficients are rational and at least one coefficient does not vanish. (Example: $c_1 = -\sqrt{2}$, $c_2 = \sqrt{3}$)

In this case the only solutions to Eq. (26) are $n_1 = n_2 = n_3 = 0$, as is easily seen by dividing Eq. (26) by $\frac{\sin(\theta_2-\theta_1)}{d_3}$, the result being

$$c_2 n_1 + c_1 n_2 + n_3 = 0. \tag{39}$$

Since the only solution for the n's is $n_1 = n_2 = n_3 = 0$, it follows from Eqs. (22) and (23) that there are no array ambiguities. Note that in Case 1 both c_1 and c_2 must be irrational, although this is not a sufficient condition for Case 1 to apply.

Case 2: There exists a non-zero solution to Eq. (39), and not both c_1 and c_2 are rational. (Example: $-c_1 = c_2 = \sqrt{2}$).

Let (n_1, n_2, n_3) denote a non-zero solution. It may be assumed that n_1, n_2, n_3 have no factor in common, for one can always divide Eq. (39) by the greatest common factor of n_1, n_2, n_3.

If (n_1', n_2', n_3') is another non-zero solution, then $n_1'n_2 - n_1n_2'$ must vanish, for otherwise the equations

$$c_2n_1 + c_1n_2 = -n_3$$

$$c_2n_1' + c_1n_2' = -n_3'$$

would have rational solutions for both c_1 and c_2. Therefore, (n_1, n_2) is proportional to (n_1', n_2') and it then follows that (n_1', n_2', n_3') is proportional to (n_1, n_2, n_3).

Since n_1, n_2, n_3 have no factor in common, there exists an integer n such that $n_1' = nn_1$, $n_2' = nn_2$, $n_3' = nn_3$.

From Eqs. (22) and (23), it then follows that the array ambiguity plot consists of lattice points confined to the line

$$x\left[-\frac{n_1}{d_1}\cos\theta_2 + \frac{n_2}{d_2}\cos\theta_1\right] = y\left[\frac{n_1}{d_1}\sin\theta_2 - \frac{n_2}{d_2}\sin\theta_1\right] \tag{40}$$

where the distance between neighboring lattice points is

$$\lambda\ \left|\csc(\theta_2-\theta_1)\right| \sqrt{\left(\frac{n_1}{d_1}\right)^2 + \left(\frac{n_2}{d_2}\right)^2 - 2\left(\frac{n_1}{d_1}\right)\left(\frac{n_2}{d_2}\right)\cos(\theta_2-\theta_1)}.$$

Case 3: Both c_1 and c_2 are rational.

Let $c_1 = \frac{a_1}{b_1}$ and $c_2 = \frac{b_2}{b_2}$, both in lowest terms, and let L be the least common multiple of b_1 and b_2. Multiplying Eq. (39) by L results in

$$I_1n_1 + I_2n_2 + Ln_3 = 0 \tag{41}$$

where I_1, I_2, L are integers having no factor in common.

Let g be the greatest common factor of I_2 and L. Then clearly g must divide n_1. From the theory of congruences, Eq. (41) is then equivalent to

$$\frac{I_2}{g} n_2 \equiv -I_1 \frac{n_1}{g} \left(\mathrm{mod}\ \frac{L}{g}\right) . \tag{42}$$

Since $\frac{I_2}{g}$ and $\frac{L}{g}$ are relatively prime, Eq. (42) is equivalent to

$$n_2 \equiv p \frac{n_1}{g} \left(\mathrm{mod}\ \frac{L}{g}\right) \tag{43}$$

where p is some fixed integer.

The solutions of Eq. (41) for n_1 and n_2 are therefore given by

$$n_1 = ag \tag{44}$$

$$n_2 = pa + b\frac{L}{g} \tag{45}$$

where a and b are any integers.

From Eq. (21), the array ambiguity points are given by the solutions to

$$x \cos \theta_1 + y \sin \theta_1 = \frac{\lambda ag}{d_1} \tag{46}$$

and

$$x \cos \theta_2 + y \sin \theta_2 = \frac{\lambda}{d_2}\left[pa + b\frac{L}{g}\right]. \tag{47}$$

If one multiplies Eq. (46) by $\frac{pd_1}{gd_2}$, and subtracts from Eq. (47), one obtains

$$x \left[\cos \theta_2 - \frac{pd_1}{gd_2} \cos \theta_1\right] + y \left[\sin \theta_2 - \frac{pd_1}{gd_2} \sin \theta_1\right] = \frac{\lambda L}{d_2 g} b. \tag{48}$$

Note that Eqs. (46) and (48), which also define the array ambiguity plot, are each the equation of a family of equidistant parallel lines as a and b range over all possible integers. There therefore exists a placement of three antennas which produces exactly the same array ambiguity plot as Case 3 for four antennas, and the array ambiguity plot is the same as that for a triangle.

FOUR ANTENNA SUMMARY

All of the types of array ambiguity plots obtainable with three antennas, namely a single line, a grid of lines, or a two-dimensional grid of lattice points formed by intersections of two non-parallel families of a grid of lines, are obtainable with four antennas. In addition there are exactly two more possible types of plots. One is a set of equidistant lattice points on a single line, and the other is no array ambiguities at all. Figure 9 is an example of the former.

The array in Fig. 11 is an example of the latter.

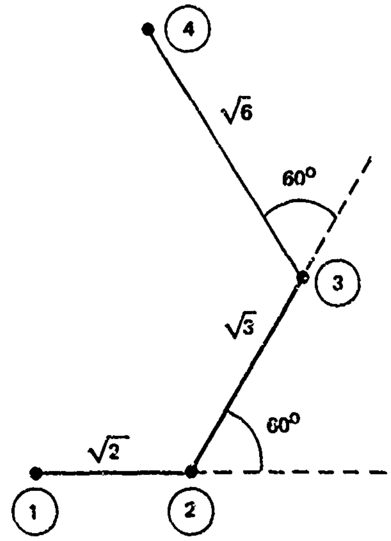

Fig. 11 FOUR ANTENNA CASE WITH NO ARRAY AMBIGUITIES

From Eqs. (37) and (38), $c_1 = -\sqrt{2}$ and $c_2 = \sqrt{3}$, which falls under Case 1 for four antennas.

n > 4 ANTENNAS

It will now be shown that the five types of plots possible for n = 4 are also possible for n > 4, and there are no others.

The proof will be by induction, that is, it is assumed that the statement is true for some n ≥ 4, and one shows that it is true for n + 1.

Begin with some arbitrary arrangement of n + 1 antennas, and let them be ordered as before. The first n antennas in this ordering constitute an n antenna case which by the induction hypothesis has an array ambiguity plot which is one of five types. The array ambiguity plot for the n + 1 antennas is the intersection of the ambiguity plot for the first n antennas with the grid of parallel lines associated with the n^{th} and $(n+1)^{th}$ antennas.

Case 1: The first n antennas have no array ambiguities.

In this case, the n + 1 antennas clearly have no array ambiguities.

Case 2: The array ambiguity plot for the first n antennas is a two-dimensional array of lattice points formed by the intersection of two sets of parallel grid lines.

In this case, one can replace the n antennas by three which have the same array ambiguity plot. Attach the $(n+1)^{th}$ antenna to the array of three in such a way that the vector joining one of the three to the $(n+1)^{th}$ has the same direction and length as that joining the n^{th} and $(n+1)^{th}$ in the original array.

The new array of four antennas has the same plot as the original $n+1$, and since the plot for four antennas must be one of the five possibilities, so is the plot for the $n+1$ antennas.

Conversely, start with an array of n antennas all lying on a line, and equally spaced. The plot is a grid of parallel lines. Adding an $(n+1)^{th}$ antenna anywhere off the line will produce a plot which is a two-dimensional array of lattice points. Such an array is therefore achievable with $n+1$ antennas.

Case 3: The array ambiguity plot for the first n antennas is a line of equally spaced lattice points.

If the grid of lines associated with the n^{th} and $(n+1)^{th}$ antennas is parallel to the above line, then one of the lines lies along it, and the intersection is the same plot.

If not, then the intersection of the line with the grid lines defines another set of lattice points on the line. If the two sets of lattice points have any point in common other than the origin, then the distances from the origin to a pair, one from each set, have a rational quotient and the intersection will be a set of equally spaced lattice points on the original line. If the two sets have no point in common except the origin, then there are no array ambiguities.

To show that a line of equally spaced lattice points is an achievable plot for $n+1$ antennas, put n of them on a line with the ratio of at least one pair of distances equal to an irrational number, and put the $(n+1)^{th}$ antenna off the line.

Case 4: The array ambiguity plot for the first n antennas is a single line.

The n antennas may be replaced by three antennas on a line having the same array ambiguity plot as the n antennas. Attach the $(n+1)^{th}$ antenna as was done in Case 2, and the plot for the resulting four antennas is the same as for the $n+1$ antennas. One of the five plot types must result.

To achieve a single line plot with $n+1$ antennas, place all $n+1$ antennas on a line, with the ratio of at least one pair of distances an irrational number.

Case 5: The array ambiguity plot for the first n antennas is a grid of parallel lines.

The n antennas may be replaced by three antennas equally spaced on a line, and one proceeds as before.

An example of $n+1$ antennas with a grid of lines for its array ambiguity plot is obtained by placing them all on a line with equal spacing.

3. PLOT TOPOLOGIES WHEN THE ANTENNAS ARE CONSTRAINED TO LIE ON A CIRCLE

POSSIBLE TYPES OF ARRAY AMBIGUITY PLOTS

The circle assumption is irrelevant for the two antenna case, and the result is the same as in Section 2; namely, the array ambiguity plot consists of a grid of equally spaced parallel lines.

For three antennas, the circle assumption implies that they cannot lie on a line, so it follows from Section 2 that there is just one type of array ambiguity plot, a two-dimensional array of lattice points formed by the intersection of two non-parallel sets of equidistant parallel lines.

For four antennas, it will be shown that the circle assumption implies that there are exactly three types of plots, the two-dimensional lattice as in the three antenna case, a set of equidistant lattice points confined to a single line, and no array ambiguities at all. In Section 2, it is shown that these are the only possibilities in general unless the four antennas lie on a line, which is ruled out by the circle assumption. It remains only to show that each of the three types is realizable by some placement of the four antennas around the circle.

The diamond array, discussed in Section 2, is an example of the two-dimensional lattice. The array in Fig. 12 is an example of a set of equidistant lattice points confined to a single line. The analysis associated with the array in Fig. 12, which is a trapezoid inscribed in a circle, is exactly the same as that for the array of Fig. 9. It is clear that

$\frac{d_3}{d_1}$ can be made irrational if one fixes the circle and antennas ① and ②, and slides ③ and ④ up or down the circle, maintaining the line joining ③ and ④ parallel to the line joining ① and ②.

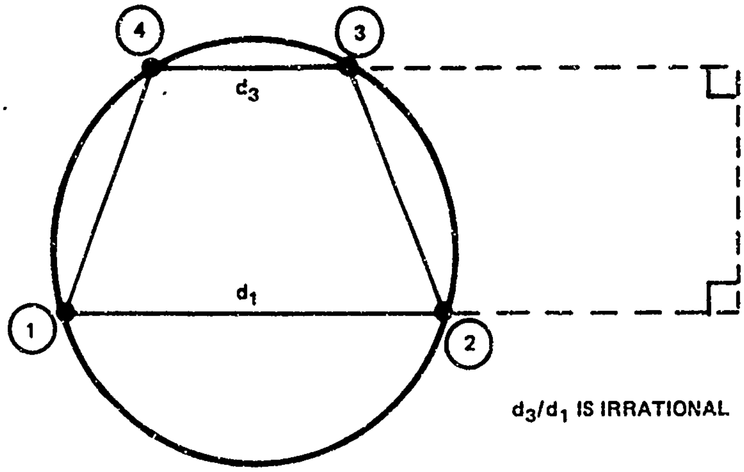

Fig. 12 ARRAY WHOSE AMBIGUITY PLOT IS A LATTICE OF POINTS ON LINE

Finally, consider the array in Fig. 13.

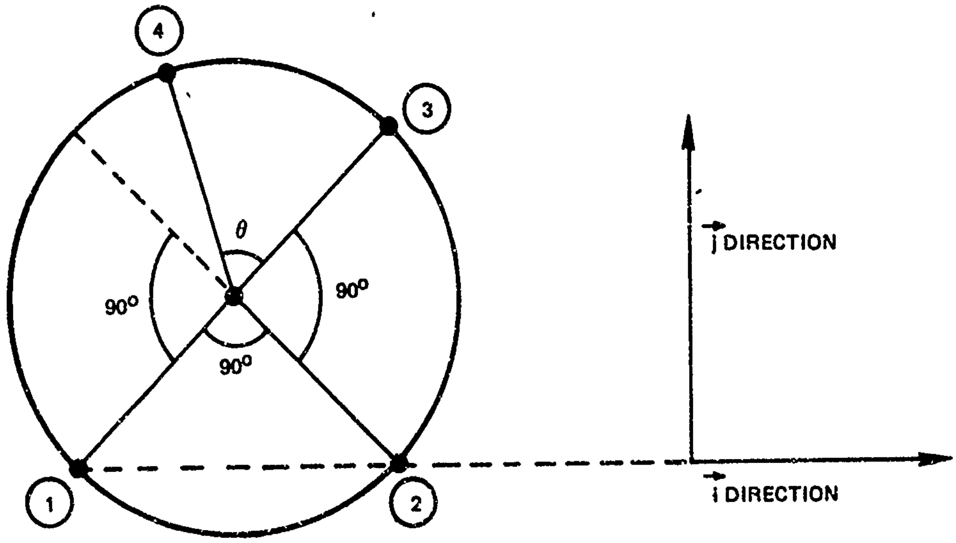

Fig. 13 AN ASYMMETRICAL ARRAY

For the ordering ①, ②, ③, ④, and letting r be the radius of the circle, one has

$$d_1 = d_2 = r\sqrt{2} \tag{49}$$

$$d_3 = 2r \sin\frac{\theta}{2} \tag{50}$$

$$\theta_1 = 0 \tag{51}$$

$$\theta_2 = 90^\circ \tag{52}$$

$$\theta_3 = 135^\circ + \frac{\theta}{2}. \tag{53}$$

From Eqs. (37) and (38),

$$c_1 = \frac{1}{2}(1 - \cos\theta - \sin\theta) \tag{54}$$

$$c_2 = \frac{1}{2}(1 - \cos\theta + \sin\theta). \tag{55}$$

Choose θ, for example, such that $\cos\theta = \frac{1}{3}\sqrt{3}$ and $\sin\theta = \frac{1}{3}\sqrt{6}$. There will then be no linear combination of c_1 and c_2 with non-zero rational coefficients which is rational. Section 2 shows that there are no array ambiguities in this case.

All three types of arrays are therefore achievable with four antennas confined to a circle.

For five or more antennas, it follows from Section 2, just as in the four antenna case, that the line grid and single line topology are ruled out since the antennas cannot lie on a line. The remaining three candidate topologies can be achieved for any given number of antennas larger than four arranged on a circle. The proof of this will be given after development of a special case which follows next.

DEFINITION OF CIRCULAR ARRAYS WITH EQUAL CENTRAL ANGLES

Figure 14 depicts the definition of this class of antenna arrays.

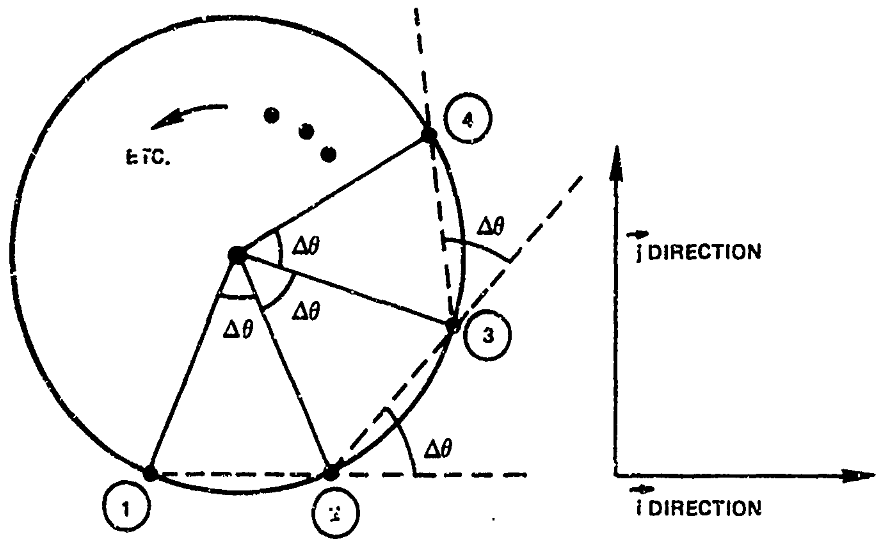

Fig. 14 CIRCULAR ARRAYS WITH EQUAL CENTRAL ANGLES

The antennas are ordered as in Fig. 14. For reference to the $\vec{i}$, $\vec{j}$ coordinate system, the vector from ① to ② is arbitrarily chosen to be in the positive $\vec{i}$ direction.

In the notation of Section 2,

$$\theta_k = (k-1)\Delta\theta \tag{56}$$

for the vector joining antennas k and k+1, k ≥ 1, and

$$d_k = 2r \sin \frac{\Delta\theta}{2} \tag{57}$$

for all k, where r is the radius of the circle.

If Δθ, expressed in radians, is equal to $2\pi \frac{p}{q}$, where $\frac{p}{q}$ is a fraction in lowest terms, then eventually the sequence will result in an antenna which lies on a previous one, and the array is the same as a regular polygon with q sides, repetition occurring thereafter. If $\frac{\Delta\theta}{2\pi}$ is an irrational number, eventually every point on the circle will be approached arbitrarily closely.

In either case, one may wish to stop the sequence at some point, say after h antennas. If $\Delta\theta = 2\pi \frac{p}{q}$, obviously the array ambiguity plot does not depend on h if $h \geq q$.

It will be shown that, if the number of distinct antennas exceeds four, then the array ambiguity plot is either a two-dimensional lattice or there are no array ambiguities, and either case can occur for any such number of antennas. For four antennas, the plot is either a one- or two-dimensional lattice, and either case can occur.

EQUATIONS FOR ARRAY AMBIGUITY PLOTS FOR CIRCULAR ARRAYS WITH EQUAL CENTRAL ANGLES

Unless $\Delta\theta$ is a multiple of π, which are the trivial cases of one or two antennas, it follows from Section 2 that the points (x,y) of the array ambiguity plot for $h > 3$ are given by (referring to Eqs. (22), (23), and (27))

$$x = \frac{n_1\lambda}{d} \tag{58}$$

$$x \cos \Delta\theta + y \sin \Delta\theta = \frac{n_2\lambda}{d} \tag{59}$$

$$n_k \sin \Delta\theta - n_{k+1} \sin 2\Delta\theta + n_{k+2} \sin \Delta\theta = 0, \quad 1 \leq k \leq h-3 \tag{60}$$

where Eq. (27) has been multiplied by d, the common value of all the d_k's.

If $h=3$, then the solution to Eqs. (58) and (59) for arbitrary integers n_1, n_2, yields the array ambiguity plot. This case has been thoroughly investigated previously, so henceforth it will be assumed that $h>3$.

Dividing each equation of the system Eq. (60) by $\sin \Delta\theta$, there results

$$n_k - 2n_{k+1} \cos \Delta\theta + n_{k+2} = 0, \qquad 1 \le k \le h-3. \tag{61}$$

So for $h>3$, the array ambiguity plot is determined by all solutions for (x,y) of Eqs. (58) and (59), where $n_1, n_2, \ldots, n_{h-1}$ are any integer solutions of the system of Eq. (60).

ARRAY AMBIGUITY PLOT WHEN $\cos \Delta\theta$ IS IRRATIONAL

If there are just four antennas, then Eq. (61) is just one equation, the solution of which is

$$n_2 = 0 \tag{62}$$

$$n_3 = -n_1. \tag{63}$$

So n_2 must be zero and n_1 can be any integer. From Eqs. (58) and (59), the array ambiguity plot is a one-dimensional set of lattice points confined to the line

$$x \cos \Delta\theta + y \sin \Delta\theta = 0. \tag{64}$$

If there are five antennas, then Eq. (61) is a system of two equations, and in addition to Eqs. (62) and (63), one has

$$n_3 = 0 \tag{65}$$

$$n_4 = -n_2. \tag{66}$$

Equations (62), (63), (65), (66) imply $n_1 = n_2 = n_3 = n_4 = 0$, so that for five antennas there are no array ambiguities.

For more than five antennas, the conclusion is obviously the same as for five, as adding antennas cannot result in adding array ambiguities.

THE REGULAR PENTAGON

In this case, $\Delta\theta = \frac{2\pi}{5}$ rad. = 72°, and $\cos \Delta\theta = \frac{\sqrt{5}-1}{4}$, an irrational number. Therefore the regular pentagon array has no array ambiguities.

Figure 15 depicts this case. In Fig. 15, the $\vec{i}$ or x axis is horizontal, and the $\vec{j}$ or y axis is vertical. The dots represent the array ambiguity plot for antennas ①, ②, ③. The line grid associated with antennas ③ and ④ has been drawn, as well as the line grid associated with antennas ④ and ⑤. The symbol ▲ denotes the array ambiguity points for antennas ①, ②, ③, ④, a one-dimensional lattice, and the symbol △ those for ②, ③, ④, ⑤, another one-dimensional lattice. Note that the symbols ▲ lie on a line, the symbols △ lie on another line, and the two sets of lattice points have only the origin (the assumed true source coordinates) in common.

ARRAY AMBIGUITIES WHEN 2 cos Δθ IS AN INTEGER

In this case Eqs. (61) allow any integer values whatsoever for n_1 and n_2, so that the array ambiguity plot is a two-dimensional lattice given by the intersection of the two sets of grid lines defined by Eqs. (58) and (59).

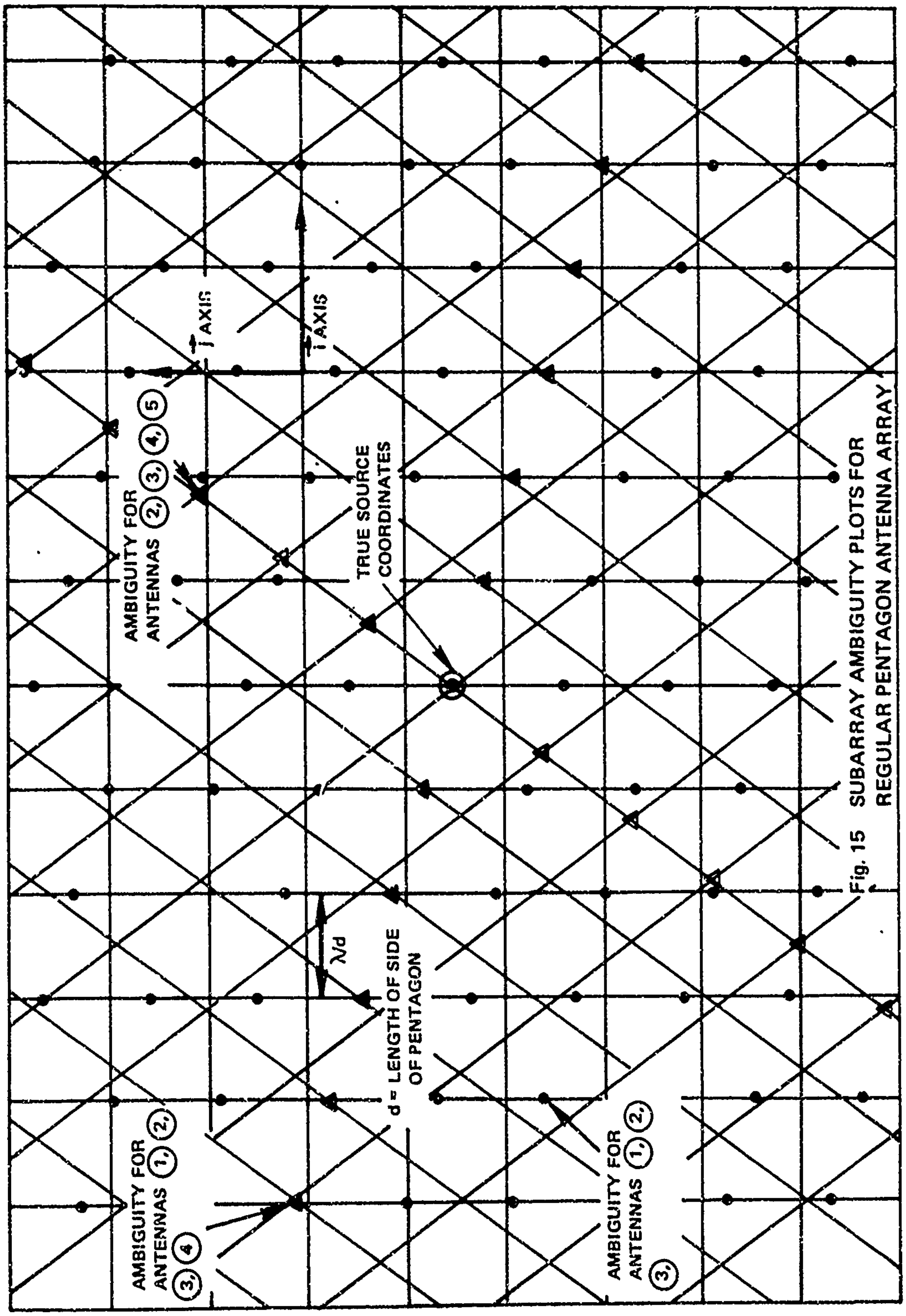

Fig. 15 SUBARRAY AMBIGUITY PLOTS FOR REGULAR PENTAGON ANTENNA ARRAY

Assuming $h > 3$, there are exactly two possibilities for this case, namely $\cos \Delta\theta = 0$ or $\frac{1}{2}$, corresponding to a square or a regular hexagon, respectively.

The regular polygon cases for 3, 4, 5, or 6 sides have now been completely solved.

ARRAY AMBIGUITIES WHEN $\cos \Delta\theta$ IS RATIONAL BUT $2 \cos \Delta\theta$ IS NOT AN INTEGER

Let $2 \cos \Delta\theta = \frac{a}{b}$, where a and b are integers, $\frac{a}{b}$ is in lowest terms, and b is not unity. The system of Eqs. (61) can be written for $h > 3$ as

$$b(n_k + n_{k+2}) - an_{k+1} = 0, \qquad 1 \leq k \leq h-3. \tag{67}$$

Case 1: h is odd. $(h > 3)$

In this case the number of equations in Eq. (67) is $h-3$, which is even. Instead of using Eqs. (58) and (59), we shall use

$$x \cos \left[\frac{h-3}{2} \Delta\theta\right] + y \sin \left[\frac{h-3}{2} \Delta\theta\right] = \frac{\lambda}{d} n_{\frac{h-1}{2}} \tag{68}$$

$$x \cos \left[\frac{h-1}{2} \Delta\theta\right] + y \sin \left[\frac{h-1}{2} \Delta\theta\right] = \frac{\lambda}{d} n_{\frac{h+1}{2}} . \tag{69}$$

Equations (67), (68), (69) constitute an equivalent set of equations which define the array ambiguity plot.

It will be shown that the array ambiguity plot is determined by Eqs. (68) and (69) where $n_{\frac{h-1}{2}}$ and $n_{\frac{h+1}{2}}$ are arbitrary multiples of $b^{\frac{h-3}{2}}$. Thus the array ambiguity plot is a two-dimensional lattice. If one compares this lattice with that for $h=3$, one sees that the ambiguities have been moved away from the origin by a factor of $b^{\frac{h-3}{2}}$.

For proof, Eqs. (67) are written as follows:

$$\frac{h-3}{2}\text{ equations}\left\{\begin{array}{lr} b(n_1+n_3) - an_2 = 0 & (70)\\ b(n_2+n_4) - an_3 = 0 & (71)\\ \vdots & \vdots \\ b\left(n_{\frac{h-5}{2}} + n_{\frac{h-1}{2}}\right) - an_{\frac{h-3}{2}} = 0 & (72)\\ b\left(n_{\frac{h-3}{2}} + n_{\frac{h+1}{2}}\right) - an_{\frac{h-1}{2}} = 0 & (73)\end{array}\right.$$

$$\frac{h-3}{2}\text{ equations}\left\{\begin{array}{lr} b\left(n_{\frac{h-1}{2}} + n_{\frac{h+3}{2}}\right) - an_{\frac{h+1}{2}} = 0 & (74)\\ b\left(n_{\frac{h+1}{2}} + n_{\frac{h+5}{2}}\right) - an_{\frac{h+3}{2}} = 0 & (75)\\ \vdots & \vdots \\ b\left(n_{h-4} + n_{h-2}\right) - an_{h-3} = 0 & (76)\\ b\left(n_{h-3} + n_{h-1}\right) - an_{h-2} = 0 & (77)\end{array}\right.$$

First it will be shown that $n_{\frac{h+1}{2}}$ and $n_{\frac{h-1}{2}}$ must be multiples of $b^{\frac{h-3}{2}}$. Equations (73) and (74) clearly imply that both $n_{\frac{h+1}{2}}$ and $n_{\frac{h-1}{2}}$ are divisible by b. If there is a solution in which the two n's are not zero, then there is some largest power of b which divides them both, and one may write

$$n_{\frac{h-1}{2}} = \alpha_1 b^k \tag{78}$$

and

$$n_{\frac{h+1}{2}} = \beta_1 b^k \tag{79}$$

where at least one of α_1 and β_1 is not divisible by b. If α_1 is not divisible by b, one works backward to the first equation, otherwise one works forward to the last. Without loss of generality, suppose α_1 is not divisible by b.

Solve Eq. (73) for $n_{\frac{h-3}{2}}$, and it follows that

$$n_{\frac{h-3}{2}} = \alpha_2 b^{k-1} \tag{80}$$

where α_2 is not divisible by b. Proceeding all the way to the first equation, one finally has

$$n_1 = \alpha_{\frac{h-1}{2}} b^{k-\left(\frac{h-3}{2}\right)} \tag{81}$$

where $\alpha_{\frac{h-1}{2}}$ is not divisible by b. In order for n_1 to be an integer, k cannot be smaller than $\frac{h-3}{2}$, which proves the result. Conversely, if $n_{\frac{h-1}{2}}$ and $n_{\frac{h+1}{2}}$ are arbitrary multiples of $b^{\frac{h-3}{2}}$, then proceeding from the middle equations up and down to the first and last equations, just as in the proof of necessity, one finds integer solutions for all of the other n's. The proof is now complete.

Case 2: h is even. $(h>3)$

The number of equations in Eq. (67) is $h-3$ which is now odd.

Instead of Eqs. (58) and (59), we shall use

$$x \cos\left[\frac{h-4}{2}\Delta\theta\right] + y \sin\left[\frac{h-4}{2}\Delta\theta\right] = \frac{\lambda}{d} n_{\frac{h-2}{2}} \tag{82}$$

$$x \cos\left[\frac{h-2}{2}\Delta\theta\right] + y \sin\left[\frac{h-2}{2}\Delta\theta\right] = \frac{\lambda}{d} n_{\frac{h}{2}} \tag{83}$$

Equations (67) are written as follows:

$$\frac{h-4}{2}\text{ equations}\begin{cases} b(n_1+n_3) - an_2 = 0 & (84) \\ b(n_2+n_4) - an_3 = 0 & (85) \\ \quad\vdots & \vdots \\ b\left(n_{\frac{h-6}{2}} + n_{\frac{h-2}{2}}\right) - an_{\frac{h-4}{2}} = 0 & (86) \\ b\left(n_{\frac{h-4}{2}} + n_{\frac{h}{2}}\right) - an_{\frac{h-2}{2}} = 0 & (87) \end{cases}$$

$$b\left(n_{\frac{h-2}{2}} + n_{\frac{h+2}{2}}\right) - an_{\frac{h}{2}} = 0 \tag{88}$$

$$\frac{h-4}{2}\text{ equations}\begin{cases} b\left(n_{\frac{h}{2}} + n_{\frac{h+4}{2}}\right) - an_{\frac{h+2}{2}} = 0 & (89) \\ \vdots & \vdots \\ b\left(n_{h-4} + n_{h-2}\right) - an_{h-3} = 0 & (90) \\ b\left(n_{h-3} + n_{h-1}\right) - an_{h-2} = 0 & (91) \end{cases}$$

The solutions to Eqs. (82)-(91) define the array ambiguity plot. It will be shown that the general solution for the array ambiguity plot is given by Eqs. (82) and (83) when $n_{\frac{h-2}{2}}$ is an arbitrary multiple of $b^{\frac{h-4}{2}}$ and $n_{\frac{h}{2}}$ is an arbitrary multiple of $b^{\frac{h-2}{2}}$. Therefore the array ambiguity plot is a two-dimensional lattice. If one compares this lattice with that for h - 1 antennas, it follows from Case 1 that the ambiguities have been moved out from the origin by a factor of b in one dimension, but are unaltered in the other.

The proof of sufficiency follows nearly the same as that for Case 1. Assuming that $n_{\frac{h-2}{2}}$ is an arbitrary multiple of $b^{\frac{h-4}{2}}$ and $n_{\frac{h}{2}}$ is an arbitrary multiple of $b^{\frac{h-2}{2}}$, one starts with Eqs. (87) and (88), working up and down equationwise, and finds that integer solutions for all of the n's result.

For necessity, one must show that if $n_{\frac{h-2}{2}}$ and $n_{\frac{h}{2}}$ are non-zero solutions to Eqs. (84)-(91), then $n_{\frac{h-2}{2}}$ is divisible by $b^{\frac{h-4}{2}}$ and $n_{\frac{h}{2}}$ is divisible by $b^{\frac{h-2}{2}}$.

Unless $h = 4$ in which case Eq. (88) is the only equation and the stated result follows easily, Eqs. (87) and (88) imply that $n_{\frac{h-2}{2}}$ and $n_{\frac{h}{2}}$ are both divisible by b.

There is therefore a highest power of b which divides both $n_{\frac{h-2}{2}}$ and $\frac{1}{b}\, n_{\frac{h}{2}}$. One can therefore write

$$n_{\frac{h-2}{2}} = \alpha_1 b^k \tag{92}$$

and

$$n_{\frac{h}{2}} = \beta_1 b^{k+1} \tag{93}$$

where not both α_1 and β_1 are divisible by b. Proceeding as in Case 1, one starts with Eqs. (87) and (88), working up or down, and finds that k cannot be less than $\frac{h-4}{2}$ or else one of n_1 or n_{h-1} will not be an integer. This completes the proof, and exhausts all possibilities for $\Delta\theta$.

ARRAY AMBIGUITY TOPOLOGIES FOR ANTENNAS ARRANGED IN A REGULAR POLYGON

The equilateral triangle, square, pentagon, and hexagon cases have already been completely solved. The array ambiguity plots for the equilateral triangle, square, and hexagon are all two-dimensional lattices. The pentagon has no array ambiguities at all. It will now be shown that any regular polygon with n sides for $n > 6$ has no array ambiguities at all.

From the previous discussion, the proof depends on proving that $\cos \frac{2\pi}{n}$ is irrational for $n > 6$.

If $\cos \frac{2\pi}{n}$ were rational, it is clear that $2 \cos \frac{2\pi}{n}$ is not an integer for $n > 6$. If $2 \cos \frac{2\pi}{n} = \frac{a}{b}$ in the lowest terms, then $b > 1$. As one adds antennas with $\Delta\theta = \frac{2\pi}{n}$, the previous proof shows that with each added antenna, the array ambiguity plot is always a two-dimensional lattice in which the plot has been altered from the previous step. On the other hand, after n steps, added antennas are only repetitions, and the array ambiguity plot cannot change thereafter.

This contradiction proves that $\cos \frac{2\pi}{n}$ is irrational for $n > 6$ and that therefore there are no array ambiguities for regular polygon antenna arrangements with more than 6 sides.

n ANTENNAS ON A CIRCLE WITH $n > 4$

The remaining loose end is to show that all three candidate topologies can be achieved for this arrangement for any such n.

Examples have already been given which show that the two-dimensional lattice and no ambiguity cases can be achieved. So all that remains is the one-dimensional lattice topology.

Let $n = h + 1$, and select an equal central angle case with h antennas such that $\sin \frac{1}{2} \Delta\theta = \frac{1}{3}\sqrt{3}$ and $\cos \frac{1}{2} \Delta\theta = \frac{1}{3}\sqrt{6}$. From the h^{th} antenna, move around the circle through a central angle ν where the n^{th} antenna will be placed. Choose $\sin \frac{1}{2} \nu = \frac{1}{2}\sqrt{1 - \frac{1}{3}\sqrt{6}}$ and $\cos \frac{1}{2} \nu = \frac{1}{2}\sqrt{3 + \frac{1}{3}\sqrt{6}}$. (See Fig. 16.) Some other useful numbers are

$$\sin \nu = \frac{1}{2}\sqrt{2} - \frac{1}{6}\sqrt{3}, \qquad \cos \nu = \frac{1}{2} + \frac{1}{6}\sqrt{6},$$

$$\sin \Delta\theta = \frac{2}{3}\sqrt{2}, \qquad \cos \Delta\theta = \frac{1}{3},$$

$$\sin \frac{3}{2} \Delta\theta = \frac{5}{9}\sqrt{3}, \qquad \cos \frac{3}{2} \Delta\theta = -\frac{1}{9}\sqrt{6}.$$

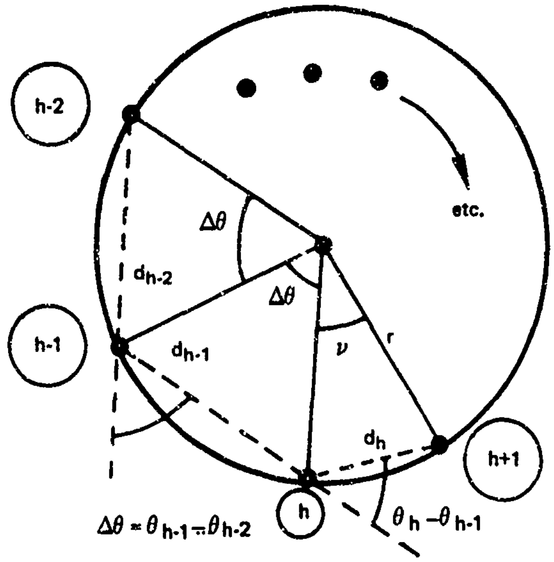

Fig. 16 AN ARRAY WITH n = h+1 ANTENNAS

Since $\cos \Delta\theta$ is rational and $2 \cos \Delta\theta$ is not an integer, the array ambiguity plot of the first h antennas is a two-dimensional lattice given by the solution to (68)-(77) or (82)-(91), where a = 2, b = 3. Also, by an argument similar to the regular polygon argument, there are no antenna duplications.

For the last antenna, another equation must be added, namely the last one of the system Eq. (27), which in this case is

$$\frac{n_{h-2}}{d_{h-2}} \sin\left(\frac{1}{2}\,\Delta\theta + \frac{1}{2}\,\nu\right) - \frac{n_{h-1}}{d_{h-1}} \sin\left(\frac{3}{2}\,\Delta\theta + \frac{1}{2}\,\nu\right) + \frac{n_h}{d_h} \sin \Delta\theta = 0. \tag{94}$$

From Fig. 16,

$$d_{h-2} = d_{h-1} = 2r \sin \frac{1}{2} \Delta\theta = \frac{2r}{3} \sqrt{3} \tag{95}$$

and

$$d_h = 2r \sin \frac{1}{2} \nu = r\sqrt{1 - \frac{1}{3}\sqrt{6}}. \tag{96}$$

Multiplying Eq. (94) by r, there results

$$\frac{n_{h-2}}{4}\left[\sqrt{3+\frac{1}{3}\sqrt{6}} + \sqrt{2-\frac{2}{3}\sqrt{6}}\right] + \frac{n_{h-1}}{12}\left[-5\sqrt{3+\frac{1}{3}\sqrt{6}} + \sqrt{2-\frac{2}{3}\sqrt{6}}\right]$$

$$+ \frac{4n_h}{3}\left[\frac{1}{\sqrt{2-\frac{2}{3}\sqrt{6}}}\right] = 0. \tag{97}$$

Multiplying through by $\frac{3}{4}\sqrt{2 - \frac{2}{3}\sqrt{6}}$, there results

$$\frac{3}{16} n_{h-2} (4 - \sqrt{6}) + \frac{1}{16} n_{h-1} (-8 + \sqrt{6}) + n_h = 0. \tag{98}$$

As in Section 2, it follows that there exists a set of three antennas which have the same array ambiguity plot as the first h antennas of Fig. 16. The equations for the plot are given by either Eqs. (68) and (69) or Eqs. (82) and (83) depending on whether h is odd or even, and where the n's in those formulas are arbitrary multiples of the appropriate powers of 3, since b = 3 in this case. The equivalent set of three antennas will not in general lie on the circle.

A new four antenna array is constructed as follows: throw away the first h - 1 antennas in Fig. 16. Order the three antennas in the equivalent three antenna array such that

Eqs. (68) and (69) or Eqs. (82) and (83) hold, where the subscripts in those equations are changed to the appropriate values. Translate the three antenna array until the third antenna lies atop the h^{th} antenna of Fig. 16.

This four antenna array has the same array ambiguity plot as that for Fig. 16, the equations for which are given by Eqs. (68), (69), (98) or (82), (83), (98), where the subscripts are changed to the appropriate values. In the notation of Section 2, c_1 and c_2 are both irrational, and Eq. (98) has non-zero solutions, for example $n_{h-2} = 4$, $n_{h-1} = 12$, $n_h = 3$.

This is an example of Case 2 for four antennas in Section 2, where it is shown that the array ambiguity plot is a one-dimensional lattice confined to a single line.

All possible array ambiguity plot topologies have now been determined for the cases examined.

Figure 17 is a flow chart which summarizes the topology results for Sections 2 and 3.

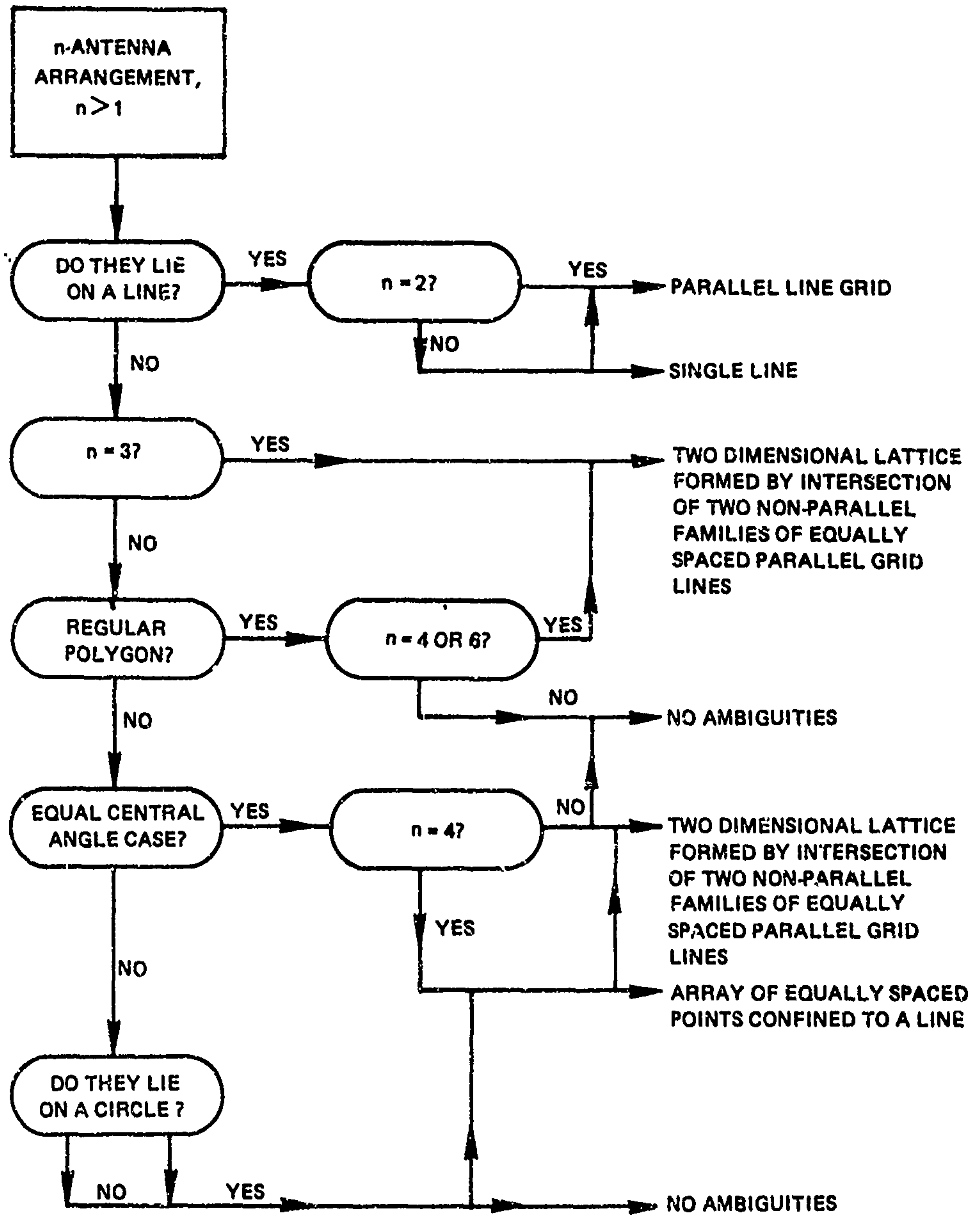

Fig. 17 FLOW CHART FOR ACHIEVABLE ARRAY AMBIGUITY TOPOLOGIES FOR CERTAIN ARRAY TYPES

4. INVARIANTS OF ARRAYS WITH TRIANGULAR ARRAY AMBIGUITY PLOTS

AREA OF A TRIANGLE FORMED BY TWO VECTORS

If $\vec{d}_1$ and $\vec{d}_2$ are two vectors, then triangles can be formed from them in four ways; put the tails together and connect the heads, put the heads together and connect the tails, or put the tail of either at the head of the other and connect for the third side. (See Fig. 18.)

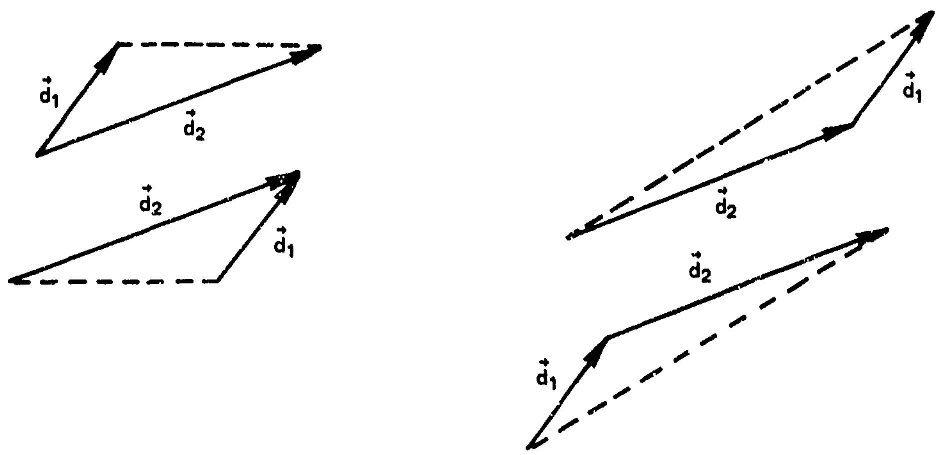

Fig. 18 TRIANGLES FORMED BY TWO VECTORS

The same four triangles are generated by putting the tails of $\pm\vec{d}_1$ and $\pm\vec{d}_2$ together and connecting the heads, where the four combinations of signs give the four triangles.

All of the triangles have the same area, given by

$$A = \frac{1}{2} \left|\vec{d}_1 \times \vec{d}_2\right| . \tag{99}$$

The proof is simple and is omitted.

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NECESSARY AND SUFFICIENT CONDITIONS THAT TWO TRIANGULAR ANTENNA ARRAYS HAVE THE SAME ARRAY AMBIGUITY PLOT

The primary result of this sub-section is the following theorem:

Let $\vec{d}_1$, $\vec{d}_2$ be vectors along two of the sides of one triangle, and $\vec{d}_1'$, $\vec{d}_2'$ vectors along two of the sides of another. Then a necessary and sufficient condition that the two triangles have the same array ambiguity plot is that there exist four integers α_1, β_1, α_2, β_2 such that

$$\vec{d}_1' = \alpha_1\vec{d}_1 + \beta_1\vec{d}_2 \tag{100}$$

$$\vec{d}_2' = \alpha_2\vec{d}_1 + \beta_2\vec{d}_2 \tag{101}$$

where

$$\alpha_1\beta_2 - \alpha_2\beta_1 = \pm 1. \tag{102}$$

To prove this, let $\vec{U}$ denote an arbitrary vector in the ambiguity plot plane which joins the location of the true source coordinates to an arbitrary array ambiguity.

It follows from Section 2 that

$$\vec{d}_1 \cdot \vec{U} = m\lambda \tag{103}$$

$$\vec{d}_2 \cdot \vec{U} = n\lambda \tag{104}$$

where all array ambiguities are generated as m and n range over all integers.

Using vector algebra, one can solve the above two equations for the vector $\vec{U}$, the solution being

$$\vec{U} = \lambda \frac{(m\vec{d}_2 - n\vec{d}_1) \times (\vec{d}_1 \times \vec{d}_2)}{(\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_1 \times \vec{d}_2)}. \tag{105}$$

Since $\vec{d}_1$ and $\vec{d}_2$ are not collinear, any vector in the antenna array plane can be expressed as a linear combination of $\vec{d}_1$ and $\vec{d}_2$. In particular one may write Eqs. (100) and (101), where α_1, β_1, α_2, β_2 are some numbers.

Since the triangle formed by $\vec{d}_1'$ and $\vec{d}_2'$ has the same array ambiguity plot, one must have $\vec{d}_1' \cdot \vec{U} = m'\lambda$, $\vec{d}_2' \cdot \vec{U} = n'\lambda$, where m' and n' are integers. Dotting Eqs. (100) and (101) with $\vec{U}$, one obtains

$$m' = \alpha_1 m + \beta_1 n \tag{106}$$

$$n' = \alpha_2 m + \beta_2 n. \tag{107}$$

Take $m = 1$, $n = 0$, and one sees that α_1 and α_2 must be integers. Similarly, take $m = 0$, $n = 1$, and one sees that β_1 and β_2 must be integers.

If one takes $m' = 1$, $n' = 0$ and solves for m and n, one sees that $\frac{\beta_2}{\Delta}$ and $\frac{\alpha_2}{\Delta}$ are integers, where

$$\Delta = \alpha_1\beta_2 - \alpha_2\beta_1. \tag{108}$$

Similarly, taking $m' = 0$, $n' = 1$, one concludes that $\frac{\beta_1}{\Delta}$ and $\frac{\alpha_1}{\Delta}$ are integers. The case $\Delta = 0$ is excluded, as this would imply no solution for these two cases.

Since Δ must be an integer, if it were not plus or minus unity it would follow that α_1, β_1, α_2, β_2 are all multiples of Δ, and hence from Eqs. (106) and (107), so are m' and n', which is a contradiction. Therefore, $\Delta = \pm 1$, and the necessity is proved.

For proof of sufficiency, assume Eqs. (100), (101), (102) hold with α_1, β_1, α_2, β_2 integers. Dotting Eqs. (100)

and (101) with $\vec{U}$ produces Eqs. (106) and (107), so that whenever m and n are integers, so are m′ and n′.

If one solves Eqs. (100) and (101) for $\vec{d}_1$ and $\vec{d}_2$, one has

$$\vec{d}_1 = \frac{\beta_2}{\Delta}\vec{d}_1' - \frac{\beta_1}{\Delta}\vec{d}_2' \tag{109}$$

$$\vec{d}_2 = -\frac{\alpha_2}{\Delta}\vec{d}_1' + \frac{\alpha_1}{\Delta}\vec{d}_2' \tag{110}$$

where the coefficients of $\vec{d}_1'$ and $\vec{d}_2'$ are integers, and the determinant of the coefficients is $\frac{1}{\Delta}$, which is also ±1.

Dotting Eqs. (109) and (110) with $\vec{U}$ shows that whenever m′ and n′ are integers, so are m and n. This completes the proof of the theorem.

First Corollary: There are two corollaries to the theorem just proved which are of interest. The first is as follows: if two triangular arrays of antennas have the same array ambiguity plot, then the areas of the two triangles are the same.

To prove this, form the cross product of $\vec{d}_1'$ and $\vec{d}_2'$ using Eqs. (100) and (101). One has

$$\vec{d}_1' \times \vec{d}_2' = \Delta\vec{d}_1 \times \vec{d}_2 . \tag{111}$$

The result follows from Eq. (99) and the fact that $\Delta = \pm 1$.

It is shown in Section 2 that there are five distinct topologies for array ambiguity plots. It is also shown that the two-dimensional lattice point topology can always be generated by some triangular antenna array, and that every triangular antenna array generates such a topology.

It will be convenient to refer to a plot with that topology as a <u>triangular array ambiguity plot</u> (TAAP). The above corollary shows that if P is a TAAP, then there exists a number T(P) uniquely determined by P, such that the area of any triangle formed by three antennas which generate P is T(P). T(P) will be called the <u>triangular array area</u> (TAA) of the array ambiguity plot P.

<u>Second Corollary</u>: Let P be a TAAP, and let $\vec{d}_1$ and $\vec{d}_2$ be two vectors in the antenna array plane which form a triangle generating P.

Two vectors in the array ambiguity plot plane are formed as follows: $\vec{U}_1$ is a vector joining the true source coordinates to the nearest ambiguity along a direction perpendicular to $\vec{d}_1$, and $\vec{U}_2$ is a vector joining the true source coordinates to the nearest ambiguity along a direction perpendicular to $\vec{d}_2$.

Expressions for $\vec{U}_1$ and $\vec{U}_2$ are obtained by setting $m = 0$, $n = 1$ and $m = 1$, $n = 0$ respectively, in Eq. (105). One has

$$\vec{U}_1 = \frac{-\lambda\, \vec{d}_1 \times (\vec{d}_1 \times \vec{d}_2)}{(\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_1 \times \vec{d}_2)} \tag{112}$$

$$\vec{U}_2 = \frac{\lambda\, \vec{d}_2 \times (\vec{d}_1 \times \vec{d}_2)}{(\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_1 \times \vec{d}_2)} . \tag{113}$$

The area of the triangle formed by $\vec{U}_1$ and $\vec{U}_2$ is $\frac{1}{2} |\vec{U}_1 \times \vec{U}_2|$. From Eqs. (112) and (113), using vector algebra, one has

$$\vec{U}_1 \times \vec{U}_2 = -\lambda^2 \frac{\vec{d}_1 \times \vec{d}_2}{(\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_1 \times \vec{d}_2)} \tag{114}$$

from which one has

$$\frac{1}{2}\,|\vec{U}_1 \times \vec{U}_2| = \frac{\lambda^2}{4T(P)}. \tag{115}$$

Therefore the area of the triangle formed by $\vec{U}_1$ and and $\vec{U}_2$ is also a plot invariant. Denoting the area by E(P), one then has

$$E(P) = \frac{\lambda^2}{4T(P)}. \tag{116}$$

This last result is the second corollary. E(P) will be called the elemental generating area (EGA) of the array ambiguity plot P.

This result is in agreement with the fact that if the antennas are moved closer together, the ambiguities move further out, and vice versa.

NECESSARY AND SUFFICIENT CONDITIONS THAT AN ARRAY WITH h + 1 ANTENNAS (h ≥ 2) HAVE A GIVEN TAAP

It requires no more than h vectors joining various antenna pairs to describe the array for ambiguity purposes. This follows from Section 2 and the discussion of ordering of the antennas. If they are ordered in an arbitrary way 1, 2, ..., h+1, then $\vec{d}_k'$ (k= 1,2,...,h) may be taken to be the vector with tail at k and head at k+1. The Eqs. (13) can be rewritten as expressions involving these vectors.

One could also take $\vec{d}_k'$ to be the vector with tail at 1 and head at k+1, as either of the two sets of vectors will produce equivalent sets of the Eqs. (13).

So let $\vec{d}_k'$ be any set of h vectors (there may be duplications) which describe the antenna array, and assume that the array has a TAAP. Let $\vec{d}_1$ and $\vec{d}_2$ be two vectors describing a triangle which has the same TAAP. Since any vector in the plane can be expressed uniquely as a linear combination of $\vec{d}_1$ and $\vec{d}_2$, one can write

$$\vec{d}'_k = \alpha_k \vec{d}_1 + \beta_k \vec{d}_2, \qquad k = 1, 2, \ldots, h. \tag{117}$$

The necessary and sufficient conditions that the array and the triangle array have the same TAAP is that the α_k's and β_k's are all integers and that there exists no integer other than ± 1 which divides all of the $\frac{h(h-1)}{2}$ determinants

$$\Delta_{jk} = \alpha_j \beta_k - \alpha_k \beta_j, \tag{118}$$

as will be proved.

Let $\vec{U}$ be the vector in Eq. (105) with $m=1$ and $n=0$, so that from Eqs. (103) and (104), $\vec{d}_1 \cdot \vec{U} = \lambda$ and $\vec{d}_2 \cdot \vec{U} = 0$. Since the two antenna arrays have the same TAAP, $\vec{d}'_k \cdot \vec{U}$ must be an integral multiple of λ. If one dots Eq. (117) with $\vec{U}$, one then sees that a necessary condition for the two arrays to have the same TAAP is that the α_k's are integers. Similarly, taking $m=0$ and $n=1$ implies the same thing about the β_k's. So for the rest of the proof, it is assumed that the α_k's and β_k's are integers.

If one dots Eq. (117) with any vector such that Eqs. (103) and (104) hold, one then sees that any array ambiguity of the triangle array is an array ambiguity of the $h+1$ antenna array. The integer condition on the α_k's and β_k's is therefore both necessary and sufficient that this be the case.

To complete the proof, one must show that the determinant conditions are necessary and sufficient that every array ambiguity of the $h+1$ antennas is an array ambiguity of the triangle.

Note first that for $h=2$ the theorem has already been proved in the previous sub-section. So assume $h \geq 3$.

Order the equations so that $\vec{d}_1'$ and $\vec{d}_2'$ are not collinear. If this were not possible, all of the antennas would lie on a line and the array could not have a TAAP.

By making appropriate linear combinations of pairs of equations of Eq. (117), one can write

$$\Delta_{1j}\vec{d}_1 = \beta_j\vec{d}_1' - \beta_1\vec{d}_j' \tag{119}$$

and

$$\Delta_{1j}\vec{d}_2 = -\alpha_j\vec{d}_1' + \alpha_1\vec{d}_j' \tag{120}$$

for $j = 2, 3, \ldots, h$.

Next, multiply Eqs. (119) by α_2, (120) by β_2, and add, obtaining

$$\Delta_{1j}\vec{d}_2' = \Delta_{2j}\vec{d}_1' + \Delta_{12}\vec{d}_j' \tag{121}$$

for $j = 3, 4, \ldots, h$.

Since $\Delta_{12} \neq 0$ ($\vec{d}_1'$ and $\vec{d}_2'$ are not collinear), it is clear that all of the $\vec{d}_j'$'s for $j = 3, \ldots, h$ can be expressed in terms of $\vec{d}_1'$ and $\vec{d}_2'$ using the Δ's alone.

If $\vec{U}$ is a vector in the array ambiguity plot plane connecting the true source coordinates with an arbitrary array ambiguity, then $\vec{U}$ dotted with all of the $\vec{d}_k'$'s are integer multiples of λ.

If one sets

$$\vec{d}_j' \cdot \vec{U} = n_j \lambda, \qquad j = 1, 2, \ldots, h \tag{122}$$

and dots Eq. (121) with $\vec{U}$, one sees that all array ambiguities can be found as follows:

First, find all integer solutions to

$$\Delta_{2j} n_1 + \Delta_{j1} n_2 + \Delta_{12} n_j = 0 \tag{123}$$

for $j = 3, 4, \ldots, h$.

Then (as in Eq. (105)) all possible vectors $\vec{U}$ defining the array ambiguities are given by

$$\vec{U} = \lambda \frac{(n_1 \vec{d}_2' - n_2 \vec{d}_1') \times (\vec{d}_1' \times \vec{d}_2')}{(\vec{d}_1' \times \vec{d}_2') \cdot (\vec{d}_1' \times \vec{d}_2')} \tag{124}$$

where n_1 and n_2 are found from the first step.

Taking the first two equations of Eq. (117) and performing the cross product, one sees that one can write $\vec{U}$ in terms of $\vec{d}_1$ and $\vec{d}_2$, resulting in

$$\vec{U} = \lambda \frac{\left[(n_1 \alpha_2 - n_2 \alpha_1)\vec{d}_1 + (n_1 \beta_2 - n_2 \beta_1)\vec{d}_2\right] \times (\vec{d}_1 \times \vec{d}_2)}{\Delta_{12} (\vec{d}_1 \times \vec{d}_2) \cdot (\vec{d}_1 \times \vec{d}_2)} \tag{125}$$

and hence

$$\vec{d}_1 \cdot \vec{U} = \lambda \frac{n_1 \beta_2 - n_2 \beta_1}{\Delta_{12}} \tag{126}$$

$$\vec{d}_2 \cdot \vec{U} = \lambda \frac{n_2 \alpha_1 - n_1 \alpha_2}{\Delta_{12}} \tag{127}$$

In addition to $\Delta_{12} \neq 0$, it also follows that at least one of Δ_{2j} and Δ_{j1} is not zero, for otherwise $\vec{d}_j'$ would be collinear with both $\vec{d}_1'$ and $\vec{d}_2'$, an impossibility. From here on, separate cases will be considered.

Case 1: $h = 3$.

Here Eq. (123) is

$$\Delta_{23}n_1 + \Delta_{31}n_2 + \Delta_{12}n_3 = 0. \tag{128}$$

Let g be the greatest common factor of the Δ's, and write

$$\Delta_{jk} = g\delta_{jk}, \qquad j, k = 1, 2, 3, \tag{129}$$

so that

$$\delta_{23}n_1 + \delta_{31}n_2 + \delta_{12}n_3 = 0 \tag{130}$$

where $\delta_{12} \neq 0$, at least one of δ_{23} and δ_{31} is not zero, and the three δ's have no common factor.

Suppose without loss of generality that $\delta_{23} \neq 0$. Then let g' be the greatest common factor of δ_{23} and δ_{12}. (g' will be unity if δ_{31} is zero.)

From the theory of congruences, the solutions for n_1 and n_2 can be found by solving

$$\left(\frac{\delta_{23}}{g'}\right) n_1 \equiv \frac{-\delta_{31}n_2}{g'} \left(\mathrm{mod}\ \frac{\delta_{12}}{g'}\right) \tag{131}$$

n_2 must be a multiple of g'. If $\delta_{31} = 0$ this follows since $g' = 1$. If $\delta_{31} \neq 0$, it follows since δ_{31} can have no factor in common with g'.

So for n_2 an arbitrary multiple of g', since $\frac{\delta_{23}}{g'}$ and $\frac{\delta_{12}}{g'}$ are relatively prime, one can multiply Eq. (131) through by the inverse of $\frac{\delta_{23}}{g'}\left(\mathrm{mod}\ \frac{\delta_{12}}{g'}\right)$ and solve for n_1. If p is this inverse, then

$$n_1 = -\delta_{31}\, p\left(\frac{n_2}{g'}\right) + n\,\frac{\delta_{12}}{g'} \qquad (132)$$

where n can be any integer.

Assume that every array ambiguity of the four antenna array is also one for the equivalent triangle.

Let $\frac{n_2}{g'}$ be zero and n = 1. Substitution into Eqs. (126) and (127) then results in concluding $\frac{\beta_2}{gg'}$ and $\frac{\alpha_2}{gg'}$ are integers. Hence gg' divides both α_2 and β_2.

If $\vec{U}$ is an arbitrary vector representing an array ambiguity of the four antenna array, dotting Eq. (101) with $\vec{U}$ then implies that $\vec{d}_2' \cdot \vec{U}$ is always a multiple of $\lambda gg'$. Unless $g = 1$, this contradicts the result that n_2 can be any multiple of g'. Therefore, the determinant condition is necessary.

Conversely, suppose the determinant condition holds with Δ_{23} assumed to be $\neq 0$ without loss of generality. Then one solves Eq. (128) as before, where now Δ_{23}, Δ_{31} and Δ_{12} have no factor in common and, setting $n_2 = mg'$, the solution for array ambiguities is given by

$$n_2 = mg' \qquad (133)$$

$$n_1 = -\Delta_{31}pm + n\,\frac{\Delta_{12}}{g'}, \qquad (134)$$

where m and n are any integers, g' is the greatest common factor of Δ_{23} and Δ_{12}, and p is the inverse of $\frac{\Delta_{23}}{g'}\left(\mathrm{mod}\ \frac{\Delta_{12}}{g'}\right)$.

Substitution into Eqs. (126) and (127) yields

$$\vec{d}_1 \cdot \vec{U} = \lambda \left[\frac{\beta_2 n}{g'} - \frac{m\,(p\Delta_{31}\beta_2 + g'\beta_1)}{\Delta_{12}}\right] \tag{135}$$

$$\vec{d}_2 \cdot \vec{U} = \lambda \left[\frac{-\alpha_2 n}{g'} + \frac{m\,(p\Delta_{31}\alpha_2 + g'\alpha_1)}{\Delta_{12}}\right] \tag{136}$$

It remains to show that the coefficients of m and n inside the brackets are integers.

If one sets $j = 2$ in Eq. (119) and forms the cross product of Eq. (119) with $\vec{d}_3'$, one obtains the identity

$$\beta_2\Delta_{31} = -\beta_3\Delta_{12} - \beta_1\Delta_{23}. \tag{137}$$

One concludes from this equation that β_2 is divisible by g'.

If one multiplies Eq. (137) by p, and notes that $p\Delta_{23} = g' + k\Delta_{12}$, where k is some integer, one sees that

$$p\Delta_{31}\beta_2 + g'\beta_1 = -\Delta_{12}[\beta_3 p + \beta_1 k]. \tag{138}$$

Hence, in Eq. (135), the coefficients of m and n in the brackets are integers. Repeating the process with Eq. (120) shows the same result for Eq. (136). The proof for h =3 is now complete.

<u>Corollary</u>: For $h = 3$, if P is the TAAP generated by $\vec{d}_1$ and $\vec{d}_2$, Eq. (117) holds with integer coefficients, and P' is the TAAP generated by the four antenna array, then $T(P') = gT(P)$, where g is the greatest common factor of the determinants.

To prove this, it will first be shown that the TAAP generated by

$$\vec{d}_1'' = \frac{g'}{\delta_{12}} \vec{d}_1' + \frac{\delta_{31} p}{\delta_{12}} \vec{d}_2' \tag{139}$$

and

$$\vec{d}_2'' = \frac{1}{g'} \vec{d}_2' \tag{140}$$

is the same as that generated by d_1', d_2', d_3', where the δ's and g' are as before when g was not necessarily unity.

First, let $\vec{U}$ denote an array ambiguity of the TAAP generated by $\vec{d}_1'$, $\vec{d}_2'$, $\vec{d}_3'$. Then $\vec{d}_1' \cdot \vec{U} = \lambda n_1$ from Eq. (132) and $\vec{d}_2' \cdot \vec{U} = \lambda n_2$ from Eq. (133). Dotting both sides of Eqs. (139) and (140) with $\vec{U}$ results in

$$\vec{d}_1'' \cdot \vec{U} = \lambda n \tag{141}$$

$$\vec{d}_2'' \cdot \vec{U} = \lambda m \tag{142}$$

which are integers.

Conversely, let $\vec{U}$ denote an array ambiguity of $\vec{d}_1''$ and $\vec{d}_2''$. As in Eq. (105), one writes

$$\vec{U} = \lambda \frac{(n\vec{d}_2'' - m\vec{d}_1'') \times (\vec{d}_1'' \times \vec{d}_2'')}{(\vec{d}_1'' \times \vec{d}_2'') \cdot (\vec{d}_1'' \times \vec{d}_2'')} . \tag{143}$$

Performing the cross product of Eqs. (139) and (140) one has

$$\vec{d}_1'' \times \vec{d}_2'' = \frac{1}{\delta_{12}} (\vec{d}_1' \times \vec{d}_2') \tag{144}$$

so that Eq. (143) may be written

$$\vec{U} = \lambda \frac{\left[\left(n \frac{\delta_{12}}{g'} - m\delta_{31} p\right) \vec{d}_2' - mg'\vec{d}_1'\right] \times (\vec{d}_1' \times \vec{d}_2')}{(\vec{d}_1' \times \vec{d}_2') \cdot (\vec{d}_1' \times \vec{d}_2')} . \tag{145}$$

If one dots Eq. (145) with both $\vec{d}_1'$ and $\vec{d}_2'$, one gets

$$\vec{d}_1' \cdot \vec{U} = \lambda \left[n \frac{\delta_{12}}{g} - m\delta_{31}p \right] \tag{146}$$

and

$$\vec{d}_2' \cdot \vec{U} = \lambda \, [mg'] \tag{147}$$

which satisfy the conditions of Eqs. (132) and (133) that $\vec{U}$ be an array ambiguity of the plot generated by $\vec{d}_1'$, $\vec{d}_2'$, $\vec{d}_3'$.

To prove the corollary, one has, using Eq. (144)

$$\vec{d}_1'' \times \vec{d}_2'' = \frac{1}{\delta_{12}} (\vec{d}_1' \times \vec{d}_2') = \frac{\Delta_{12}}{\delta_{12}} (\vec{d}_1 \times \vec{d}_2) = g\,(\vec{d}_1 \times \vec{d}_2). \tag{148}$$

The corollary then follows from the previous results on TAA's.

Case 2: $h \geq 4$.

The proof here is to show that the h equations may be replaced by h - 1 equations whose determinants have the same g.c.f., and TAAP's are preserved.

To this end, expressing $\vec{d}_1''$ and $\vec{d}_2''$ of Eqs. (139) and (140) in terms of $\vec{d}_1$ and $\vec{d}_2$, one has

$$\vec{d}_1'' = \left[\frac{g'\alpha_1 + \delta_{31}p\alpha_2}{\delta_{12}} \right] \vec{d}_1 + \left[\frac{g'\beta_1 + \delta_{31}p\beta_2}{\delta_{12}} \right] \vec{d}_2 \tag{149}$$

and

$$\vec{d}_2'' = \frac{\alpha_2}{g'} \vec{d}_1 + \frac{\beta_2}{g'} \vec{d}_2 \tag{150}$$

where $\vec{d}_1''$ and $\vec{d}_2''$ have the same TAAP as $\vec{d}_1'$, $\vec{d}_2'$, and $\vec{d}_3'$.

Dividing Eq. (137) by g results in

$$\beta_2\delta_{31} = -\beta_3\delta_{12} - \beta_1\delta_{23}. \tag{151}$$

Similarly, one derives the identity

$$\alpha_2\delta_{31} = -\alpha_3\delta_{12} - \alpha_1\delta_{23}. \tag{152}$$

From the definition of p, one has

$$p\delta_{23} = g' + k\delta_{12} \tag{153}$$

where k is some integer.

From Eqs. (151) and (152) it is seen that β_2 and α_2 are divisible by g'.

Multiplying Eqs. (151) and (152) by p, and using Eq. (153), substitution into Eq. (149) results in

$$\vec{d}_1'' = -(\alpha_3 p + \alpha_1 k)\,\vec{d}_1 - (\beta_3 p + \beta_1 k)\,\vec{d}_2. \tag{154}$$

Also,

$$\vec{d}_2'' = \frac{\alpha_2}{g'}\,\vec{d}_1 + \frac{\beta_2}{g'}\,\vec{d}_2. \tag{155}$$

The determinant of the coefficients can be found by computing the cross product $\vec{d}_1'' \times \vec{d}_2''$, which from Eq. (148) is found to be just g.

In the h original equations, select any one other than the first three, namely

$$\vec{d}_j' = \alpha_j\vec{d}_1 + \beta_j\vec{d}_2. \tag{156}$$

The three determinants associated with Eqs. (154), (155), (156) are given by

$$\Delta_1 = g \tag{157}$$

$$\Delta_2 = -p\Delta_{3j} - k\Delta_{1j} \tag{158}$$

$$\Delta_3 = \frac{\Delta_{2j}}{g'}. \tag{159}$$

Since g is the greatest common factor of Δ_{12}, Δ_{23}, and Δ_{31}, the six determinants of the equations for $\vec{d}_1'$, $\vec{d}_2'$, $\vec{d}_3'$, $\vec{d}_j'$ will have the same g.c.f. as Eqs. (157), (158), (159) if any factor of g which is a factor of Δ_2 and Δ_3 is also a factor of Δ_{1j}, Δ_{2j}, Δ_{3j}, and vice versa.

Toward this end, some more identities are needed. If one forms the cross product of Eq. (121) with $\vec{d}_3'$, divides by g and rearranges terms, one arrives at

$$\Delta_{3j}\delta_{12} + \Delta_{1j}\delta_{23} = -\Delta_{2j}\delta_{31}. \tag{160}$$

If one multiplies Eq. (158) by δ_{23}, substitutes Eqs. (153) and (160), divides by g', substitutes Eq. (159), one arrives at

$$\Delta_{3j} = -\frac{\delta_{23}}{g'}\Delta_2 + k\delta_{31}\Delta_3. \tag{161}$$

If one multiplies Eq. (158) by δ_{12}, substitutes Eqs. (153) solved for $k\delta_{12}$, substitutes Eq. (160), divides by g', substitutes Eq. (159), one also has

$$\Delta_{1j} = \frac{\delta_{12}}{g'}\Delta_2 - p\delta_{31}\Delta_3. \tag{162}$$

Now let g'' divide g, Δ_2, Δ_3. From Eqs. (161) and (162), g'' divides Δ_{3j} and Δ_{1j}. From Eq. (159), g'' divides Δ_{2j}.

Conversely, let g'' divide g, Δ_{1j}, Δ_{2j}, Δ_{3j}. From Eq. (158), g'' divides Δ_2. From Eqs. (161) and (162), g'' divides $k\delta_{31}\Delta_3$ and $p\delta_{31}\Delta_3$. If one divides Eq. (153) by g', one sees that p and k have no factor in common. Hence g'' divides $\delta_{31}\Delta_3$. From Eq. (159), g'' divides $g'\Delta_3$. But g' and δ_{31} have no factor in common, so that g'' divides Δ_3.

The equations have now been reduced to $h-1$ equations whose determinants have the same g.c.f., and TAAP's are preserved. Now repeat the process until three equations are reached. An appeal to the theorem for Case 1 then proves the theorem in general.

Corollary: If $h+1$ antennas not on a line have a TAAP denoted by P', with $\vec{d}_k'$ being h vectors representing the array, $k = 1, 2, \ldots, h$; if $\vec{d}_1$ and $\vec{d}_2$ are such that Eq. (117) holds with integer coefficients, and P is the TAAP generated by $\vec{d}_1$ and $\vec{d}_2$, then $T(P') = gT(P)$, where g is the greatest common factor of the $\frac{h(h-1)}{2}$ determinants.

The proof follows from the corollary to Case 1 and the procedure discussed in Case 2.

THE FUNDAMENTAL AREA THEOREM

If $h+1$ antennas are not all on a line, then the array has a TAAP if and only if the ratio of the areas of all pairs of triangles formed by antenna elements is a rational number. In addition, if another array of three antenna elements has the same TAAP, then the area of the triangle is the largest number which divides into the areas of all triangles of the $h+1$ antenna array with integer quotients.

For proof, assume first that the array has a TAAP. If $\vec{d}_k'$, $k = 1, 2, \ldots, h$, are h vectors which generate the array, any triangle of the array can be described by two vectors, each of which is a sum or difference of one or more of the d_k''s. The area of the triangle, from Eq. (99), is determined by sums and differences of cross products of the $\vec{d}_k'$'s. From the theorem of the previous sub-section, the area is therefore the sum of determinants of coefficients of Eq. (117) times the area of a triangle which has the same TAAP. Since the coefficients of Eq. (117) are all integers, the ratio of the areas of any two triangles of the array is a rational number. In addition, the ratio of the area of any triangle of the array to the area of a triangle having the same TAAP as the h+1 antenna array is an integer. If one orders the antennas 1, 2, ..., h+1, and lets $\vec{d}_k'$ denote the vector with tail at 1 and head at k+1 for $k = 1, 2, \ldots, h$, then the areas of triangles formed by $\vec{d}_k'$ and $\vec{d}_\ell'$ divided by the area of a triangle having the same TAAP are just the set of determinants of Eq. (117), which have no factor in common by virtue of the theorem.

Conversely, suppose the ratio of areas of triangles of the array are all rational numbers. Order the $\vec{d}_k'$'s as above in such a way that $\vec{d}_1'$ and $\vec{d}_2'$ are not collinear.

Next, write

$$\vec{d}_k' = \alpha_k'\vec{d}_1' + \beta_k'\vec{d}_2' \qquad k = 3, 4, \ldots, h. \tag{163}$$

Forming the cross product of $\vec{d}_k'$ with $\vec{d}_1'$ and $\vec{d}_2'$ respectively shows that the α_k''s and β_k''s are rational numbers.

If α_k' and β_k' are expressed in lowest terms, and one multiplies each equation by the lowest common multiple of the denominators of α_k' and β_k', one has the equivalent of Eq. (121), where for each equation the Δ's have no factor in common.

Following the same development as in the proof of the previous theorem, $\vec{d}_1'$, $\vec{d}_2'$, $\vec{d}_3'$ generate a TAAP, and may be replaced by the vectors $\vec{d}_1''$ and $\vec{d}_2''$ of Eqs. (139) and (140).

One repeats the process of elimination of equations until one arrives at just two vectors, having proved in the process that the original array has a TAAP. This completes the proof.

5. ANTENNA ARRAY CONSTRAINTS FOR AMBIGUITY RESOLUTION

AN AREA CONSTRAINT FOR ARRAYS WITH THREE ANTENNAS

Theorem I: If a source direction can be determined unambiguously by a three antenna array whenever it lies within a central cone having half-angle ρ, then the area of the triangle formed by the three antennas is no larger than $\frac{\sqrt{3}\,\lambda^2}{12 \sin^2 \rho}$, where λ is the wave length of the signals. Also, this number is the least upper bound of the areas of all such triangles.

Proof: Order the sides of such a triangle by length, so that

$$d_1 \geq d_2 \geq d_3. \tag{164}$$

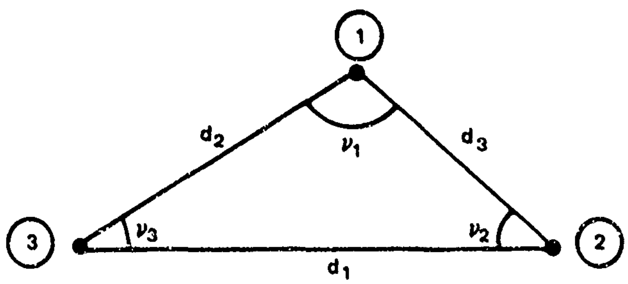

Fig. 19 TRIANGULAR ARRAY

Then

$$\nu_1 \geq \nu_2 \geq \nu_3 \tag{165}$$

and

$$\nu_1 \geq \frac{\pi}{3}, \qquad \nu_3 \leq \nu_2 \leq \frac{\pi}{2}. \tag{166}$$

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From Eqs. (22) and (23), the distances from the true source coordinates to the ambiguities are computed to be

$$D_{m,n} = \lambda \csc \nu_2 \sqrt{\left(\frac{m}{d_1}\right)^2 + \left(\frac{n}{d_3}\right)^2 - 2 \cos \nu_2 \left(\frac{m}{d_1}\right)\left(\frac{n}{d_3}\right)} \quad (167)$$

where m and n are any integers.

As in the discussion surrounding Eq. (12),

$$D_{m,n} \geq 2 \sin \rho \quad (168)$$

for all m,n not both zero.

In particular, if $m = 1$, $n = 0$, one has

$$\frac{\lambda \csc \nu_2}{d_1} \geq 2 \sin \rho \quad (169)$$

or

$$d_1 \sin \nu_2 \leq \frac{\lambda}{2 \sin \rho}. \quad (170)$$

From the law of sines, the left hand side of Eq. (170) is the same as $d_2 \sin \nu_1$, so that

$$d_2 \sin \nu_1 \leq \frac{\lambda}{2 \sin \rho}. \quad (171)$$

If $\nu_1 \leq \frac{2\pi}{3}$, it follows from Eq. (166) that $\sin \nu_1 \geq \frac{\sqrt{3}}{2}$, so from Eq. (171),

$$d_2 \leq \frac{\lambda}{\sqrt{3} \sin \rho}. \quad (172)$$

From Eq. (166), $\sin \nu_2 \geq \sin \nu_3$, so from Eq. (170),

$$d_1 \sin \nu_3 \leq \frac{\lambda}{2 \sin \rho}. \tag{173}$$

From multiplying Eqs. (172) and (173) together, one obtains

$$\frac{1}{2} d_1 d_2 \sin \nu_3 \leq \frac{\sqrt{3}\, \lambda^2}{12 \sin^2 \rho}, \tag{174}$$

the left hand side being just the area of the triangle.

If $\nu_1 \geq \frac{2\pi}{3}$, one constructs a new triangle as follows: Drop a perpendicular from (3) to the extended line joining (1) and (2) (see Fig. 20). On the extended line, lay off line segments each of length d_3 until the point where the dropped perpendicular intersects the line is encompassed by the line segment. If (k+1) coincides with the point of intersection, stop at (k+1).

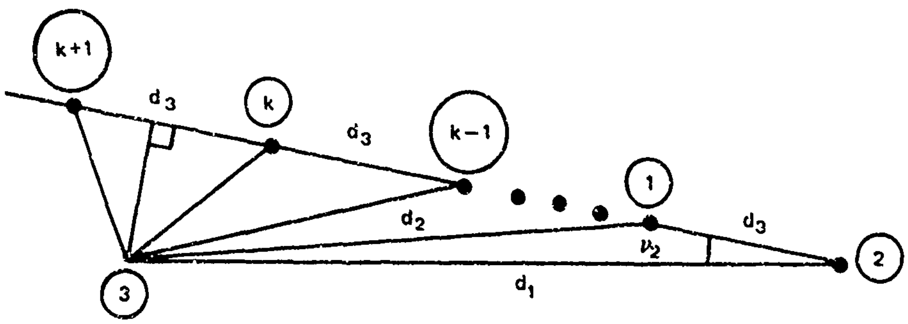

Fig. 20 CONSTRUCTION OF A NEW TRIANGLE

All of the triangles of Fig. 20 having one of the line segments as a side have the same area. In addition, they all have the same array ambiguity plot. The latter follows from the discussion in Section 4 that the vectors representing any two sides of a triangle determine its ambiguity plot. Since any two adjacent triangles in Fig. 20 have two common vectors, one along the extended line and the other along the common side, the result follows.

In the final triangle, the only possible obtuse angle is (k+1) (3) (k). If this angle is less than $\frac{2\pi}{3}$, the area inequality holds by virtue of the previous proof. If not, then one repeats the process of constructing a new triangle by starting with triangle (k+1) (3) (k) instead of (1) (2) (3), and extending the shortest side in the same manner as before. One keeps repeating the process until one reaches a triangle where all angles are less than $\frac{2\pi}{3}$, at which point the theorem is proved. The procedure must stop eventually, for in any triangle with an angle $\geq \frac{2\pi}{3}$, the ratio of the shortest side to the longest side is $\leq \frac{1}{\sqrt{3}}$, and the shortest side of the n^{th} triangle is always the longest side of the $(n+1)^{th}$ triangle unless the procedure stops. If the procedure did not stop, the length of the longest side, and hence the area, would approach zero, a contradiction. The proof of the area inequality is now complete.

To show that the expression is the least upper bound, consider an equilateral triangle with $d_1 = d_2 = d_3 = \frac{\lambda}{\sqrt{3}\sin\rho}$. Its area is $\frac{\sqrt{3}\,\lambda^2}{12\sin^2\rho}$. From Eq. (167),

$$D_{m,n} = 2\sin\rho\sqrt{(m-n)^2 + mn} \geq 2\sin\rho\sqrt{mn}, \qquad mn > 0. \quad (175)$$

If one of m or n is zero (not both) and the other is one, then $D_{m,n} = 2 \sin \rho$. If neither is zero, then Eq. (175) shows that no ambiguity can be closer to the true source coordinates than $2 \sin \rho$. The proof of Theorem I is now complete.

PROPERTIES OF UNAMBIGUOUS PHASE FOR ARRAYS HAVING A TAAP AND FOUR ANTENNAS

Preliminary Concepts and Definitions

For the purposes here, there are three antenna topologies which must be distinguished. These will be called type 1, 2, 3 quadrilateral arrays, respectively. TAAP is defined in Section 4.

Type 1: Three antennas lie on a line.

Note that not all four antennas may lie on a line since the existence of a TAAP is assumed.

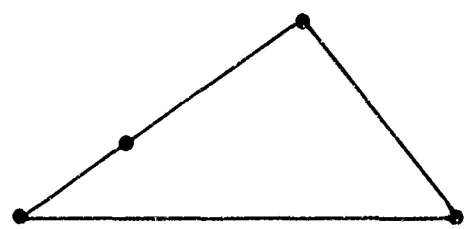

Fig. 21 TYPE 1 QUADRILATERAL ARRAY

Type 2: The triangle formed by some triplet of antennas contains the fourth antenna in its interior.

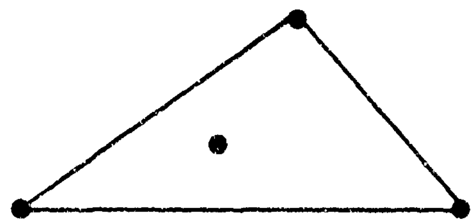

Fig. 22 TYPE 2 QUADRILATERAL ARRAY

Type 3: Given any three antennas, the fourth antenna is exterior to the triangle formed by the three.

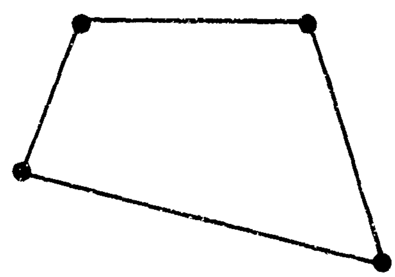

Fig. 23 TYPE 3 QUADRILATERAL ARRAY

Other Definitions

1. The area of a quadrilateral array is defined to be the area of the appropriate triangle for type 1 and type 2 arrays, and the area of the enclosed region as in Fig. 23 for type 3 arrays.

2. Label the antennas ①, ②, ③, ④ in an arbitrary fashion. Four numbers C_1, C_2, C_3, C_4 are defined

to be the areas of the triangles obtained by omitting antennas 1, 2, 3, 4, respectively. One of these numbers will be zero for a type 1 array. C is defined to be the area of the quadrilateral array.

3. Five numbers B_1, B_2, B_3, B_4, B are defined as follows: For a type 3 array, $B_1 = -C_1$. If antenna j ($j=2,3,4$) is opposite to antenna 1, then $B_j = -C_j$. If it is adjacent to antenna 1, then $B_j = C_j$. For a type 2 array, if antenna j ($j=1,2,3,4$) is the interior antenna, then $B_j = -C_j$, otherwise $B_j = C_j$. For a type 1 array, if antenna j ($j=1,2,3,4$) is the antenna not on a vertex of the triangle of Fig. 21, then $B_j = -C_j$, otherwise $B_j = C_j$. Finally, $B = C$.

4. Four numbers A_1, A_2, A_3, A_4 are defined as follows: If T(P) is the triangular array area of the TAAP generated by the four antennas, then

$$A_j = \frac{B_j}{T(P)}, \qquad j = 1, 2, 3, 4. \tag{176}$$

5. A denotes the ratio of the area of the quadrilateral to T(P).

Theorem II: For any quadrilateral array in a plane having a TAAP, the A's have the following properties:

1. A_1, A_2, A_3, A_4, A are all integers.
2. A_1, A_2, A_3, A_4 have no factor in common.
3. $A_1 + A_2 + A_3 + A_4 = 0$.
4. $|A_1| + |A_2| + |A_3| + |A_4| = 2A$.

Proof: The first two properties are a direct consequence of the fundamental area theorem. The last two follow easily from the definitions of the A's.

Theorem III: For any source direction, let μ_1, μ_2, μ_3, μ_4 denote the unambiguous absolute phase* at antennas ①, ②, ③, ④, respectively, in units of cycles. Then

$$A_1\mu_1 + A_2\mu_2 + A_3\mu_3 + A_4\mu_4 = 0. \tag{177}$$

Proof: Using the vector notation of Section 4, the vector description and antenna numbering of the antenna array is taken to be as depicted in Fig. 24.

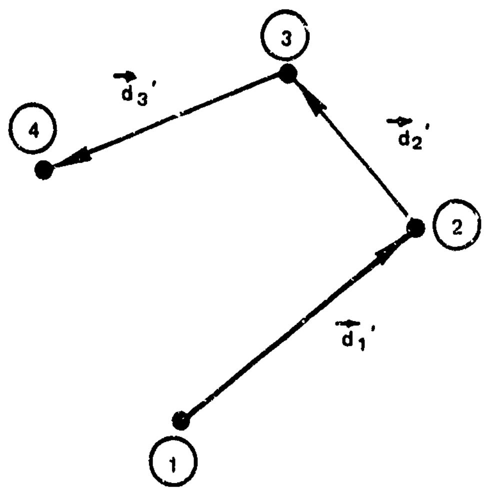

Fig. 24 ARRAY ANTENNA NUMBERING AND VECTOR DESCRIPTION

Strictly speaking, Fig. 24 depicts a type 3 array. By letting antenna ③ move downwards, one gets a type 1 array and then a type 2 array.

* See the discussion which follows the proof of Theorem III.

Let $\vec{n}$ denote a unit vector pointing up out of the plane of the paper and perpendicular to it. With the above convention, it then follows that regardless of array type, one has:

$$B_1\vec{n} = \frac{1}{2}\left(\vec{d}_3' \times \vec{d}_2'\right) \tag{178}$$

$$B_2\vec{n} = \frac{1}{2}\left[\left(\vec{d}_1' + \vec{d}_2'\right) \times \vec{d}_3'\right] \tag{179}$$

$$B_3\vec{n} = \frac{1}{2}\left[\left(\vec{d}_2' + \vec{d}_3'\right) \times \vec{d}_1'\right] \tag{180}$$

$$B_4\vec{n} = \frac{1}{2}\left(\vec{d}_1' \times \vec{d}_2'\right). \tag{181}$$

If $\vec{d}_1$ and $\vec{d}_2$ denote vectors which describe a triangle generating the same TAAP as the four antenna array, one sees from Eq. (117) that

$$\frac{1}{2}\left(\vec{d}_3' \times \vec{d}_2'\right) = \Delta_{32}\left[\frac{1}{2}\left(\vec{d}_1 \times \vec{d}_2\right)\right] = \Delta_{32}T(P)\vec{n} \tag{182}$$

where $\vec{d}_1$ and $\vec{d}_2$ are chosen so that $\vec{d}_1 \times \vec{d}_2$ lies in the positive $\vec{n}$ direction.

From Eqs. (176), (178), and (182), one then has

$$A_1 = \Delta_{32}. \tag{183}$$

Similarly,

$$A_2 = \Delta_{13} + \Delta_{23} \tag{184}$$

$$A_3 = \Delta_{21} + \Delta_{31} \tag{185}$$

$$A_4 = \Delta_{12} \tag{186}$$

where the Δ's are defined as in Eq. (118).

If $\vec{s}$ is a unit vector in the direction of the source, then from Eqs. (5), (6) and (10), the unambiguous phase differences are given by

$$\varphi_j' = \mu_{j+1} - \mu_j = \frac{1}{\lambda}\, \vec{d}_j' \cdot \vec{s}, \qquad j = 1, 2, 3. \tag{187}$$

Take $j = 3$ in Eq. (121) and dot both sides of that equation with $\vec{s}$. Using Eqs. (183), (184), (185), (186), (187), one has

$$\begin{aligned} 0 &= \Delta_{23}\varphi_1' + \Delta_{31}\varphi_2' + \Delta_{12}\varphi_3' \\ &= \Delta_{23}(\mu_2-\mu_1) + \Delta_{31}(\mu_3-\mu_2) + \Delta_{12}(\mu_4-\mu_3) \\ &= \Delta_{32}\mu_1 + (\Delta_{13}+\Delta_{23})\mu_2 + (\Delta_{21}+\Delta_{31})\mu_3 + \Delta_{12}\mu_4 \\ &= A_1\mu_1 + A_2\mu_2 + A_3\mu_3 + A_4\mu_4 \end{aligned} \tag{188}$$

which proves Theorem III.

DISCUSSION OF UNAMBIGUOUS ABSOLUTE PHASE

The absolute phase in any given antenna has meaning only when a reference phase is given, and obviously has no bearing in itself on any angular coordinates of the source. The phase difference between any pair of antennas does not depend on the reference phase, and is a measure of a function of the angular coordinates of the source.

Nonetheless, as will be apparent shortly, it is more natural for the purposes of this paper to work with the μ's instead of the φ''s. One should note that, from the third property of the A's in Theorem II, the apparent presence of the phase reference in Eq. (177) is only an illusion, for one can write

$$
\begin{aligned}
&A_1\mu_1 + A_2\mu_2 + A_3\mu_3 + A_4\mu_4 \\
&\quad = A_1\mu_1 + A_2(\mu_1+\varphi_1') + A_3(\mu_1+\varphi_1'+\varphi_2') + A_4(\mu_1+\varphi_1'+\varphi_2'+\varphi_3') \\
&\quad = (A_1+A_2+A_3+A_4)\mu_1 + (A_2+A_3+A_4)\varphi_1' + (A_3+A_4)\varphi_2' + A_4\varphi_3' \\
&\quad = (A_2+A_3+A_4)\varphi_1' + (A_3+A_4)\varphi_2' + A_4\varphi_3'. \qquad (189)
\end{aligned}
$$

So Eq. (177) is really an equation in phase differences only.

AMBIGUITY RESOLUTION AND TOLERANCE PROBLEMS WITH FOUR ANTENNAS

Suppose one wishes to arrange three antennas in a plane in such a way that any source direction within a cone having half-angle ρ can be determined unambiguously. The area constraint of Theorem I may very well negate the possibility of doing this for at least two reasons: One may be forced to put the antennas so close together that mutual coupling effects are unacceptable, or one may find that due to the short base lines involved, the errors in measuring the source direction are unacceptable. In either of these events, it is natural to consider increasing the number of antennas. The objective in so doing is to move the antennas farther apart while maintaining the ambiguity separation for the array ambiguity plot generated by the new array that one would have had with the unacceptable array with three antennas.

Although this is possible, there is a price in addition to adding antennas, and this is a new tolerance problem that has entered the picture.

Before discussing this problem in more detail, a few words about array ambiguity plot topologies are in order. In the two dimensional problem and for more than three antennas, one has the choice of three such topologies: a TAAP topology, a one-dimensional set of lattice points confined to a single line, or no array ambiguities at all. For this application, and because of the tolerance problem, it is conjectured that a TAAP topology is always best. This is based on experimentation with specific cases by the author, and a firm proof has not as yet been devised. Based on the conjecture, only antenna arrays with TAAP topologies are considered as candidates in this paper.

Prior to introducing the tolerance subject, next consider the four antenna case, and suppose all phase difference measurements are error free. As shown in Section 2, all information for ambiguity resolution is contained in three distinct unambiguous phase difference measurements, φ_1', φ_2', φ_3'. As in the derivation of Eq. (188), a certain linear combination with integer coefficients of these three quantities is zero for any source direction. It follows from Section 4 that $\varphi_1' + m_1$, $\varphi_2' + m_2$, $\varphi_3' + m_3$, where m_1, m_2, m_3 are integers, will satisfy the equation if and only if $\varphi_1' + m_1$, $\varphi_2' + m_2$, $\varphi_3' + m_3$ represent the phase differences from a source direction separated from the true source by an ambiguity of the TAAP.

This suggests the implementation for determining the true source coordinates. Let φ_1, φ_2, φ_3 be the ambiguous phase measurements, and suppose $-\frac{1}{2} < \varphi_1 \le \frac{1}{2}$, $-\frac{1}{2} < \varphi_2 \le \frac{1}{2}$, $-\frac{1}{2} < \varphi_3 \le \frac{1}{2}$. One knows that there exist integers n_1, n_2, n_3 such that $\varphi_1' = \varphi_1 + n_1$, $\varphi_2' = \varphi_2 + n_2$ and $\varphi_3' = \varphi_3 + n_3$. Further, the first expression in Eq. (188) will be an integer if one replaces φ_1', φ_2', φ_3' by φ_1, φ_2, φ_3. Thus $\Delta_{23}\varphi_1 + \Delta_{31}\varphi_2 + \Delta_{12}\varphi_3 = n$, where n is an integer which one determines when φ_1, φ_2, φ_3 are known. If one finds any integer solutions to $\Delta_{23}n_1 + \Delta_{31}n_2 + \Delta_{12}n_3 = -n$, then the φ''s so constructed are the phase differences associated with either the true source direction, or removed from it by an ambiguity of the TAAP. If the distance in the TAAP to the nearest array ambiguity is $\geq 2 \sin \rho$, then one can select the correct solution for n_1, n_2, n_3 from the cone of directions constraint. Having found the unambiguous values φ_1', φ_2', φ_3', one sees from Eq. (10) that one has redundant information (three equations in two unknowns) for x and y. One can then form x and y as linear combinations of φ_1', φ_2', φ_3' to optimize some statistic, say to provide minimum variance on x and y based on whatever statistical assumption about measurement errors is made.

It will be convenient for the purpose of analysis to work with the final expression of Eq. (188) instead of the one using phase differences. One reason for this is that there are several ways one can make phase comparisons and several resultant triplets of phase differences one can use. In general, the first expression of Eq. (188) will look different for different triplets, and their underlying equivalence is disguised. If each equation in phase differences is expressed in terms of the μ's, then one always gets the same equation. There is therefore no preferred set of triplets or ordering of phase comparisons.* (This statement is false for more than four antennas as far as which equations one should use is concerned.)

*All four antennas must be used in generating the triplet.

For example, one can write

$$\begin{aligned} & A_1\mu_1 + A_2\mu_2 + A_3\mu_3 + A_4\mu_4 \\ & \quad = -A_1(\mu_2-\mu_1) - (A_1+A_2)(\mu_3-\mu_2) + A_4(\mu_4-\mu_3) \\ & \quad = A_2(\mu_2-\mu_1) + A_3(\mu_3-\mu_1) + A_4(\mu_4-\mu_1) \\ & \quad = \text{etc.} \end{aligned} \tag{190}$$

using property 3 of Theorem II.

Another reason why the use of the equation in μ's is useful for the sake of analysis is the natural interpretation of the A's as normalized areas (plus or minus) of triangles of the quadrilateral array.

Now suppose errors are made in measuring φ_1, φ_2, φ_3. If one imagines the errors to increase from zero, one sees that one will arrive at the wrong value of at least one of n_1, n_2, n_3 when n can no longer be correctly determined.

This will not occur if the inequality

$$|A_1\epsilon_1 + A_2\epsilon_2 + A_3\epsilon_3 + A_4\epsilon_4| < \frac{1}{2} \tag{191}$$

holds, otherwise it will. The ϵ's are the errors in absolute phase at each of the four antennas. The condition Eq. (191), both necessary and sufficient that all phase differences be unambiguously resolved except for TAAP ambiguities, may be written in terms of phase difference errors for any choice of triplets of phase differences.

It is not clear what one can say in general about system performance when Eq. (191) is violated, except that the effects are bound to be undesirable. Until subsequent analysis or simulation shows performance not seriously affected, the author takes the position that one should attempt to satisfy Eq. (191) in the selection of an antenna array.

An interesting sufficient condition for Eq. (191) to hold can be derived as follows. Suppose

$$|\epsilon_i| \le \epsilon, \qquad i = 1, 2, 3, 4. \tag{192}$$

If the ϵ_i's are all equal to $\pm\epsilon$, and add up with the worst possible sign in Eq. (191), one has from Theorem II,

$$2A\epsilon < \frac{1}{2}. \tag{193}$$

The errors for phase differences will not exceed 2ϵ in absolute value. If one defines T_φ to be the value of 2ϵ when Eq. (193) is an equality, namely $\frac{1}{2A}$ (this definition will change for more than four antennas), then one has the following theorem:

Theorem IV: For any four antenna array with a TAAP, if the absolute values of the errors in phase at all antennas are less than $\frac{1}{2} T_\varphi = \frac{1}{4A}$, then, except for the TAAP ambiguities, all phase differences can be resolved unambiguously.

Combining Theorems I, IV, and the definition of A, one has the following inequality:

$$T_\varphi \le \frac{\sqrt{3}\,\lambda^2}{24 \sin^2 \rho \cdot (\text{Area of Quadrilateral Array})}. \tag{194}$$

Thus, although the area of the quadrilateral array can be made as large as possible while preserving the triangular array area, it can be done only at the expense of tolerance on the phase difference measurements.

Suppose, for example, that one needs $\frac{\sqrt{3}\,\lambda^2}{24\sin^2\rho} = \frac{1}{6}\text{ in}^2$ and one can only guarantee less than ±20° of phase difference error ($T_\varphi = \frac{1}{18}$). Then the area of the quadrilateral array should not exceed 3 in^2. If for other reasons this is unacceptable, then using four antennas does not solve the original problem. Thus, although Eq. (194) does not tell one how to construct a four antenna array with the desired properties, it serves as a bound which sometimes tells one that any four antenna array will be unacceptable.

T_φ will subsequently be referred to as the "sufficient tolerance" for the array. The mathematical expression for T_φ will in general differ from $\frac{1}{2A}$ for more than four antennas.

PROPERTIES OF UNAMBIGUOUS PHASE FOR ARRAYS HAVING A TAAP AND FIVE ANTENNAS

If four antenna arrays are unacceptable, one may wish to use five. In this case there may be considerable relaxation of the tolerance constraint over the four antenna case.

For five antennas, the discussions in Sections 2 and 4 show that two equations of the type of Eq. (177) are needed to resolve ambiguities that are not TAAP ambiguities. Each basic equation uses four antennas or three phase differences, and all five antennas are used. Since there are five ways of choosing four antennas, there are five equations. An independent proof will be given that there are exactly two independent equations, in that there exists at least one independent pair, and for any independent pair, the other three are linearly derivable from the pair. It will also be shown that in general there are two particular equations that it is best to work with.

For five antennas, twelve different antenna array topologies are distinguished, as shown in Fig. 25. Any topology is always one of the twelve.

Using the antenna numbering as given in Fig. 25, ten numbers $C_{ij} = C_{ji}$ for i, j = 1, 2, ..., 5, $i \neq j$ are defined to be the areas of the triangles formed by omitting antennas (i) and (j). Five numbers C_i are defined to be the area of the quadrilateral formed by omitting antenna (i). C is defined to be the area of the entire array, which is taken to be the area of the enclosed regions of Fig. 25.

Twenty-five numbers B_{ij}, i, j = 1, 2, ..., 5 are defined as follows: for i = j, $B_{ij} = 0$. For $j > i$, $B_{ji} = -B_{ij}$. For $i < j$, B_{ij} is given by Table 1, where the antenna numbering is given in Fig. 25. B_{ij} is always $\pm C_{ij}$. The proper sign is given in Table 1.

Table 1.

Signs for $B_{ij} = \pm C_{ij}$

	All Types	Types 1,2, 3,4,5,6,7	Types 6,8, 9,10,11	Types 6,9,12
B_{12}	-			
B_{13}	+			
B_{14}	-			
B_{15}	+			
B_{23}		+	+	-
B_{24}	+			
B_{25}	-			
B_{34}	-			
B_{35}	+			
B_{45}		+	-	-

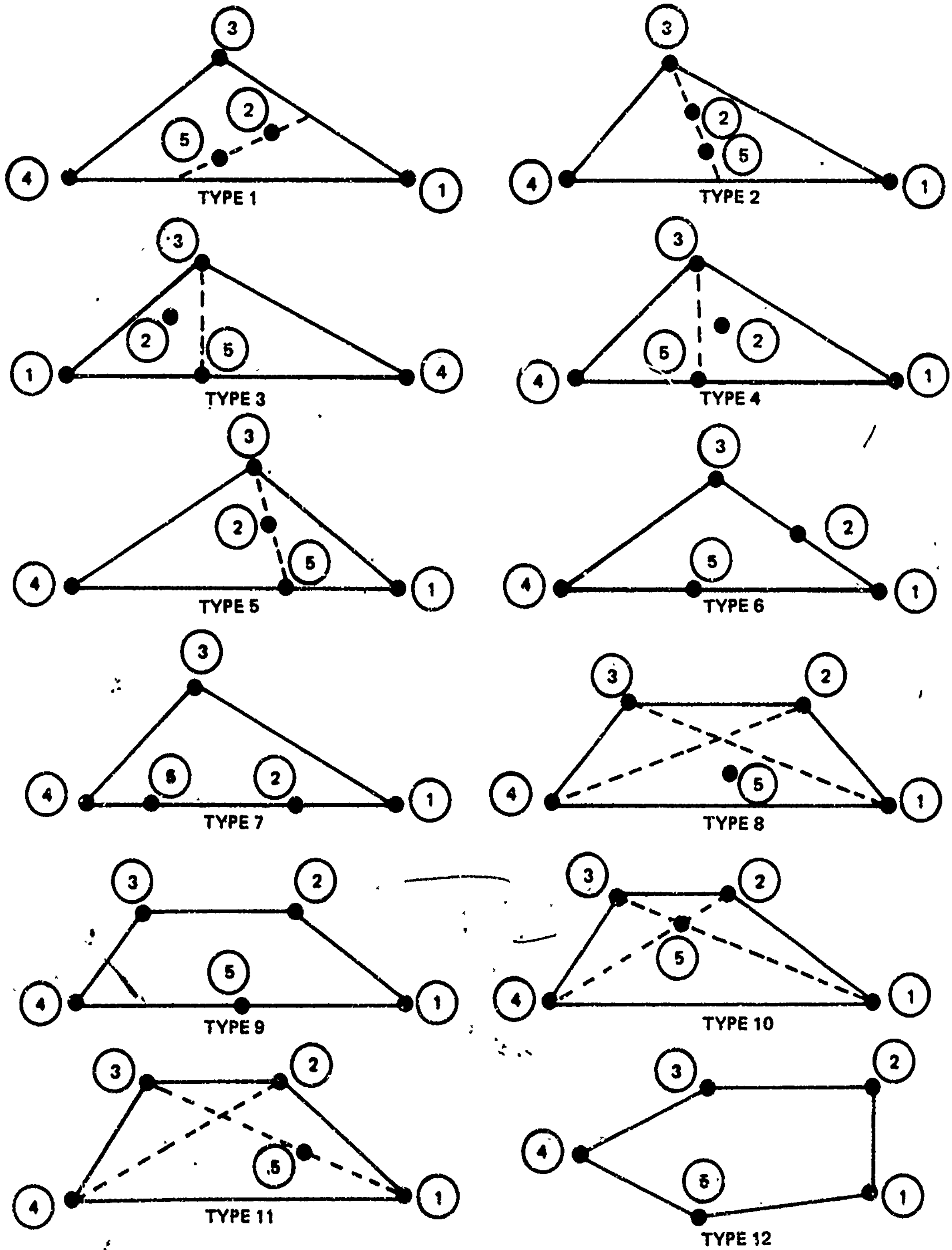

Fig. 25 THE TWELVE TOPOLOGIES FOR FIVE ANTENNAS

In addition, let P denote the TAAP of the entire array and, except for $i = 3$ in a type 7 array, let P_i denote the TAAP of the array generated when antenna (i) is omitted. Define A_{ij} by

$$A_{ij} = \frac{B_{ij}}{T(P)}, \qquad i, j = 1, 2, \ldots, 5. \tag{195}$$

Also let

$$A_i = \frac{C_i}{T(P)}, \qquad i = 1, 2, \ldots, 5 \tag{196}$$

and

$$A = \frac{C}{T(P)}. \tag{197}$$

The theorem analogous to Theorem II is now as follows:

<u>Theorem V</u>: For any antenna array in a plane having a TAAP and five antennas, the following properties hold:

1. A_{ij}, A_i, A are all integers, $i, j = 1, 2, \ldots, 5$.
2. The A_{ij} have no factor in common, $i, j = 1, 2, \ldots, 5$.
3. Except for $i = 3$ in a type 7 array, the greatest common factor of $A_{i1}, A_{i2}, \ldots, A_{i5}$ is $\frac{T(P_i)}{T(P)}$, $i = 1, 2, \ldots, 5$.

   In a type 7 array, $A_{31} = \ldots = A_{35} = 0$.
4. $\sum_{j=1}^{5} A_{ij} = 0$ for $i = 1, 2, \ldots, 5$.
5. $\sum_{j=1}^{5} |A_{ij}| = 2A_i$, for $i = 1, 2, \ldots, 5$.
6. The matrix $((A_{ij}))$ is skew symmetric.

Proof: For fixed i, the B_{ij}'s are the signed areas of the triangles of the quadrilateral (except for B_{ii}, which is zero) formed by omitting antenna (i). Except for the possibility of changing the sign of all of $A_{i1}, \ldots, A_{i5}$, the sign conventions are the same as those for the four antenna case derived earlier. So properties 4 and 5 follow from properties 3 and 4 of Theorem II. Property 6 follows directly from the definitions.

Properties 1 and 2 follow directly from the fundamental area theorem. So does property 3, because for any fixed i, the fundamental area theorem implies that the greatest common factor of $\frac{B_{i1}}{T(P_i)}, \frac{B_{i2}}{T(P_i)}, \ldots, \frac{B_{i5}}{T(P_i)}$ is unity. $\frac{T(P_i)}{T(P)}$ is thus an integer, and property 3 follows by multiplication. The type 7 array case is obvious.

Theorem VI: For any source direction, let $\mu_1, \mu_2, \ldots, \mu_5$ denote the unambiguous absolute phase at antennas (1) through (5), respectively, in units of cycles. Then

$$\sum_{j=1}^{5} A_{ij}\mu_j = 0, \qquad i = 1, 2, \ldots, 5. \tag{198}$$

Proof: For each i, one is using a quadrilateral only, and the theorem follows from Theorem III and the definition of the A_{ij}'s.

ADDITIONAL PROPERTIES OF $((A_{ij}))$

Theorem VII: For any i, j, k, ℓ = 1, 2, ..., 5 the following identity holds:

$$A_{ij}A_{k\ell} + A_{jk}A_{i\ell} + A_{ki}A_{j\ell} = 0. \tag{199}$$

Proof: If any two subscripts have the same value, Eq. (199) collapses to an obvious identity. If not, then there is exactly one missing subscript; call it m. Since it is obviously equivalent to prove Eq. (199) when the A's are replaced by the B's, note that the B's appearing in said equation are the signed areas of the six triangles of the five antenna array which have antenna m as a vertex.

Let $\vec{n}$ be a unit vector normal to the plane of Fig. 25 and pointing up out of the paper. Let $\vec{d}_{mi}$, $\vec{d}_{mj}$, $\vec{d}_{mk}$, $\vec{d}_{m\ell}$ be vectors with tails at m and heads at i, j, k, ℓ respectively. Then obviously

$$\begin{aligned} \frac{1}{2}(\vec{d}_{mi} \times \vec{d}_{mj}) &= \pm B_{k\ell}\vec{n} \\ \frac{1}{2}(\vec{d}_{mi} \times \vec{d}_{mk}) &= \pm B_{j\ell}\vec{n} \\ &\vdots \\ \frac{1}{2}(\vec{d}_{mk} \times \vec{d}_{m\ell}) &= \pm B_{ij}\vec{n}. \end{aligned} \tag{200}$$

Since all array types other than 1, 3, 8, 12, are limiting cases of one of the types 1, 3, 8, 12, it is sufficient to consider only these four types in determining the signs in Eq. (200).

Define Q_{mij} to be the coefficient of $\vec{n}$ in $\frac{1}{2}(\vec{d}_{mi} \times \vec{d}_{mj})$.

Table 2 relates Q_{mij} to $B_{k\ell}$ for antenna array types 1, 3, 8, 12. Only cases for which $i < j$ and $k < \ell$ are considered in Table 2.

Table 2

Q_{mij} vs. m,i,j

			Q_{mij}			
m	i	j	Type 1	Type 3	Type 8	Type 12
1	2	3	$-B_{45}$	$+B_{45}$	$-B_{45}$	$-B_{45}$
↓	2	4	$+B_{35}$	$-B_{35}$	$+B_{35}$	$+B_{35}$
↓	2	5	$-B_{34}$	$+B_{34}$	$-B_{34}$	$-B_{34}$
↓	3	4	$-B_{25}$	$+B_{25}$	$-B_{25}$	$-B_{25}$
↓	3	5	$+B_{24}$	$-B_{24}$	$+B_{24}$	$+B_{24}$
↓	4	5	$-B_{23}$	$+B_{23}$	$-B_{23}$	$-B_{23}$
2	1	3	$+B_{45}$	$-B_{45}$	$+B_{45}$	$+B_{45}$
↓	1	4	$-B_{35}$	$+B_{35}$	$-B_{35}$	$-B_{35}$
↓	1	5	$+B_{34}$	$-B_{34}$	$+B_{34}$	$+B_{34}$
↓	3	4	$+B_{15}$	$-B_{15}$	$+B_{15}$	$+B_{15}$
↓	3	5	$-B_{14}$	$+B_{14}$	$-B_{14}$	$-B_{14}$
↓	4	5	$+B_{13}$	$-B_{13}$	$+B_{13}$	$+B_{13}$
3	1	2	$-B_{45}$	$+B_{45}$	$-B_{45}$	$-B_{45}$
↓	1	4	$+B_{25}$	$-B_{25}$	$+B_{25}$	$+B_{25}$
↓	1	5	$-B_{24}$	$+B_{24}$	$-B_{24}$	$-B_{24}$
↓	2	4	$-B_{15}$	$+B_{15}$	$-B_{15}$	$-B_{15}$
↓	2	5	$+B_{14}$	$-B_{14}$	$+B_{14}$	$+B_{14}$
↓	4	5	$-B_{12}$	$+B_{12}$	$-B_{12}$	$-B_{12}$
4	1	2	$+B_{35}$	$-B_{35}$	$+B_{35}$	$+B_{35}$
↓	1	3	$-B_{25}$	$+B_{25}$	$-B_{25}$	$-B_{25}$
↓	1	5	$+B_{23}$	$-B_{23}$	$+B_{23}$	$+B_{23}$
↓	2	3	$+B_{15}$	$-B_{15}$	$+B_{15}$	$+B_{15}$
↓	2	5	$-B_{13}$	$+B_{13}$	$-B_{13}$	$-B_{13}$
↓	3	5	$+B_{12}$	$-B_{12}$	$+B_{12}$	$+B_{12}$
5	1	2	$-B_{34}$	$+B_{34}$	$-B_{34}$	$-B_{34}$
↓	1	3	$+B_{24}$	$-B_{24}$	$+B_{24}$	$+B_{24}$
↓	1	4	$-B_{23}$	$+B_{23}$	$-B_{23}$	$-B_{23}$
↓	2	3	$-B_{14}$	$+B_{14}$	$-B_{14}$	$-B_{14}$
↓	2	4	$+B_{13}$	$-B_{13}$	$+B_{13}$	$+B_{13}$
↓	3	4	$-B_{12}$	$+B_{12}$	$-B_{12}$	$-B_{12}$

Inspection of Table 2 shows that regardless of the value of m, for $i < j$, $k < \ell$, one always has

$$Q_{mij}Q_{mk\ell} = B_{ij}B_{k\ell} \tag{201}$$

But Eq. (201) is valid in general, as interchanging i and j or k and ℓ changes the sign of both sides of Eq. (201).

Equation (199) will therefore be valid if and only if (dropping the subscript m on the $\vec{d}$'s)

$$(\vec{d}_i \times \vec{d}_j)\cdot(\vec{d}_k \times \vec{d}_\ell) + (\vec{d}_j \times \vec{d}_k)\cdot(\vec{d}_i \times \vec{d}_\ell) + (\vec{d}_k \times \vec{d}_i)\cdot(\vec{d}_j \times \vec{d}_\ell) = 0. \tag{202}$$

Using vector algebra, the following string of identities proves the result:

$$\begin{aligned}
&(\vec{d}_i \times \vec{d}_j)\cdot(\vec{d}_k \times \vec{d}_\ell) + (\vec{d}_j \times \vec{d}_k)\cdot(\vec{d}_i \times \vec{d}_\ell) + (\vec{d}_k \times \vec{d}_i)\cdot(\vec{d}_j \times \vec{d}_\ell) \\
&\quad = -\vec{d}_\ell \cdot \left\{ \vec{d}_k \times (\vec{d}_i \times \vec{d}_j) + \vec{d}_i \times (\vec{d}_j \times \vec{d}_k) + \vec{d}_j \times (\vec{d}_k \times \vec{d}_i) \right\} \\
&\quad = -\vec{d}_\ell \cdot \left\{ (\vec{d}_k \cdot \vec{d}_j)\vec{d}_i - (\vec{d}_k \cdot \vec{d}_i)\vec{d}_j + (\vec{d}_i \cdot \vec{d}_k)\vec{d}_j \right. \\
&\qquad\qquad - (\vec{d}_i \cdot \vec{d}_j)\vec{d}_k + (\vec{d}_j \cdot \vec{d}_i)\vec{d}_k \\
&\qquad\qquad \left. - (\vec{d}_j \cdot \vec{d}_k)\vec{d}_i \right\} \\
&\quad = -\vec{d}_\ell \cdot (\text{zero vector}) = 0.
\end{aligned} \tag{203}$$

<u>Theorem VIII</u>: The matrix $\left((A_{ij})\right)$ has rank two.

<u>Proof</u>: It has rank at least one since not all the A_{ij}'s vanish (not all antennas lie on a line).

For the antenna numbering in Fig. 25, A_{25} never vanishes for any array type. Therefore the second and fifth row vectors

$$(A_{21}, 0, A_{23}, A_{24}, A_{25})$$

and

$$(A_{51}, A_{52}, A_{53}, A_{54}, 0)$$

are linearly independent, and the rank is at least two.

Let i = 1, 3, or 4, and one has

$$A_{5i} (A_{21}, 0, A_{23}, A_{24}, A_{25}) + A_{i2} (A_{51}, A_{52}, A_{53}, A_{54}, 0)$$

$$= (A_{5i}A_{21} + A_{i2}A_{51}, A_{i2}A_{52}, A_{5i}A_{23} + A_{i2}A_{53}, A_{5i}A_{24} + A_{i2}A_{54}, A_{5i}A_{25}). \qquad (204)$$

Applying Theorem VII three times, the right hand side of Eq. (204) can be written as

$$A_{52} (A_{i1}, A_{i2}, A_{i3}, A_{i4}, A_{i5}).$$

Since $A_{52} \neq 0$, it follows that every row of $((A_{ij}))$ is a linear combination of the second and fifth rows, thus proving Theorem VIII.

The following theorem will be needed for the discussion of the tolerance problem for five antennas.

<u>Theorem IX</u>: Let (A_{aj}) and (A_{bj}) $\left(\text{or } (A_{ja}) \text{ and } (A_{jb})\right)$ denote two linearly independent rows (or columns) for j = 1, 2, ..., 5 of the matrix $((A_{ij}))$, i, j = 1, 2, ..., 5. Let g_a and g_b denote the greatest common factor of A_{aj} for j = 1, 2, ..., 5, and A_{bj} for j = 1, 2, ..., 5, respectively. Then $A_{ab} \neq 0$ and A_{ab} is divisible by $g_a g_b$.

Proof: Since $((A_{ij}))$ is skew symmetric, it is sufficient to prove the theorem for rows.

If $A_{ab} = 0$, then, since $A_{aa} = A_{ab} = A_{ba} = A_{bb} = 0$, it follows that the a^{th} and b^{th} columns would all be zero because the a^{th} and b^{th} rows are linearly independent and $((A_{ij}))$ has rank two from Theorem VIII. Since $((A_{ij}))$ is skew symmetric, the a^{th} and b^{th} rows would also all be zero, a contradiction. Hence $A_{ab} \neq 0$.

Set $i = a$ and $j = b$ in Eq. (199), divide by $g_a g_b$, obtaining

$$\frac{A_{ab}A_{k\ell}}{g_a g_b} + \left(\frac{A_{bk}}{g_b}\right)\left(\frac{A_{a\ell}}{g_a}\right) + \left(\frac{A_{ka}}{g_a}\right)\left(\frac{A_{b\ell}}{g_b}\right) = 0. \tag{205}$$

From Theorem VII, Eq. (205) holds for any $k, \ell = 1, 2, \ldots, 5$, and the four expressions within the parentheses are all integers by virtue of the definition of g_a and g_b. Hence $A_{ab}A_{k\ell}$ is divisible by $g_a g_b$ for all $k, \ell = 1, 2, \ldots, 5$.

If p is some prime number which divides either g_a or g_b, let p^{n_1} be the highest power of p which divides g_a and p^{n_2} be the highest power of p which divides g_b, Hence $p^{n_1+n_2}$ divides $A_{ab}A_{k\ell}$ for all $k, \ell = 1, 2, \ldots, 5$.

From property 2 of Theorem V, one may choose k and ℓ such that p does not divide $A_{k\ell}$. Hence $p^{n_1+n_2}$ divides A_{ab}, and Theorem IX follows by decomposing g_a and g_b into their prime factors.

Corollary: For each i, $i = 1, 2, \ldots, 5$, let g_i denote the greatest common factor of the numbers A_{ij}, $j = 1, 2, \ldots, 5$. Then, for $i, j = 1, 2, \ldots, 5$, A_{ij} is divisible by $g_i g_j$. (g_i is undefined for type 7 arrays with $i = 3$.)

Proof: If $A_{ij} = 0$ for some i, j, the corollary is obvious. If $A_{ij} \neq 0$, the proof follows exactly as in the proof of Theorem IX, where one replaces a and b by i and j respectively in Eq. (205).

AMBIGUITY RESOLUTION AND TOLERANCE PROBLEMS WITH FIVE ANTENNAS

Instead of the single Eq. (177) in the four antenna case, there are now the five Eqs. (198). From Theorem VIII, however, there are only two independent ones. Proceeding as in the four antenna case, it follows that if one knows the values of four ambiguous phase differences φ_i utilizing all five antennas, and if one finds four integers n_1, n_2, n_3, n_4 such that

$$\varphi_i' = \varphi_i + n_i \tag{206}$$

is a solution to Eq. (198) when Eq. (198) is rewritten in terms of the φ_i''s, then one has all possible unambiguous phase differences resulting either from the source direction or a direction separated from it by a TAAP ambiguity. If the TAAP ambiguities are far enough apart, then as before the correct solution may be chosen.

Again, it does not matter in what order phase differences are taken. Unlike the four antenna case when there was just one equation, one now has a choice of two out of five, and from the point of view of measurement tolerances, it does matter how they are selected.

Although the A_{ij}'s have no factor in common, the coefficients of any one equation may have a factor in common. Except for $i = 3$ in a type 7 array, the coefficients of the i^{th} equation have a greatest common factor $g_i \left(= \frac{T(P_i)}{T(P)}\right.$ from

Theorem V). So instead of using Eq. (198) one uses

$$\sum_{j=1}^{5} \left(\frac{A_{ij}}{g_i}\right) \mu_j = 0, \qquad i = 1, 2, \ldots, 5, \tag{207}$$

where the coefficients of Eq. (207) in any equation have no factor in common.

Select two equations from Eq. (207), say the a^{th} and b^{th} equations, which are linearly independent. Then consider all pairs of independent equations each of which is a linear combination of these two and has integer coefficients.

Formally, let c, d, u, v, I_j, J_j be defined as follows:

$$\frac{cA_{aj}}{g_a} + \frac{dA_{bj}}{g_b} = I_j, \qquad j = 1, 2, \ldots, 5 \tag{208}$$

$$\frac{uA_{aj}}{g_a} + \frac{vA_{bj}}{g_b} = J_j, \qquad j = 1, 2, \ldots, 5 \tag{209}$$

where I_j, J_j are integers, and one works with the two independent expressions $\sum_{j=1}^{5} I_j \mu_j$ and $\sum_{j=1}^{5} J_j \mu_j$.

As in Eq. (191), non-array ambiguities are resolved correctly if and only if

$$\left| \sum_{j=1}^{5} I_j \epsilon_j \right| < \frac{1}{2} \tag{210}$$

$$\left| \sum_{j=1}^{5} J_j \epsilon_j \right| < \frac{1}{2}. \tag{211}$$

The concept of the sufficient tolerance T_φ carries over to the five antenna case, but the formal definition is quite a bit different.

Set

$$\sum_{j=1}^{5} |I_j| = I \tag{212}$$

and

$$\sum_{j=1}^{5} |J_j| = J. \tag{213}$$

If T_{cduv} denotes the smaller of $\frac{1}{I}$ and $\frac{1}{J}$, then T_φ is the maximum of T_{cduv} over all selections from Eqs. (208) and (209) with the prescribed properties.

The next task will be to find bounds on T_φ. To this end, note that, by setting $j = a, b$ in Eqs. (208) and (209), it follows that c, d, u, v are rational numbers.

From Eqs. (208) and (209), direct computation yields

$$I_j J_k - I_k J_j = (cv-du)\left(\frac{A_{aj}A_{bk}}{g_a g_b} - \frac{A_{bj}A_{ak}}{g_a g_b}\right), \quad j, k = 1, 2, \ldots, 5. \tag{214}$$

From Theorem VII, Eq. (214) becomes

$$I_j J_k - I_k J_j = (cv-du)\ NA_{jk}, \quad j, k = 1, 2, \ldots, 5 \tag{215}$$

where

$$N = \frac{A_{ab}}{g_a g_b}, \tag{216}$$

a non-zero integer by virtue of Theorem IX.

Next it will be shown that (cv-du)N is a non-zero integer. It is non-zero since $N \neq 0$ and $\sum_{j=1}^{5} I_j \mu_j$ and $\sum_{j=1}^{5} J_j \mu_j$ are independent. Since c, d, u, v are rational, set $(cv-du)N = \frac{q_1}{q_2}$, a fraction in lowest terms. If p is any prime factor of q_2, then property 2 of Theorem V allows selection of A_{jk} in Eq. (215) such that p does not divide A_{jk}. Since p does not divide q_1, and the left hand side of Eq. (215) is an integer, p could not have existed, and thus $(cv-du)N$ is a non-zero integer. Hence

$$|(cv-du)N| \geq 1. \tag{217}$$

One now has, from Eqs. (215) and (217),

$$|A_{jk}| \leq |I_j||J_k| + |I_k||J_j|, \quad j, k = 1, 2, \ldots, 5. \tag{218}$$

Summing Eq. (218) over k, one has

$$2A_j \leq J|I_j| + I|J_j|, \quad j = 1, 2, \ldots, 5 \tag{219}$$

from property 5 of Theorem V.

Summing Eq. (219) over j, and dividing by 2, one has

$$\sum_{j=1}^{5} A_j \leq I \cdot J. \tag{220}$$

The larger of I and J cannot be less than $\sqrt{\sum_{j=1}^{5} A_j}$, so one has the inequality

$$T_\varphi \leq \frac{1}{\sqrt{\sum_{j=1}^{5} A_J}}. \tag{221}$$

An inequality the other way can be obtained by observing from Fig. 25 that one can always choose two independent equations of Eq. (198) such that $A_i < A$ and $A_j < A$. So, since A_i, A_j and A are integers,

$$\frac{1}{2(A-1)} \leq T_\varphi. \tag{222}$$

Combining with Theorem I, the analogous formula to Eq. (194) is Eq. (221) where one substitutes for A_j the expression

$$F_j = \frac{(\text{Area of } j^{\text{th}} \text{ quadrilateral}) \times 12 \sin^2 \rho}{\sqrt{3}\,\lambda^2} \tag{223}$$

in Eq. (221).

One can replace Eq. (221) by a weaker inequality by observing from Fig. 25 that

$$\sum_{j=1}^{5} A_j \geq 3A \tag{224}$$

where equality occurs only for a type 10 array, so that

$$T_\varphi \leq \frac{1}{\sqrt{3A}} \tag{225}$$

and, combining with Theorem I,

$$T_\varphi \leq \sqrt{\frac{\sqrt{3}\,\lambda^2}{36 \sin^2 \rho \cdot (\text{area of five antenna array})}}. \tag{226}$$

As before, none of Eqs. (221), (225), (226) tells one how to construct a five antenna array with the desired properties, but they do serve as bounds which can sometimes tell one how close a given array approaches the best in terms of tolerance, or in other cases they may serve to show that a five antenna array is inadequate.

Consider again the example that was used in the four antenna case, where $T_\varphi = \frac{1}{18}$ and $\frac{\sqrt{3}\,\lambda^2}{24 \sin^2 \rho} = \frac{1}{6}$ in^2. In this case one concludes from Eq. (226) that the area of the five antenna array should not exceed 36 in^2, which is a considerable improvement over the 3 in^2 result for four antennas. One should also bear in mind that the 36 in^2 number will rarely, if ever, be achievable, but it gives one a goal to shoot for. In fact, it is conjectured by the author that none of Eqs. (221), (225), or (226) are ever achievable as equalities.

SOME EXAMPLES OF ARRAYS WITH FIVE ANTENNAS

Consider the five antenna array of Fig. 26. The antennas, arranged on a circle, form an equal central angle case according to the definition in Section 3. The array is also type 12 of Fig. 25.

The array ambiguity plot is given in Fig. 27. In this plot, some of the sub-array ambiguities are also shown.

The following data on the array of Fig. 26 are computed according to techniques developed here and in the previous sections, and are given without proof.

$$T(P) = \frac{d^2}{32} \sin \Delta\theta \cong .029d^2 \cong .0091 \times (\text{diameter})^2. \quad (227)$$

The matrix $\left(\left(A_{ij}\right)\right)$ is given by

$$\left(\left(A_{ij}\right)\right) = \begin{pmatrix} 0 & -16 & 28 & -28 & 16 \\ 16 & 0 & -21 & 33 & -28 \\ -28 & 21 & 0 & -21 & 28 \\ 28 & -33 & 21 & 0 & -16 \\ -16 & 28 & -28 & 16 & 0 \end{pmatrix}. \quad (228)$$

The A_i's and A are given by

$$A_1 = 44, A_2 = A_3 = A_4 = 49, A_5 = 44, A = 65. \quad (229)$$

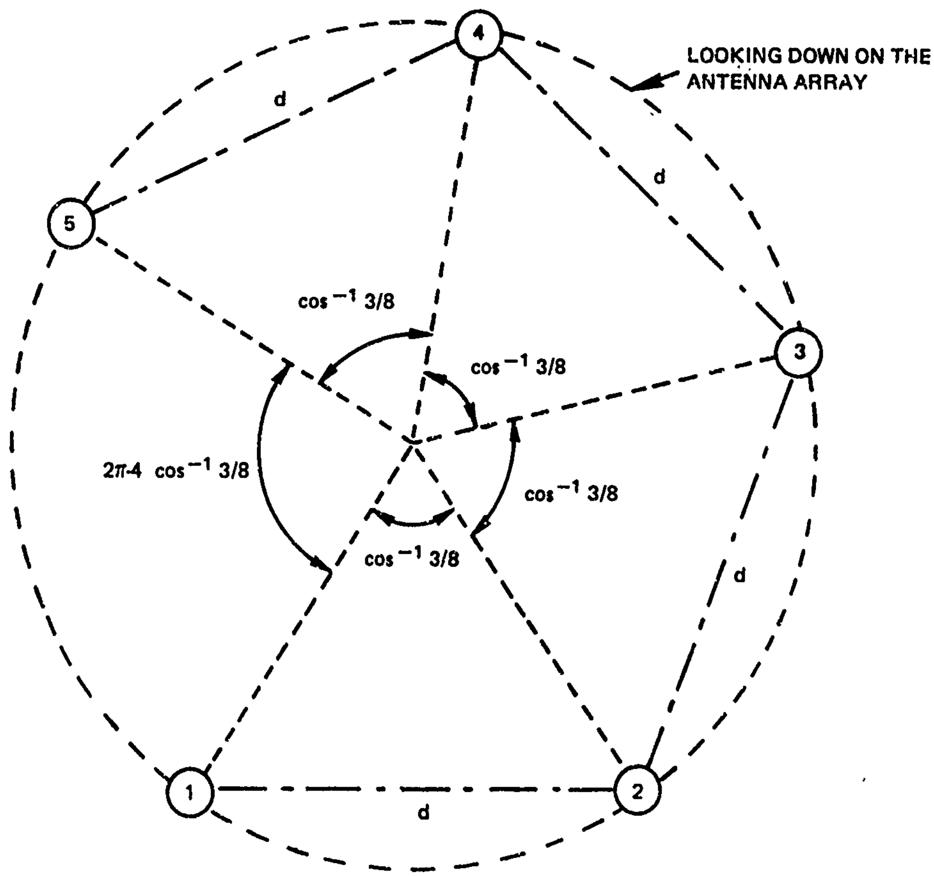

Fig. 26 FIVE INTERFEROMETER ARRAY CONFIGURATION

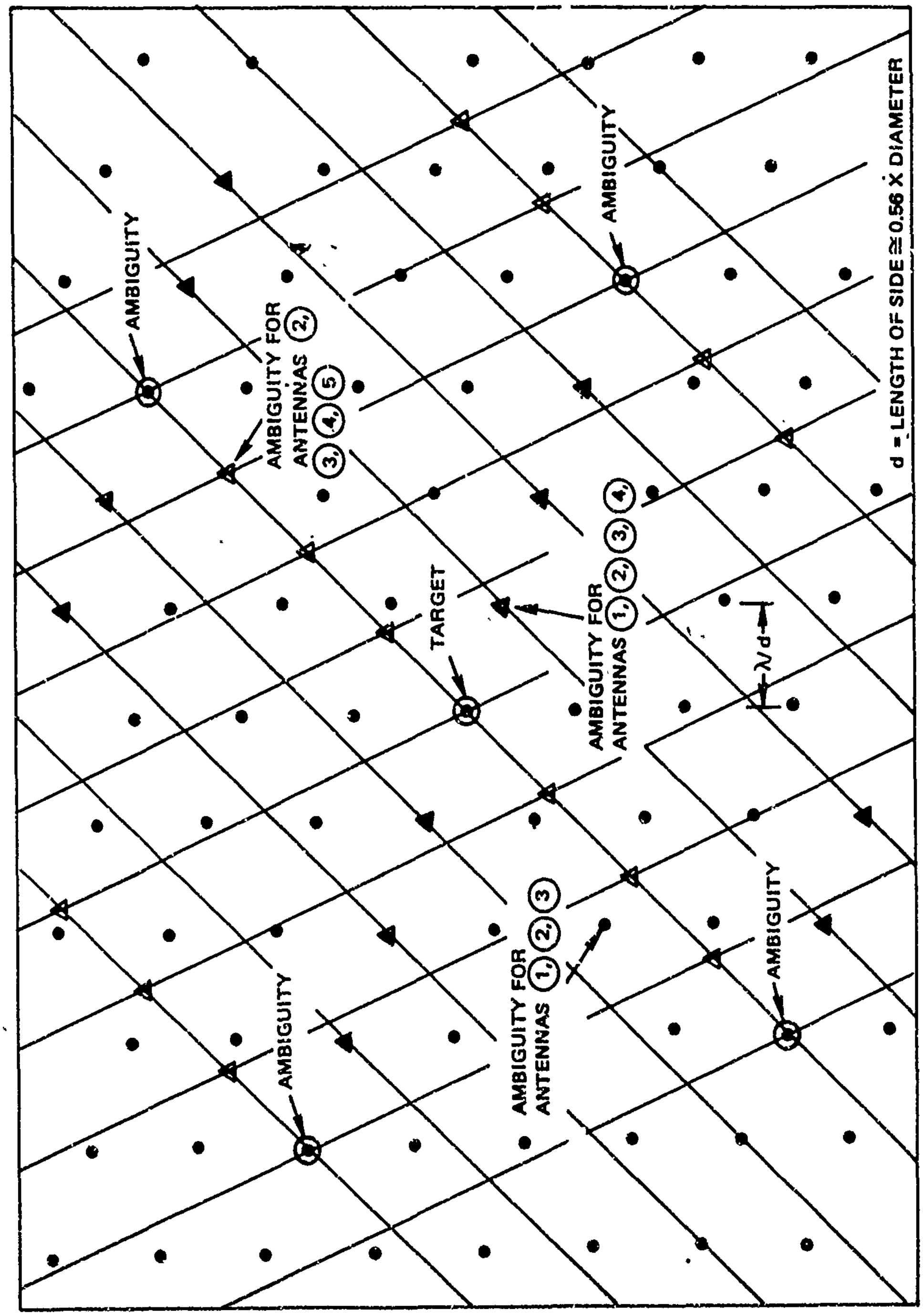

Fig. 27 AMBIGUITY PLOT FOR FIVE INTERFEROMETER ARRAY CONFIGURATION

Also,

$$\sum_{i=1}^{5} A_i = 235. \tag{230}$$

The five Eqs. (207) are

$$-4\mu_2 + 7\mu_3 - 7\mu_4 + 4\mu_5 = 0 \tag{231}$$

$$16\mu_1 - 21\mu_3 + 33\mu_4 - 28\mu_5 = 0 \tag{232}$$

$$-4\mu_1 + 3\mu_2 - 3\mu_4 + 4\mu_5 = 0 \tag{233}$$

$$28\mu_1 - 33\mu_2 + 21\mu_3 - 16\mu_5 = 0 \tag{234}$$

$$-4\mu_1 + 7\mu_2 - 7\mu_3 + 4\mu_4 = 0. \tag{235}$$

Any two equations selected from the above five are linearly independent.

One can show in this case that one can do no better as far as T_φ is concerned than to choose Eqs. (208) and (209) to be any pair of equations selected from Eqs. (231), (233), (235). (Open question unresolved by the author: Is it always true that for some pair a, b in Eqs. (208) and (209), one gets the largest possible T_{cduv} for $\begin{pmatrix} c & d \\ u & v \end{pmatrix}$ = the unit matrix? It is conjectured that the answer is yes.)

Hence one has for this array

$$T_\varphi = \frac{1}{22} \cong .045. \tag{236}$$

Inequality (222) says

$$T_\varphi \geq .0078 \tag{237}$$

so that the array is considerably better than the worst possible array for T_φ.

Inequality (225) (the weakest) from Eq. (229) says

$$T_\varphi \leq .072. \tag{238}$$

The next strongest inequality comes from combining Eqs. (230) and (221), giving

$$T_\varphi \leq .065. \tag{239}$$

A stronger inequality is derivable from inequality (220). Since I and J are integers, and $\sum_{i=1}^{5} A_i = 235$, I and J cannot both be less than 16; so

$$T_\varphi \leq \frac{1}{16} = .0625. \tag{240}$$

The next array example is given in Fig. 28.

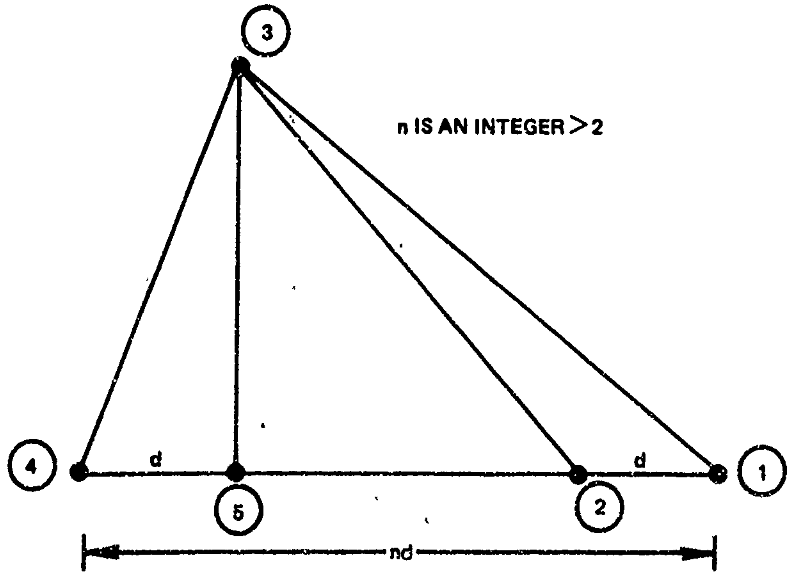

Fig. 28 A TYPE 7 ARRAY

The matrix $((A_{ij}))$ is given by

$$((A_{ij})) = \begin{pmatrix} 0 & -1 & 0 & 2-n & n-1 \\ 1 & 0 & 0 & n-1 & -n \\ 0 & 0 & 0 & 0 & 0 \\ n-2 & 1-n & 0 & 0 & 1 \\ 1-n & n & 0 & -1 & 0 \end{pmatrix} \qquad (241)$$

Also, one has

$$A_1 = n-1, \quad A_2 = n, \quad A_3 = 0, \quad A_4 = n-1,$$

$$A_5 = n, \qquad A = n, \qquad \sum_{i=1}^{5} A_i = 4n-2. \qquad (242)$$

One can show that as far as T_φ is concerned, one can do no better than to use the equations associated with the first and fourth rows, namely

$$-\mu_2 + (2-n)\mu_4 + (n-1)\mu_5 = 0 \qquad (243)$$

$$(n-2)\mu_1 + (1-n)\mu_2 + \mu_5 = 0. \qquad (244)$$

Hence $T_\varphi = \frac{1}{2(n-1)}$, which from Eq. (222) is the worst possible result. Equation (222) is therefore achievable, and Fig. 28 represents a poor choice for $n > 3$ of antenna locations from the point of view of tolerance.

THE ONE-DIMENSIONAL CASE FOR THREE AND FOUR ANTENNAS

It turns out that the one-dimensional problem is a special case for the two-dimensional situation, and is equivalent to the two-dimensional problem when all but one of the antennas lie on a line.

Consider first the case of three antennas on a line. Introducing a fictitious antenna off of the line, one has a type 1 quadrilateral array (Fig. 29).

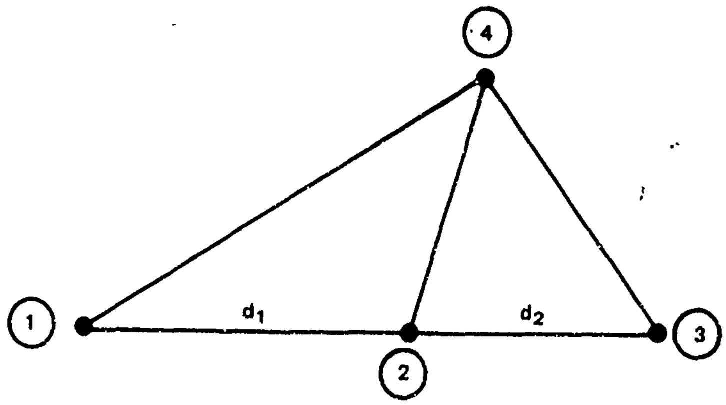

Fig. 29 TYPE 1 QUADRILATERAL ARRAY

From the definition of the A's for quadrilaterals, one has

$$A_1 = \frac{C_1}{T(P)} \tag{245}$$

$$A_2 = \frac{-C}{T(P)} \tag{246}$$

$$A_3 = \frac{C_3}{T(P)} \tag{247}$$

$$A_4 = 0 \tag{248}$$

$$A = -A_2 = \frac{C}{T(P)} \tag{249}$$

where C_1 and C_3 are the areas of the triangles formed by omitting antennas 1 and 3, respectively, and C is the area of the entire triangle.

Since all triangles in Fig. 29 have the same altitude, one has

$$C_1 : C_3 : C = d_2 : d_1 : (d_1+d_2). \qquad (250)$$

Assuming the array has a TAAP, it follows from the fundamental area theorem that the A's of Eqs. (245) – (249) depend only on the d's and not at all on the location of antenna 4. Furthermore, it is seen that, since $A_4 = 0$, Eq. (177) depends only on the phases at antennas 1, 2, 3.

If one sets

$$\frac{d_1}{d_2} = \frac{p}{q} \qquad (251)$$

where p and q are integers having no factor in common, Eq. (177) becomes

$$q\mu_1 - (p+q)\mu_2 + p\mu_3 = 0. \qquad (252)$$

One then has

$$T_\varphi = \frac{1}{2(p+q)}. \qquad (253)$$

This might be more intuitive if one introduces the quantity R which is defined to be the ratio of the separation of the lines in the line grid array ambiguity plot for antennas 1, 2, 3 divided by $\frac{\lambda}{d_1 + d_2}$, the separation of the lines in the plot for only antennas 1 and 3. From Section 2, one has

$$R = p + q. \qquad (254)$$

So

$$T_\varphi = \frac{1}{2R}. \qquad (255)$$

For a given integer value of R, this analysis also tells one where one should put antenna ②. If, for example, one wished to expand the ambiguities due to antennas ① and ③ by a factor of 10, then ② should be placed at any of $\frac{1}{10}$, $\frac{3}{10}$, $\frac{7}{10}$, or $\frac{9}{10}$ of the way between antennas ① and ③, and one gets precisely the same answer in any of the four cases. The sufficient phase difference tolerance in any of these cases is $\frac{1}{20}$, or ±18°.

Next, consider four antennas on a line. Introducing a fifth fictitious antenna off the line, one has a type 7 array as in Fig. 25. From the definition of the A_{ij}'s, one has

$$\left(\left(A_{ij}\right)\right) = \begin{pmatrix} 0 & \frac{-C_{12}}{T(P)} & 0 & \frac{-C_{14}}{T(P)} & \frac{C_{15}}{T(P)} \\ \frac{C_{12}}{T(P)} & 0 & 0 & \frac{C_{24}}{T(P)} & \frac{-C}{T(P)} \\ 0 & 0 & 0 & 0 & 0 \\ \frac{C_{14}}{T(P)} & \frac{-C_{24}}{T(P)} & 0 & 0 & \frac{C_{45}}{T(P)} \\ \frac{-C_{15}}{T(P)} & \frac{C}{T(P)} & 0 & \frac{-C_{45}}{T(P)} & 0 \end{pmatrix} \tag{256}$$

Again, the ratios of the areas of any pair of triangles depend only on the ratios of corresponding line segments along the base, since all triangles have the same altitude. As before, assuming a TAAP for the five antenna array, the fundamental area theorem implies that the entries in Eq. (256) do not depend on the location of antenna ③.

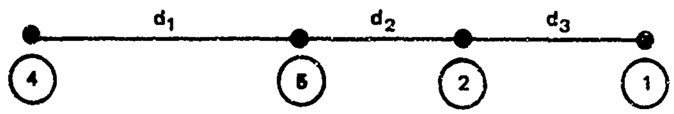

Fig. 30 FOUR ANTENNAS ON A LINE

If, in Fig. 30, one defines p, q, r to be three integers having no factor in common such that

$$d_1 : d_2 : d_3 = p : q : r \tag{257}$$

then the matrix $\left(\left(A_{ij}\right)\right)$ is given by

$$\left(\left(A_{ij}\right)\right) = \begin{pmatrix} 0 & -p & 0 & -q & p+q \\ p & 0 & 0 & q+r & -(p+q+r) \\ 0 & 0 & 0 & 0 & 0 \\ q & -(q+r) & 0 & 0 & r \\ -(p+q) & p+q+r & 0 & -r & 0 \end{pmatrix} \tag{258}$$

Also, Eqs. (198) depend only on the phases at antennas 4, 5, 2, 1, and any two of the four non-zero equations determine the others.

Suppose one used the first and fourth rows in the attempt to choose good antenna locations. One would then have

$$-p\mu_2 - q\mu_4 + (p+q)\mu_5 = 0 \tag{259}$$

and

$$q\mu_1 - (q+r)\mu_2 + r\mu_5 = 0. \tag{260}$$

If g_1 is the greatest common factor of p and q, and g_4 is the greatest common factor of q and r, then from Theorem IX, g_1g_4 divides q. If one works with Eqs. (259) and (260), inequality (217) reduces to $\frac{q}{g_1g_4} \geq 1$, and from Eqs. (218) – (221), it is clear that for good tolerance one should have $\frac{q}{g_1g_4}$ as small as possible, preferably unity. (Note that in the example of Fig. 28, $\frac{q}{g_1g_4} = n - 2$, which is only unity if $n = 3$.)

So, suppose one sets

$$q = g_1g_4 \tag{261}$$

$$p = g_1\alpha \tag{262}$$

$$r = g_4\beta \tag{263}$$

where, in this case, g_1 and g_4 can have no factor in common (otherwise p, q, r would have a factor in common), α and g_4 have no factor in common, and β and g_1 have no factor in common, since g_1 and g_4 are greatest common factors.

One then replaces Eqs. (259) and (260) by

$$-\alpha\mu_2 - g_4\mu_4 + (\alpha + g_4)\mu_6 = 0 \tag{264}$$

and

$$g_1\mu_1 - (g_1 + \beta)\mu_2 + \beta\mu_5 = 0. \tag{265}$$

T_φ will then be no smaller than the smaller of the two numbers $\frac{1}{2(\alpha + g_4)}$ and $\frac{1}{2(\beta + g_1)}$. It is also clear from Eqs. (218) – (221) that one should strive to make these numbers close together. One should also bear in mind that in some cases, another pair of equations might be better.

To obtain an estimate of how good T_φ is when choosing antenna locations, one needs the inequalities (221) and

(225), which, in the present notation, are

$$T_\varphi \le \frac{1}{\sqrt{3(p+q+r) + q}} \tag{266}$$

and

$$T_\varphi \le \frac{1}{\sqrt{3(p+q+r)}} \tag{267}$$

respectively.

Before considering some examples, these results will be recast in terms of R, the ratio of the separation of grid lines for the array ambiguity plot of antennas ④, ⑤, ②, ① to that for antennas ④ and ①.

From the discussion in Section 2, the number L is found to be just p, and R is p+q+r.

From Eqs. (221), (222), (225), (266), (267), for four antennas on a line one has

$$\frac{1}{2(R-1)} \le T_\varphi \le \frac{1}{\sqrt{3R+q}} < \frac{1}{\sqrt{3R}}. \tag{268}$$

For a fixed R, the problem now is to choose p, q, r to make T_φ as large as possible within the constraints of Eq. (268). From Eq. (255), one sees that a considerable potential improvement in T_φ results by going from three to four antennas, at least for large R.

The following examples serve to illustrate that in some cases considerable improvement can in fact be achieved.

It should be borne in mind that this paper is not about the question of a formal procedure for best choice of antenna locations, if indeed there is such a procedure. In the examples to follow, the antenna location choices were guesses on the part of the author, and there may in fact be better choices. They were the best ones the author found in an attempt to get as close as possible to the upper bounds on T_φ given by Eq. (268).

Case 1: Suppose R is of the form $3n^2+3n+1$. R would then be one of the set of numbers, 7, 19, 37, ... Then let $p=n^2$, $q=n(n+1)$, $r=(n+1)^2$. One then has $g_1=n$, $g_4=n+1$, $\alpha=n$, $\beta=n+1$, and Eq. (261) holds. The two numbers $\frac{1}{2(\alpha+g_4)}$ and $\frac{1}{2(\beta+g_1)}$ both equal $\frac{1}{2(2n+1)}$, and one concludes that $T_\varphi \geq \frac{1}{2(2n+1)}$. Inequality (268) becomes

$$\frac{1}{6n(n+1)} \leq T_\varphi \leq \frac{1}{\sqrt{10n^2+10n+3}} < \frac{1}{\sqrt{9n^2+9n+3}} . \tag{269}$$

The ratio of

$$\frac{1}{2(2n+1)} \quad \text{to} \quad \frac{1}{\sqrt{10n^2+10n+3}}$$

and the ratio of

$$\frac{1}{2(2n+1)} \quad \text{to} \quad \frac{1}{\sqrt{9n^2+9n+3}}$$

asymptotically approaches $\frac{\sqrt{10}}{4}$ and $\frac{3}{4}$ respectively, as $n \to \infty$, and the latter ratio always exceeds $\frac{3}{4}$. Thus the choice of antenna locations for R of the assumed form can be thought of as being at least 75% efficient for tolerance, using the weakest inequality.

Had one chosen $p=1$, $q=3n^2+3n-1$, $r=1$ as in Fig. 28, the analysis there shows that one would have the worst possible antenna arrangement, and the efficiency would approach zero as $n \to \infty$. If $n=3$, say, so that $R=37$, this worst case would have a sufficient tolerance T_φ of only $\frac{1}{72}$, or ±5° of errors on phase difference measurements. In the previous choice of antenna locations, however, T_φ is no worse than $\frac{1}{14}$, an improvement of better than a factor of 5. If one wanted $R=37$ with only three antennas, the analysis for that case shows that one would be stuck with (from Eq. (255)) $T_\varphi = \frac{1}{74}$, not much worse than the worst case four antenna location, but at least five times worse than the best case four antenna location.

Case 2: $R = n^2$, a perfect square. In this case, let $p=1$, $q=n-1$, $r=n^2-n$. One has $g_1=1$, $g_4=n-1$, $\alpha=1$, $\beta=n$. The two numbers $\frac{1}{2(\alpha+g_4)}$ and $\frac{1}{2(\beta+g_1)}$ are $\frac{1}{2n}$ and $\frac{1}{2(n+1)}$, respectively, and one concludes $T_\varphi \geq \frac{1}{2(n+1)}$. One can do better than this in this case, for it turns out that one should not work with the first and fourth rows of Eq. (258), but rather the first and fifth rows. The equations equivalent to Eqs. (264) and (265) are then

$$-\mu_2 - (n-1)\mu_4 + n\mu_5 = 0 \tag{270}$$

and

$$-\mu_1 + n\mu_2 - (n-1)\mu_4 = 0. \tag{271}$$

Both equations have the same tolerance $\frac{1}{2n}$, so that $T_\varphi \geq \frac{1}{2n}$, a slight improvement over using the first and fourth rows.

Inequality (268) becomes

$$\frac{1}{2(n^2-1)} \leq T_\varphi \leq \frac{1}{\sqrt{3n^2+n-1}} < \frac{1}{n\sqrt{3}}. \tag{272}$$

So the above antenna placement is at least $\frac{\sqrt{3}}{2} \cong 87\%$ efficient for tolerance when $R = n^2$.

ANALOGOUS THEOREMS PERTAINING TO THE TWO-DIMENSIONAL CASE FOR SIX ANTENNAS WITH A TAAP

First, one needs the theorems analogous to Theorems II (V) and III (VI) for four (five) antennas. Before this can be done, the definition of the A's, B's, C's must be extended.

For six antennas, three subscripts are needed, so C_{ijk} ($i, j, k = 1, 2, \ldots, 6$) is defined to be zero if any two subscripts are equal; otherwise it is equal to the area of the triangle remaining when antennas (i), (j), (k) are ignored. For any i, j, k, B_{ijk} will equal $\pm C_{ijk}$, and A_{ijk} will equal $\frac{B_{ijk}}{T(P)}$, where T(P) is the TAA of the array of six antennas. So all that remains in the definition of the B's and A's is which sign to use in $B_{ijk} = \pm C_{ijk}$.

Begin by numbering the antennas so that when antenna (6) is ignored, one of the twelve types in Fig. 25 results, with the antenna numbering as shown there. This can always be done since it is assumed that the six antennas have a TAAP, and cannot therefore all lie on a line. Then the sign relating B_{6jk} and C_{6jk} is defined to be the same as in Table 1 for $j < k$ just as if the subscript 6 were missing, and for $k < j$, $B_{6jk} = -B_{6kj}$, as before.

Next, B_{526} is defined to be $-B_{625}$, and is not zero.

Theorems V and VI and subsequent theorems for five antennas would hold equally well if every sign in Table 1 were changed. With antenna (5) removed, there are thus two possible tables for the remaining five antennas analogous to Table 1. Select the table for which B_{526} has the sign as defined above. All signs for B_{5jk} are then uniquely determined except for cases where the area of the triangle is zero, where it doesn't matter. So F_{5jk} is defined, and $B_{5jk} = -B_{5kj}$ for j, k = 1, 2, ..., 6.

Similarly, define B_{256} to be $-B_{526}$. Again one selects the one of the two tables with antenna ② removed for which B_{256} has the defined sign. So B_{2jk} is defined, and $B_{2kj} = -B_{2jk}$ for j, k = 1, 2, ..., 6.

Proceeding in the same fashion, B_{1jk} is defined by choosing the proper table so that $B_{126} = -B_{621}$. From Fig. 25, $B_{621} \neq 0$. Again $B_{1kj} = -B_{1jk}$, for j, k = 1, 2, ..., 6.

Suppose that the configuration with antenna ⑥ removed is not type 10 of Fig. 25. Then B_{4jk} is defined in the same manner, where $B_{426} = -B_{624}$, with $B_{624} \neq 0$ from Fig. 25. For a type 10 array when antenna ⑥ is removed, define B_{4jk} as before with $B_{416} = -B_{614}$, and $B_{614} \neq 0$ from Fig. 25. In either case, $B_{4kj} = -B_{4jk}$ for j, k = 1, 2, ..., 6.

Finally, suppose all antennas except ③ lie on a line. Then $B_{3jk} = 0$ for j, k = 1, 2, ..., 6. If the array is type 7 after removing antenna ⑥, and ⑥ is not on the line joining ① and ④, then B_{3jk} is defined as before with $B_{352} = -B_{253}$, where $B_{253} \neq 0$. If the array is not type 7 when antenna ⑥ is removed, then B_{3jk} is defined as before with $B_{365} = -B_{635}$, and $B_{635} \neq 0$. As before $B_{3kj} = -B_{3jk}$ for j, k = 1, 2, ..., 6.

The B_{ijk}'s and A_{ijk}'s are now all defined, and $B_{ikj} = -B_{ijk}$, j, k = 1, 2, ..., 6.

The quantities B_{ij} for i, j = 1, 2, ..., are defined to be zero if i = j, and equal to the area (always taken as positive) of the quadrilateral formed by omitting antennas ⓘ and ⓙ otherwise.

B_i for $i = 1, 2, \ldots, 6$ is defined as the area (always taken as positive) of the five-sided figure formed by omitting antenna (i).

B is defined as the area of the entire array (always taken as the area of a convex region containing all points of the array, the region having three, four, five, or six sides.

A_{ij}, A_i, A are obtained from the B's by dividing each appropriate B by T(P), the triangular array area of the entire array.

P_{ij} is defined as the array ambiguity plot obtained by omitting antennas (i) and (j).

<u>Theorem X</u>: For any antenna array in a plane having a TAAP and six antennas, the following properties hold:

1. A_{ijk}, A_{ij}, A_i, A are all integers, $i, j, k = 1, 2, \ldots, 6$.
2. The A_{ijk} have no factor in common, $i, j, k = 1, 2, \ldots, 6$.
3. Except for those i and j for which $A_{ij1} = A_{ij2} = \ldots = A_{ij6} = 0$, the greatest common factor of $A_{ij1}, A_{ij2}, \ldots, A_{ij6}$ is $\frac{T(P_{ij})}{T(P)}$, $i, j = 1, 2, \ldots, 6$.
4. $\sum_{k=1}^{6} A_{ijk} = 0$, $i, j = 1, 2, \ldots, 6$.
5. $\sum_{k=1}^{6} |A_{ijk}| = 2A_{ij}$, $i, j = 1, 2, \ldots, 6$.
6. Interchanging <u>any</u> two subscripts of A_{ijk} changes its sign. That is, for any $i, j, k = 1, 2, \ldots, 6$, one has $A_{ijk} = -A_{ikj} = A_{kij} = -A_{kji} = A_{jki} = -A_{jik}$.

Proof: Except for property 6, the proof of Theorem X follows from the definition of the A's, and is proved in the same way as Theorem V. The proof for property 6 follows the proof of the next theorem.

Theorem XI: For any source direction, let $\mu_1, \mu_2, \ldots, \mu_6$ denote the unambiguous absolute phase at antennas ① through ⑥, respectively, in units of cycles. Then

$$\sum_{k=1}^{6} A_{ijk}\mu_k = 0, \qquad i, j = 1, 2, \ldots, 6. \tag{273}$$

Proof: This follows directly from Theorem VI and the definition of the A's.

For property 6 of Theorem X, if i and j are such that $A_{ij1} = A_{ij2} = \ldots = A_{ij6} = 0$, property 6 is obvious. If not, then

$$\sum_{k=1}^{6} A_{ijk}\mu_k = 0$$

and

$$\sum_{k=1}^{6} A_{jik}\mu_k = 0$$

are equations in which the coefficients of the μ_k's are signed areas of triangles of the same quadrilateral. It follows from the derivation of the A's for five antennas that the two sets of coefficients are either identical, or one set has all opposite signs from the other set.

For i = 5, j = 6, since $B_{562} = -B_{652}$, it follows that $A_{56k} = -A_{65k}$ for k = 1, 2, ..., 6.

For $i = 2$, $j = 5$, since $B_{256} = -B_{526}$, it follows that $A_{25k} = -A_{52k}$ for $k = 1, 2, \ldots, 6$.

For $i = 2$, $j = 6$, since $B_{265} = -B_{625}$, it follows that $A_{26k} = -A_{62k}$ for $k = 1, 2, \ldots, 6$.

Similarly, for $i = 1$, $j = 6$, the result follows since $B_{162} = -B_{612}$. By setting $k = 1$ in the previous step, one can show that $B_{162} = -B_{216}$, so the result follows for $i = 1$, $j = 2$. (A's and B's are interchangeable as far as sign is concerned.) Also from Fig. 25, B_{156} cannot vanish. Setting $k = 1$ in the third previous step and $k = 5$ in the $i = 1$, $j = 6$ case, one can show that $B_{156} = -B_{516}$, so the result follows for $i = 1$, $j = 5$.

So for any pair i, j selected from 1, 2, 5, 6, $A_{ijk} = -A_{jik}$ for $k = 1, 2, \ldots, 6$. For the remainder of the proof it will be assumed that the array is none of types 2, 4, 5, 6, 7, 9, 10, 11 if antenna 6 is removed. If the desired results follow under this assumption, they must follow in general, as these types are all limiting cases of types 1, 3, 8, 12.

This assumption implies that perhaps $B_{623} = B_{632} = 0$, but B_{6ij} does not vanish for any other pair i, j such that no two subscripts are equal.

Since $B_{365} = -B_{635}$, the result follows for $i = 3$, $j = 6$. Similarly, since $B_{356} = -B_{536}$, the result follows for $i = 3$, $j = 5$. Setting $k = 3$ in the $i = 1$, $j = 6$ case gives $B_{163} = -B_{613}$. This, combined with setting $k = 1$ in the $i = 3$, $j = 6$ case gives $B_{136} = -B_{631} = B_{361} = -B_{316}$, so the result holds for $i = 3$, $j = 1$.

If the array is not type 3 with antenna 6 removed, then $B_{623} \neq 0$. Setting $k = 2$ in the $i = 3$, $j = 6$ case and

$k = 3$ in the $i = 2$, $j = 6$ case, one has $B_{326} = -B_{362} = B_{632} = -B_{623} = B_{263} = -B_{236}$, so the result holds for $i = 2$, $j = 3$. If the array is type 3 with antenna 6 removed, then suppose that $B_{123} \neq 0$. Setting $k = 3$ in the $i = 1$, $j = 2$ case and $k = 2$ in the $i = 3$, $j = 1$ case leads to $B_{231} = -B_{213} = B_{123} = -B_{132} = B_{312} = -B_{321}$, so the result follows for $i = 2$, $j = 3$. If $B_{123} = 0$, then antennas 1, 4, 5, 6 all lie on a line and $B_{23k} = 0$ for $k = 1, 2, \ldots, 6$.

It has now been shown that for any pair i, j selected from 1, 2, 3, 5, 6, $A_{ijk} = -A_{jik}$, $k = 1, 2, \ldots, 6$.

Finally, since $B_{462} = -B_{642}$, the result follows for $i = 4$, $j = 6$. Also, since $B_{426} = -B_{246}$, it follows for $i = 4$, $j = 2$.

Set $k = 4$ in the $i = 1$, $j = 6$ case and $k = 1$ in the $i = 4$, $j = 6$ case. This leads to $B_{416} = -B_{461} = B_{641} = -B_{614} = B_{164} = -B_{146}$, so the result holds for $i = 4$, $j = 1$.

Set $k = 4$ in the $i = 3$, $j = 6$ case and $k = 3$ in the $i = 4$, $j = 6$ case. This leads to $B_{436} = -B_{463} = B_{643} = -B_{634} = B_{364} = -B_{346}$, so the result holds for $i = 4$, $j = 3$.

Set $k = 4$ in the $i = 5$, $j = 6$ case and $k = 5$ in the $i = 4$, $j = 6$ case. This leads to $B_{456} = -B_{465} = B_{645} = -B_{654} = B_{564} = -B_{546}$, so the result holds for $i = 4$, $j = 5$. This completes the proof of property 6 of Theorem X.

The theorem analogous to Theorem VII is the following:

<u>Theorem XII</u>: For any subscripts $i, j, k, \ell, m, n = 1, 2, \ldots, 6$,

$$A_{ijk}A_{\ell mn} + A_{ikn}A_{j\ell m} + A_{ik\ell}A_{jmn} + A_{ikm}A_{\ell jn} = 0. \qquad (274)$$

Before proving this, it is noted that, if any two subscripts are equal, the expression reduces either to an obvious identity, or is a special case of Theorem VII. For example, if $j = n$, it reduces to $A_{ijk}A_{\ell mj} - A_{ijk}A_{\ell mj} = 0$. If $i = \ell$, it reduces to $A_{ijk}A_{imn} + A_{ikm}A_{ijn} + A_{imj}A_{ikn} = 0$ which follows from Eq. (199), since this is the set of Eqs. (199) for a five antenna array where antenna ⓘ is omitted.

<u>Proof:</u> Consider the matrix

$$M_1 = \begin{pmatrix} A_{\ell mk} & A_{\ell mn} & A_{\ell mj} \\ A_{i\ell k} & A_{i\ell n} & A_{i\ell j} \\ A_{imk} & A_{imn} & A_{imj} \end{pmatrix} \tag{275}$$

Expand the determinant of M_1 by the first row, and use Theorem VII and property 6 of Theorem X. One has

$$\begin{aligned} D(M_1) &= A_{\ell mk}\left(A_{i\ell n}A_{imj} + A_{inm}A_{i\ell j}\right) \\ &\quad + A_{\ell mn}\left(A_{i\ell j}A_{imk} + A_{ijm}A_{i\ell k}\right) \\ &\quad + A_{\ell mj}\left(A_{i\ell k}A_{imn} + A_{ikm}A_{i\ell n}\right) \\ &= A_{\ell mk}A_{i\ell m}A_{inj} + A_{\ell mn}A_{i\ell m}A_{ijk} + A_{\ell mj}A_{i\ell m}A_{ikn} \\ &= A_{i\ell m}\left(A_{\ell mk}A_{inj} + A_{\ell mn}A_{ijk} + A_{\ell mj}A_{ikn}\right). \end{aligned} \tag{276}$$

Similarly if one expands the determinant of M_1 by the first column, one has

$$\begin{aligned}
D(M_1) &= A_{\ell mk}\left(A_{i\ell n}A_{imj} + A_{inm}A_{i\ell j}\right) \\
&\quad + A_{i\ell k}\left(A_{min}A_{m\ell j} + A_{mn\ell}A_{mij}\right) \\
&\quad + A_{imk}\left(A_{\ell mn}A_{\ell ji} + A_{\ell mj}A_{\ell in}\right) \\
&= A_{\ell mk}A_{i\ell m}A_{inj} + A_{i\ell k}A_{mi\ell}A_{mnj} + A_{imk}A_{\ell mi}A_{\ell jn} \\
&= A_{i\ell m}\left(A_{\ell mk}A_{inj} + A_{i\ell k}A_{mnj} + A_{imk}A_{\ell jn}\right). \qquad (277)
\end{aligned}$$

Equating Eqs. (276) and (277), one has

$$A_{i\ell m}\left(A_{ijk}A_{\ell mn} + A_{ikn}A_{j\ell m} + A_{ik\ell}A_{jmn} + A_{ikm}A_{\ell jn}\right) = 0. \qquad (278)$$

The expression within the brackets is the expression in Eq. (274). One may assume that $A_{i\ell m} \neq 0$ for the sake of the proof, for if $A_{i\ell m} = 0$, one could imagine moving an antenna slightly to cause $A_{i\ell m} \neq 0$, and in that case the theorem follows. It follows in general by an appeal to continuity.

There are at most fifteen distinct equations (except for changing the sign of all terms by inverting the first two subscripts) in Eq. (273). The next theorem will prove that there are exactly three independent ones.

Theorem XIII: Given six antennas not all on a line, then the following six by fifteen matrix has rank three:

$$M = \begin{pmatrix} A_{121} & A_{122} & \cdots & A_{126} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ A_{561} & A_{562} & \cdots & A_{566} \end{pmatrix} \qquad (279)$$

Proof: From Fig. 25, $A_{256} \neq 0$ since antennas ①, ③, ④ never lie on a line. The rows defined by (A_{25i}), (A_{26i}) and (A_{56i}) are therefore independent since $A_{252} = A_{255} = A_{262} = A_{266} = A_{565} = A_{566} = 0$. The matrix M therefore has rank at least three.

Consider an arbitrary row defined by (A_{ikm}), where i and k are fixed and m = 1, 2, ..., 6. Setting $\ell = 2$, $j = 5$, $n = 6$ in Eq. (274), one has

$$A_{256}A_{ikm} = -A_{ik5}A_{26m} + A_{ik6}A_{25m} + A_{ik2}A_{56m}. \qquad (280)$$

Since $A_{256} \neq 0$, the row (A_{ikm}) is a linear combination of the three rows shown to be independent, and the theorem is proved.

THE TOLERANCE PROBLEM FOR SIX ANTENNAS WITH A TAAP

The machinery has now been developed for proceeding with the tolerance question.

Of the at most fifteen distinct equations in Eq. (273) there are only three linearly independent ones. In an analogous manner to the five antenna case, one seeks three equations which maximize T_φ.

If g_{ij} is the greatest common factor of the row having i and j as the first two subscripts $\left(= \frac{T(P_{ij})}{T(P)} \text{ from Theorem X}\right)$, then the equations corresponding to Eqs. (208) and (209) are

$$\frac{C_{11}A_{25j}}{g_{25}} + \frac{C_{12}A_{26j}}{g_{26}} + \frac{C_{13}A_{56j}}{g_{56}} = I_j, \quad j = 1, 2, \ldots, 6 \qquad (281)$$

$$\frac{C_{21}A_{25j}}{g_{25}} + \frac{C_{22}A_{26j}}{g_{26}} + \frac{C_{23}A_{56j}}{g_{56}} = J_j, \quad j = 1, 2, \ldots, 6 \qquad (282)$$

$$\frac{C_{31}A_{25j}}{g_{25}} + \frac{C_{32}A_{26j}}{g_{26}} + \frac{C_{33}A_{56j}}{g_{56}} = K_j, \quad j = 1, 2, \ldots, 6 \qquad (283)$$

where the determinant of the C's is $\neq 0$, and the I's, J's and K's are integers.

One then works with the three independent expressions

$$\sum_{j=1}^{6} I_j\mu_j, \quad \sum_{j=1}^{6} J_j\mu_j, \quad \text{and} \quad \sum_{j=1}^{6} K_j\mu_j.$$

Set

$$\sum_{j=1}^{6} |I_j| = I \qquad (284)$$

$$\sum_{j=1}^{6} |J_j| = J \qquad (285)$$

$$\sum_{j=1}^{6} |K_j| = K. \qquad (286)$$

If $T_{C_{ij}}$ denotes the smallest of $\frac{1}{I}$, $\frac{1}{J}$, $\frac{1}{K}$, then T_φ is the maximum of $T_{C_{ij}}$ over all choices of the C's with the prescribed properties.

The next task is to find bounds for T_φ. To this end, note that, by setting in turn j = 2, 5, 6 in Eqs. (281)-(283), it follows that all of the C's are rational numbers.

Consider the matrix equation

$$\begin{pmatrix} \frac{C_{11}}{g_{25}} & \frac{C_{12}}{g_{26}} & \frac{C_{13}}{g_{56}} \\ \frac{C_{21}}{g_{25}} & \frac{C_{22}}{g_{26}} & \frac{C_{23}}{g_{56}} \\ \frac{C_{31}}{g_{25}} & \frac{C_{32}}{g_{26}} & \frac{C_{33}}{g_{56}} \end{pmatrix} \begin{pmatrix} A_{25j} & A_{25n} & A_{25\ell} \\ A_{26j} & A_{26n} & A_{26\ell} \\ A_{56j} & A_{56n} & A_{56\ell} \end{pmatrix}$$

$$= \begin{pmatrix} I_j & I_n & I_\ell \\ J_j & J_n & J_\ell \\ K_j & K_n & K_\ell \end{pmatrix} \qquad j, n, \ell = 1, 2, \ldots, 6. \quad (287)$$

The next step is to equate determinants of Eq. (287). If D(C), D(A), D(I) denote the three determinants, one has

$$D(C)D(A) = D(I). \qquad (288)$$

From Theorems VII, XII, and property 6 of Theorem X, one has, expanding by the first row,

$$D(A) = A_{25j}\,(A_{62n}A_{65\ell} + A_{6n5}A_{62\ell})$$

$$+ A_{25n}\,(A_{62\ell}A_{65j} + A_{6\ell 5}A_{62j})$$

$$+ A_{25\ell}\,(A_{62j}A_{65n} + A_{6j5}A_{62n})$$

$$= A_{25j}A_{625}A_{6n\ell} + A_{25n}A_{625}A_{6\ell j} + A_{25\ell}A_{625}A_{6jn}$$

$$= -A_{625}\,(A_{2j5}A_{\ell 6n} + A_{25n}A_{j\ell 6} + A_{25\ell}A_{j6n})$$

$$= (A_{625})(A_{256}A_{\ell jn}) = (A_{256})^2 A_{\ell jn}. \tag{289}$$

So

$$\left[(A_{256})^2 D(C)\right]\ A_{\ell jn} = D(I). \tag{290}$$

Now $D(I)$ is an integer, and $(A_{256})^2D(C)$ is a rational number. If one sets $(A_{256})^2D(C)$ equal to a fraction in lowest terms, the denominator must divide $A_{\ell jn}$ for all ℓ, j, n. This contradicts property 2 of Theorem X, which proves that $(A_{256})^2D(C)$ is an integer. Since it is not zero, one has

$$\left|(A_{256})^2 D(C)\right| \geq 1. \tag{291}$$

One then has the following string of inequalities:

For all ℓ, j, n = 1, 2, ..., 6,

$$|A_{\ell jn}| \leq |D(I)| \leq |I_j||J_n||K_\ell| + |I_j||J_\ell||K_n| + |I_n||J_j||K_\ell|$$
$$+ |I_n||J_\ell||K_j| + |I_\ell||J_j||K_n| + |I_\ell||J_n||K_j|. \tag{292}$$

Summing over n, and using property 5 of Theorem X, one has

$$2A_{\ell j} \le \left[|I_j||K_\ell| + |I_\ell||K_j|\right]J + \left[|I_j||J_\ell| + |I_\ell||J_j|\right]K + \left[|J_j||K_\ell| + |J_\ell||K_j|\right]I. \tag{293}$$

Summing over ℓ and j, dividing by two, one has

$$\sum_{j,\ell=1}^{6} A_{\ell j} \le 3IJK. \tag{294}$$

Since the largest of I, J, and K cannot be less than $\sqrt[3]{\frac{1}{3}\sum_{\ell,j=1}^{6} A_{\ell j}}$, one has

$$T_\varphi \le \frac{1}{\sqrt[3]{\frac{1}{3}\sum_{\ell,j=1}^{6} A_{\ell j}}} \tag{295}$$

The expression $\sum_{\ell,j=1}^{6} A_{\ell j}$ is just twice the sum of the (normalized by $\frac{1}{T(P)}$) areas of all quadrilaterals that can be formed by taking the antennas four at a time.

The analogous expression for Eq. (224) carries over, and is

$$\sum_{\ell=1}^{6} A_{\ell j} \ge 3A_j, \qquad j = 1, 2, \ldots, 6 \tag{296}$$

so that a weaker inequality than (295) is

$$T_\varphi \leq \frac{1}{\sqrt[3]{\sum_{j=1}^{6} A_j}} \tag{297}$$

where $\sum_{j=1}^{6} A_j$ is just the (normalized by $\frac{1}{T(P)}$) sum of the areas of all pentagons that can be formed by taking the antennas five at a time.

It can be shown (the proof is omitted) that

$$\sum_{j=1}^{6} A_j \geq 4A \tag{298}$$

so that a still weaker inequality is

$$T_\varphi \leq \frac{1}{\sqrt[3]{4A}} \tag{299}$$

where A is the (normalized by $\frac{1}{T(P)}$) area of the hexagonal array.

It is conjectured that none of the inequalities of Eqs. (295), (297), or (299) are ever achievable.

It is an open question (probably answered in the affirmative) as to whether or not one can achieve the best tolerance by some selection of three of the equations (273) without having to form additional linear combinations of them.

For an inequality the other way, the lower bound on T_φ is probably larger than $\frac{1}{2(A-1)}$, but the point is not belabored here, as arrays with T_φ this small are poor arrangements and not therefore very interesting. An obvious lower bound follows from property 5 of Theorem X and the observation that the area of a quadrilateral of the array cannot exceed the area of the entire array. One then has

$$\frac{1}{2A} \le T_\varphi. \tag{300}$$

Inequality (226) carries over as follows:

$$T_\varphi \le \sqrt[3]{\frac{\sqrt{3}\,\lambda^2}{48 \sin^2 \rho \cdot (\text{area of six antenna array})}}. \tag{301}$$

For the example considered after inequality (226), where $T_\varphi = \frac{1}{18}$ and $\frac{\sqrt{3}\,\lambda^2}{24 \sin^2 \rho} = \frac{1}{6}\ \text{in}^2$, one has a bound on on the area of a six antenna array of 486 in^2, a considerable improvement over the 36 in^2 bound for five antennas. Of course the bound can probably not be achieved, but as examples to follow will show, one can approach it. An important observation to make is that each time one adds an antenna, starting with four, the tolerance-ambiguity-area dilemma is eased by an order of magnitude if achievable measurement errors are not too large and if one arranges the antennas properly.

THE GENERAL PLANAR ARRAY CASE FOR ARRAYS WITH A TAAP

Consider the case of n antennas in a plane. Assume $n \ge 4$ and that the array has a TAAP. (They therefore do not all lie on a line.)

Let Q_k denote the sum of the areas of all possible subarrays with k antennas, where $4 \le k \le n$. Q_n is just the area of the entire array.

Let P denote the array ambiguity plot of the entire array, and T(P) its triangular array area.

The inequalities for T_φ that have been proven for $n \le 6$ are special cases of the following inequalities:

$$\frac{T(P)}{2Q_n} \le T_\varphi \le \sqrt[n-3]{\frac{(n-3)T(P)}{2Q_4}} \le \dots \le \sqrt[n-3]{\frac{\binom{n-3}{k-3}T(P)}{(k-2)Q_k}}$$

$$\le \dots \le \sqrt[n-3]{\frac{T(P)}{Q_{n-1}}} \le \sqrt[n-3]{\frac{T(P)}{(n-2)Q_n}}. \qquad (302)$$

Inequality (302) for $n > 6$ is only a conjecture at this time, since the author has not yet proved it. It is also conjectured that none of the upper bounds can be reached by T_φ (except for the case $n = 4$).

The innermost inequality is the strongest bound, and the outermost is the weakest bound. Note also that for $n = 4$, the left and right hand sides collapse to the same value so that inequality (302) becomes an equality (the same result as previously derived for this case).

AN EXAMPLE OF AN ARRAY WITH SIX ANTENNAS

The first array example is given in Fig. 31. The six antennas are arranged on a circle in a hexagon which is not regular. The array ambiguity plot for this array, which is a TAAP, is given in Fig. 32.

The following data for this array, computed according to techniques developed here and in the previous sections, are given without proof.

$$T(P) \cong \frac{1}{2}\left(\frac{d_1}{5}\right)^2 \sin(52.615^\circ) \cong \frac{d^2}{2}(.0805)^2(.7946) \cong .0026d^2 \qquad (303)$$

where d is the diameter of the circle.

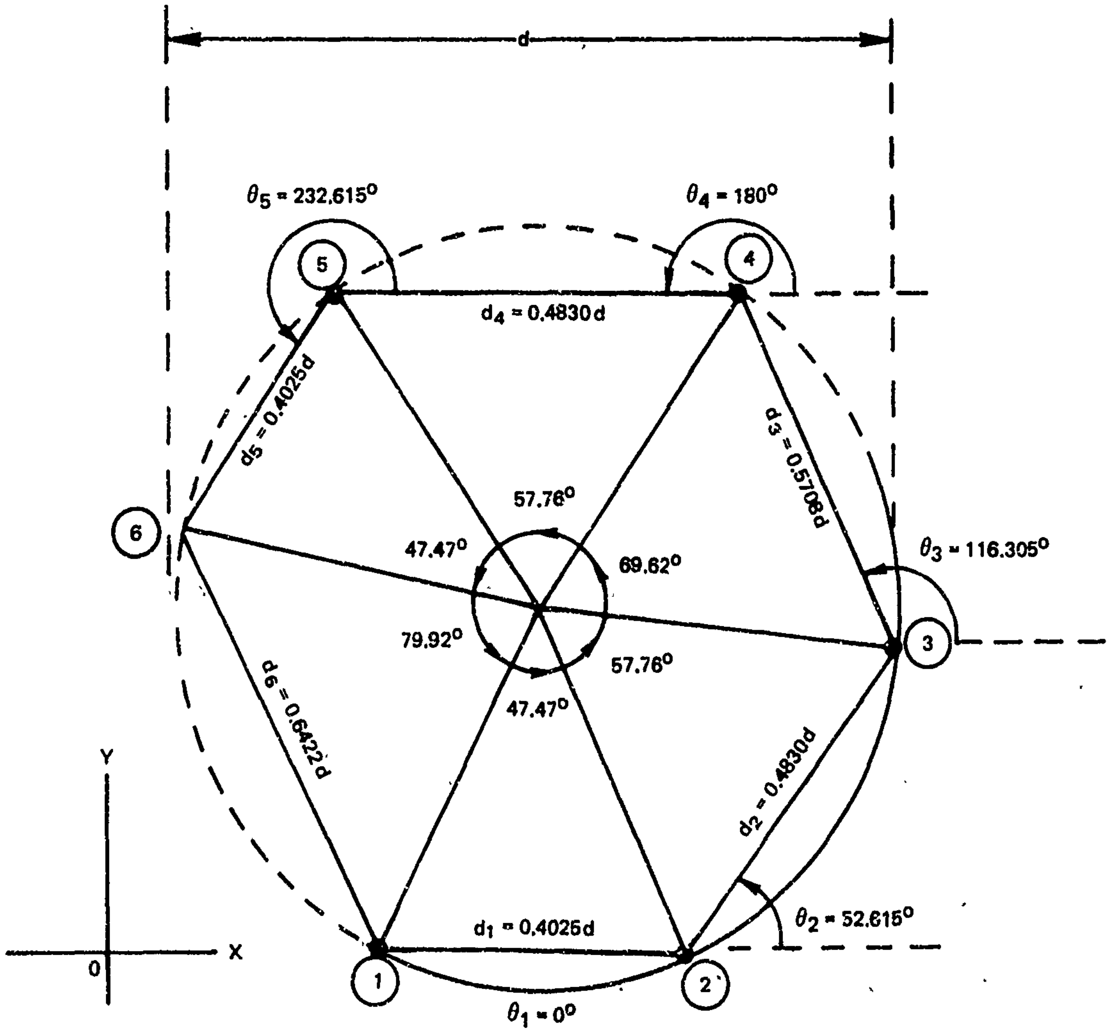

Fig. 31 HEXAGONAL ARRAY OF COPLANAR ANTENNAS

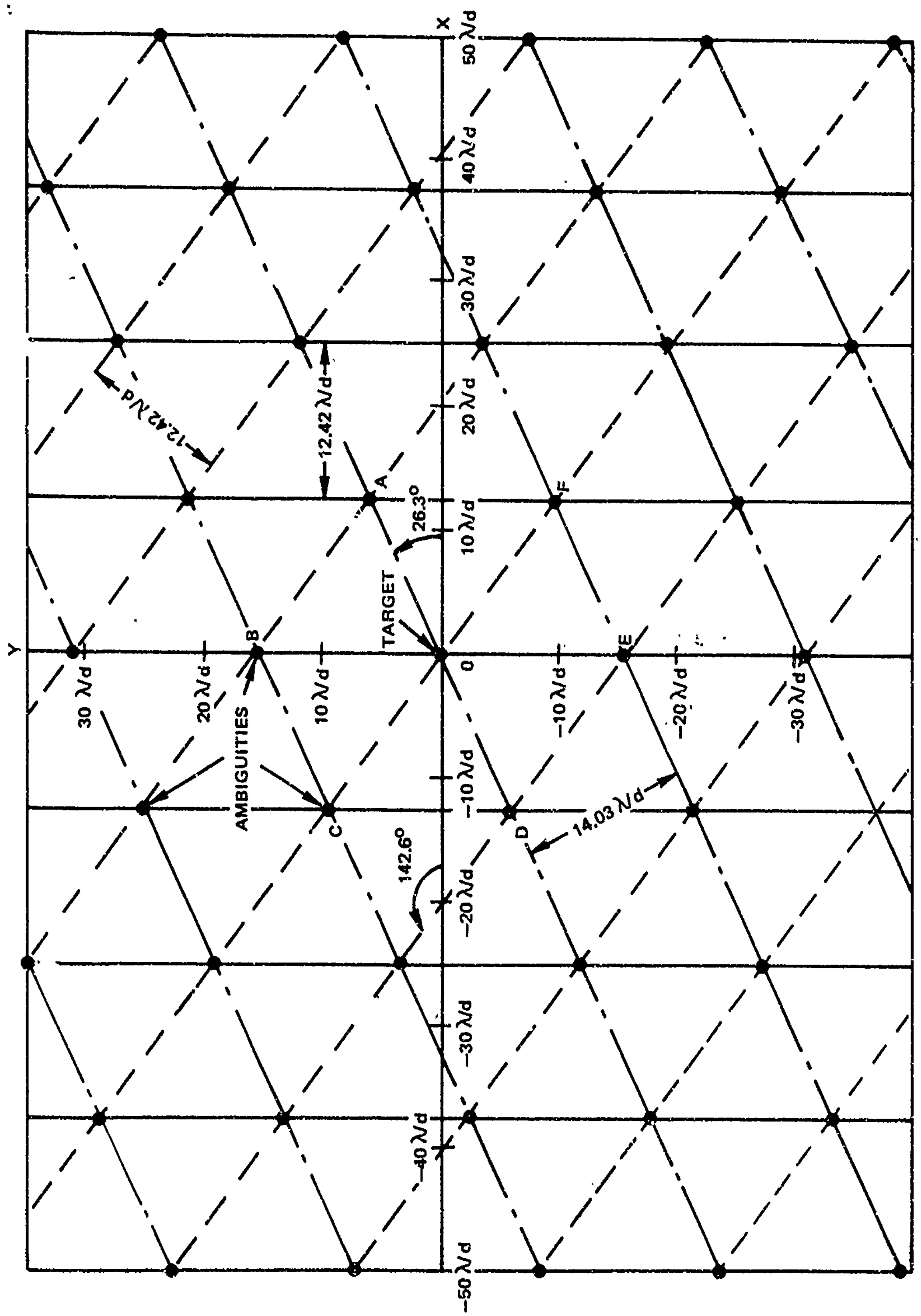

Fig. 32 AMBIGUITY PLOT FOR HEXAGONAL ARRAY OF COPLANAR ANTENNAS

The values for A_{ijk} with $i < j < k$ are given in Table 3.

Table 3

A_{ijk} for Antenna Array of Fig. 31

i	j	k	A_{ijk}
1	2	3	30
		4	-70
		5	88
		6	-48
	3	4	70
		5	-124
		6	84
	4	5	84
		6	-84
	5	6	48
2	3	4	-45
		5	99
		6	-84
	4	5	-99
		6	124
	5	6	-88
3	4	5	45
		6	-70
	5	6	70
4	5	6	-30

The values for the normalized areas of the quadrilaterals (A_{ij}) formed by omitting antennas (i) and (j) are given for $i < j$ in Table 4.

Table 4

A_{ij} for Antenna Array of Fig. 31

i	j	A_{ij}
1	2	118
	3	154
	4	154
	5	172
	6	132
2	3	129
	4	169
	5	187
	6	172
3	4	115
	5	169
	6	154
4	5	129
	6	154
5	6	118

The values for the normalized areas of the six pentagons of the array (A_i) formed by omitting antenna i are given in Table 5.

Table 5

A_i for Antenna Array of Fig. 31

i	A_i
1	202
2	217
3	199
4	199
5	217
6	202

Also, the normalized area of the entire array is

$$A = 247. \tag{304}$$

$\frac{Q_4}{T(P)}$ and $\frac{Q_5}{T(P)}$ are given by the sum of the entries in Tables 4 and 5, respectively. $\frac{Q_6}{T(P)}$ is just A, which is 247. Performing the calculations, one has

$$\frac{Q_4}{T(P)} = 2226 \tag{305}$$

$$\frac{Q_5}{T(P)} = 1236. \tag{306}$$

Inequality (302) for n = 6 then becomes

$$\frac{1}{494} \leq T_\varphi \leq \sqrt[3]{\frac{1}{1484}} \leq \sqrt[3]{\frac{1}{1236}} \leq \sqrt[3]{\frac{1}{988}} \tag{307}$$

or

$$.002 \leq T_\varphi \leq .088 \leq .093 \leq .100. \tag{308}$$

If one works with the three best equations (from the tolerance point of view) of Eq. (273), one finds that they are given by the (i,j) pairs (3,6), (1,4) and (1,6). After dividing each of the three equations by the greatest common factor of its coefficients, the resulting equations are

$$6\mu_1 - 6\mu_2 + 5\mu_4 - 5\mu_5 = 0 \tag{309}$$

$$5\mu_2 - 5\mu_3 + 6\mu_5 - 6\mu_6 = 0 \tag{310}$$

$$4\mu_2 - 7\mu_3 + 7\mu_4 - 4\mu_5 = 0. \tag{311}$$

These three equations all have the same tolerance, and the others all have smaller tolerance. So from the definition of T_φ,

$$T_\varphi \geq \frac{1}{22} \cong .045. \tag{312}$$

As before one uses ≥ instead of = in Eq. (312) as there may a priori be a set of equations which are linear combinations of Eqs. (309)-(311) and which allow a larger tolerance. As in the discussion preceding Eq. (236), Eq. (312) may be an equality. The author has been unable to find any better linear combinations for this array.

As in Eq. (240), a smaller upper bound can be obtained by observing that, from Eqs. (294) and (305) the largest of I, J, K cannot be smaller than 12, since I, J, K are integers. So

$$T_\varphi \leq \frac{1}{12} \cong .083. \tag{313}$$

COMPARISON OF ARRAYS OF FIGS. 26 AND 31

If one carries over the concept of tolerance efficiency as the ratio of T_φ to the weakest (largest) upper bound, as was introduced in the discussion succeeding Eq. (269), and assumes that T_φ is indeed $\frac{1}{22}$ for the six antenna array (and not greater), one finds an efficiency of 62% for the five antenna array, and 45% for the six antenna array.

From Eqs. (236) and (312), both arrays have the same sufficient tolerance.

If $\frac{d_6}{d_5}$ is defined as the ratio of the diameters of the circles inscribing the six and five antenna arrays, one sees by comparing Eqs. (227) and (303) that if $\frac{d_6}{d_5} = \sqrt{\frac{91}{26}} \cong 1.87$, then the triangular array areas are the same. Roughly speaking, if $\frac{d_6}{d_5} \cong 1.87$, then the two arrays will have about the same unambiguous coverage in terms of a cone of source directions. If $\frac{d_6}{d_5} < 1.87$, the six antenna array will have greater unambiguous coverage than the five antenna array.

Concerning the efficiency numbers, it has been mentioned before that for more than four antennas, tolerance efficiencies of 100% are undoubtedly not achievable, although how close one can come is unknown. Also, tolerance efficiency is not the whole story as far as the relation between tolerance and ambiguity remoteness is concerned. Since the tolerance efficiency is a function of T(P), which is an area, it is a priori possible to have two arrays with the same number of antennas such that the most efficient array actually has the closest nearest ambiguity. The way this might happen would be for the lattice points in one dimension of the TAAP of the more efficient array to be more widely spaced than those of the other dimension.

Consider next the following problem: Assuming $T_\varphi = \frac{1}{22}$, is it possible to arrange five antennas more or less uniformly around a circle with 8-inch diameter if* $\lambda = 1.75 \sin \rho$ inches?

Three efficiencies are defined, all ≤ 1, as follows:

$$E_P = \frac{12\, T(P) \sin^2 \rho}{\sqrt{3}\, \lambda^2} \tag{314}$$

$$E_A = \frac{\text{enclosed area}}{\text{area of regular pentagon on same circle}} \tag{315}$$

*ρ is the half-angle of the cone of unambiguous coverage.

and

$$E_\varphi = T_\varphi \sqrt{\frac{3 \times (\text{enclosed area})}{T(P)}} . \qquad (316)$$

E_φ is the tolerance efficiency just defined. E_A is a measure of the uniformity of distribution of the antennas around the circle. E_P (Theorem I shows that $E_P \le 1$) is a measure of achieving a good balance in the location of the ambiguities in the array ambiguity plot.

Using the numbers assumed here, Eqs. (314)-(316) become

$$E_P = 2.26\, T(P) \qquad (317)$$

$$E_A = \frac{\text{enclosed area}}{38} \qquad (318)$$

$$E_\varphi = \frac{1}{22}\sqrt{\frac{3 \times (\text{enclosed area})}{T(P)}}. \qquad (319)$$

Eliminating T(P) and enclosed area from Eqs. (317)-(319), one has

$$E_\varphi^2 = .53\,\frac{E_A}{E_P}. \qquad (320)$$

The array in Fig. 26 has an E_A in excess of .98. If one assumes that one would not want an E_A any less than .80, one has

$$E_\varphi^2 \ge \frac{.42}{E_P} \qquad (321)$$

or

$$E_\varphi \ge .65\,\frac{1}{\sqrt{E_P}}. \qquad (322)$$

Now E_P is less than unity, so practically one would undoubtedly be required to exceed 70% in tolerance efficiency. This is considered by the author to be a very marginal situation, and the answer to the original question is undoubtedly no unless one relaxes the restriction on E_A, which will probably result in arrays which are undesirable for other reasons.

For six antennas, Eq. (317) remains the same, Eq. (318) becomes (using a regular hexagon for the denominator)

$$E_A = \frac{\text{enclosed area}}{41.6} \qquad (323)$$

and Eq. (319) becomes

$$E_\varphi = \frac{1}{22} \sqrt[3]{\frac{4 \times (\text{enclosed area})}{T(P)}} . \qquad (324)$$

Eliminating T(P), the equivalent of Eq. (320) is

$$E_\varphi^3 = .035 \frac{E_A}{E_P} \qquad (325)$$

or

$$E_\varphi = .33 \sqrt[3]{\frac{E_A}{E_P}}. \qquad (326)$$

The array of Fig. 31 has an E_A in excess of .98, so that practically, six antennas solves the problem with margin to spare.

One can in fact show that the array of Fig. 26 will solve the problem for a 6-inch diameter circle, and the array of Fig. 31 will solve it for a 12-inch diameter circle.

A SECOND EXAMPLE WITH SIX ANTENNAS

Consider the array in Fig. 33.

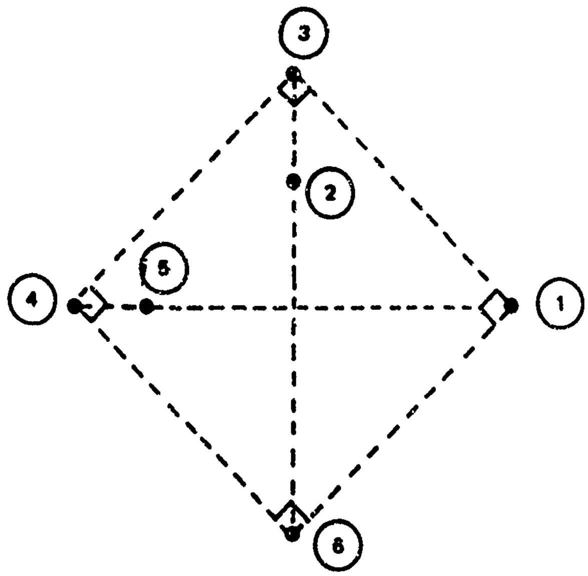

Fig. 33 AN ARRAY

In Fig. 33, antennas ①, ③, ④, ⑥ are arranged on a square, the ratio of the distance from ① to ④ to the distance from ④ to ⑤ is n, an integer > 2, and the ratio of the distance from ③ to ⑥ to the distance from ② to ③ is also n.

Omitting antenna ⑥ leaves a type 4 array of Fig. 25. Proceeding as before, the A_{ijk}'s are given in Table 6 for the case where n is odd.

All of the A_{ijk}'s in Table 6 are to be divided by 2 to give the A_{ijk}'s for the even case.

Table 6

A_{ijk} for Antenna Array of Fig. 33 With n Odd

i	j	k	A_{ijk}
1	2	3	$-n$
		4	$-n(n-2)$
		5	n^2
		6	$-n$
	3	4	$(n-1)(n-2)$
		5	$-n(n-1)$
		6	$n-2$
	4	5	0
		6	$-(n-2)$
	5	6	n
2	3	4	$-n(n-1)$
		5	n^2
		6	0
	4	5	$-n^2$
		6	$n(n-1)$
	5	6	$-n^2$
3	4	5	$n(n-1)$
		6	$-(n-1)(n-2)$
	5	6	$n(n-2)$
4	5	6	n

For n odd, the A_{ij}'s are given in Table 7. To obtain the A_{ij}'s for n even, one divides the entries for A_{ij} in Table 7 by 2.

For n odd, the A_i's are given in Table 8. To obtain the A_i's for n even, divide the entries for A_i in Table 8 by 2.

Table 7

A_{ij} for Antenna Array of Fig. 33 With n Odd

i	j	A_{ij}
1	2	n^2
	3	$n(n-1)$
	4	$n(n-2)$
	5	n^2
	6	$2(n-1)$
2	3	n^2
	4	$2n(n-1)$
	5	$2n^2$
	6	n^2
3	4	$2(n-1)^2$
	5	$2n(n-1)$
	6	$n(n-2)$
4	5	n^2
	6	$n(n-1)$
5	6	n^2

Table 8

A_i for Antenna Array of Fig. 33 With n Odd

i	A_i
1	n^2
2	$2n^2$
3	$2n(n-1)$
4	$2n(n-1)$
5	$2n^2$
6	n^2

Finally, $A = 2n^2$ for n odd and $A = n^2$ for n even.

For n odd, inequalities (302) are

$$\frac{1}{4n^2} \leq T_\varphi \leq \sqrt[3]{\frac{1}{4n(3n-2)}} \leq \sqrt[3]{\frac{1}{2n(5n-2)}} \leq \sqrt[3]{\frac{1}{8n^2}}. \tag{327}$$

For n even,

$$\frac{1}{2n^2} \leq T_\varphi \leq \sqrt[3]{\frac{1}{2n(3n-2)}} \leq \sqrt[3]{\frac{1}{n(5n-2)}} \leq \sqrt[3]{\frac{1}{4n^2}}. \tag{328}$$

If one now lists the equations of (273) and selects the best three (from the tolerance point of view), one finds that they are (after dividing each equation through by the greatest common factor of its coefficients):

$$\mu_1 - \mu_3 + \mu_4 - \mu_6 = 0 \tag{329}$$

$$n\mu_2 - (n-1)\mu_3 - \mu_6 = 0 \tag{330}$$

$$\mu_1 + (n-1)\mu_4 - n\mu_5 = 0 \tag{331}$$

for odd n. Actually, there are other choices for Eqs. (330) and (331), but none has any larger tolerance than Eqs. (330) and (331). If one assumes, as previously conjectured, that one cannot improve tolerancewise by any linear combination of these equations, one then has, for n odd,

$$T_\varphi = \frac{1}{2n}. \tag{332}$$

Note that for this case, Eqs. (330) and (331) are each one-dimensional cases, allowing resolution of non-array ambiguities in two perpendicular directions individually.

Equation (329) ties the two dimensions together, allowing resolution of non-array ambiguities for <u>any</u> antenna pair.

For n even, one can improve slightly on Eqs. (330) and (331). The best three equations in this case are:

$$\mu_1 - \mu_3 + \mu_4 - \mu_6 = 0 \tag{333}$$

$$\frac{n}{2}\mu_2 - \frac{(n-2)}{2}\mu_3 + \frac{(n-2)}{2}\mu_4 - \frac{n}{2}\mu_5 = 0 \tag{334}$$

$$\frac{(n-2)}{2}\mu_1 - \frac{n}{2}\mu_2 + \frac{n}{2}\mu_5 - \frac{(n-2)}{2}\mu_6 = 0. \tag{335}$$

In this case, all other equations have a smaller tolerance than Eqs. (334) and (335).

Again, if one makes the same assumption as made after Eq. (331), one has

$$T_\varphi = \frac{1}{2(n-1)} \tag{336}$$

for n even.

Now suppose that one wished to compare the antenna configurations of Figs. 31 and 33, assuming they have the same tolerance. Then, since $T_\varphi = \frac{1}{22}$, one has n = 11 or n = 12. Suppose also that one assumes that the arrays of Figs. 31 and 33 are constrained to lie on the same circle (except for antennas 2 and 5 in Fig. 33, which will lie in the interior).

Let P_1 and P_2 denote the array ambiguity plots of Figs. 31 and 33, respectively. Then

$$\frac{\text{Area of Fig. 31}}{\text{Area of Fig. 33}} \cdot \frac{T(P_2)}{T(P_1)}$$

$$= \text{ratio of A's for the two cases} = \left.\begin{matrix} \frac{247}{242}, & n = 11 \\ \frac{247}{144}, & n = 12 \end{matrix}\right\} \qquad (337)$$

If d is the diameter of the circle, then

$$\frac{\text{Area of Fig. 31}}{\text{Area of Fig. 33}} = \frac{(.0026)(247)d^2}{0.5d^2} \cong 1.3. \qquad (338)$$

So from Eqs. (337) and (338),

$$\frac{T(P_2)}{T(P_1)} \cong \left.\begin{matrix} 0.8, & n = 11 \\ 1.3, & n = 12 \end{matrix}\right\}. \qquad (339)$$

One would therefore expect the nearest ambiguities to be farthest away for the n = 11 case of Fig. 33 as is the case. The n = 12 case has the closest nearest ambiguities, and the array of Fig. 31 is in between. In this sense, the n = 11 case of Fig. 33 is superior.

One should note however that some pairs of antennas are much closer together in the Fig. 33 array than they are in the Fig. 31 array. This could make a difference for other reasons, for example mutual coupling.

In addition, if the two arrays were compared on the basis of occupying the same area, the array of Fig. 31 would be superior, although only slightly so for the n = 11 case of the array of Fig. 33.

The tolerance efficiency for the array of Fig. 33 is computed to be 45% for n = 11 and 38% for n = 12. This compares with 45% for the array of Fig. 31.

m ANTENNAS ON A LINE, $m \geq 3$

Just as before, the one-dimensional problem for m antennas is a special case of the two-dimensional problem for m + 1 antennas.

Given m antennas on a line, one inserts a fictitious antenna off the line, as in Fig. 34.

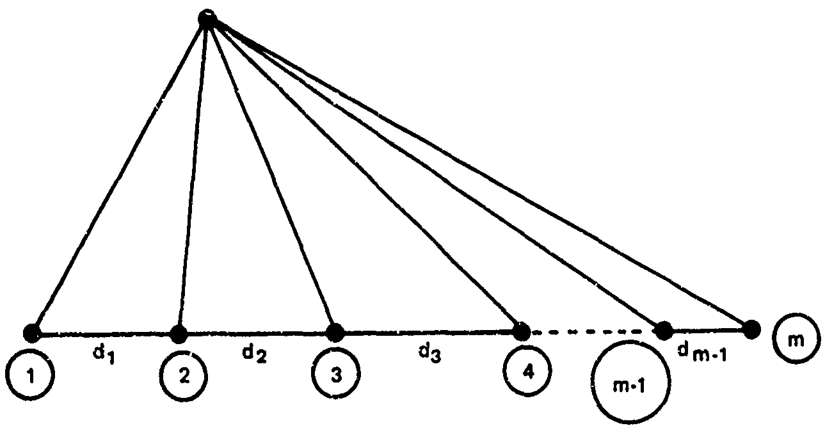

Fig. 34 m ANTENNAS ON A LINE

Let the lengths of the m - 1 line segments be related by

$$d_1 : d_2 : \ldots : d_{m-1} = p_1 : p_2 : \ldots : p_{m-1} \tag{340}$$

where $p_1, p_2, \ldots, p_{m-1}$ are integers with no factor in common.

As before, the ratios of the areas of any pair of triangles depend only on the ratios of corresponding line segments along the base, since all triangles have the same altitude.

Again, let R denote the ratio of the separation of grid lines in the array ambiguity plot for the antennas (1), (2), ..., (m) to that for antennas (1) and (m).

From the discussion in Section 2, it follows that

$$R = \sum_{j=1}^{m-1} p_j. \tag{341}$$

The inequalities (302) carry over (as an established theorem for m = 3, 4, 5 and as a conjecture for $m \geq 6$). The proof for m = 5 is quite similar to the previous proofs for m = 3 and 4, and is omitted.

Ignoring the intermediate inequalities, inequality (302) in this case becomes

$$\frac{1}{2R} \leq T_\varphi \leq \sqrt[m-2]{\frac{1}{(m-1)R}}. \tag{342}$$

The tolerance efficiency is defined as

$$E_\varphi = T_\varphi \left[\sqrt[m-2]{(m-1)R}\right] \tag{343}$$

a number not exceeding unity (a theorem for m = 3, 4, 5, a conjecture for $m \geq 6$). $E_\varphi = 1$ for m = 3, and it is conjectured that it is strictly less than unity for m > 3. It is not known how close one can approach it.

Consider the case where

$$p_j = p^{j-1}, \qquad j = 1, 2, \ldots, m-1 \tag{344}$$

where p is a positive integer > 1.

In this case

$$R = \frac{p^{m-1} - 1}{p - 1}. \tag{345}$$

Equation (342) becomes

$$\frac{p - 1}{2(p^{m-1} - 1)} \leq T_\varphi \leq \sqrt[m-2]{\frac{(p-1)}{(m-1)(p^{m-1}-1)}} . \tag{346}$$

In this case one set of equations in the μ_i's that one could use to resolve ambiguities other than array ambiguities can be shown by the methods of Section 2 and this section to be

$$p\mu_i - (1+p)\mu_{i+1} + \mu_{i+2} = 0, \qquad i = 1, 2, \ldots, (m-2). \tag{347}$$

All of these equations have the same tolerance, so that (assuming a better set of equations is not obtainable for this array)

$$T_\varphi = \frac{1}{2(1+p)}. \tag{348}$$

The tolerance efficiency, assuming the conjecture, is given by

$$E_\varphi = \frac{1}{2(1+p)} \left[\sqrt[m-2]{\frac{(m-1)(p^{m-1}-1)}{p - 1}} \right]. \tag{349}$$

As $m \to \infty$, $E_\varphi \to \frac{p}{2(1+p)}$, a rather interesting result that illustrates that the tolerance ambiguity problem improves exponentially by adding antennas at appropriate locations. In addition, the smaller tolerance one can meet, the larger one can make p, and therefore the fewer number of antennas one needs to accomplish the same expansion factor of the ambiguities.

Another interesting example is the following one. In this case an attempt is made not to let the antennas get too close to each other.

Let p be a positive integer, and

$$p_j = p^{m-j-1} (1+p)^{j-1}, \qquad j = 1, 2, \ldots, m-1 \tag{350}$$

so that

$$R = (p+1)^{m-1} - p^{m-1}. \tag{351}$$

The equations for the μ_i's can be taken to be

$$(1+p)\mu_i - (1+2p)\mu_{i+1} + p\mu_{i+2} = 0, \quad i = 1, 2, \ldots, (m-2). \tag{352}$$

All of these equations have the same tolerance, so that (assuming one can find no better set of equations for this array)

$$T_\varphi = \frac{1}{2(1+2p)}. \tag{353}$$

The tolerance efficiency is then

$$E_\varphi = \frac{(p+1)}{2(1+2p)} \left\{ \sqrt[m-2]{(m-1)(p+1)\left[1 - \left(\frac{p}{1+p}\right)^{m-1}\right]} \right\} \quad (354)$$

which approaches $\frac{p + 1}{2(1+2p)}$ as $m \to \infty$.