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THE USE OF LAGRANGIAN MULTIPLIERS FOR OPTIMUM RELIABILITY ALLOCATION

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February 1972

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RESEARCH REPORT

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by

Robert A. Davison

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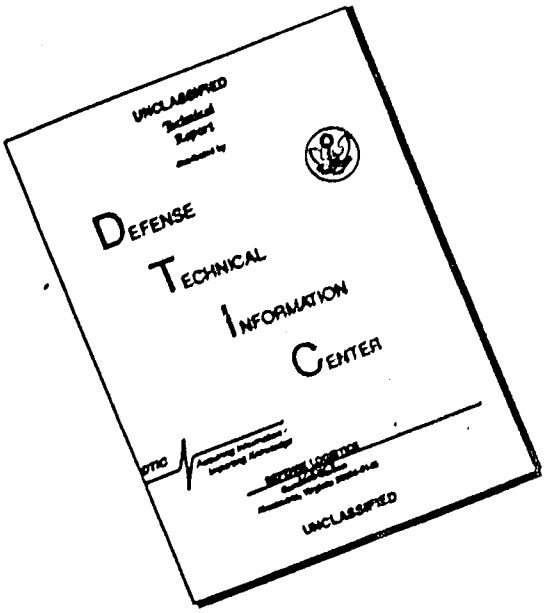

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ABSTRACT

Research Performed by Robert Davison
Under The Supervision of Dr. R. J. McNichols

This paper develops a method to allocate reliability to system elements where a system reliability requirement must be met or a total cost constraint is imposed. Parallel and serially arranged elements are considered with each element's probability density function of time to failure being approximated by the two parameter Weibull distribution. The problem is approached as a minimization of total system cost or system unreliability and Lagrange multipliers prove to be useful as a solution technique. A cost function is developed to relate the cost of and element to the reliability level achieved in that element. Both the element's cost function and reliability expression must be continuous variables. Numerical examples are shown for each allocation model.

ACKNOWLEDGMENTS

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The ideas, concepts, and results herein presented are those of the author(s) and do not necessarily reflect approval or acceptance by the Department of the Army.

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CHAPTER I

INTRODUCTION TO RELIABILITY ALLOCATION

During several phases of design and development of systems for Army use, it is necessary to translate the system's and subsystems' availability, maintainability and readiness requirements into quantitative reliability requirements for the many system sublevels. It is mandatory that both realistic and consistent reliability specifications be assigned to all the components and or subsystems that comprise the desired operational package. The process of this assignment is most commonly termed reliability apportionment or reliability allocation.

The prime factor to be considered in any reliability allocation is that of assuring the total system reliability requirement is met once the component reliabilities are obtained and the system assembled for operation. More frequently than not it is a cumbersome if not impossible task to exactly express the reliability of the system as a function of the components' and subsystems' reliabilities in a mathmatical statement. Several factors contribute to this difficulty. Early in the design phase some subsystem designs are not "frozen" with respect to configuration and complexity. Essentiality of components to system success is not well known. Certain subsystems are only needed intermittantly or for a portion of the mission while others

might be required for the duration. The usage environment is not always well defined and indeed the system might impose additional stress on its components as a consequence of its operation. Where a system utilizes state-of-the-art components, the foregoing problems are compounded by the lack of historical data concerning reliability. To make the reliability allocation even more complicated is one additional constraint; economy. Today, with the financial scrutiny being exercised in all categories of government expenditure, it is important that the most effective system possible be obtained for the resources available.

In the past two decades much work has gone into the development of methods to allocate reliability to system sublevels. Earlier allocation methods concerned themselves solely with the problem of satisfying a reliability or failure rate specification. No specific consideration for cost was reflected in the models used to allocate system reliability or failure rate. Although design specifications might have been met, the expenditure in scarce resources was greater than necessary. As systems became more complex and the costs associated with supporting such systems astronomical, it was evident that resource expenditure must be a variable included in an allocation model. Due to the importance of the cost consideration, more and more emphasis has been given to this quantity in allocation procedures. This increased emphasis has culminated in the use of optimiztion

techniques for seeking and defining the most resource effective methods of designing and developing a piece of equipment. By applying optimization, a best alternative can often be found that will save time, effort and money in the long run. A brief description follows of the more popular optimization methods now in use.

METHODS OF OPTIMIZATION

Of the several techniques used for the optimization of reliability constrained by scarce resources the more popular methods are Lagrange multipliers, linear programming and dynamic programming. As would be expected, each of these methods varies in difficulty of formulation and solution according to the function to be optimized and the nature of the constraint relationships. Here an attempt will be made to give a brief explanation of how each method is employed.

The Lagrange multiplier method is analytical in that it attempts to simultaneously zero all the partial derivatives of the Lagrangian function. A linear combination of the objective and constraint equations is first formed, then the partial derivative is successively taken with respect to the variables of the objective function and the Lagrangian multipliers. The resulting simultaneous equations are solved in terms of the variables of interest which will then optimize the objective function. Two principle difficulties are encountered in this method. The first is the difficulty

associated with the differentiability of the objective and constraint equations, while the second is the simultaneous solution of the thus obtained equations.

The linear programming method considers objective and constraint functions which are linear in nature. The linear constraints expressed as inequalities are made equations by the use of slack variables and are arranged in matrix notation. This matrix is then iterated successively (by an operational algorithm such as the Simplex algorithm) in the direction which tends to optimize the objective function. Other solution techniques are available in linear programming such as graphical solutions and the inverse matrix technique. In the graphical solution, for instance, the region of feasible solutions (those which satisfy the resource constraints) is first found, then the optimum solution(s) for the objective function is examined in this region. There are several drawbacks to the linear programming model. First, the objective and constraint relationships must be linear combinations of the problem variables unless a suitable nonlinear to linear transformation is found. Secondly, the easily understood graphical technique is limited to three variables (three dimensional representation). Lastly, matrix operations in more than three or four variables are quite cumbersome for longhand computation and usually require digital computer solution.

In dynamic programming a return function is defined as

the sum of the values of the alternative functions of return. The value of each of these alternative functions is determined by the amount of resource which is expended on that particular task. In addition it is permissible to have several stages at which portions of the total resource can be expended. Thus, the total return is the sum of the return functions at each stage where resources are expended. The problem is then to optimize the total return function subject to the resources available. Employing existence and uniqueness theroms which have been developed it is possible to make successive approximations to the optimum return and approach a desired optimum policy. The main problem with this technique is that a digital computer is required in all but the simplest cases. Also, the number of numeric iterations required in problems where an iterative algorithm is used is largely dependent on the closeness of the initial approximations to the optimum policy.

In the chapter that follows, a portion of the literature available on solution of the reliability apportionment problem is reviewed. The work in these several papers considers the use of the aforementioned techniques for various cases of optimum apportionment. Chapter III considers the optimal allocation for a series system where the system reliability or total cost is specified. Two cases are considered; elements with a constant failure rate and elements whose failure rates can be described by a Weibull distribution.

CHAPTER II

LITERATURE REVIEW

As was mentioned in the introduction, many allocation methods have evolved from work done in reliability over the past two decades. The scope of these methods ranges from simple to quite complex depending on assumptions made and the variables included in the development. With respect to resources expended in an allocation, the methods so far developed can be divided into two distinct catagories; basic allocations and optimal allocations.

BASIC ALLOCATIONS

Basic allocations are so called because the model used to allocate element reliabilities or failure rates only considers the overall system specification to be met; no consideration is given to the resources expended in meeting this specification.

The most elementary method used is the equal allocation method (1).* Assumptions for this method are:

1. independently failing components and
2. serially arranged components.

If a system reliability goal is defined as R' and this goal

---

* Number in parentheses refers to numbered references in the List of References.

has not been met, then

$$R' > \prod_{i=1}^{n} R_i$$

and the component reliabilities must be set at

$$R_i = \sqrt[n]{R'} .$$

Although this method is straightforward and easily applied, it is only economically suitable when the elements are similar enough that the cost of reliability increase among them is nearly equal. If this is not the case, large sums of money could be spent in increasing the reliability of a very complex component when it could be spent more effectively on the lower cost components.

ARINC (2), has developed a slightly improved allocation technique by defining a weighting factor. Assumptions in the development of this technique are:

1. independently failing, serially arranged components,
2. constant component failure rates and
3. initial failure rate λ, is known for each component.

The weighting factor is defined as

$$\omega_i = \frac{\lambda_i}{\sum^{n} \lambda_i}$$

for n components so that if a system failure rate of λ' is

desired, the new failure rate of the jth component is

$$\lambda_i = \omega_i \lambda' \quad .$$

Thus, the decrement (or increment) of each component failure rate is proportional to its initial value. Two specific problems arise from using the ARINC method. First, if a component failure rate is unreasonably high or low initially, the allocated failure rate will not tend to remedy the situation. Secondly, no consideration is made for the effort to decrease a components failure rate. Less total effort could be involved in arriving at the same system failure rate if other component failure rates were allocated.

A method slightly more complex is the AGREE (7) allocation technique. The assumptions for the use of this method are:

1. constant element failure rates,
2. serially arranged components where the ith component contains n_i elements which are required for time t_i in each mission, and
3. the ith element has conditional probability w_j that its failure will result in system failure.

If overall system reliability R' is desired, the mean life allocated to the ith component is

$$\theta_i = \frac{N\, w_i\, t_i}{n_i(-\ln R')}$$

where

$$N = \sum_{i=1}^{n} n_i \quad .$$

This method is considerably more realistic than the aforementioned techniques. It not only considers the complexity of the component, but also the essentiality and mission operating time. In this sense it does, to a degree, consider the difficulty involved in reducing a component's failure rate. The failure rate is in inverse proportion to the component complexity. If, however, the elements in one component are not similar to the elements in another component, the complexity is not a good comparative measure of the difficulty to decrease failure rate. Additionally, it is hard to define a component's essentiality in the early design phase.

For considering redundancy within subsystems, the NAVWEPS (6) allocation is applicable. Assumption for its use are:

1. subsystems are serially arranged, each with constant failure rate, and
2. the failure probability of each subsystem is independent of other subsystems.

The first step is to reduce the redundant configuration(s) to a series system (i.e., any active element results in system failure). The redundant failure rate is estimated by

$$\lambda_r = \frac{R_s(t)}{R_r(t)} \lambda_s$$

where $R_s(t)$ is the serial system reliability over time t,

$R_r(t)$ is the redundant system reliability and λ_s is the serial system failure rate. If a system reliability of $R'(t)$ is required the reliability of the ith subsystem is given by

$$R_i(t) = R'(t)^{\lambda_i/\lambda_o}$$

where λ_i is either a redundant or series failure rate of a redundant or series subsystem and λ_o is the sum of all subsystem failure rates. This is the same result as obtained by the ARINC (2) technique.

As with the other basic methods, no consideration is given to the effort to increase a susbsystem reliability. In addition it is necessary to have at least a good estimate of the series and redundant reliabilities for a particular time as well as series failure rate. The results, as with some earlier techniques, are directly dependent on prior knowledge of component's reliabilities.

OPTIMAL ALLOCATIONS

Up to this point only basic allocation techniques have been reviewed. Methods which will now be discussed are not basic allocations in that parameters consicered in the allocation are not just those that quantitatively describe the reliability, essentiality or complexity of the systems elements. Procedures have been developed to not only allocate in some manner to satisfy a reliability requirement, but to do this while satisfying some constraint on resource expen-

ditures. Several of these methods will now be discussed.

One of the earliest attempts at finding an optimum allocation with a minimum of resource expenditures was developed by Albert (1). An effort function is assumed that reflects the expenditure (of effort, time, money, etc.) necessary to increase the reliability of a component. The assumptions in the development are:

1. serially arranged, independently failing components,
2. initial component reliabilities are well established,
3. all components share the same effort function, and
4. the effort increases monotonically with component reliability; an increment in component reliability requires as much or more effort at higher initial reliability levels.

With these restrictions, the minimum effort allocation for an n component system is

$$R = \left(\frac{R'}{\prod_{j=1}^{m} R_j} \right)^{1/k}$$

where

R= reliability allocated to components with initial reliability less than $\sqrt[n]{R'}$.

R'= system reliability goal .

$\prod_{j=1}^{m} R_j$= reliability product over all elements with initial reliability greater than $\sqrt[n]{R'}$ ($\prod_{j=1}^{m} R_j$ =1 when no components meet this requirement).

k= number of components that have reliability less than $\sqrt[n]{R'}$.

Where a system is encountered that meets the assumptions of this model, the allocation procedure is simple and straightforward. The assumption of a common effort function is rather restrictive in that this will probably occur only when the components are identical, which is seldom the case.

For considering redundant configurations, Brier (3) has developed an optimization model for switched components in parallel standby. The author considers a system with N identical subsystems connected in a parallel switching arrangement. Each subsystem is composed of M identical elements. The assumptions made in the development are

1. all elements are identical and have constant, independent failure rates,
2. elements do not fail when not energized,
3. successful operation of one element is adequate for subsystem operation,
4. switches between subsystems are identical with time independent probability of success, and
5. switch failure is due only to non-switching when required; premature switching is not considered.

The system begins operation on the first subsystem until all M elements have failed then switches sequentially until all N-1 redundancies are failed or a switch fails. An optimum

N and M are found from the Lagrangian

$$L = (1-P)^{N} \sum_{i=1}^{M} \frac{1}{i} - \sigma N (1+rM)$$

where P is the switch reliability, σ is a Lagrange multiplier, and r is the ratio of element cost to switch cost. From algebraic rearrangement the system mean time between failure can be found from

$$MTBF_{system} = \frac{1}{\lambda} (1-P)^{N-1} \sum_{i=1}^{M} \frac{1}{i} ,$$

where $1/\lambda$ is the MTBF of each element. The total cost is found from the second part of the Lagrangian function and is

$$C \left[N(1+rM) - 1\right] ,$$

where C is the cost of a switch. Since a closed form solution was not obtainable for the above Lagrangian, the author provides a digital computer program and shows a table of optimum redundancies for ranges on the problem variables of $1 \leq N \leq 50$, $1 \leq M \leq 50$, $.95 \leq P \leq .99$ and $.25 \leq r \leq 2.00$. The cost model developed is a powerful tool to the designer considering a switched parallel redundancy. Several drawbacks exist in this model:

1. Failure rates must be constant.
2. All elements and switches must be identical.
3. It is necessary to have access to the author's computer program or tables of optimum redundancies.

A field very much akin to reliability is availability. Availability considers not only reliability or failure rate

of an equipment, but also the time required to bring the equipment from a failed state to an operable state. McNichols and Messer ( ) have considered the minimization of cost to achieve a level of inherent availability. Although not expressly designed for use as a reliability optimization technique, a form of the equation is presented that can be used as a tool for failure rate allocation. The assumptions of the model are

1. serially arranged components with independent failures,
2. each component has a constant failure rate, and
3. the cost of a decrement $\Delta\lambda$ in element failure rate is $C_\lambda\left[\frac{1}{\lambda+\Delta\lambda} - \frac{1}{\lambda}\right]$ where C_λ is a constant.

Inherent availability can be expressed as

$$A = \frac{1}{1 + \sum_{i=1}^{n} M_i \lambda_i}$$

for n elements in series where M_i is the mean time to repair the ith item, and λ_i is the failure rate of the ith item. If we let the M_i equal unity for each element, the inherent availability reduces to

$$A = \frac{1}{1 + \sum_{i=1}^{n} \lambda_i} .$$

For a series system, however, $\sum_{i=1}^{n} \lambda_i$ is the system failure rate. Using a Lagrangian of the form

$$L = \sum_{i=1}^{n}\left[\frac{C_{\lambda_i}}{\lambda_i + \Delta\lambda_i} - \frac{C_{\lambda_i}}{\lambda_i}\right] - K\left[\sum_{i=1}^{n}\Delta\lambda_i - \frac{A - A'}{AA'}\right]$$

where A is the present availability, A' is the desired availability (determined in this case by $1/(1+\lambda_s')$, λ_s' being the desired system failure rate), and K is the Lagrangian multiplier. Rearranging the results of the author, the minimum cost failure rate allocation for the ith element is

$$\lambda_i = \frac{\lambda_s' \sqrt{C_{\lambda_i}}}{\sum_{i=1}^{n}\sqrt{C_{\lambda_i}}} \quad .$$

This allocation is considerably more flexible than that devised by Albert (1) in that each element can have a different cost function. Two possible drawbacks are the necessity of a series assumption and the accuracy of "fitting" the cost function given to the actual cost versus failure rate data. The cost function proposed for the series system in the next chapter is more powerful in that it is a two parameter family of curves and can therefore more accurately describe a general cost versus failure rate curve.

A paper written by Spivey (8) compares and contrasts the results obtained by using three different optimization methods on a specific system configuration. The methods considered are dynamic programming, mathmatical programming and Lagrange multipliers. Due to the complexity of the system considered, a series-parallel network, the author

elected to employ a digital computer solution. The specific model used with each technique will not be repeated here, only the results found. It was determined that the optimum solution for the system considered was found by dynamic programming. The solutions obtained by mathmatical programming and Lagrangian multipliers were within .6% and .5% of the optimum. These errors might be attributible to arithmetic operations in the digital computer or the nature of the model developed.

The following chapter will discuss and develop a series model and some constraints applicable to the allocation problem. For the series configuration, two forms of the cost function are considered. The preference of one development over the other should be based on the adequacy of a particular cost function in describing actual improvement of reliability or failure rate with cost. In Chapter IV an optimum allocation is developed for the parallel system.

CHAPTER III

SERIES SYSTEMS

INTRODUCTION

In this chapter only series systems will be considered. This configuration is a result of the following assumptions:

1. The probability of failure of any element is independent of failures in any other elements.
2. All elements are required to operate for system success.

The first two sections deal with a series system with one additional assumption, constant element failure rates. A specific cost function is developed for which the optimum allocation is found for both a total cost constraint and a system failure rate specification. Constant element failure rates are not assumed for the second series model considered.

CONSTANT FAILURE RATES - TOTAL COST CONSTRAINT

A special case of the series model is where each element displays an exponentially distributed time to failure. When this is true and each element failure is independent of all others, the statement

$$\text{maximize} \quad \prod_{i=1}^{m} R_i$$

is equivalent to

$$\text{minimize} \quad \sum_{i=1}^{m} \theta_i$$

where θ_i is the failure rate of the ith element and R_i is

the reliability of the ith element. In this model the object will be to minimize the system failure rate subject to an overall expenditure in achieving this failure rate. Thus, it is necessary to express the cost of each element as a function of the failure rate achieved in that element. Summing over the costs of these elements, the total system cost can be determined.

One reasonable assumption in expressing the cost of an element as a function of its failure rate is that additional expenditures for failure rate improvement result in diminishing returns of failure rate reduction. One function that displays this characteristic is an exponential cost function in which the cost increases exponentially with a reduction in failure rate. Expressed mathematically, the cost of failure rate θ_i in element i is

$$A_i \, e^{B_i(\theta_{oi}-\theta_i)}$$

where A_i is the cost of obtaining the lowest possible failure rate θo_i in element i, and B_i is the cost gradient. To completely define this cost function, two estimates are required; 1) the cost A at the minimum failure rate obtainable and 2) a cost $\hat{C}$ at the present (or any) failure rate level $\hat{\theta}$. For a cost $\hat{C}$ at failure rate level $\hat{\theta}$, B can be estimated as

$$B = \ln(\hat{C}/A)/(\theta_o-\hat{\theta})$$

With a cost function defined for each element of the series

system, the total cost for m elements is

$$\sum_{i=1}^{m} A_i \, e^{B_i(\theta_{oi}-\theta_i)} \quad .$$

If for the development of this system the upper bound on expenditure is H, the Lagrangian function can be written

$$L = \sum_{i=1}^{m} \theta_i - \lambda\left[\sum_{i=1}^{m} A_i \, e^{B_i(\theta_{oi}-\theta_i)} - H\right] \quad .$$

Taking partial derivatives, the equations to satisfy are

$$1 + \lambda A_i B_i \, e^{B_i(\theta_{oi}-\theta_i)} = 0 \qquad i=1,2,\ldots,m \qquad \text{Eq. 1}$$

and $$\sum_{i=1}^{m} A_i \, e^{B_i(\theta_{oi}-\theta_i)} - H = 0. \qquad \text{Eq. 2}$$

One additional restriction is that

$$\theta_i \geq \theta_{oi} \qquad i=1,2,\ldots,m$$

to insure that a failure rate is not allocated that is better than can be obtained. Summing Equations 1 over all m elements gives

$$\sum_{i=1}^{m} \frac{1}{B_i} + \lambda \sum_{i=1}^{m} A_i \, e^{B_i(\theta_{oi}-\theta_i)} = 0.$$

Substituting Equation 2,

$$\lambda = -\frac{1}{H} \sum_{i=1}^{m} \frac{1}{B_i} \quad .$$

Using this in Equation 1 and solving for θ_i, the optimum failure rate allocation for the ith element is

$$\theta_i = \theta_{oi} + \frac{1}{B_i} \ln \left[\frac{A_i B_i \sum_{i=1}^{m} \frac{1}{B_i}}{H} \right] \geq \theta_{oi} \, . \qquad \text{Eq. 3}$$

In the case where this last inequality is not met by the allocation for one or more elements, the failure rate of these elements is set equal to their respective minimum failure rate. This will occur when the quantity

$$\frac{A_i B_i \sum_{i=1}^{m} \frac{1}{B_i}}{H}$$

is found to be less than one in a particular element. At this point a reallocation must be made to the remaining elements with the remaining resources available for allocation. The new maximum expenditure is

$$H_{new} = H - \sum^{k} A_i$$

for the remaining elements, where the summation is over the k elements set at their minimum failure rate.

EXAMPLE 1

During the design phase of a vibration recording system it is desired to find the most cost effective allocation of failure rate to the system. The system is composed of accelerometer, amplifier and recorder. All elements must function for system operation (series assumption), and element failures are independent. The elements are assumed to have a constant failure rate and the applicable cost parameters are:

	Accelerometer(1)	Amplifier(2)	Recorder(3)
θ_o	.0001	.001	.002
A	$250	$1800	$1400
$\hat{\theta}$	.00025	.003	.007
$\hat{c}$	$100	$900	$500

The total expenditure for system development should not exceed $2,500. What failure rates should be allocated to the system elements?

Using $\hat{\theta}$ and $\hat{c}$ for each element, the cost gradient B_i can be found as B_1=6110, B_2=346, B_3=199. The initial allocation using Equation 3 is θ_1= .000365, θ_2= .00308 and θ_3= .00146. At this point the system failure rate is .004905. However, since θ_3 is less than that obtainable, it is set to its lowest value of .002. This results in an expenditure of $1,400 on element three. Using the allocation equation again with H_{new}= $2,500- $1,400 = $1,100 gives

θ_1= .000337 and θ_2= .00258 .

The design criteria for minimum system failure rate is

	Accelerometer	Amplifier	Recorder
θ	.000337	.00258	.002
Cost	$58.60	$1,041.40	$1,400

System failure rate = .004917

CONSTANT FAILURE RATES - SYSTEM FAILURE RATE CONSTRAINT

The problem considered here is essentially the same as

the one just presented except that in this case allocation must satisfy some requirement on system failure rate. Stated mathematically,

$$\text{minimize} \quad \sum_{i=1}^{m} A_i e^{B_i(\theta_{oi}-\theta_i)}$$

$$\text{subject to} \quad \sum_{i=1}^{m} \theta_i = \theta_s \quad \text{and } \theta_i \geq \theta_s \quad i=1,2,..,m.$$

The Lagrangian function can be written as

$$L = \sum_{i=1}^{m} A_i e^{B_i(\theta_{oi}-\theta_i)} - \lambda\left(\sum_{i=1}^{m} \theta_i - \theta_s\right)$$

where the conditions to satisfy in this case are

$$A_i B_i e^{B_i(\theta_{oi}-\theta_i)} + \lambda = 0 \qquad i=1,2, \ldots,m \qquad \text{Eqs. 4}$$

and

$$\sum_{i=1}^{m} \theta_i - \theta_s = 0. \qquad \text{Eq. 5}$$

Taking the natural logarithm of Equation 4 and summing over all i yields

$$\sum_{i=1}^{m} \theta_i = \sum_{i=1}^{m} \theta_{oi} - \ln(-\lambda)\sum_{i=1}^{m} \frac{1}{B_i} + \sum_{i=1}^{m} \frac{1}{B_i} \ln(A_i B_i).$$

Substituting Equation 5 into this equation and solving in terms of the Lagrange multiplier give

$$\ln(-\lambda) = \frac{\sum_{i=1}^{m} \theta_{oi} + \sum_{i=1}^{m} \frac{1}{B_i} \ln(A_i B_i) - \theta_s}{\sum_{i=1}^{m} \frac{1}{B_i}} .$$

Using Equation 4 with this relationship gives the optimal failure rate allocation to the ith element as

$$\theta_i = \theta_{oi} + \frac{1}{B_i}\left[\ln(A_i B_i) - \ln(-\lambda)\right] \qquad i=1,2,..,m \qquad \text{Eq. 6}$$

and $\theta_i \geq \theta_{oi}$ $i=1,2,\ldots,m$.

Again, if this last inequality is not met by the first allocation, then for those elements for which it is not satisfied the minimum failure rate should be allocated. This situation will occur whenever the quantity A_iB_i is less than $-\lambda$. With these elements allocated, the new system failure rate to satisfy is

$$\theta_{new} = \theta_s - \sum_{i=1}^{k} \theta_{oi}$$

summed over all k elements at their minimum failure rate.

EXAMPLE 2

For the system presented in Example 1, it is decided after further investigation that a system failure rate of .005 would be adequate to meet design specifications. What amount should be spent on each element for development and what failure rates can be expected?

Using the A_i, B_i and θ_{oi} as defined in Example 1, $\ln(-\lambda) = 12.64$. This is greater than $\ln(A_3B_3)$ so that θ_3 is again set to .002. The new failure rate constraint is $\theta_{new} = .005 - .002 = .003$. Allocating .003 between the remaining elements yields

$$\theta_1 = .000330 \qquad \text{and} \qquad \theta_2 = .00267.$$

The new design criteria are

	Accelerometer	Amplifier	Recorder
θ_i	.000330	.00267	.002
Cost	$61.30	$1,010	$1,400

The system failure rate is now .005 at a total cost of $2,471.30.

Because many systems exist which do not exhibit a constant failure rate the section which follows will consider a series system where the elements can have other than a constant failure rate. It is assumed that each elements probability density function can be adequately described by the Weibull distribution.

A MORE GENERAL SERIES MODEL -- DISCUSSION

Up to this point, the only series system considered was one which consisted of elements which all displayed exponentially distributed times to failure. The optimum allocation derived is only valid where the failure rate for each element is constant and the form of the cost function assumed can adequately describe the cost of acheiving a failure rate for an element. Even though the constant failure rate assumption is not too far-fetched for most electronic equipments, it is definitely not applicable to certain other equipments. In practice it is found that certain equipment may display times to failure that are distributed as a normal, gamma, chi-square or other familiar probability density function (p.d.f.). For such equipment, it is believed that money and effort spent in design and development will result in a more reliable piece of equipment. This being the case, it should be possible in certain instances to find the basic

relationship between the level of reliability achieved and the effort expended in the process. In the case of the exponential p.d.f. it is possible to relate the reliability to cost by defining a functional relationship between the failure rate and its associated cost. Such a function was defined in the allocation method derived in the first part of this chapter. Generally, for a p.d.f. other than the exponential or chi-square, it is necessary to specify more than one parameter in order to specify the distribution. The resulting functional relationship between cost and reliability in an element will be more complicated since it may now be a function of all the distribution parameters. At this point if the discussion is restricted to the family of two parameter p.d.f.'s, a cost function for obtaining distribution parameters α and β would be $C(\alpha, \beta)$. An example of such parameters might be the mean and standard deviation of the normal distribution.

Under the assumptions of a series system, the system reliability is the product of the element reliabilities. The system reliability is thus

$$R_s(t) = \prod_{i=1}^{n} R_i(t)$$

for n elements and t hours of operation. This is equivalent to the system failure rate being the sum of the element failure rates, or

$$\theta_s(t) = \sum_{i=1}^{n} \theta_i(t) \ . \qquad \text{Eq. 7}$$

For any p.d.f., the reliability can be written

$$R_i(t) = \exp \quad -\int_0^t \theta_i(u)du \qquad \text{Eq. 8}$$

where $\theta_i(t)$ is the failure rate of the ith element. In order to maximize the system reliability at time t it is necessary to find the minimum value of

$$\int_0^t \theta_s(u)du \ = \int_0^t \left[\sum_{i=1}^{n} \theta_i(u)\right] du \qquad \text{Eq. 9}$$

for the system elements. To greatly simplify the task of minimizing Equation 9 with respect to the distribution parameters of each element, a common and integrable p.d.f. could be used for each element. A p.d.f. which is quite flexible in its ability to approximate other distributions is the Weibull distribution which can be expressed as

$$f(t;\alpha,\beta) = (\beta/\alpha)\ t^{\beta-1} e^{-t^{\beta}/\alpha} \quad , \quad t \geq 0.$$

For judicious choices of α and β, accurate approximations to the normal, exponential, gamma, chi-square, and other distributions can be obtained. The Appendix considers the values of α and β that can be estimated to describe other p.d.f.'s. Since the failure rate of the Weibull distribution can be expressed as

$$\theta(t) = \frac{\beta}{\alpha}\ t^{\beta-1} \qquad \text{Eq. 10}$$

the right hand side of Equation 9 can be written

$$\int_0^t \left[\sum_{i=1}^{n} \theta_i(u)\right] du = \int_0^t \left[\sum_{i=1}^{n} \frac{\beta_i}{\alpha_i} u^{\beta_i - 1}\right] du$$

which when evaluated yields for the system

$$\int_0^t \left[\sum_{i=1}^{n} \theta_i(u)\right] du = \sum_{i=1}^{n} \frac{t^{\beta_i}}{\alpha_i} \qquad \text{Eq. 11}$$

Now that the integral of the system failure rate in the interval of interest (0,t) has been expressed as a function of the time interval and Weibull parameters for each element, it is of interest to define the costs associated with designing and developing the Weibull parameters α and β into each element. Although there are many mathematical possibilities for expressing the cost of achieving parameters α and β in an element as a function of these parameters, it might be helpful to understand how α and β affect the mean and variance of time to failure for the system. It can be seen from Equation 11 that each element's contribution to the value of the integral increases with increasing β and decreases with increasing α. Qualitatively then, increasing α and decreasing β for any element will result in a smaller value of the integral in Equation 11 and thus a higher system reliability. Examination of the equations in the Appendix reveals the changes in the distribution mean and variance with changes in α and β. It is intuitive that for a given element the cost of development will increase if α is increased and/or β is decreased in the Weibull density describing the elements p.d.f. of time to failure.

A cost relationship which is proposed for use in the following analyses is

$$C(\alpha,\beta) = A\alpha^{a}\beta^{-b} \qquad \text{Eq. 12}$$

where A, a and b are positive constants which appropriately describe the cost of a particular element. Equation 12 is flexible enough that a wide variety of actual cost versus reliability curves could be approximated. The Appendix considers methods to approximate parameters in this cost function for an element. One such set of cost curves for cost divided by A versus element reliability is shown in Figure 1 for b=1.0 and a=.25.

ALLOCATION OF α AND β FOR A SERIES SYSTEM---TOTAL COST REQUIREMENT

Using the cost function proposed in the last section, the allocation problem for the series system is to minimize $\sum_{i=1}^{n} \frac{t_o^{\beta_i}}{\alpha_i}$ subject to $\sum_{i=1}^{n} A_i \alpha_i^{a_i} \beta_i^{-b_i} = H$, where H is the maximum permissible cost and n is the number of system elements. The Lagrangian function for this problem is

$$L = \sum_{i=1}^{n} \frac{t_o^{\beta_i}}{\alpha_i} - \lambda\left[\sum_{i=1}^{n} A_i \alpha_i^{a_i} \beta_i^{-b_i} - H\right] .$$

Taking partial derivatives, the equations of interest are

$$\frac{t_o^{\beta_i}}{\alpha_i^2} - \lambda a_i A_i \alpha_i^{a_i-1} \beta_i^{-b_i} = 0 \qquad i=1,2,..,n \qquad \text{Eqs. 13}$$

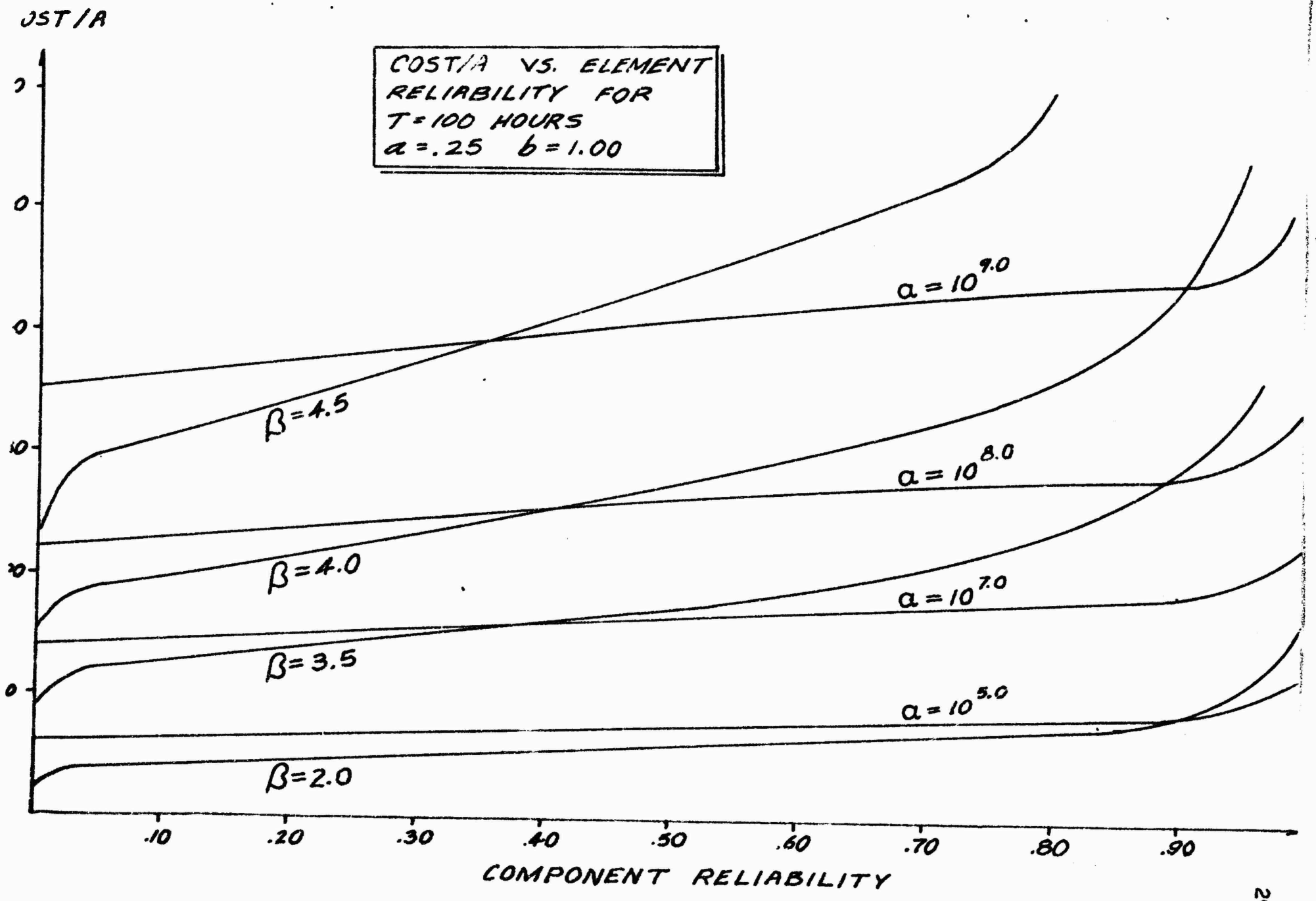
COST/A
COST/A VS. ELEMENT RELIABILITY FOR T=100 HOURS a=.25 b=1.00
a = 10^9.0
a = 10^8.0
a = 10^7.0
a = 10^5.0
β=4.5
β=4.0
β=3.5
β=2.0
.10
.20
.30
.40
.50
.60
.70
.80
.90
COMPONENT RELIABILITY

FIGURE 1

$$\frac{1}{\alpha_i} t_o^{\beta_i} \ln(t_o) + \lambda b_i A_i \alpha_i^{a_i} \beta_i^{-b_i-1} = 0 \quad i=1,2,\ldots,n \qquad \text{Eqs. 14}$$

and $\sum_{i=1}^{n} A_i \alpha_i^{a_i} \beta_i^{-b_i} - H = 0$.

Equations 13 and 14 can be combined to give

$$\beta_i = \frac{b_i}{a_i \ln(t_o)} \quad . \qquad \text{Eq. 15}$$

Substitution of Equation 15 into Equation 13 or 14 yields

$$\alpha_i = \left[\frac{-t_o^{\frac{b_i}{a_i \ln(t_o)}} \left(\frac{b_i}{a_i \ln(t_o)} \right)^{b_i}}{\lambda a_i A_i} \right]^{\frac{1}{a_i+1}}$$

or

$$\alpha_i^{a_i} = \left[\frac{-e^{\frac{b_i}{a_i}} \left(\frac{b_i}{a_i \ln(t_o)} \right)^{b_i}}{\lambda a_i A_i} \right]^{\frac{a_i}{a_i+1}} \qquad \text{Eq. 16}$$

The cost constraint can now be written:

$$H = \sum_{i=1}^{n} -\left(\frac{1}{\lambda}\right)^{a_i/(a_i+1)} \left[A_i \left(\frac{b_i}{a_i \ln(t_o)} \right)^{-b_i} \left(\frac{e^{b_i/a_i}}{a_i} \right)^{a_i} \right]^{\frac{1}{a_i+1}} \qquad \text{Eq. 17}$$

Equation 17 can be solved for the Lagrange multiplier λ by trial and error choice of the multiplier. Once a suitable multiplier is determined it can be used in Equation 16 to find the allocation of α_i to the ith element. If an element is allocated an α and β which is better than can be obtained for that element, the α is set at the highest and β to the lowest possible values for the element. A new allocation is now made over the remaining elements with the remaining funds.

To find the appropriate multiplier in Equation 17 with any degree of accuracy would require a digital computer solution. A situation where the solution of Equation 17 for the multiplier is simplified is considered in the following section.

OPTIMUM ALLOCATION FOR EQUAL a_i

When a situation is encountered in which the a_i's can be set to a common value in the element cost functions, Equation 17 can be solved explicitly for the multiplier. With $a_i = a$ $i=1,2,\ldots,n$, Equation 17 becomes

$$-\left(\frac{1}{\lambda}\right)^{\frac{a}{a+1}} = \frac{H}{\left(\frac{1}{a}\right)^{\frac{a}{a+1}} \sum_{i=1}^{n} \left[A_i \left(\frac{b_i}{a_i \ln(t_o)}\right)^{-b_i} e^{b_i}\right]^{\frac{1}{a}}} .$$

Equation 16 can now be written

$$\alpha_i = -\left(\frac{1}{\lambda}\right)^{\frac{a}{a+1}} \left[\frac{e^{b_i/a} \left(\frac{b_i}{a \ln(t_o)}\right)^{b_i}}{a A_i}\right]^{\frac{1}{a+1}} . \qquad \text{Eq. 18}$$

From Equation 15 the β_i can be written

$$\beta_i = \frac{b_i}{a \ln(t_o)} . \qquad \text{Eq. 19}$$

Some mention should be made of the situation in which the a_i's are equal. One obvious instance is where the p.d.f. of time to failure for all elements is similar enough that the ratio of cost increase to increase in α is approximately equal. It is possible however for cases to occur where the

a_i's are nearly equal although the p.d.f. of time to failure for elements is not the same. The example which follows will illustrate an approach to the case where the p.d.f. of element time to failure is not the same but the a_i's are nearly equal.

EXAMPLE 3

It is desired to find the optimum distribution parameters to allocate three elements which comprise a series system in order to maximize the system reliability for 20 hours operation with the funds available. The p.d.f.'s of time to failure for the first, second and third elements are normal, gamma and exponential. Two estimates of cost versus mean life for the elements are as follows:

Normal	$\sigma^2/\mu^2 = .01$	$\mu_1 = 50$ hrs.	$C_1 = \$1,000$
		$\mu_2 = 75$ "	$C_2 = \$3,000$
Gamma	$\sigma^2/\mu^2 = .0625$	$\mu_1 = 20$ "	$C_1 = \$500$
		$\mu_2 = 30$ "	$C_2 = \$800$
Exponen.	$\sigma^2/\mu^2 = 1.0$	$\mu_1 = 100$ "	$C_1 = \$200$
		$\mu_2 = 150$ "	$C_2 = \$220$.

What α and β parameters should be allocated each element if total cost should be $1,500?

Using the methods of the Appendix to estimate the cost parameters, the cost functions for elements one, two and three can now be written

$$e^{16.47}\, a_1^{.226}\, \beta_1^{-8.20}, \qquad e^{8.20}\, a_2^{.226}\, \beta_2^{-3.49}$$

and $$e^{4.25} \; \alpha_3^{.228} \; \beta_3^{-.685} .$$

Using Equation 18, the optimal α_i are found to be

$$\alpha_1 = e^{40.55} \qquad \alpha_2 = e^{18.15} \qquad \alpha_3 = e^{7.137} .$$

Substituting these into their appropriate cost equations, the amount to be spent on components one, two and three is

$$C_1 = \$202 \qquad C_2 = \$940 \qquad C_3 = \$360 ,$$

for a total cost of $1,502. The resultant mean life of each element will be

$$\mu_1 = 31.2 \text{ hrs.} \qquad \mu_2 = 34.7 \text{ hrs.} \qquad \mu_3 = 1,250 \text{ hrs.}$$

The system reliability can be written as

$$\exp \; -\sum_{i=1}^{n} \frac{t^{\beta_i}}{\alpha_i} ,$$

so for this allocation and t_o = 20 hours, the system reliability is .933. An improved allocation could now be accomplished by making new cost parameter estimates in the region of the initial allocation.

The section which follows considers a problem very similar to that just discussed. Now, however, the system reliability will be held constant and minimum cost found to satisfy this reliability.

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MINIMUM COST FOR SYSTEM RELIABILITY REQUIREMENT

The last section considered maximizing system reliability subject to a total expenditure. A typical problem analagous to this is finding the minimum design and development

cost associated with meeting the system reliability requirement. Suppose a system reliability requirement of $R_s(t_o)$ is specified for operating time t_o. For a series system then

$$R_s(t_o) = \exp -\sum_{i=1}^{n} \frac{t_o^{\beta_i}}{\alpha_i}$$

so that

$$\sum_{i=1}^{n} \frac{t_o^{\beta_i}}{\alpha_i} = - \ln R_s(t_o) \quad .$$

In the following discussion, the term $- \ln R_s(t_o)$ will be considered the requirement to meet and will be denoted by K. Therefore $R_s(t_o)$ is equal to exp(-K). For the system composed of n elements in series the Lagrangian function is

$$L = \sum_{i=1}^{n} A_i \alpha_i^{a_i} \beta_i^{-b_i} - \lambda \left[\sum_{i=1}^{n} \frac{t_o^{\beta_i}}{\alpha_i} - K \right]$$

where the object is now to minimize the total cost term

$$\sum_{i=1}^{n} A_i \alpha_i^{a_i} \beta_i^{-b_i} \quad .$$

The equations resulting from partial differentiation are

$$a_i A_i \alpha_i^{a_i - 1} \beta_i^{-b_i} + \lambda \frac{t_o^{\beta_i}}{\alpha_i^2} = 0 \qquad i=1,2,..,n \qquad \text{Eqs. 19}$$

$$b_i A_i \alpha_i^{a_i} \beta_i^{-b_i - 1} + \lambda \frac{t_o^{\beta_i}}{\alpha_i} \ln (t_o) \qquad i=1,2,..,n \qquad \text{Eqs. 20}$$

and

$$\sum_{i=1}^{n} \frac{t_o^{\ i}}{\alpha_i} - K = 0.$$

Solution of Equations 19 and 20 for β_i gives

$$\beta_i = \frac{b_i}{a_i \ln(t_o)} .$$

From this it is seen that the optimal level for β in an element is the same whether optimizing in terms of total cost or system reliability and is only a function of the cost parameters a and b and the operating time t_o. Equation 19 can be used with the equation for β_i to find the α_i as

$$\left[\frac{-\lambda\left(\frac{b_i}{a_i \ln(t_o)}\right)^{b_i} e^{b_i/a_i}}{a_i A_i}\right]^{\frac{1}{a_i+1}} .$$

The reliability constraint can now be written

$$\sum_{i=1}^{n} (-\lambda)^{\frac{-1}{a_i+1}} \left(e^{b_i/a_i}\right)^{\frac{a_i}{a_i+1}} \left[\frac{\left(\frac{b_i}{a_i \ln(t_o)}\right)^{b_i}}{a_i A_i}\right]^{\frac{-1}{a_i+1}} = K. \qquad \text{Eq. 21}$$

From this relationship a trial and error technique can be used to find the required multiplier. As before an accurate allocation would require a digital computer solution. Once the multiplier is determined, the α_i can be found from

$$\alpha_i = (-\lambda)^{\frac{1}{a_i+1}} \left[\frac{\left(\frac{b_i}{a_i \ln(t_o)}\right)^{b_i} e^{b_i/a_i}}{a_i A_i}\right]^{\frac{1}{a_i+1}} . \qquad \text{Eq. 22}$$

The following section deals with the case of equal a_i and the resulting straightforward solution.

ALLOCATION FOR EQUAL a_i

Inspection of Equation 21 reveals that when the a_i are equal the α_i can be found directly. Letting all a_i equal

the common value a, Equation 21 can be written

$$(-\lambda)^{\frac{-1}{a+1}} = \frac{K}{\sum_{i=1}^{n} e^{b_i/a+1} \left[\frac{\left(\frac{b_i}{a_i \ln(t_o)}\right)^{b_i}}{a\ A_i}\right]^{\frac{-1}{a+1}}} .$$

The allocation for α and β in the ith element becomes

$$\alpha_i = \left[\frac{\left(\frac{b_i}{a\ \ln(t_o)}\right)^{b_i} e^{b_i/a}}{a\ A_i}\right]^{\frac{1}{a+1}} \frac{\sum_{i=1}^{n} e^{b_i/a+1} \left[\frac{\left(\frac{b_i}{a\ \ln(t_o)}\right)^{b_i}}{a\ A_i}\right]^{\frac{-1}{a+1}}}{K}$$

$$\beta_i = \frac{b_i}{a\ \ln(t_o)} . \qquad \text{Eqs. 23}$$

EXAMPLE 4

For the system of Example 3 it was found that the allocation resulted in a system reliability of .933 for 20 hours operation at a cost of $1,500. What savings could be made if the system reliability requirement for 20 hours operation was .90 (K=.1052)?

Using the cost functions as developed for elements one, two and three, Equation 23 gives the following α allocations:

$$\alpha_1 = e^{40.13} \qquad \alpha_2 = e^{17.73} \qquad \alpha_3 = e^{6.717} .$$

These values yield a system reliability of .900. The resulting element costs are:

$$C_1 = \$181 \qquad C_2 = \$850 \qquad C_3 = \$325 .$$

The new total cost is $1,356; a savings of $144.

Analysis of the type just presented could be a valuable tool to examine cost effective reliability specifications.

A trade-off could be performed to determine a suitable balance between cost and achieved reliability. In this way a more effective system could be defined.

Chapter 3 has restricted the discussion to series systems. The first part considered the allocation when the elements displayed a constant failure rate. The second part of this chapter considered the series system where the failure rate of each element was not necessarily constant, but the distribution of times to failure for each element could be adequately described by the Weibull distribution. An optimum allocation for a parallel system will be treated in the next chapter. The Weibull approximation will again be used to aid in the mathematical development.

CHAPTER IV

PARALLEL SYSTEMS

INTRODUCTION

This chapter will be devoted to the consideration of parallel systems. For the purpose of discussion, a parallel system will be defined here as one which requires the successful operation of at least one element of two or more elements in active redundancy. It is also assumed that the failure of any element or elements in the redundancy places no additional stress on the remaining elements. With the restrictions stated above, the system reliability for time t_o can be written

$$R_s(t_o) = 1 - \prod_{i=1}^{m} (1-R_i(t_o)),$$

where $1-R_i(t_o)$ is the unreliability of the ith element of m in active redundancy. If the object is to maximize system reliability for time t_o, then it is necessary to minimize the quantity $\prod_{i=1}^{m} (1-R_i(t_o))$ for the system elements.

Since for Weibull distributed elements the unreliability can be expressed as

$$1 - e^{t_o^{\beta}/a},$$

the object is to minimize

$$\prod_{i=1}^{m} (1- e^{t_o^{\beta_i}/a_i}) \qquad \text{Eq. 24}$$

where, as in the last chapter, the total expenditure con-

straint is

$$\sum_{i=1}^{m} A_i \alpha_i^{a_i} \beta_i^{-b_i} = H. \qquad \text{Eq. 25}$$

If Equations 24 and 25 are used to form the Lagrangian function, the equations which result from partial differentiation are simultaneous, non-linear in the α_i. These equations are not ameable to closed form solution. One possibility in reducing the complexity of these equations is to make the first order approximation

$$\exp -t_o^{\beta}/\alpha = 1 - t_o^{\beta}/\alpha \ .$$

For element reliabilities greater than .72, this results in less than 5% error. With this approximation the term to minimize for the system is

$$\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} \ .$$

Forming the Lagrange function as

$$\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} - \lambda\left[\sum_{i=1}^{m} A_i \alpha_i^{a_i} \beta_i^{-b_i} - H\right] \ ,$$

the equations to satisfy are

$$-\frac{1}{\alpha_i} \prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} - \lambda a_i A_i \alpha_i^{a_i - 1} \beta_i^{-b_i} = 0 \qquad i=1,2,..,m \qquad \text{Eqs. 26}$$

$$\ln(t_o) \prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} + \lambda b_i A_i \alpha_i^{a_i} \beta_i^{-b_i - 1} = 0 \qquad i=1,2,..,m \qquad \text{Eqs. 27}$$

and $$\sum_{i=1}^{m} A_i \alpha_i^{a_i} \beta_i^{-b_i} - H = 0. \qquad \text{Eq. 28}$$

Combining Equations 26 and 27 yields

$$\beta_i = \frac{b_i}{a_i \ln(t_o)} \qquad \text{Eq. 28}$$

which is a familiar result. Summing Equation 26 over all m elements gives

$$\left[\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i}\right]\left[\sum_{i=1}^{m} \frac{1}{a_i}\right] + \lambda \sum_{i=1}^{m} a_i \alpha_i^{a_i} \beta_i^{-b_i} = 0.$$

With $\prod_{i=1}^{m} A_i \alpha_i^{a_i} \beta_i^{-b_i}$

replaced by H, the multiplier λ is

$$\frac{-\left[\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i}\right]\left[\sum_{i=1}^{m} \frac{1}{a_i}\right]}{H} .$$

This multiplier in Equation 27 yields:

$$\ln(t_o)\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} - \frac{\left[\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i}\right]\left[\sum_{i=1}^{m} \frac{1}{a_i}\right]}{H} b_i A_i \alpha_i^{a_i} \beta_i^{-b_i-1} = 0.$$

Rearranging in terms of α_i,

$$\alpha_i = \left[\frac{H \ln(t_o) \beta_i^{b_i+1}}{b_i A_i \sum_{i=1}^{m} \frac{1}{a_i}}\right]^{\frac{1}{a_i}} .$$

Since $\beta_i = b_i / a_i \ln(t_o)$,

$$\alpha_i = \left[\frac{H (b_i)^{b_i}}{A_i a_i^{b_i+1} \ln(t_o)^{b_i} \sum_{i=1}^{m} \frac{1}{a_i}}\right]^{\frac{1}{a_i}} , \qquad \text{Eq. 29}$$

which is the optimal α allocation for each element when m different elements comprise the system.

EXAMPLE 5

A design is being considered for a programmable electronic calculator. Because a problem can be programmed or inputed via a manual keyboard, the programmer and keyboard are considered to be parallel elements. Either element can fail and a problem can still be processed using the other element as an input device. The cost function for the electronic programmer has been estimated as

$$C_1 = e^{5.0} \alpha_1^{.25} \beta_1^{-1.55} .$$

The cost function for the keyboard has similarly been estimated as

$$C_2 = e^{4.0} \alpha_2^{.30} \beta_2^{-4.5} .$$

If \$2,000 is to be spent on the input system, what α and β should be allocated to each element to maximize reliability for 500 hours operation?

Using Equation 28 with the above cost parameters, the optimum β_i for elements one and two are

$$\beta_1 = 1.00 \quad \text{and} \quad \beta_2 = 2.42 .$$

The optimum α_i can be found by Equation 29 and are

$$\alpha_1 = e^{8.91} \quad \text{and} \quad \alpha_2 = e^{22.65} .$$

These α and β substituted in their respective cost functions

yield $C_1 = \$1,070$ and $C_2 = \$930$.

The system unreliability is $\prod_{i=1}^{m}(t_o^{\beta_i}/\alpha_i)$, or .000076 for 500 hours operation.

MINIMUM COST FOR RELIABILITY REQUIREMENT

The object of the following analysis is to minimize

$$\sum_{i=1}^{m} A_i \alpha_i^{a_i} \beta_i^{-b_i}$$

when a system reliability requirement must be met. If the system reliability requirement is $R_s(t_o) = 1 - K(t_o)$ for t_o hours operation, the object is to minimize

$$\sum_{i=1}^{m} A_i \alpha_i^{a_i} \beta_i^{-b_i} , \qquad \text{subject to}$$

$$\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} = K(t_o).$$

For this case the Lagrangian is

$$\sum_{i=1}^{m} A_i \alpha_i^{a_i} \beta_i^{-b_i} - \lambda\left[\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} - K(t_o)\right]$$

so that the equations to satisfy are

$$a_i A_i \alpha_i^{a_i-1} \beta_i^{-b_i} + \frac{\lambda}{\alpha_i} \prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} = 0 \qquad i=1,2,..,m , \qquad \text{Eqs. 30}$$

$$b_i A_i \alpha_i^{a_i} \beta_i^{-b_i-1} + \lambda \ln(t_o) \prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} = 0 \qquad i=1,2,..,m , \qquad \text{Eqs. 31}$$

and $$\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} - K(t_o) = 0.$$

As before, combining Equations 30 and 31 results in

$$\beta_i = \frac{b_i}{a_i \ln(t_o)} \qquad i=1,2,\ldots,m \quad .$$

Substituting the reliability constraint in equation 30 and multiplying by $t_o^{\beta_i a_i}$

gives

$$\frac{t_o^{\beta_i}}{\alpha_i} = \left(-\frac{1}{\lambda}\right)^{1/a_i} \left[\frac{a_i A_i t_o^{a_i \beta_i}}{K(t_o)\beta_i^{b_i}}\right]^{1/a_i} \quad .$$

Taking the product of this expression over all m elements yields

$$\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i} = K(t_o) = \prod_{i=1}^{m} \left(-\frac{1}{\lambda}\right)^{\frac{1}{a_i}} \left[\frac{a_i \quad A_i t_o^{a_i \beta_i}}{K(t_o)\beta_i^{b_i}}\right]^{\frac{1}{a_i}}$$

or

$$[-\lambda]^{\sum_{i=1}^{m} \frac{1}{a_i}} = \frac{1}{K(t_o)} \prod_{i=1}^{m} \left[\frac{a_i^{b_i+1} A_i e^{b_i} (\ln(t_o))^{b_i}}{K(t_o)\, b_i^{b_i}}\right]^{\frac{1}{a_i}} \quad .$$

The optimum multiplier is thus

$$-\lambda = \left[\frac{1}{K(t_o)} \prod_{i=1}^{m} \left[\frac{a_i^{b_i+1} A_i e^{b_i} (\ln(t_o))^{b_i}}{K(t_o)\, b_i^{b_i}}\right]^{\frac{1}{a_i}}\right]^{\frac{1}{\sum_{i=1}^{m} \frac{1}{a_i}}} \qquad \text{Eq. B 32}$$

With this result the optimum α for the ith element can be found as

$$(-\lambda)^{\frac{1}{a_i}} \left[\frac{K(t_o)\ b_i^{\ b_i}}{A_i a_i^{\ b_i+1} (\ln(t_o))^{b_i}} \right]^{\frac{1}{a_i}} \qquad \text{Eq. 33}$$

where t_o refers to the time for which the system reliability has been specified.

EXAMPLE 6

For the programmable calculator of Example 5, what minimum cost could be expected to achieve a .99 reliability for 500 hours operation? What α and β should be allocated the two elements?

The optimum level for the β_i can again be found from Equation 26 and are

$$\beta_1 = 1.00 \qquad \text{and} \qquad \beta_2 = 2.42 \ .$$

With a K(500) value of .01, the optimum multiplier $-\lambda$ can be found in Equation 32 and is exp 9.575. The resultant allocation for α_1 and α_2 are exp(5.42) and exp(20.38) from Equation 33. This results in an unreliability of .0046 for element one and 2.22 for element two. In the development of the allocation for the parallel system no restriction was imposed on the value of unreliability an element could be allocated. However a value of unreliability greater than one is physically meaningless. When this occurs a close look should be taken at the elements envolved to see whether or not they are actually required in the system. If so, the lowest reliability permissable should be allocated these

elements in order to maximize overall system reliability. For the example under discussion it is decided that .80 is the lowest admissible reliability for element two; it is necessary for element one to have .95 reliability at 500 hours to satisfy the system requirement. With the β_i held constant the new α_i are

$$\alpha_1 = e^{7.82} \quad \text{and} \quad \alpha_2 = e^{16.00} .$$

These α and β result in element costs of $1,030 for element one and $230 for element two for a system cost of $1,260.

For the situation where more than two elements comprise the system and one is allocated an unadmissable level of unreliability the analysis would proceed as follows. The elements which were allocated too large an unreliability would be set to their minimum reliability. The new system reliability requirement could be found by factoring out the unreliability due to the elements set at their minimum reliability. A new allocation can now be made over the remaining elements.

OPTIMAL REDUNDANCY FOR IDENTICAL ELEMENTS

The situation is often encountered where identical elements are arranged in active redundancy to improve the system reliability. When this is the case it might be advantageous to know how many elements should be used to achieve the most reliable system for a fixed expenditure H.

If a cost function of the form $A\alpha^{a}\beta^{-b}$ is known for these elements, it is possible to find the optimum redundancy m. For m elements comprising the system, the unreliability

$$\prod_{i=1}^{m} \frac{t_o^{\beta_i}}{\alpha_i}$$

reduces to

$$\left[\frac{t_o^{\beta}}{\alpha}\right]^{m}$$

for identical elements. The cost associated with this system is

$$mA\alpha^{a}\beta^{-b} .$$

Thus for this case the Lagrangian may be formed as

$$\left[\frac{t_o^{\beta}}{\alpha}\right]^{m} - \lambda\left(mA\alpha^{a}\beta^{-b} - H\right) .$$

The simultaneous equations to consider are

$$\left[\frac{t_o^{\beta}}{\alpha}\right]^{m} \ln\left[\frac{t_o^{\beta}}{\alpha}\right] - \lambda A\alpha^{a}\beta^{-b} = 0 , \qquad \text{Eq. } 34$$

$$\frac{m}{\alpha}\left[\frac{t_o^{\beta}}{\alpha}\right]^{m} + \lambda amA\alpha^{a-1}\beta^{-b} = 0 , \qquad \text{Eq. } 35$$

$$m \ln(t_o)\left[\frac{t_o^{\beta}}{\alpha}\right]^{m} + bmA\alpha^{a}\beta^{-b-1} = 0 \qquad \text{Eq. } 36$$

and $\quad mA\alpha^{a}\beta^{-b} - H = 0 .$

Simultaneous solution of Equations 35 and 36 in terms of β

yields

$$\beta = \frac{b}{a \ln(t_o)} \, . \qquad \text{Eq. 37}$$

Equations 34 and 35 can be combined to give

$$\ln\left[\frac{t_o^{\beta}}{\alpha}\right] = -\frac{1}{a}$$

or

$$\alpha^{a} = e t_o^{a\beta} = e^{b+1} \, . \qquad \text{Eq. 38}$$

Using the cost constraint with the optimum α and β, the expression for the optimum redundancy can be written as

$$m = \frac{H\, b^{b}}{A e^{b+1}\, a^{b} \left(\ln(t_o) \right)^{b}} \, . \qquad \text{Eq. 39}$$

There is no guarantee that m will assume an integer value in Equation 39 although anything but an integer number of redundant elements has no physical significance. Unless the optimum m is found to be and integer, the integer just smaller and just larger should be tested to see which is the true optimum. This is accomplished by dividing the total expenditure into equal parts for each element at each integer tested. Using Equation 37 and the cost equation for each element the optimal α and β can be found. With those two parameters known for each element the system reliability can be determined and the largest one chosen for the integers tested.

EXAMPLE 7

A multi-engine, all terrain vehicle is being considered

where the estimated cost of achieving Weibull parameters α and β for each power train is expressed as

$$c = e^{8.5} \alpha^{.5} \beta^{-10.0} .$$

For preliminary analysis, one power train operating successfully is considered adequate for mission success. If it is decided that $20,000 should be spent on the power train(s), what is the optimum number of power trains and what power train reliability can be expected for 1000 hours operation?

The optimum α , β and m can be found from Equations 37, 38 and 39 with the cost parameters given above. The required values are

$$\beta = 2.90 \qquad \alpha = e^{22.0} \qquad m = 2.72 .$$

It is necessary to test integers on both sides of m to see which yields the lowest value of unreliability. With m equal to 2, each element has a reliability of .933 for a system reliability of .9955. With m equal to 3, the element reliability is .848 for a system reliability of .9964. Therefore three power trains is the optimal level of redundancy from a cost standpoint. Figure 2 illustrates the variation in system reliability with the number of redundant elements when the total expenditure is $20,000.

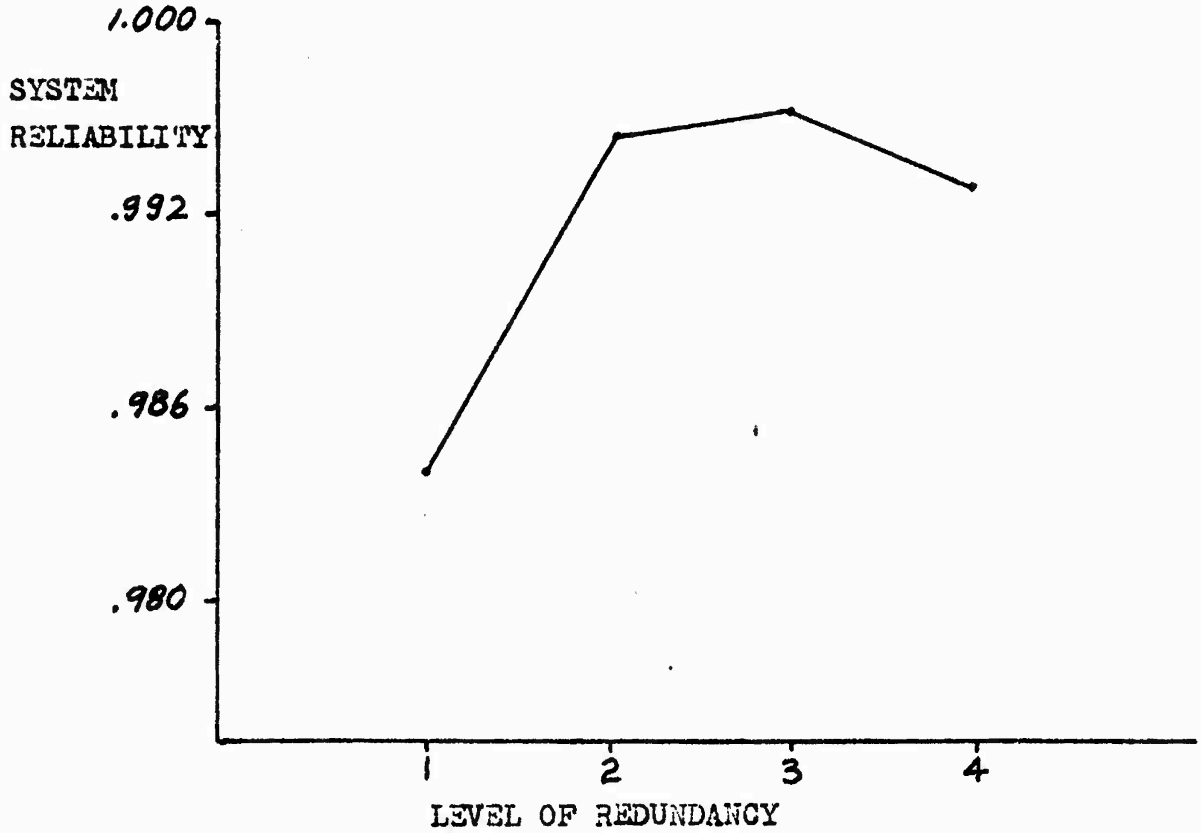
1.000
SYSTEM
RELIABILITY
.992
.986
.980
1
2
3
4
LEVEL OF REDUNDANCY

FIGURE 2

CHAPTER V

SUMMARY AND CONCLUSIONS

A cost based optimum reliability allocation technique has been developed. Two basic configurations have been considered for allocation, series systems and parallel systems. The technique is based on the minimization of system unreliability subject to a total expenditure to achieve system reliability improvement. The dual of this problem is also considered, namely the minimum expenditure necessary to achieve a predetermined system reliability requirement. The applicability of this technique is contingent on the accuracy with which each element's probability density function can be approximated by the two parameter Weibull distribution. In addition it is necessary that parameters can be found for the propo[illegible] function which discribes the cost associated with achieving a reliability level in an element. A method is outlined in the Appendix for estimating parameters for the cost function. The solution to the reliability allocation problem is found by the application of the Lagrange multiplier method.

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For the case of the series configuration, two specific systems are treated. The first is the system where each element has an exponential distribution of time to failure (constant failure rate), and the second is where the p.d.f. of time to failure for each element can be adequately

described by the Weibull distribution. Where the elements have a constant failure rate, an optimum failure rate allocation is shown where the total expenditure is specified or the system failure rate is specified. For the case where all the elements have a Weibull failure rate a closed form solution is not possible with the proposed cost function. An iterative technique is required to find a suitable multiplier and thus the optimum allocation. A modification of the cost function is presented which allows the explicit solution of optimum Weibull parameters for each element and thus the money to be spent for the improvement of each element. To illustrate the use of the allocation technique a numerical example is shown for each section. Since the exponential p.d.f. is a special case of the Weibull p.d.f., either approach can be employed to allocate to a constant failure rate, series system. The choice should depend on the accuracy of the cost function in defining true cost versus element failure rate.

For the parallel system a method is developed to allocate Weibull parameters to each element to minimize system unreliability for a given expenditure. As with the series system it is assumed that the p.d.f. of each element can be adequately described by the Weibull distribution. The cost function is the same as used with the previous series model. The dual of the above problem is also considered where a system reliability requirement must be met.

One additional situation is considered; finding the optimum number of identical elements in parallel to maximize system reliability with an imposed expenditure. To allow an analytical solution of the allocation problem a first order approximation is made for element reliability. This approximation results in less than 5% error for element reliabilities greater than .72. Following each development is a numerical example.

One general conclusion that can be made regarding reliability allocation is that for anything but very simple system configurations the mathematics involved becomes quite complicated. Systems of any complexity (series-parallel, dependent element) in most cases require a dynamic programming approach and subsequent digital computer solution. Reliability allocation is an important step in the conceptual, development and design phases of a system. As such it is a task which must be performed. If a model is available which can be used to treat the specific system under consideration, it should be tested.

SUGGESTIONS FOR FURTHER RESEARCH

Work on the development of the models presented in this paper could be expanded in several directions.

In the series model using the Weibull approximation, it was found that explicit analytical solution was not possible for the optimum Weibull parameters. An iterative technique is required to locate the optimum Lagrange multiplier. A

digital computer program would enhance the usefulness of the allocation technique and serve as a tool for sensitivity analysis and for study of alternative expenditure policies. Solution is not limited to the digital computer. The simultaneous equations resulting from the Lagrangian function could be solved by the use of an analog computer and sensitivity of the optimum allocation could be studied by the variation of problem parameters.

The optimum allocation for the parallel system was found only after a first order approximation was made on element reliabilities. Perhaps an analytical solution exists for this problem which does not necessitate this simplifying assumption. If not, it would be worth while to test the closeness of the allocation developed in this paper to the true optimum. This could be done by solving the original non-linear, simultaneous equations of the Lagrangian by an iterative technique. Again, a digital computer could be employed to solve these original simultaneous equations.

One further area of research is in defining suitable parameters for the proposed cost function. The method presented in the Appendix is very limited in that it is necessary to assume a fixed ratio between elements mean life and standard deviation of time to failure. This is equivalent to holding the beta or shape parameter in the Weibull distribution constant. This might not be a realistic cost versus reliability assumption for certain elements. A

valuable contribution to reliability work would be the research and cataloging of distributional forms of various system elements and components, and the cost associated with the reliability levels achieved. Such a listing would greatly aid in the development of more accurate cost functions for reliability allocation.

APPENDIX

APPENDIX

ESTIMATION OF WEIBULL PARAMETERS

The accuracy of the optimum allocation models that were presented in Chapters III and IV are strongly dependent on two assumptions. The first is the assumption that the p.d.f.'s of the system elements can be adequately described by the Weibull p.d.f. If a Weibull distribution can be found it is still necessary to define a cost function which relates the achievement of improved reliability to the cost for improvement. The key to overcoming both of these problems lies in the accurate estimation of the Weibull parameters, alpha and beta, for an element's distribution. Several techniques are available for finding suitable Weibull parameters to describe a given distribution. Three possible techniques are 1) equating the mean and variance of the Weibull p.d.f. to the corresponding parameters of the element's distribution, 2) defining alpha and beta such that the area under the element's p.d.f. is equivalent to the area under the Weibull p.d.f. for some time period and 3) use maximum liklihood estimators with points from the element's distribution. Each of these methods will be discussed presently.

A very straightforward approach to obtaining estimates of alpha and beta is to equate distribution means and variances. For a Weibull distribution of the form

$$f(t;\alpha,\beta) = \frac{\beta}{\alpha} t^{\beta-1} e^{-t^{\beta}/\alpha},$$

the mean (μ) and variance (σ^2) can be written as

$$\mu = \alpha^{1/\beta}\Gamma\left[\frac{\beta+1}{\beta}\right]$$

and $$\sigma^2 = \alpha^{2/\beta}\left(\Gamma\left[\frac{\beta+2}{\beta}\right] - \left(\Gamma\left[\frac{\beta+1}{\beta}\right]\right)^2\right) .$$

These equations when rearranged in terms of the Weibull parameters become

$$\frac{\Gamma\left[\frac{\beta+2}{\beta}\right]}{\left(\Gamma\left[\frac{\beta+1}{\beta}\right]\right)^2} = 1 + (\sigma/\mu)^2$$

and $$\alpha = \left(\frac{\mu}{\Gamma\left[\frac{\beta+1}{\beta}\right]}\right)^{\beta} .$$

Thus if the ratio of standard deviation to mean is known, beta can be estimated. With beta and the distribution mean the second equation can be used to estimate alpha.

Another possibility for estimating an alpha and beta is to equate area under the Weibull p.d.f. for a given time interval to the area under the element's p.d.f. If for time t_i the system element has reliability R_i, an equation of the form

$$e^{-t_i^{\beta}/\alpha} = R_i$$

can be found for all points $R_i(t_i)$ along the reliability curve. At least two data points are required to find estimates of alpha and beta. For each pair of points the two

simultaneous equations

$$e^{-t_1^{\beta}/\alpha} = R_1$$

and

$$e^{-t_2^{\beta}/\alpha} = R_2$$

are solved. The result of this solution is

$$\beta = \frac{\ln\,(\ln R_1/\ln R_2)}{\ln\,(t_1/t_2)}$$

and

$$\alpha = \frac{-t_1}{\ln R_1} \quad .$$

If more than two data points are available for parameter estimation, the points can be taken in all combinations of two at a time and equations solved for several estimates of alpha and beta. For n data points it is possible to find $n!/((n-2)!(2)!)$ estimates of the Weibull parameters. An overall estimate of these parameters can be made from the average of the so obtained estimates.

Maximum liklihood estimators can be used to estimate alpha and beta if the distribution of the element times to failure is initially assumed to be a Weibull p.d.f. For a sample of n times to failure (t_i) the maximum liklihood estimators for alpha and beta are (4)

$$n\alpha - \sum_{i=1}^{n} t_i^{\beta} = 0$$

and

$$n/\beta + \sum_{i=1}^{n} \ln t_i - 1/\alpha \sum_{i=1}^{n} t_i \ln t_i = 0 \ .$$

Solving these equations for the Weibull parameters would of course require an iterative procedure. To by-pass this difficulty beta can be found by solving the equations

$$n\alpha - \sum_{i=1}^{n} t_i = 0$$

and

$$\frac{\sum_{i=1}^{n} t_i}{n} = \alpha^{1/\beta}\Gamma\left[\frac{\beta+1}{\beta}\right]$$

simultaneously. This second equation is an unbiased estimate of the distribution mean. Solution of these equations leads to

$$n\frac{\left(\sum_{i=1}^{n} t_i\right)^{\beta}}{n^{\beta}\left(\Gamma\left[\frac{\beta+1}{\beta}\right]\right)^{\beta}} - \sum_{i=1}^{n} t_i^{\beta} = 0.$$

From this equation an estimate of beta can be found. An estimate of alpha is then found from

$$\alpha = \frac{\sum_{i=1}^{n} t_i^{\beta}}{n} .$$

The use of one particular parameter estimation technique over another is dependent on several factors. If the p.d.f. is known for the element under question, the first method should be used. The mean and variance of the distribution can be equated and the desired α and β parameters determined. When only two points are available on the element reliability curve, the second estimation method is easily used and accurate enough for preliminary allocation.

When a sufficient volume of data is available and the Weibull p.d.f. is assumed then maximum liklihood estimators are the most accurate estimators available.

ESTIMATION OF COST FUNCTION PARAMETERS

The following is a technique whereby the parameters A, a and b can be estimated for the cost function

$$A\alpha^{a}\beta^{-b} \quad .$$

One simplifying assumption of this technique is that the ratio of element mean life to standard deviation of times to failure is constant. This is equivalent to holding the "shape" of the distribution constant. For a preliminary allocation this is not considered a serious drawback. It seems reasonable that even though the mean life of an equipment increases the general shape of its failure probability distribution should not be altered greatly. The accuracy of this simplifying assumption, however, will be born out by the Weibull parameters estimated for the element in question. In addition, at least two cost estimates are required for each element on the cost versus element reliability curve. If the preceeding requirements are met the procedure is as follows. Using an estimating technique mentioned earlier, Weibull parameters α_1, α_2 and β are found for the element. Only one β is defined since the ratio of mean life to standard deviation remains constant. Associated with these two reliability levels, $R(\alpha_1,\beta,t)$ and $R(\alpha_2,\beta,t)$, for oper-

ating time t are two cost estimates C_1 and C_2. The cost equations resulting from these points on the reliability curve are

$$C_1 = A\alpha_1^a\beta^{-b}$$

and

$$C_2 = A\alpha_2^a\beta^{-b}$$

where A is the position parameter and a and b are the slope parameters. Taking the natural logarithm of both equations and combining yields

$$a = \frac{\ln(C_1/C_2)}{\ln(\alpha_1/\alpha_2)} \quad . \qquad \text{Eq. 1A}$$

Since it was found for the analyses in Chapters III and IV that the optimal level of beta for all elements will be b/a $\ln(t_o)$, b can be estimated as

$$b = \beta a \ln(t_o), \qquad \text{Eq. 2A}$$

where as before t_o is the system operating time over which reliability is being optimized. Using an estimation technique previously discussed, α and β can be determined for two reliability and cost levels of the element. With two α values defined, the slope parameter a can be found from Equation 1A. Once a is determined, b can be found from Equation 2A. With both a and b determined, the equation for the position parameter A can be written as

$$A = \frac{C_1}{\alpha_1^a\beta^{-b}} = \frac{C_2}{\alpha_2^a\beta^{-b}} \quad .$$

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