R-WV-T-6-22-73

RELIABILITY DATA ANALYSIS MODEL

R. RACICOT

U.S. ARMY WEAPONS COMMAND

BENET WEAPONS LABORATORY
WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189

JULY 1973

TECHNICAL REPORT

AMCMS No. 611102.11.85300.01

DA Project No. 1T061102B14A

Pron No. B9-2-67162-01-M2-M7

DISPOSITION

Destroy this report when it is no longer needed. Do not return it to the originator.

DISCLAIMER

The findings in this report are not to be construed as an official Department of the Army position unless so designated by other authorized documents.

Unclassified
Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Watervliet Arsenal Watervliet, N.Y. 12189	2a. REPORT SECURITY CLASSIFICATION Unclassified 2b. GROUP

3. REPORT TITLE

RELIABILITY DATA ANALYSIS MODEL

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Technical Report

5. AUTHOR(S) (First name, middle initial, last name)

R. Racicot

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
July 1973	89	20

8a. CONTRACT OR GRANT NO. AMCMS No. 611102.11.85300.01 b. PROJECT NO. DA Project No. 1T061102B14A c. Pron No. B9-2-67162-01-M2-M7 d.	9a. ORIGINATOR'S REPORT NUMBER(S) R-WV-T-6-22-73 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	U. S. Army Weapons Command

13. ABSTRACT

A computer model has been prepared to assist in the analysis of reliability test data. Essentially, the model computes point estimates and confidence limits for mission reliability of components, subsystems, and systems from component failure data. The main features of the model are: 1. Performs goodness-of-fit tests to determine the best fit probability distribution of component failure times, 2. Computes maximum likelihood estimates of distribution parameters, 3. Computes point estimates of reliability for the renewal nonconstant failure rate case, and 4. Computes lower confidence limits for component, subsystem, and system reliability for the constant failure rate case.

DD FORM 1 NOV 65 1473 REPLACES DD FORM 1473, 1 JAN 64, WHICH IS OBSOLETE FOR ARMY USE.

Unclassified
Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Reliability						
Statistics						
Probability						
Distribution Selection						
Computer Model						
Failure Rate						
Bayesian Statistics						

R-WV-T-6-22-73

AD

RELIABILITY DATA ANALYSIS MODEL

R. RACICOT

U.S. ARMY WEAPONS COMMAND

BENET WEAPONS LABORATORY
WATERVLIET ARSENAL
WATERVLIET, N.Y. 12189

JULY 1973

TECHNICAL REPORT

AMCMS No. 611102.11.85300.01

DA Project No. 1T061102B14A

Pron No. B9-2-67162-01-M2-M7

APPROVED FOR PUBLIC RELEASE: DISTRIBUTION UNLIMITED

RELIABILITY DATA ANALYSIS MODEL

ABSTRACT

A computer model has been prepared to assist in the analysis of reliability test data. Essentially, the model computes point estimates and confidence limits for mission reliability of components, subsystems, and systems from component failure data. The main features of the model are: 1. Performs goodness-of-fit tests to determine the best fit probability distribution of component failure times, 2. Computes maximum likelihood estimates of distribution parameters, 3. Computes point estimates of reliability for the renewal nonconstant failure rate case, and 4. Computes lower confidence limits for component, subsystem, and system reliability for the constant failure rate case.

Cross-Reference Data

Reliability

Statistics

Probability

Distribution Selection

Computer Model

Failure Rate

Bayesian Statistics

FOREWORD

This project was supported in part by the Project Manager's Office for Close Support Weapon Systems and by USASASA for the Future Rifle System.

The author would like to thank R. Haggerty and R. Scanlon of the Benet Weapons Laboratory for preparing a number of the computer subroutines used in the model.

TABLE OF CONTENTS

Figures

Tables

RELIABILITY DATA ANALYSIS MODEL

INTRODUCTION

Current Army regulations [1] require the specification of quantified reliability goals in the development of new weapon systems. These reliability goals must also be verified by tests prior to final acceptance and fielding of the system. Reliability, by definition, is a probabilistic quantity. In addition, the amount of testing that can be conducted is limited by cost and time considerations and consequently reliability assessment must be conducted within a statistical framework. To assist in this assessment effort a statistical data analysis computer model has been prepared and is described in this report. Essentially, the model computes point estimates and confidence limits for reliability of components, subsystems and system from component test data.

The specific reliability index considered is the mission reliability defined as the probability that a given system will successfully perform its intended function without failure for a specified period of time under a given set of conditions. Time can be given in terms of clock time, rounds, cycles, miles, etc. The system may be required to perform a number of missions over its expected life.

[1] Army Regulation AR705-50, "Army Materiel Reliability and Maintainability," Headquaters Dept. of the Army, Washington, D.C., 8 January 1968.

PURPOSE OF THE RELIABILITY COMPUTER MODEL

The reliability model is intended to serve a number of useful purposes. First, reliability analyses for the realistic situations encountered in weapon system testing and for all but the simplest assumptions made for computation entail rather complicated mathematical formulation and tedious computation. The computer model consequently provides a systematic means of incorporating more complex analyses with less restrictive assumptions into a routine data analysis procedure. Second, the model provides greater speed in obtaining results of data analyses. Third, the mathematical procedures and formulations incorporated into the model provide a means of standardizing reliability assessment among design agencies, test agencies and the user. Finally, the model provides a basis for determining the type and amount of test data required to perform a particular kind of analysis. This provides insight into planning of tests and data collection procedures required.

TYPE OF DATA CONSIDERED FOR ANALYSIS

The type of data considered for analysis is failure and suspension data on a component or a number of components making up a subsystem or system. Definition of failure depends on the particular requirements and the mission profile of a given system and can include part breakage, out-of-tolerance performance, incipient failure, safety hazard, etc. Suspended data is obtained when a component is removed from the test without failure, the test is terminated without failure or when the failure of a component is attributable to the failure of another component or cause.

Since failure rates and resulting mission reliabilities are generally transient, all data is assumed collected as a function of system age. Specifically, the age given in terms such as rounds, time or miles on individual components within a system at failure or suspension is required. In general it is not enough to record only the total test time and number of failures except in the case of constant failure rate.

TYPE OF INFORMATION GENERATED

The type of information generated by the model includes results of analyses for each individual component within a subsystem, for each subsystem making up a given system and for the entire system. Component information includes results of distribution selection procedures, Kolmogorov-Smirnov goodness-of-fit tests, maximum likelihood estimates of distribution parameters, point estimates of the mission reliability as a function of system age for the non-constant failure rate case and confidence limits on average mission reliability over the system life for the constant failure rate case. Subsystem and system information generated includes point estimates and confidence intervals on mission reliability for the constant failure rate case.

A more complete description of required input data for the model and the resulting output information will be presented in subsequent sections dealing with the computer program.

USUAL ASSUMPTIONS FOR DATA ANALYSIS AND THEIR LIMITATIONS

It is worthwhile at this point to review the usual assumptions made in performing a reliability assessment for a given system and the desirability of relaxing these assumptions for some of the realistic cases encountered in weapon system testing. Undoubtedly, the most common assumption made in reliability data analysis is the assumption of constant failure rate for components and systems. The most significant implication of this assumption is that the probability of failure of a component or system is independent of its age. Components are assumed to fail at completely random points in time. This, of course, does not permit the full treatment of the cases of early failures and wear-out failures common to mechanical components.

There are a number of reasons why the constant failure rate assumption is so often made. First, the constant failure rate is often assumed primarily to simplify analysis and computation. Second, straightforward techniques for determining confidence intervals on component reliability from test data are readily available for constant failure rate. Third, testing procedures are greatly simplified since only the total number of failures and total test time are required for data analysis. Also, the amount of data required for each component is not as critical as in those cases where theoretical distribution selection must be considered. Finally, for system reliability verification tests, individual component data is not required with all failures being treated as system failures. This is true only if all components have equal test times.

Assuming a constant failure rate can lead to three types of errors; the first is in the computation of reliability for fixed MTBF (mean time between failures), the second is in the computation of confidence limits, and the third is in the estimation of MTBF when suspension or censored data has been generated. The magnitude of the error in computing reliability for fixed MTBF cannot be determined in general. For illustrative purposes, however, consider a hypothetical case in which a system is made up of equal components in series. Assume further that each component has a Weibull distribution and an MTBF (mean time between failure) equal to twice the expected system life. The number of missions over system life is assumed to be 150. The magnitude of the error in assuming constant failure rate can now be determined for this particular case. Table I lists the results of average system reliability for different numbers of components in the system and for various values of the Weibull shape parameter β. For a shape parameter β equal to 1.0 the Weibull distribution reduces to the exponential which describes the distribution for constant failure rate. For values of β greater than 1.0 the failure rate is an increasing function of time characteristic of wear-out phenomena. The greater the value of β the more peaked the distribution is about the mean. As can be seen from Table I, large differences can be obtained when

TABLE I

AVERAGE SYSTEM RELIABILITY FOR EQUAL COMPONENTS HAVING WEIBULL DISTRIBUTION OF FAILURE TIMES.

COMPONENT MTBF	=	1.000
SYSTEM LIFE	=	0.500
NUMBER OF MISSIONS	=	150
MISSION TIME	=	0.003333

AVERAGE SYSTEM MISSION RELIABILITY.....

Weibull Shape Parameter β	NUMBER OF COMPONENTS				
	1	10	50	100	500
1.0	0.996672	0.967216	0.846482	0.716531	0.188875
2.0	0.998771	0.987781	0.940379	0.884313	0.540791
3.0	0.999430	0.994311	0.971877	0.944545	0.751819
4.0	0.999724	0.997248	0.986315	0.972817	0.871277
5.0	0.999865	0.998655	0.993293	0.986630	0.934916
6.0	0.999934	0.999339	0.996700	0.993411	0.967488
7.0	0.999968	0.999675	0.998377	0.996757	0.983889
8.0	0.999984	0.999841	0.999208	0.998416	0.992104
9.0	0.999993	0.999934	0.999672	0.999345	0.996727
10.0	0.999996	0.999962	0.999812	0.999625	0.998124

assuming exponential or constant failure rate for Weibull components that in actuality have β values greater than 1.0. For example, a system of 50 equal components, each component having a Weibull shape parameter equal to 3.0 would have an average system reliability of 0.972. Assuming a constant failure rate on the other hand would yield a computed reliability of 0.846 which is significantly different than the true value for this case.

Reliability for the Weibull distribution with $\beta > 1$ was computed using renewal theory where, in this case, a component is replaced or renewed upon failure. The mathematical formulation for the renewal case is described in the Analytical Methods section of this report.

The second source of error in assuming constant failure rate is in the computation of confidence limits. Consider as an example problem a test sample of failure times given as 2000, 2200, 1800, 1900 and 2100. For this problem the MTBF is 2000 and the standard deviation is 158. Lower confidence limits on the true MTBF can readily be computed assuming different underlying distributions of failure times. For example, the lower 90% confidenced means for the exponential (constant failure rate)and the normal (increasing failure rate) distributions are 1250 and 1758 respectively. It is common in reliability verification tests to require that the lower confidenced MTBF (or reliability) exceed a given fixed value for acceptance. If the requirement in the above problem were 1500, the assumption of the exponential distribution would lead to rejection and the assumption of the normal distribution would lead to acceptance. Although the above is an oversimplification, it does indicate the possible error in assuming the wrong distribution in determining confidence limits.

The third source of error in assuming constant failure rate lies in the unrealistic handling of suspension data in estimating population parameters. Consider as an example problem a test sample of failure times given as 3800, 3900, 4100 and 4200 and a sample of suspension times given as 3500, 4000, 4000 and 4500. In this example, eight different components were tested but only four failed. The point estimates of MTBF assuming exponential and Weibull distributions are 8000 and 4130 respectively. As can be seen the MTBF for constant failure rate is nearly twice the value for the nonconstant failure rate Weibull distribution.

In the first two sources of error discussed above the computed reliability is generally conservative for components and systems that wear out. Reliability and lower confidence limits are conservative in that lower than true values are computed. In the third source of error the resulting reliability is generally nonconservative. The unrealistic handling of suspension data seems to be the most significant source of error in assuming constant failure rate for testing of mechanical systems.

A second assumption or requirement which is often made in reliability analysis is that the data sample be complete. Consequently only the failure times in a given test are considered for analysis. The main reason for this assumption is that statistical methods that treat suspended data along with the failure data are in many cases restrictive, complicated or not available. It is clear, however, that suspended data, particularly data involving suspension times that are greater than some or all of the failure times should contribute significant information toward determining the unknown population parameters. This was observed in the above discussion of the constant failure rate assumption. Suspended data can result in many ways. In weapon system testing, for example, it can arise from the removal of a component from test without failure, completion of a system test prior to failure or through failure resulting directly from the failure of another component or cause such as accident.

A third assumption which is often made in determining confidence limits on system reliability from system tests is that the system can be treated as if it were a single component with no differentiation being made as to which component within the system actually fails. All failures are treated as system failures. A necessary underlying assumption for this approach is that either all components have constant failure rate or all components have failed a number of times so that steady state conditions prevail. In this instance confidence intervals on system reliability are readily derived using the theory applicable to the constant failure rate case. A limitation of this assumption is that results of individual component or subsystem tests cannot be included with the system test results. Also, if components are redesigned during the course of a test, which is often the case in large weapon system testing, total test time on the redesigned components are not the same as the total system test time. The usual methods of analyzing system test data consequently do not apply in this case.

Finally, large sample theory is often assumed in computing confidence intervals since methods to handle small samples may not exist or are too difficult to use.

MAIN FEATURES AND STATUS OF THE PRESENT MODEL

The reliability data analysis model presented in this report contains a number of general solutions to overcome many of the limitations of the usual assumptions discussed in the previous section. The main features of the model are summarized as follows:

a. Performs goodness-of-fit test to determine the best fit probability distribution of component failure times. The theoretical distributions considered are the exponential, normal, lognormal, Weibull and gamma.

b. Computes maximum likelihood estimates of population parameters for general theoretical distribution of failure times.

c. Can handle suspended data which results when a component is tested without failure.

d. Computes point estimates of reliability for the renewal case; that is, for the non-constant failure rate case.

e. Computes lower confidence limits on component, subsystem and system reliability for the constant failure rate case.

The present version of the reliability data analysis computer model does not contain all of the features ultimately planned for in the final version. Most significant of the limitations is the assumption of constant failure rate in determining confidence intervals although point estimates of reliability are determined for the general non-constant failure rate case. In addition components are assumed to be in series for system reliability computation and no provision is presently made for preventive maintenance parts replacement of components. The analytical methods section of this report describes the techniques to be employed in removing these limitations with work on future versions of the model presently being undertaken.

The remainder of this report contains a discussion of the analytical methods used for computation followed by a presentation of the computer program with example input and output information.

ANALYTICAL METHODS

The purpose of this section is to briefly outline the mathematical and statistical methods used or planned in the reliability data analysis model. A complete discussion of probability and statistical theory will not be presented in this report and the reader is referred to the references cited for more complete discussions. Much of the theory presented is straightforward and readily found in the literature. However, some of the statistical and computational methods used are the result of research efforts at the Watervliet Arsenal and will be the subject of forthcoming reports and publications.

There are a number of good texts on the general subject of reliability. The texts found particularly useful to this writer are those by Lloyd and Lipow [2], Barlow and Proschan [3], Gnedenko, Belyayev and Solovyev [4] and Pieruschka [5].

Computation of Component Mission Reliability.

Although one of the computational goals in the model is system reliability it is necessary to develop the analytical methods for computing system reliability by considering the individual components or elements making up the system. This is particularly true in the case of non-constant failure rate components.

At the present time there is considerable confusion among design engineers on the definition of component reliability when the component is part of a system. The confusion lies primarily on the time reference used in computing reliability. Many texts on reliability consider time in terms of component age. However, for system reliability the system age is the important time reference. Since components that fail within a system are replaced or renewed at random points in

---

[2] Lloyd, D.K., and Lipow, M., "Reliability: Management, Methods, and Mathematics," Prentice-Hall, Englewood Cliffs, New Jersey (1962).

[3] Barlow, R.E., and Proschan, F., "Mathematical Theory of Reliability," John Wiley & Sons, New York (1965).

[4] Gnedenko, B.V., Belyayev, Yu.K., and Solovyev, A.D., "Mathematical Methods of Reliability Theory," Academic Press, New York (1969).

[5] Pieruschka, E., "Principles of Reliability," Prentice-Hall, Englewood Cliffs, New Jersey (1963).

time, component ages are generally not known a priori as a function of system age. In this instance renewal theory must be employed to determine the transient mission reliability as a function of system age. This aspect of the problem will be considered in the following sections.

(a) Component Reliability Based on Component Age.

Consider first a single component where the time reference is component age. Let

$f(t)$ = Probability density distribution of component failure times or time between failures.

$F(t) = \int_0^t f(t)dt$ = cumulative distribution function.

= Probability that the component will fail in the time interval $(0,t)$.

$R(t)$ = Reliability defined as the probability that the component will not fail in the time interval $(0,t)$ where the component is assumed new at time 0.

$= 1 - F(t)$

$\lambda(t)$ = Conditional failure rate where $\lambda(t)dt$ is the probability of failure in the time interval $(t,t+dt)$ given that the component has survived to time t.

The conditional failure rate $\lambda(t)$ defined above is a useful and descriptive quantity in reliability theory. It can be determined from the probability distribution of failure times as follows:

$$\lambda(t) = f(t)/(1-F(t))$$

$$= f(t)/R(t) \qquad (1)$$

Figure 1 shows descriptively the failure rate for three typical cases in reliability; decreasing, constant and increasing failure rate.

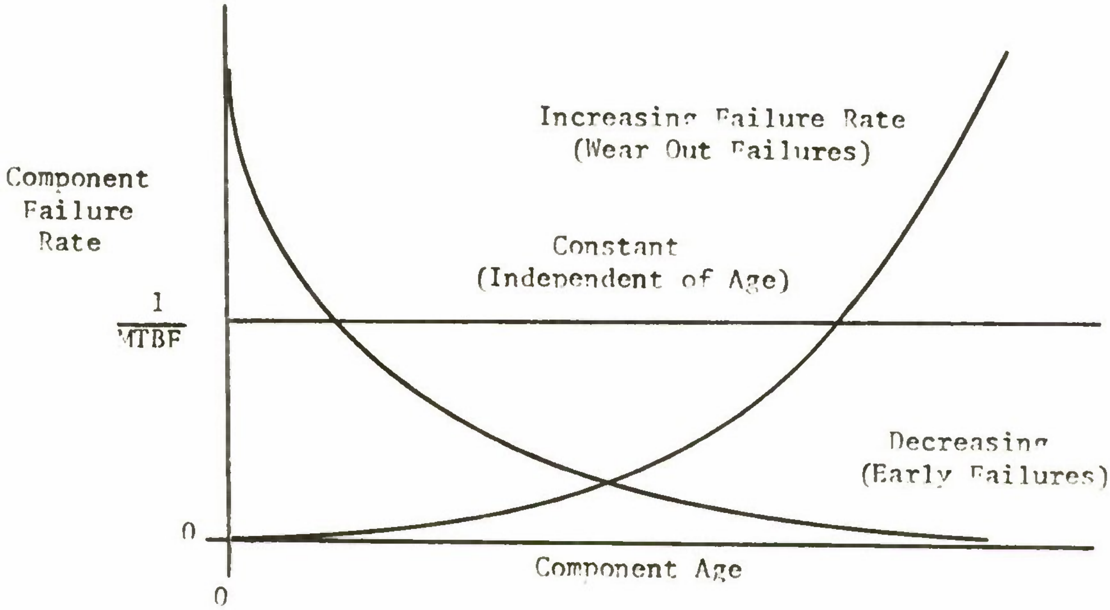


Figure 1. Component Failure Rate vs Component Age

The decreasing failure rate implies that the longer a component survives the lower its probability of failure. This is typical of early failures in which components containing manufacturing or material flaws tend to fail early. The increasing failure rate is typical of components that wear out where the longer a component survives the higher the probability of failure. Mechanical components in which early failures have been eliminated generally have increasing failure rates caused by such phenomena as fatigue, corrosion, erosion and abrasion. The constant failure rate is typical of components which have a probability of failure independent of its age. This can result, for example, from a situation in which excessive loads can cause failure where load fluctuations are purely random in time such as wind loads.

Examples:

1. Exponential Distribution

For this case

$$f(t) = \lambda e^{-\lambda t}$$

$$F(t) = 1-e^{-\lambda t}$$

$$R(t) = e^{-\lambda t}$$

$$\lambda(t) = \lambda = \text{constant} \tag{2}$$

The exponential distribution describes the constant failure rate case.

2. Weibull Distribution

$$f(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} e^{-(t/\eta)^{\beta}}$$

$$F(t) = 1 - e^{-(t/\eta)^{\beta}}$$

$$R(t) = e^{-(t/\eta)^{\beta}}$$

$$\lambda(t) = \frac{\beta}{\eta}\left(\frac{t}{\eta}\right)^{\beta-1} \tag{3}$$

in which β and η are distribution or population parameters. Note that for the Weibull distribution β values less than, equal to and greater than unity yield failure rates which are decreasing, constant and increasing respectively. This characteristic makes the Weibull distribution a useful one in reliability analysis.

(b) Mission Reliability of a Component Within a System [3, 4, 5].

Consider next the situation of a component within a system. The quantity of interest here is the reliability of the component for

[3]Barlow, R. E., and Proschan, F., "Mathematical Theory of Reliability," John Wiley & Sons, New York (1965).

[4]Gnedenko, B. V., Belyayev, Yu.K., and Solovyev, A. D., "Mathematical Methods of Reliability Theory," Academic Press, New York (1969).

[5]Pieruschka, E., "Principles of Reliability," Prentice-Hall, Englewood Cliffs, New Jersey (1963).

a mission time interval of $(t, t + \tau)$ where t is the system age and τ is the mission length. Prior to time t the component could have failed and been replaced one or more times. Figure 2 depicts the conditional failure rate $\lambda(t)$ of a component within a system showing failure points for an increasing failure rate.

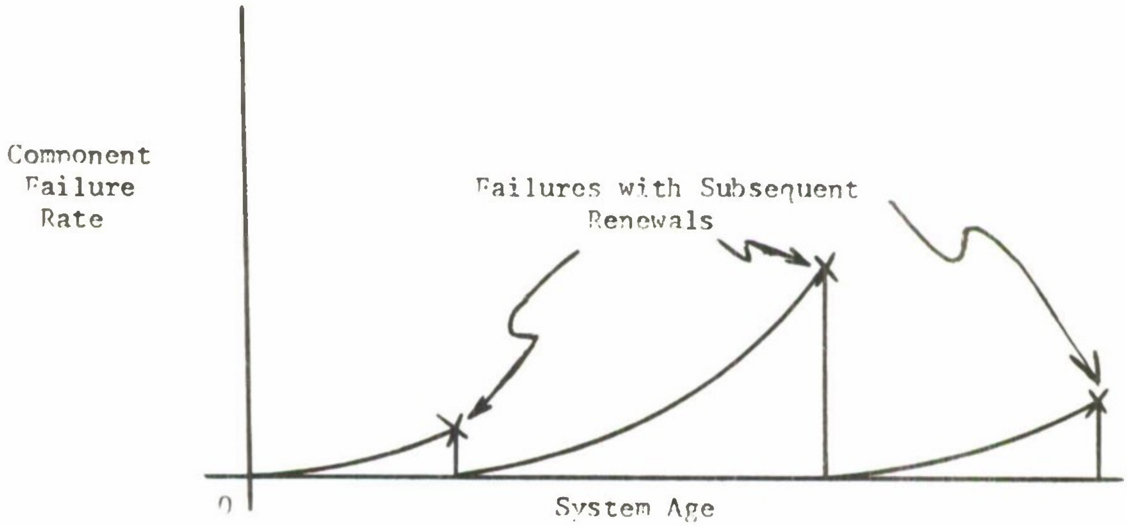


Figure 2. Component Failure Rate vs System Age For The Case Of Ideal Repair

In general the failure times of components within a system are not known in advance as depicted in Figure 2 and consequently must be treated probabilistically. This is accomplished by defining another quantity h(t) called the renewal rate or unconditional failure rate which is an ensemble average of the failure rate over the population of all systems. The renewal rate is defined such that

$h(t)dt$ = unconditional probability of component failure in system time interval $(t,t+dt)$.

The renewal rate for a given component is a function of the underlying failure distribution given by the following equation:

$$h(t) = f(t) + \int_0^t h(x)f(t-x)dx.$$

Derivation of this equation is presented in references [2], [3] and elsewhere. Logically, equation (4) can be derived from the theorems of total and conditional probabilities [6] along with the definitions of the terms in equation (4). Multiplying both sides of equation (4) by dt and considering the integral as a sum we have

$f(t)dt$ = Probability that the original component fails in the time interval $(t, t + dt)$.

$h(x)dx$ = Probability that a failure and subsequent renewal occurred in time interval $(x, x + dx)$.

$f(t-x)dt$ = Conditional probability that a component which was renewed at time x fails in the time interval $(t, t + dt)$.

$h(x)f(t-x)dxdt$ = Unconditional probability of failure in time interval $(t, t + dt)$ for a component which could fail at time x.

$f(t)dt + \int_0^t h(x)f(t-x)dxdt$ = Total probability of failure which is the sum of all possible conditions which could lead to failure in the time interval $(t, t + dt)$,

= $h(t)dt$ by definition.

The mission reliability can now be determined from the renewal

[2]Lloyd, D.K., and Lipow, M., "Reliability: Management, Methods, and Mathematics," Prentice-Hall, Englewood Cliffs, New Jersey (1962).

[3]Barlow, R. E., and Proschan, F., "Mathematical Theory of Reliability," John Wiley & Sons, New York (1965).

[6]Papoulis, A., "Probability, Random Variables, and Stochastic Processes," McGraw-Hill, New York (1965).

rate h(t) using the relation from reference [4]

$$R(t,\tau) = 1-F(t+\tau) + \int_0^t [1-F(t+\tau-x)]h(x)dx \qquad (5)$$

in which

$R(t,\tau)$ = Mission reliability at system time t for a mission length τ,

= Probability of no failure in $(t, t+\tau)$.

$1-F(t+\tau)$ = Probability that the original component in the system has not failed at time $t+\tau$.

$1-F(t+\tau-x)$ = Probability that a component which failed at time x has not failed at time $t+\tau$.

$h(x)dx$ = Probability of failure at time x.

Typical examples of the renewal rate and mission reliability of an increasing failure rate component are depicted in Figures 3 and 4. As can be seen the renewal rate increases with system age until about a system time equal to the MTBF (mean time between failure) of the component. The renewal rate then approaches a constant value equal to the reciprocal of the MTBF. The reliability on the other hand decreases from a value of 1.0 at t=0 and approaches a constant value. The asymptotic values of h(t) and $R(t,\tau)$ can be derived directly from equations (4) and (5) by passing t to the limit infinity. These values are given by the relations

$$h(\infty) = 1/MTBF$$

$$R(\infty,\tau) = \frac{1}{MTBF}\int_\tau^\infty [1-F(x)]dx. \qquad (6)$$

[4] Gnedenko, B.V., Belyayev, Yu.K., and Solovyev, A.D., "Mathematical Methods of Reliability Theory," Academic Press, New York (1969).

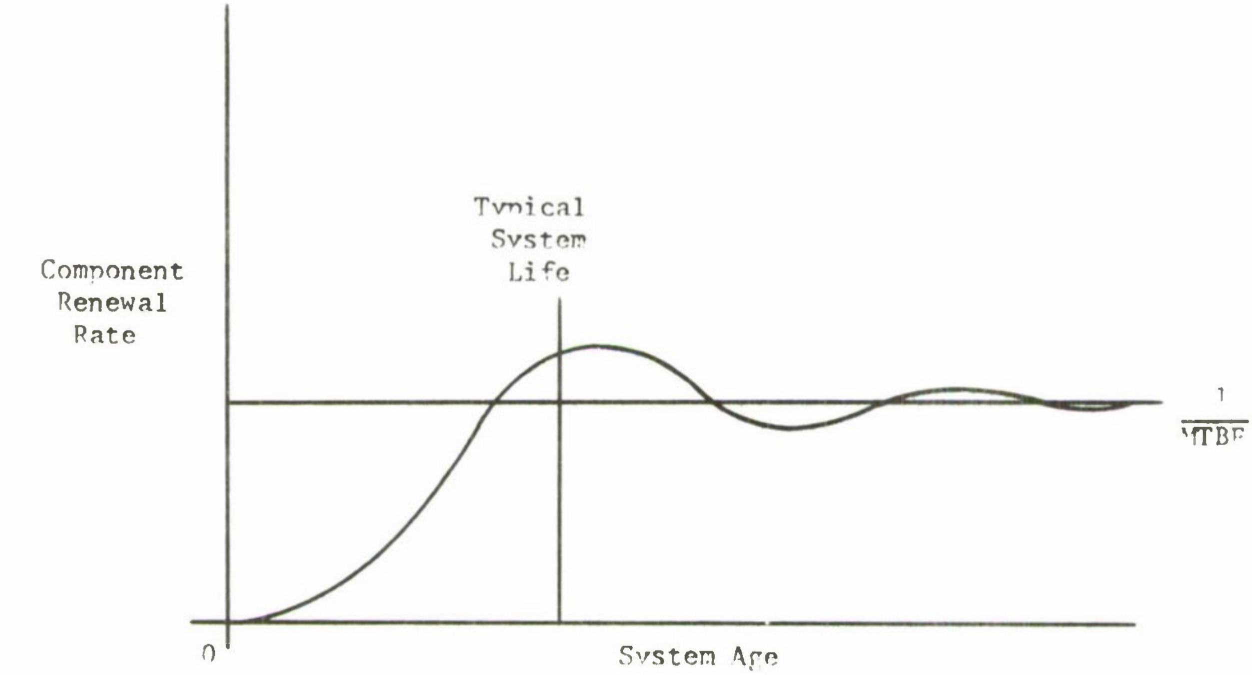


Figure 3. Component Renewal Rate vs System Age.

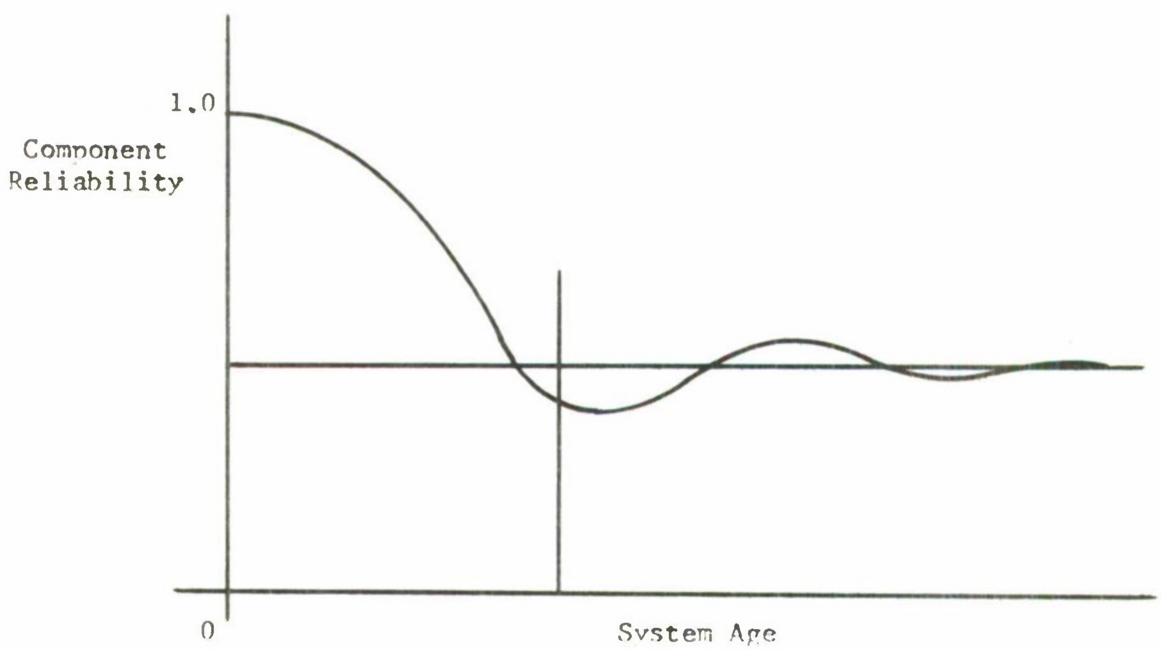


Figure 4. Component Reliability vs System Age.

For the exponential distribution (constant failure rate) the renewal rate and reliability are both constant throughout system life and are equal to λ and $e^{-\lambda\tau}$ respectively.

It should be noted at this point that for a high reliability system (say .85 or greater) the MTBF of the components comprising the system should be of the same order of magnitude as the expected or required system life. Consequently, the components within a system are essentially exercised within the transient portion of their lives thus indicating the general importance of considering renewal theory in determining component reliability for high reliability systems.

(c) Numerical Solution for the Renewal Rate.

A number of numerical techniques were investigated to solve the integral equation (4) for the renewal rate h(t). The most computationally efficient solution investigated thus far was through the use of finite difference methods [7]. Finite difference yields a direct solution for the renewal rate. Other more general techniques investigated which give complete solutions to the renewal problem (e.g. probability distribution of total number of failures or time to nth failure) involved the use of orthogonal expansions of Laplace transforms. Two sets of orthogonal functions considered were trigonometric and Laguerre polynomials. The finite difference solution, although limited to solution of the renewal rate, generally required less computer time for given accuracy. The solution is also quite general in that it can be used to determine renewal rate for most theoretical distributions of failure time applicable to reliability

[7] McKelvey, R. W., "An Analysis of Approximate Methods for Fredholm's Integral Equation of the First Kind," December 1956, AD650530.

theory.

In solving equation (4) using finite differences, the density $f(t)$ and the renewal rate $h(t)$ are discretized over a fixed time interval generally the system life. In this case $f(t)$ and $h(t)$ are written as

$$f(t) = f(k\Delta t) = f_k$$
$$h(t) = h(k\Delta t) = h_k$$
$$t = k\Delta t \text{ where } k=0,1,\ldots,n.$$

The integral equation (4) can then be written as

$$\int_0^t f(t-x)h(x)dx \simeq \frac{\Delta x}{2} \sum_{j=0}^{k-1} [f(k\Delta t-j\Delta x)h(j\Delta x) + f(k\Delta t-(j+1)\Delta x)h((j+1)\Delta x)]$$

$$\simeq \frac{\Delta x}{2} \sum_{j=1}^{k-1} [f_{k-j}h_j + f_{k-j-1}h_{j+1}] \quad \text{for } k = 1,\ldots,n \qquad (7)$$

$$= 0 \text{ for } k = o.$$

Substituting into equation (4) then gives

$$h_k = f_k + \frac{\Delta x}{2} \sum_{j=0}^{k-1} [f_{k-j}h_j + f_{k-j-1}h_{j+1}] \quad \text{for } k = 1,\ldots,n \qquad (8)$$

$$= f_o \quad \text{for } k = 0.$$

Equation (8) represents n+1 equations with n+1 unknowns which can be readily solved using Gaussian elimination.

A difficulty is encountered in some situations where $f(0) = \infty$ such as in the case of the Weibull distribution with shape parameter $\beta<1$. In this instance $f(0)$ is fixed at a finite value such that the actual area under $f(t)$ in the interval $(0, \Delta t)$ (i.e. $F(\Delta t)$) is made equal to the finite difference area $(f(0) + f(\Delta t))\Delta t/2$.

(d) Numerical Solution for Mission Reliability.

Once the renewal rate is determined the transient mission reliability can then be computed from equation (5):

$$R(t,\tau) = 1-F(t+\tau) + \int_0^t [1-F(t+\tau-x)]h(x)dx.$$

Numerically, this equation is difficult to solve since the integral must be evaluated over the whole interval $(0,t)$ for each different value of t. By making use of the renewal equation an equivalent equation for $R(t,\tau)$ can be derived which is computationally more efficient. Integrating the renewal equation (4) from 0 to T gives

$$H(T) = F(T) + \int_0^T H(T-x)f(x)dx \tag{9}$$

$$\text{where } H(T) = \int_0^T h(t)dt.$$

Making the substitution $T=t+\tau$ in equation (9) gives the following equation:

$$H(t+\tau) = F(t+\tau) + \int_0^{t+\tau} H(t+\tau-x)f(x)dx. \tag{10}$$

Integrating by parts and using the fact that $H(0) = 0$ and $F(0) = 0$ then gives

$$H(t+\tau) = F(t+\tau) + \int_0^{t+\tau} F(x)h(t+\tau-x)dx$$

$$= F(t+\tau) + \int_0^{t+\tau} F(t+\tau-x)h(x)dx. \quad (11)$$

From equation (11) the following equation can be readily derived:

$$\int_0^t [1-F(t+\tau-x)]h(x)dx = F(t+\tau) - \int_t^{t+\tau} [1-F(t+\tau-x)]h(x)dx \quad (12)$$

where the definition $H(t) = \int_0^t h(x)dx$ was used.

Substituting equation (12) into equation (5) for mission reliability finally yields

$$R(t,\tau) = 1 - \int_t^{t+\tau} [1-F(t+\tau-x)]h(x)dx \quad (13)$$

Note that in this equation the integral is evaluated only over the interval $(t, t+\tau)$ and that the value of $F(y)$ is required only over the interval $(0,\tau)$. This simplifies the numerical solution for mission reliability.

(e) Average Mission Reliability.

Interval reliability as discussed in the previous section is a transient function of system age. Weapon system requirements, however, generally specify one value of mission reliability for the system. This specified value could represent the lowest reliability to be experienced by the system or an average value over system life. In the computer model average system reliability is computed using the relation

$$\overline{R}(\tau) = \frac{1}{n} \sum_{i=1}^{n} R_i(\tau) \tag{14}$$

in which

$R_i(\tau)$ = Reliability for the ith mission

n = Expected number of missions over system life.

Computation of System Mission Reliability.

In general, system reliability can be computed directly from the component reliabilities using an appropriate reliability model [2,8]. For example, for the series reliability model, system reliability $R_s(t)$ is determined from

$$R_s(t) = \prod_{i=1}^{n} R_i(t) \tag{15}$$

in which

$R_i(t)$ = Reliability of the ith component

n = Total number of components.

A series model is applicable whenever the failure of a single component within a system results in a failure of the system. This model has been initially chosen for the data analysis computer model.

In the more general case where redundancy and load sharing are inherent in the system, reliability models are more complicated but can be derived in most given situations [2,8].

[2] Lloyd, D. K., and Lipow, M., "Reliability: Management, Methods, and Mathematics," Prentice Hall, Englewood Cliffs, New Jersey (1962).

[8] Bazovsky, I., "Reliability Theory and Practice," Prentice-Hall, Englewood Cliffs, New Jersey (1962).

Probability Distribution Selection.

One of the more difficult and questionable aspects of data analysis is the selection of a theoretical distribution applicable to a given set of data. This is particularly true for small data samples where information is ultimately required at the tails of the distribution. In the case of component failure data, theoretical considerations, prior history and experience play a large part in distribution selection. For example, for particular failure modes such as fatigue the lognormal and Weibull distributions have been found to yield good characterizations of the data [9, 10].

The data analysis computer model includes a distribution fitting program to assist in the selection of a best-fit theoretical distribution for use in reliability computation. Essentially, this program computes the standard error for a number of different candidate distributions. The standard error is a measure of the deviation of the data from the theoretical cumulative distributions.

The Kolmogorov-Smirnov statistical goodness-of-fit test is also used to determine which of the theoretical distributions can be rejected for given confidence level. Final selection of the distribution can then be made based on the computer results, theoretical considerations and/or personal experience and judgment.

(a) Candidate Distributions.

Following are the candidate distributions considered for characterizing component failure times. In order to standardize terminology used to describe the distribution parameters, the terms

[9]Dolan, T. J., "Basic Concepts of Fatigue Damage in Metals," Metal Fatigue, McGraw-Hill, New York (1959), Chapter 3, pp. 3 -67.

[10]Freudenthal, A. M., and Shinozuka, M., "Structural Safety Under Conditions of Ultimate Load Failure and Fatigue," Columbia University, New York, N.Y., October 1961, WADD Technical Report 61-177.

scale, shape and location parameters are used for all distributions and are defined below.

(1) Exponential

$$f(t) = \frac{1}{\eta} e^{-t/\eta} \tag{16}$$

where η = MTBF = scale parameter

$= 1/\lambda$

λ = constant failure rate.

(2) Normal

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2} \tag{17}$$

where μ = MTBF = scale parameter

σ = Standard deviation = shape parameter.

(3) Weibull

$$f(t) = \left(\frac{\beta}{\eta}\right) \left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}} \tag{18}$$

where μ = MTBF $= \gamma + \eta\Gamma(1+1/\beta)$

σ^2 = Variance $= \eta^2[\Gamma(1+2/\beta)-\Gamma^2(1+1/\beta)]$

η = Scale parameter

β = Shape parameter

γ = Location parameter.

(4) Lognormal

$$f(t) = \frac{1}{(t-\gamma)\sqrt{2\pi}\,\beta} e^{-\frac{1}{2\beta^2}(\ell n(t-\gamma)-\eta)^2} \tag{19}$$

where μ = MTBF = $\gamma+e^{(\eta+\beta^2/2)}$

σ^2 = Variance = $e^{(2\eta+\beta^2)}(e^{\beta^2}-1)$

η = Average $\{\ell n\ t\}$ = Scale parameter

β = Standard Deviation $\{\ell n\ t\}$ = Shape parameter

γ = Location parameter.

(5) Gamma

$$f(t) = \frac{(t-\gamma)^{\beta-1}}{\Gamma(\beta)\eta^{\beta}} e^{-(\frac{t-\gamma}{\eta})} \qquad (20)$$

where μ = MTBF = $\gamma+\beta\eta$

σ^2 = Variance = $\beta\eta^2$

η = Scale parameter

β = Shape parameter

γ = Location parameter.

More detailed discussions of these distributions can be found in the literature [6,11].

(b) Least Squares Fit of Data [12].

For each distribution other than Gamma a least squares fit of the data to the distribution is made. Generally, the theoretical distribution is linearized as far as possible to simplify computation. In the case of the Gamma distribution maximum likelihood estimates of parameters are used rather than solving the more difficult nonlinear problem.

The cumulative distribution function F(t) is used in fitting the

[6] Papoulis, A., "Probability, Random Variables, and Stochastic Processes," McGraw-Hill, New York (1965).

[11] Ireson, W. G., Editor, "Reliability Handbook" McGraw-Hill, New York (1966), Chapter 4.

[12] Draper, N. R., and Smith, H., "Applied Regression Analysis," John Wiley & Sons, New York (1966).

data with the median ranks being used as the true values of the cumulative distribution associated with each ordered data point [13]. The median ranks for a complete data sample are computed using the relation

$$MR_j = \frac{j-0.3}{n-0.4} \tag{21}$$

where n = Data sample size

j = $1,\ldots,n$

= Order numbers for the data where the data is sorted in increasing order.

When suspended data has been generated along with failure data, the median ranks are determined for the failure data only with the suspension times being used to modify the median ranks [13]. In this instance the failure and suspension data are sorted in increasing order. The sample size n in equation (21) is now the total number of failure and suspension data items. The order number j associated with each failure item is determined as follows; initially j is set equal to zero. An increment is then computed which is to be added to j to give the order number for the first failure item using the general relation

$$\text{New Increment} = \frac{n + 1 - (\text{Previous Order Number})}{1+(\text{Number of Items Following the Present Suspension Set}).} \tag{22}$$

If there are no suspension data prior to the first failure, then the increment is 1.0 and the order number for the first failure item is 1.0. This procedure is repeated for each subsequent failure item

[13] Johnson, L.G., "Theory and Technique of Variation Research," Elsevier, New York (1964), Chapter 8.

using equation (22) in each case. The median ranks are then computed using equation (21).

The median ranks are considered to be the true values of $F(t)$ associated with the ordered failure times t_i. The distribution parameters for a given theoretical form of $F(t)$ are then determined by minimizing the sum of the squares of the error between the theoretical distribution evaluated at the failure times and the median ranks or between the transformed distribution and median ranks.

Consider as an example the Weibull distribution. In this case

$$F(t) = 1 - e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}} . \qquad (23)$$

Rearranging equation (23) and then taking double logs gives

$$\ln\ln(1/(1-F(t))) = \beta(\ln(t-\gamma)-\ln\eta). \qquad (24)$$

For fixed γ, this equation is linear in $\ln(t-\gamma)$. The error between the transformed theoretical distribution and the data is then given by

$$\varepsilon_i = \beta(\ln(t_i-\gamma) - \ln\eta) - \ln\ln(1/(1-MR_i)) \qquad (25)$$

where t_i = ith failure time

MR_i = ith median rank.

The parameters β, η and γ are then found which minimize the total square error $\varepsilon^2 = \sum\varepsilon_i^2$. A similar approach is used for other distributions.

(c) <u>Computation of the Standard Error</u> [12].

In general, the standard error in the fitting of

---

[12] Draper, N.R., and Smith, H., "Applied Regression Analysis" John Wiley & Sons, New York (1966).

a theoretical distribution to data is not the same as the error discussed in the previous section. The standard error is a measure of the difference between the untransformed theoretical distribution and the median ranks. The standard error is defined by the following equation:

$$\text{s.e.} = \text{Standard Error} = \frac{\left(\sum_{i=1}^{n} (F(t_i)-MR_i)^2\right)^{1/2}}{n - p - 1} \tag{26}$$

where $F(t_i)$ = Theoretical distribution evaluated at the failure times t_i.

MR_i = Median ranks

n = Number of failure points

p = Number of population parameters estimated in determining $F(t)$.

The smaller the s.e. the closer the data fits the theoretical distribution.

(d) Kolmogorov-Smirnov Goodness of Fit [14].

A non-parametric distribution has been derived by Kolmogorov [15] and Smirnov [16] for a particular statistic d which is a measure of the fluctuation of sample data about the theoretical distribution from which the sample is drawn. The statistic d is defined as the maximum absolute deviation between the theoretical and observed cumulative probability distributions. The theoretical distribution is fixed by specifying both the functional form and the parameters.

[14] Siegel, S., "Nonparametric Statistics for the Behavioral Sciences," McGraw-Hill, New York (1956), Chapter 4.

[15] Kolmogorov, A., "Confidence Limits for an Unknown Distribution Function," Annals Mathematical Statistics, Vol. 12, (1941), pp. 461-463.

[16] Smirnov, N. V., "Table for Estimating the Goodness of Fit of Empirical Distributions," Annals Mathematical Statistics, Vol. 19, (1948), pp. 279-281.

In application of the K-S statistic, a given set of data is hypothesized to have been drawn from a given theoretical distribution. The statistic d is then determined from the data and compared to the K-S distribution of d. That is, for given significance level α or confidence level $(1-\alpha)$ the theoretical value d_α is determined from K-S tables and compared to the computed d. If $d>d_\alpha$, the hypothesis is rejected. If $d<d_\alpha$ then there is no sufficient reason to reject the given theoretical distribution and the hypothesis may be accepted.

The statistic d is determined for the candidate distributions in the computer model to provide bases for rejecting distributions and to help indicate the best-fit distribution.

Maximum Likelihood Estimates of Parameters [17].

Maximum likelihood estimates of population parameters are determined for each component within a system for which failure data has been generated. There are a number of reasons for generating maximum likelihood estimators as part of the reliability model. First, maximum likelihood yields estimates of parameters which have a number of desirable attributes. For example, if an efficient estimator for small samples exists (i.e. one with minimum variance), then maximum likelihood provides such an estimator. In general then minimum confidence intervals can be derived in this case. Maximum likelihood estimators are also consistent in that they approach the true parameter values as sample size increases. In addition, maximum likelihood estimators are asymptotically normal as sample size increases

[17]Mood, A.M., and Graybill, F.A., "Introduction to the Theory of Statistics," McGraw-Hill, New York (1963), Chapter 8.

for most cases. This simplifies determination of confidence intervals for large samples.

A second reason for determining maximum likelihood estimates is that they can be computed for relatively general underlying probability distributions through the use of routine mathematical procedures. Finally, the numerical procedures planned to be used in the computer model to determine confidence intervals require the maximum likelihood estimates of parameters to improve computational efficiency.

The basic idea behind maximum likelihood estimation is relatively straightforward. One assumes first that a sample of n_f failures and n_s suspensions have been generated from a given theoretical distribution with parameters $\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_p)$. The failure and suspension data are designated as $t_{fi}(i=1,\ldots,n_f)$ and $t_{si}(i=1,\ldots,n_s)$ respectively. The joint probability distribution of the random sample of failure and suspension times can be written as

$$L(\underline{t};\underline{\alpha}) = g_{\underline{t}}(t_{f1},\ldots,t_{fn_f},t_{s1},\ldots,t_{sn_s};\underline{\alpha}) \tag{27}$$

where $L(\underline{t};\underline{\alpha})$ = Defined as the likelihood function

$g_{\underline{t}}$ = Joint distribution of the sample outcome

$\underline{t}$ = Vector of the sample data values

$\underline{\alpha}$ = Vector of parameters for given underlying distributions of failure times.

It is next assumed that each failure and suspension time is a statistically independent outcome. Equation(27) can then be written in terms of the failure density f(t) and the cumulative distribution

F(t) as follows [6]:

$$L(\underline{t};\underline{\alpha}) = C \prod_{i=1}^{n_f} f(t_{fi};\underline{\alpha}) \prod_{j=1}^{n_s} [1-F(t_{sj};\underline{\alpha})] \tag{28}$$

where C = Normalizing constant such that the area under $L(\underline{t};\underline{\alpha})$ is unity

$f(t;\underline{\alpha})$ = Theoretical density distribution of failure times.

$F(t;\underline{\alpha})$ = Cumulative distribution function

$= \int_0^t f(x;\underline{\alpha})dx$

$[1-F(t;\underline{\alpha})]$ = Probability that the failure time is greater than t. This term is used to represent the probability of obtaining the suspension times.

As can be seen, the likelihood function L is defined as the multivariate probability distribution of the random failure and suspension times. Solving for the parameters $\underline{\alpha}$ such that L is maximized consequently gives the highest probability density for the given sample outcome $\underline{t}$. The parameter values which maximize L are denoted as $\underline{\hat{\alpha}}$ and are called the maximum likelihood estimates of $\underline{\alpha}$.

In practice ℓn L given by equation (29) is maximized rather than L to simplify computation. This can be done since the maximum of any positive function and its log are equivalent.

$$\ell n\, L(\underline{t};\underline{\alpha}) = \ell n\, C + \sum_{i=1}^{n_f} \ell n\, f(t_{fi};\underline{\alpha}) + \sum_{j=1}^{n_s} \ell n[1-F(t_{xj};\underline{\alpha})] \tag{29}$$

[6] Papoulis, A., "Probability, Random Variables, and Stochastic Processes," McGraw-Hill, New York (1965).

Consider the Weibull distribution as an example. For this case

$$f(t;\underline{\alpha}) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

$$F(t;\underline{\alpha}) = 1 - e^{-\left(\frac{t-\gamma}{\eta}\right)^{\beta}}$$

where $\underline{\alpha} = (\eta,\beta,\gamma)$.

The likelihood function can then be written as

$$\ell n\, L = \ell n\, C + \sum_{i=1}^{n_f} [\ell n\beta - \ell n\eta + (\beta-1)(\ell n(t_{f_i}-\gamma) - \ell n\eta) - \left(\frac{t_{f_i}-\gamma}{\eta}\right)^{\beta}] - \sum_{j=i}^{n_s} \left(\frac{t_{s_j}-\gamma}{\eta}\right)^{\beta} \qquad (30)$$

Solving for η,β and γ which maximize ℓn L yields the maximum likelihood estimators for the Weibull parameters.

Likelihood functions such as given by equation (30) are maximized in the computer model using Rosenbrock's algorithm [18] which is a general solution of the unconstrained minimization problem for non-linear functions. Minimizing the negative of ℓn L maximizes the function.

Once point estimates of the failure distribution parameters are generated for a given component, the mission reliability can be computed using the methods previously presented.

---

[18] Rosenbrock, H.H., "Automatic Method for Finding the Greatest or Least Value of a Function," Computer Journal, Vol. 3, (1960), pp. 175-184.

Bayesian and Classical Confidence Intervals for Component Reliability.

Classical methods of confidencing component mission reliability for small samples are generally available only for the constant failure rate case. In this instance the χ^2 distribution represents the inferencing distribution for the MTBF from which confidenced reliability can be derived [2]. Bayesian methods will consequently be used in the computer model for the non-constant failure rate components since this method provides a systematic means of determining confidence intervals for more general problems. A complete discussion of Bayesian statistics will not be presented in this report and the reader is referred to the books by Lindley [19] for a more complete discussion of this topic.

Bayesian inferencing is based, as the name implies, on Bayes' probability theorem which is essentially a conditional probability statement:

Bayes' Theorem:

$$f(y,x) = f(y;x)f(x) = f(x;y)f(y) \tag{31}$$

in which

$f(y,x)$ = Joint probability density of random variables x and y.

$f(y;x)$ = Conditional probability of y given x

$f(x;y)$ = Conditional probability of x given y

[2] Lloyd, D. K., and Lipow, M., "Reliability: Management, Methods, and Mathematics," Prentice-Hall, Englewood Cliffs, New Jersey (1962).

[19] Lindley, D.V., "Introduction to Probability and Statistics from a Bayesian Viewpoint," Part 1: Probability and Part 2: Inference, Cambridge University Press, Cambridge (1965).

$f(x),\ f(y) \quad = \quad$ Marginal distribution of x and y respectively.

From equation (31)

$$f(y;x) = f(x;y)\,\frac{f(y)}{f(x)}\,. \tag{32}$$

In the Bayesian approach the unknown parameters of the distribution of failure times are considered to be random variables. The sample outcome $\underline{t}$ from a given test are also random variables which are dependent on the distribution parameters. Bayes' theorem, equation (32), can be written in this instance as

$$f(\underline{\alpha};\underline{x}) = f(\underline{x};\underline{\alpha})\,\frac{f(\underline{\alpha})}{f(\underline{x})} \tag{33}$$

in which

$f(\underline{\alpha};\underline{x}) \quad = \quad$ Density distribution of parameters given the test sample $\underline{x}$.

$f(\underline{x};\underline{\alpha}) \quad = \quad$ Density distribution of the sample outcome given the parameters $\underline{\alpha}$.

$f(\underline{\alpha}),f(\underline{x}) \quad = \quad$ Marginal densities of $\underline{\alpha}$ and $\underline{x}$ respectively.

In equation (33) the distribution $f(\underline{x};\underline{\alpha})$ is just the likelihood function $L(\underline{x};\underline{\alpha})$ by definition. The distribution $f(\underline{\alpha})$ is called the prior of $\underline{\alpha}$ and $f(\underline{\alpha};\underline{x})$ is called the posteriori distribution of $\underline{\alpha}$ since it is determined after the sample $\underline{x}$ is determined and is consequently conditioned by the test results.

The prior distribution $f(\underline{\alpha})$ generally reflects prior knowledge or prior data on the population parameters $\underline{\alpha}$ before the test results $\underline{x}$ have been generated. For the case of weapon system components there is at the present time little or no prior information and consequently $f(\underline{\alpha})$ must reflect in some way our prior ignorance of the values of $\underline{\alpha}$. Choosing the prior distribution to reflect degrees of ignorance is one of the more difficult and questionable aspects of Bayesian statistics. Much research work is presently being conducted on the question of prior distribution selection. Priors which represent maximum ignorance for some of the simpler problems have been found [19,20]. It has also been found, however, that a uniform prior on the parameters yields confidence intervals that are generally close to exact in a classical frequency sense for a number of reliability indices. The uniform distribution assigns equal probability to all values of a random variable over a given range. The robustness of uniform priors is discussed somewhat in reference [19]. Uniform priors have consequently been used in our reliability statistics work and have also been found to be generally robust.

The distribution $f(\underline{x})$ in equation (33) is considered to be a constant and is determined so that the area under $f(\underline{\alpha};\underline{x})$ is unity.

Equation (33) can now be rewritten in the following form:

$$f(\underline{\alpha};\underline{x}) = C\, L(\underline{x};\underline{\alpha}) \tag{34}$$

where $f(\underline{\alpha};\underline{x})$ = Posteriori distribution of $\underline{\alpha}$ given $\underline{x}$.

C = Constant such that area under $f(\underline{\alpha};\underline{x})$ is unity. The constant C contains the constant terms $f(\underline{x})$ and $f(\underline{\alpha})$.

[19] Lindley, D. V., "Introduction to Probability and Statistics from a Bayesian Viewpoint," Part 1: Probability and Part 2: Inference, Cambridge University Press, Cambridge (1965).

[20] Jaynes, E. T., "Prior Probabilities," IEEE Transactions on Systems Science and Cybernetics, Vol. SSC-4, No. 3, Sept. 1968, pp. 227-241.

$L(\underline{x};\underline{\alpha})$ = Likelihood function.

In computing confidenced reliability our interest is not on the parameters themselves but rather on a function of the parameters. From probability theory [6] the cumulative distribution of any function z of the random variables $\underline{\alpha}$ is determined by the relation

$$F_Z(z;x) = \iiint_{D_Z} f(\underline{\alpha};\underline{x})d\underline{\alpha} \tag{35}$$

in which

$F_Z(z;\underline{x})$ = Posteriori distribution of Z

D_Z = Domain of $\underline{Z}$ such that $Z(\underline{\alpha}) \leq z$

$f(\underline{\alpha};\underline{x})$ = Posteriori distribution of $\underline{\alpha}$ given by equation (34).

Once the posteriori distribution $F(z;\underline{x})$ is known confidence intervals can be constructed on z for any given confidence level CL by determining the appropriate percentage points directly from $F(z;\underline{x})$. For example, a lower confidence limit is determined by solving the following equation for z_L:

$$F_Z(z_L;\underline{x}) = 1 - CL \tag{36}$$

in which

z_L = Lower confidence limit.

CL = Confidence level.

Since component mission reliability is a direct function of the failure distribution and hence of the parameters, confidence intervals

[6] Papoulis, A., "Probability, Random Variables, and Stochastic Processes," McGraw-Hill, New York (1965).

on mission reliability can be determined using the above formulation. Solution of equations (34), (35) and (36) for confidence limits can be accomplished numerically using computer routines.

Bayesian Confidence Intervals for System Reliability.

(a) Constant Failure Rate Components In Series.

Confidence intervals on component reliability for constant failure rate can be determined using the χ^2 distribution [2]. For a system of components in series, however, classical confidence intervals can be readily determined only if all components have equal test times. In this instance the system can be treated as if it were a component with all failures being counted as system failures regardless of which components fail. The χ^2 distribution can again be used to determine confidence limits.

For series components with unequal test times, however, classical methods do not generally apply. For this case Bayesian statistics, as discussed in the previous section, can be employed to determine confidence intervals.

Consider the series system

$$R_s = \prod_{i=1}^{n_c} R_i \tag{37}$$

in which R_s = System reliability.

n_c = Number of components.

R_i = Reliability of the ith component.

---

[2] Lloyd, D. K., and Lipow, M., "Reliability: Management, Methods, and Mathematics," Prentice-Hall, Englewood Cliffs, New Jersey (1962).

For constant failure rate, $R_i = e^{-\lambda_i \tau}$ and the system reliability becomes

$$R_s = \prod_{i=1}^{n_c} e^{-\lambda_i \tau} = e^{-\lambda_s \tau} \tag{38}$$

where

$$\lambda_s = \sum_{i=1}^{n_c} \lambda_i$$

The system failure rate λ_s is equal to the sum of the individual component failure rates. The posteriori probability density of the individual component failure rates can be determined using equation (34) by letting $\underline{\alpha} = \lambda_i$. Since λ_s is equal to the sum of independent random variables, the distribution for λ_s is equal to the convolution of the individual component distributions [6]. Performing the required convolution is generally tedious and it is at this point that a simplifying assumption is made. Using the Central Limit Theorem for sums of random variables [6] it is assumed that the posteriori distribution for λ_s is Gaussian. Consequently, all that is required is the mean and variance of λ_s to define the posteriori distribution. These are determined using the relations

$$E(\lambda_s) = \sum_{i=1}^{n_c} E(\lambda_i) \tag{39}$$

$$Var(\lambda_s) = \sum_{i=1}^{n_c} Var(\lambda_i) \tag{40}$$

Where $E(\cdot)$ = Expected value

$Var(\cdot)$ = Variance

---

[6] Papoulis, A., "Probability, Random Variables, and Stochastic Processes," McGraw-Hill, New York (1965).

Confidence intervals can now be constructed for λ_s and hence for system reliability using equations (39) and (40).

The accuracy of making the Gaussian assumption was checked and it was found that a system with about ten components and one to three failures experienced per component yielded relatively accurate confidence intervals in comparison to true values. In the example problem considered test times for each component were assumed equal which permitted the computation of classical confidence limits.

(b) Non-constant Failure Rate Components In Series.

For the series system equation (37) applies:

$$R_s = \prod_{i=1}^{n_c} R_i$$

Taking logs of both sides of this equation yields

$$\ln R_s = \sum_{i=1}^{n_c} \ln R_i$$

Equation (41) represents a sum of independent random variables and consequently the posteriori distribution of $\ln R_s$ can be determined using the Gaussian assumption as discussed in the previous section. Confidence intervals can then be constructed from this distribution.

COMPUTER PROGRAM

Two computer programs have been prepared to perform the computations required for reliability data analysis. One program called

DISTSEL performs goodness-of-fit computations and is used to assist in distribution selection given component failure and suspension data. Included in this program are computations of the standard error and Kolmogorov-Smirnov statistic. At the present time, the theoretical distributions that can be handled are the exponential, two and three parameter Weibull, two parameter lognormal, two parameter gamma and the normal.

The second computer program called RELIAB performs computations required to determine reliability and confidence limits for components, subsystems and system given component data. The component data includes failure and suspension times and the associated theoretical distributions to be assumed.

The fortran listings of the computer programs are lengthy; DISTSEL and RELIAB contain 17 and 25 subroutines respectively. These listings are consequently not contained in this report but can be made available upon request. The input data format for both programs is presented in Appendix I and output data for a sample problem are given in Appendix II.

As a final note, it should be emphasized that the computer models are flexible. Input and output data formats can readily be changed for particular situations, perhaps to render the programs compatible with a given computer data file system. Improved versions of the programs will also be generated in the future.

REFERENCES

1. Army Regulation AR705-50, "Army Materiel Reliability and Maintainability," Headquaters Dept. of the Army, Washington, D. C., 8 January 1968.

2. LLOYD, D. K., and LIPOW, M., "Reliability: Management, Methods, and Mathematics," Prentice-Hall, Englewood Cliffs, New Jersey (1962).

3. BARLOW, R. E., and PROSCHAN, F., "Mathematical Theory of Reliability," John Wiley & Sons, New York (1965).

4. GNEDENKO, B. V., BELYAYEV, Yu.K., and SOLOVYEV, A. D., "Mathematical Methods of Reliability Theory," Academic Press, New York (1969).

5. PIERUSCHKA, E., "Principles of Reliability," Prentice-Hall, Englewood Cliffs, New Jersey (1963).

6. PAPOULIS, A., "Probability, Random Variables, and Stochastic Processes," McGraw-Hill, New York (1965).

7. McKELVEY, R. W., "An Analysis of Approximate Methods for Fredholm's Integral Equation of the First Kind," December 1956, AD650530.

8. BAZOVSKY, I., "Reliability Theory and Practice," Prentice-Hall, Englewood Cliffs, New Jersey (1062).

9. DOLAN, T. J., "Basic Concepts of Fatigue Damage in Metals," Metal Fatigue, McGraw-Hill, New York (1959), Chapter 3, pp. 39-67.

10. FREUDENTHAL, A. M., and SHINOZUKA, M., "Structural Safety Under Conditions of Ultimate Load Failure and Fatigue," Columbia University, New York, N.Y., October 1961, WADD Technical Report 61-177.

11. IRESON, W. G., Editor, "Reliability Handbook," McGraw-Hill, New York (1966), Chapter 4.

12. DRAPER, N. R., and SMITH, H., "Applied Regression Analysis," John Wiley & Sons, New York (1966).

13. JOHNSON, L. G., "Theory and Technique of Variation Research," Elsevier, New York (1964), Chapter 8.

14. SIEGEL, S., "Nonparametric Statistics for the Behavioral Sciences," McGraw-Hill, New York (1956), Chapter 4.

15. KOLMOGOROV, A., "Confidence Limits for an Unknown Distribution Function," Annals Mathematical Statistics, Vol. 12, (1941), pp. 461-463.

16. SMIRNOV, N. V., "Table for Estimating the Goodness of Fit of Empirical Distributions," Annals Mathematical Statistics, Vol. 19, (1948), pp. 279-281.

17. MOOD, A. M., and GRAYBILL, F. A., "Introduction to the Theory of Statistics," McGraw-Hill, New York (1963), Chapter 8.

18. ROSENBROCK, H. H., "Automatic Method for Finding the Greatest or Least Value of a Function," Computer Journal, Vol. 3, (1960), pp. 175-184.

19. LINDLEY, D. V., "Introduction to Probability and Statistics from a Bayesian Viewpoint," Part 1: Probability and Part 2: Inference, Cambridge University Press, Cambridge (1965).

20. JAYNES, E. T., "Prior Probabilities," IEEE Transactions on Systems Science and Cybernetics, Vol. SSC-4, No. 3, Sept. 1968, pp. 227-241.

APPENDIX I

INPUT FORMAT FOR PROGRAMS DISTSEL AND RELIAB

The following listings give the card by card description of the input data and format required by the reliability data analysis programs.

(a) Overall System And Subsystem Data Cards

Card No.	Column No.	Description	Format
1	1-80	Overall system identification.	Alphanumeric
2	1-10	Number of missions over system life.	I10
	11-20	Specified system life.	F10.0
	21-30	Desired confidence level for reliability.	F10.0
3	1-40	Number of components per subsystem for which data is supplied. Up to four subsystems can be handled.	4I10
4 to 4 +NSUB	1-80	Subsystem identification. NSUB is the number of subsystems where one card is used to describe each subsystem.	Alphanumeric

(b) Element or Component Data Cards

The following data cards are required for each element starting with the elements of the first subsystem down to the elements of the last subsystem:

Card No.	Column No.	Description	Format
1	1	Theoretical distribution code number*.	I1
	2-10	Element identification code number.	I9
	11-80	Description of element.	Alphanumeric
2 to $n_f + n_s$	1	Code number for determining if data on card is a failure time, a suspension time or last card for current element**. n_f is the number of failures and n_s is the number of suspensions.	I1
	2-11	Failure or suspension time. If this is the last card then the remaining columns are blank.	F10.0
	12-80	Description of the failure or suspension.	Alphanumeric

* Distribution Code Number:
1 - Exponential
2 - Two parameter Weibull
3 - Three parameter Weibull
4 - Two parameter lognormal
5 - Three parameter lognormal
6 - Two parameter gamma
7 - Three parameter gamma
8 - Normal

** Failure or Suspension Code Number:
1 - Suspension
2 - Failure
3 - End of data

(c) Sample Problem

The following is a listing of data input for a hypothetical reliability data analysis problem:

```
SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL
         10 2000.0    0.70
          3          1          1          0
SAMPLE PROBLEM SUBSYSTEM A
SAMPLE PROBLEM SUBSYSTEM B
SAMPLE PROBLEM SUBSYSTEM C
1100000001    SAMPLE PROBLEM ELEMENT A1
2 1200.0      SPACE FOR FAILURE DESCRIPTION
2 900.0       SPACE FOR FAILURE DESCRIPTION
2 1500.0      SPACE FOR FAILURE DESCRIPTION
2 1300.0      SPACE FOR FAILURE DESCRIPTION
2 1350.0      SPACE FOR FAILURE DESCRIPTION
2 1000.0      SPACE FOR FAILURE DESCRIPTION
2 1300.0      SPACE FOR FAILURE DESCRIPTION
2 1100.0      SPACE FOR FAILURE DESCRIPTION
2 850.0       SPACE FOR FAILURE DESCRIPTION
3
1100000002    SAMPLE PROBLEM ELEMENT A2
2 1500.0      SPACE FOR FAILURE DESCRIPTION
1 8000.0      SPACE FOR SUSPENSION DESCRIPTION
3
1100000003    SAMPLE PROBLEM ELEMENT A3
1 9000.0      SPACE FOR SUSPENSION DESCRIPTION
3
2200000001    SAMPLE PROBLEM ELEMENT B1
2 1200.0      SPACE FOR FAILURE DESCRIPTION
2 900.0       SPACE FOR FAILURE DESCRIPTION
2 1500.0      SPACE FOR FAILURE DESCRIPTION
2 1300.0      SPACE FOR FAILURE DESCRIPTION
2 1350.0      SPACE FOR FAILURE DESCRIPTION
2 1000.0      SPACE FOR FAILURE DESCRIPTION
2 1300.0      SPACE FOR FAILURE DESCRIPTION
2 1100.0      SPACE FOR FAILURE DESCRIPTION
2 850.0       SPACE FOR FAILURE DESCRIPTION
3
4300000001    SAMPLE PROBLEM ELEMENT C1
2 1200.0      SPACE FOR FAILURE DESCRIPTION
2 900.0       SPACE FOR FAILURE DESCRIPTION
2 1500.0      SPACE FOR FAILURE DESCRIPTION
2 1300.0      SPACE FOR FAILURE DESCRIPTION
2 1350.0      SPACE FOR FAILURE DESCRIPTION
2 1000.0      SPACE FOR FAILURE DESCRIPTION
2 1300.0      SPACE FOR FAILURE DESCRIPTION
2 1100.0      SPACE FOR FAILURE DESCRIPTION
2 850.0       SPACE FOR FAILURE DESCRIPTION
3
```

APPENDIX II

OUTPUT DATA FOR SAMPLE PROBLEM

The following pages contain output data generated by programs DISTSEL and RELIAB for the sample input data presented in Appendix I.

RESULTS FROM PROGRAM DISTSEL

FAILED SPECIMENS AND MEDIAN RANKS...

850.0	0.07447	SPACE FOR FAILURE DESCRIPTION
900.0	0.18085	SPACE FOR FAILURE DESCRIPTION
1000.0	0.28723	SPACE FOR FAILURE DESCRIPTION
1100.0	0.39362	SPACE FOR FAILURE DESCRIPTION
1200.0	0.50000	SPACE FOR FAILURE DESCRIPTION
1300.0	0.60638	SPACE FOR FAILURE DESCRIPTION
1300.0	0.71277	SPACE FOR FAILURE DESCRIPTION
1350.0	0.81915	SPACE FOR FAILURE DESCRIPTION
1500.0	0.92553	SPACE FOR FAILURE DESCRIPTION

STANDARD ERROR OF ESTIMATE...

DISTRIBUTION	S.E.
EXPONENTIAL	0.280808E 00
NORMAL	0.566338E-01
LOG-NORMAL	0.657507E-01
WEIBULL-2 PARM.	0.530806E-01
WEIBULL-3 PARM.	0.642205E-01
GAMMA-2 PARM.	0.832009E-01

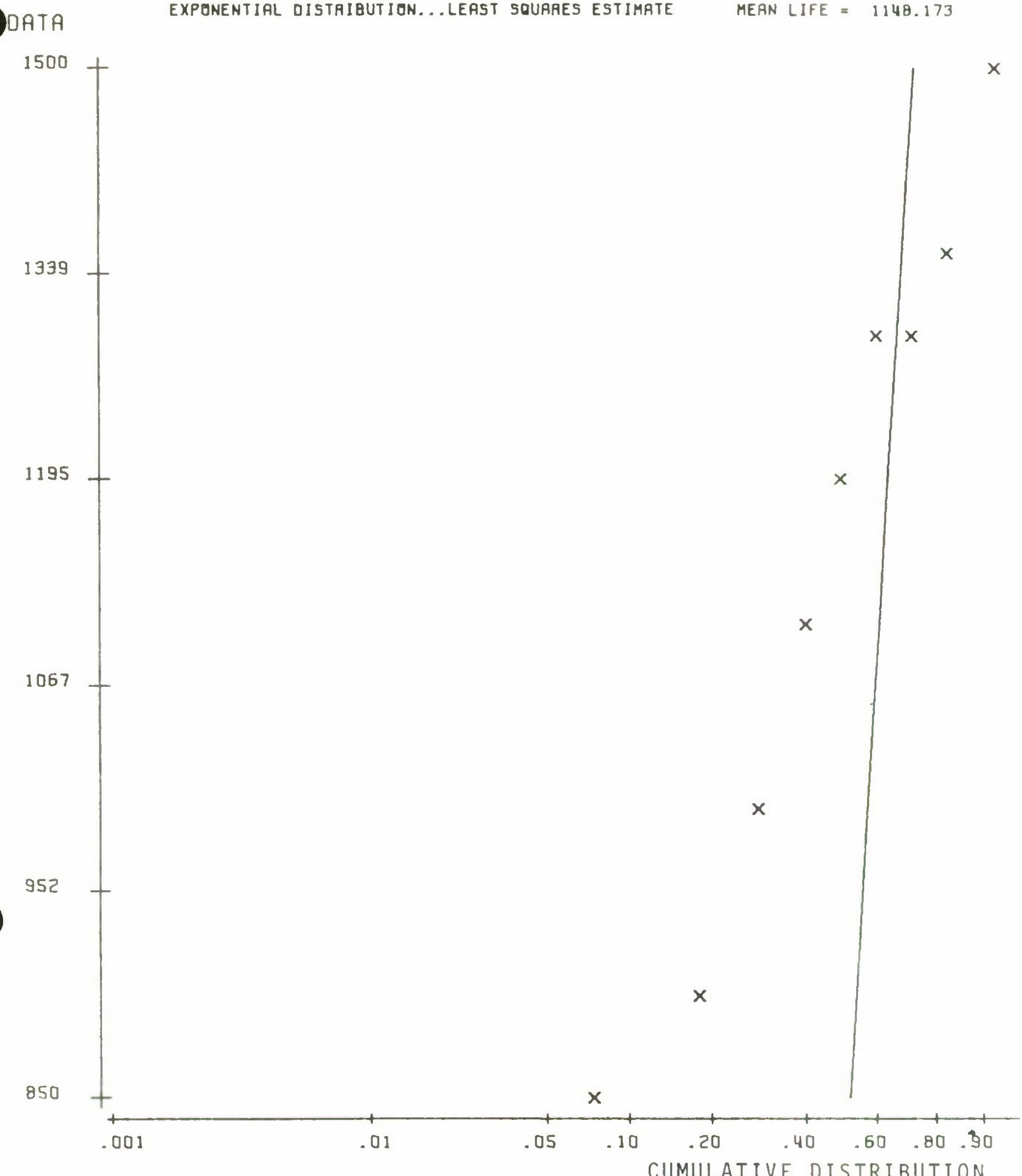
DISTRIBUTION SELECTED... WEIBULL-2 PARAMETER

K-S STATISTIC (0.141) IS LESS THAN TABLE VALUE (0.388)
THEREFORE MAY ACCEPT HYPOTHETICAL DISTRIBUTION AT 10 PERCENT LEVEL

SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL MARCH 1973

100000000 SAMPLE PROBLEM ELEMENT A1

EXPONENTIAL DISTRIBUTION...LEAST SQUARES ESTIMATE MEAN LIFE = 1148.173

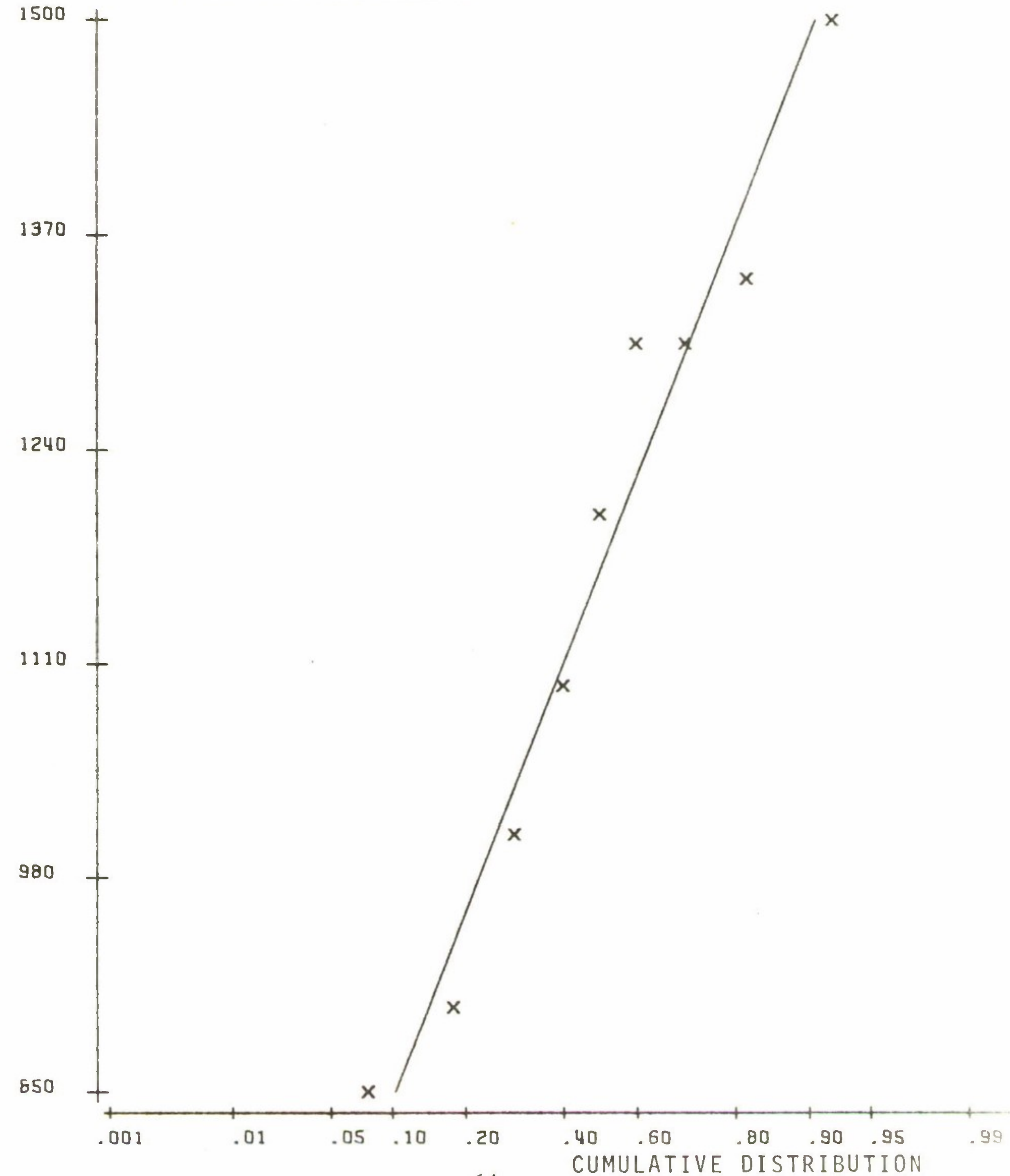


SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL MARCH 1973

100000000 SAMPLE PROBLEM ELEMENT A1

NORMAL DISTRIBUTION...LEAST SQUARES ESTIMATES MEAN LIFE = 1166.665

STANDARD DEVIATION =245.0730

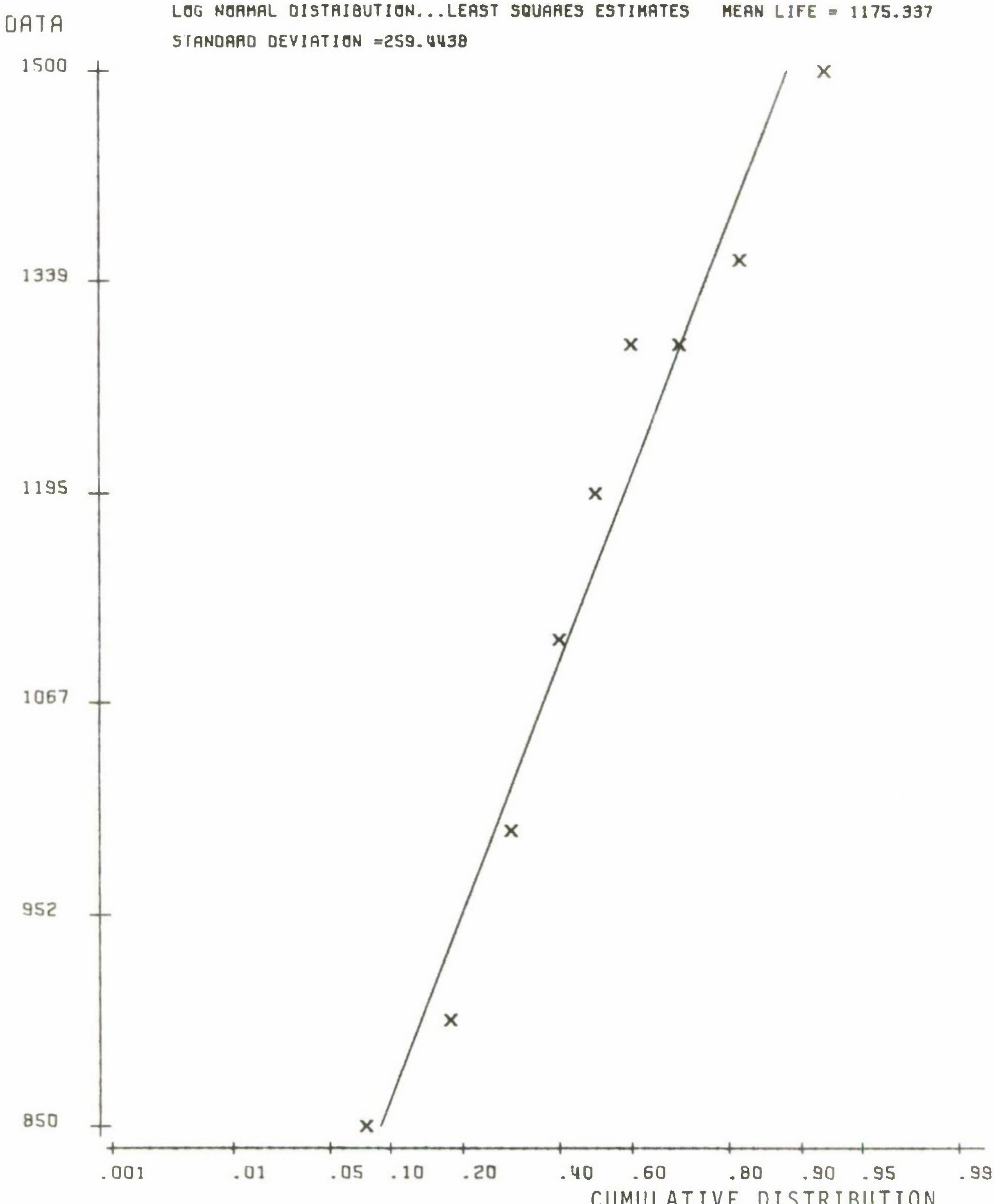


SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL   MARCH 1973

100000000   SAMPLE PROBLEM ELEMENT A1

LOG NORMAL DISTRIBUTION...LEAST SQUARES ESTIMATES   MEAN LIFE = 1175.337

STANDARD DEVIATION =259.4438

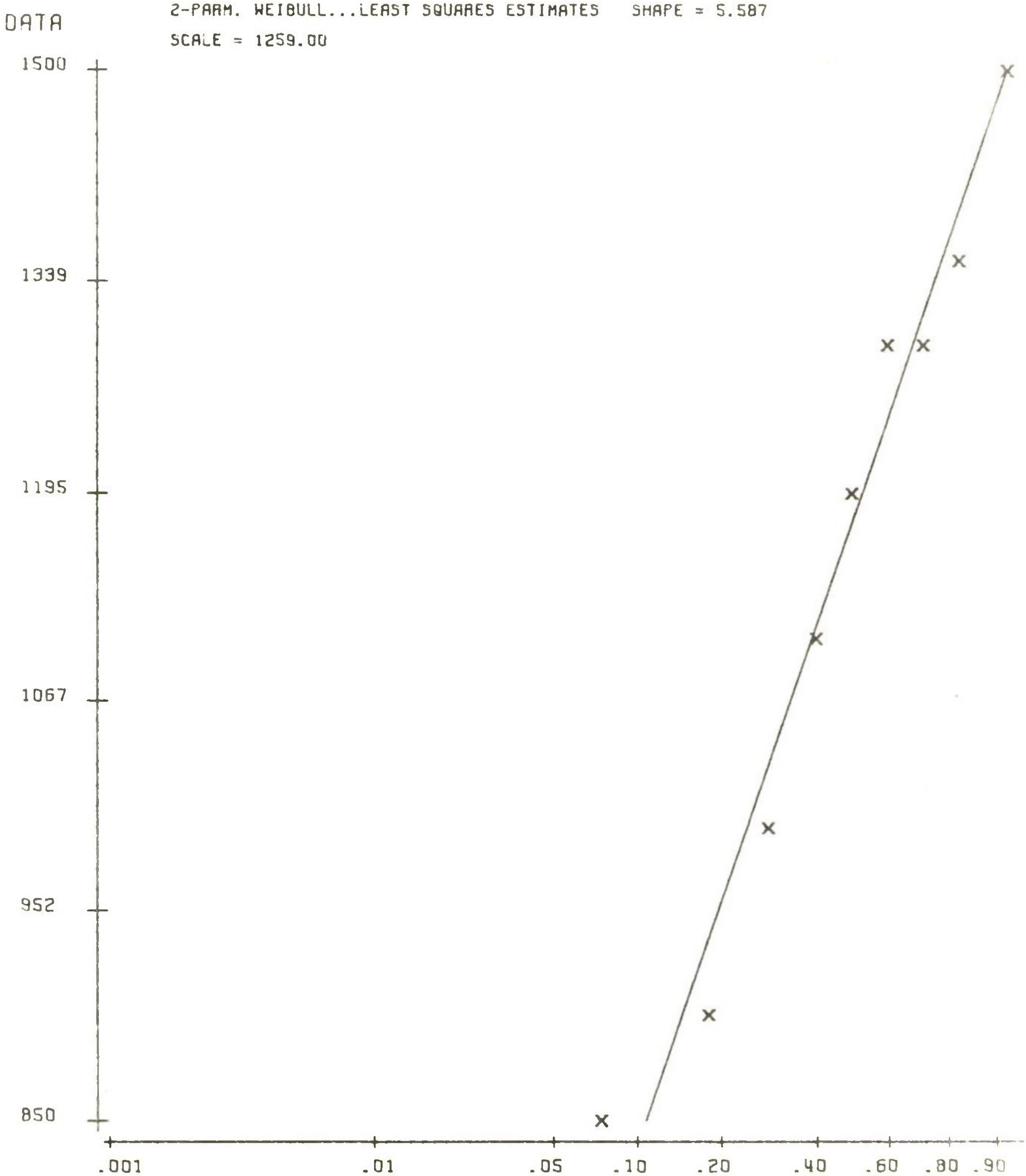


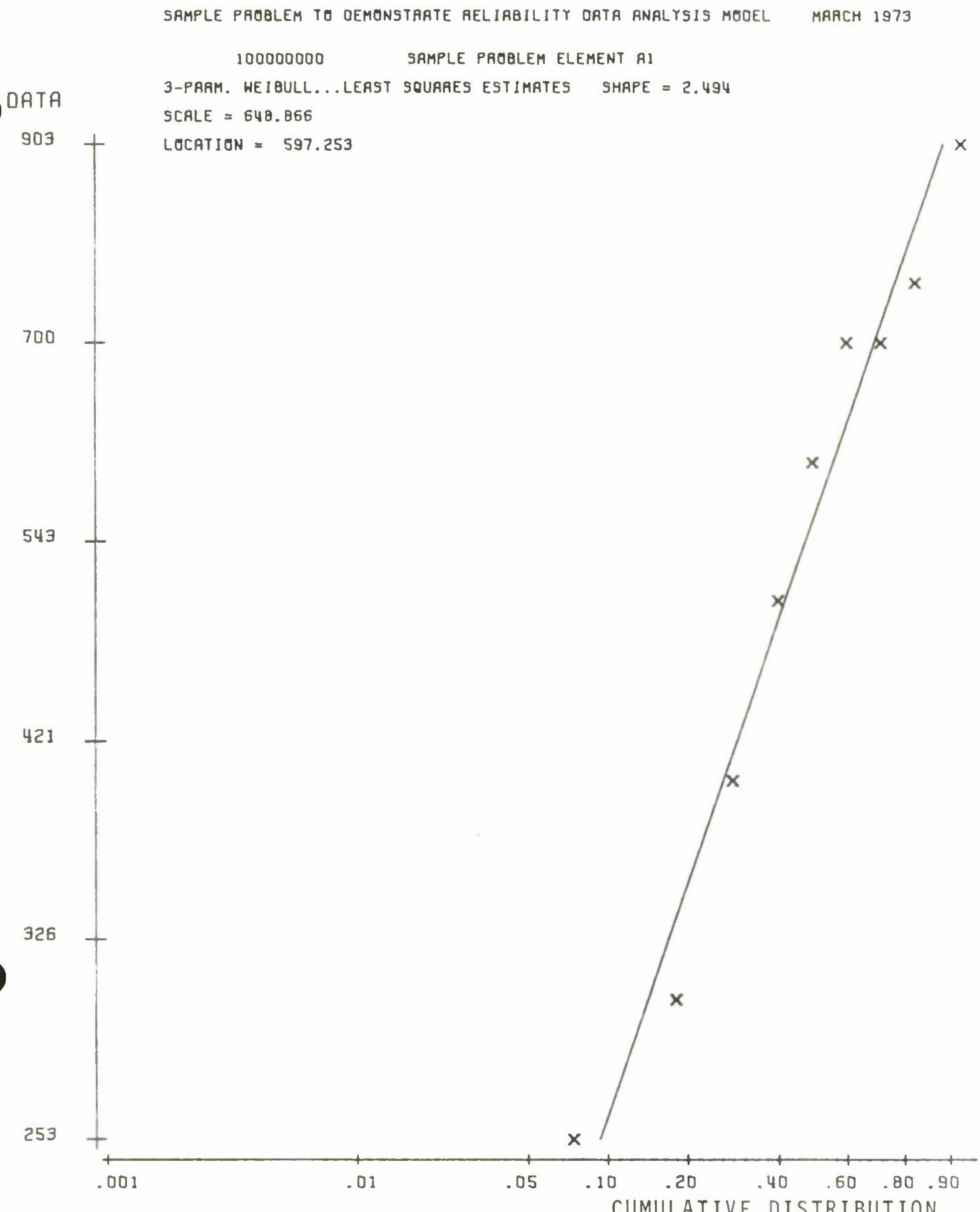
SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL MARCH 1973

100000000 SAMPLE PROBLEM ELEMENT A1

2-PARM. WEIBULL...LEAST SQUARES ESTIMATES SHAPE = 5.587

SCALE = 1259.00



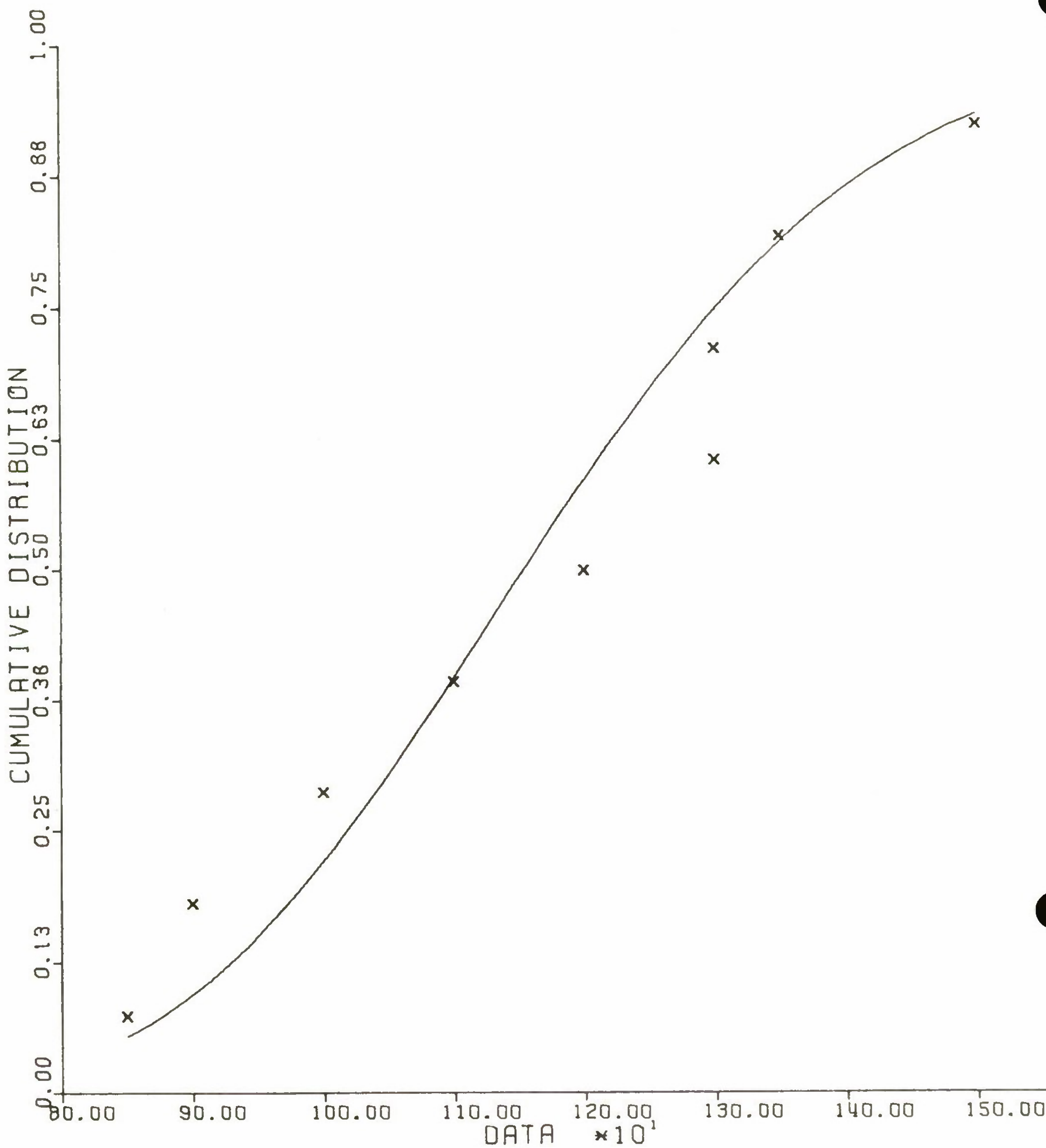

SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL MARCH 1973
100000000 SAMPLE PROBLEM ELEMENT A1
3-PARM. WEIBULL...LEAST SQUARES ESTIMATES SHAPE = 2.494
SCALE = 648.866
LOCATION = 597.253
DATA
903
700
543
421
326
253
.001
.01
.05
.10
.20
.40
.60
.80
.90
CUMULATIVE DISTRIBUTION

SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL   MARCH 1973

100000000   SAMPLE PROBLEM ELEMENT A1

GAMMA DISTRIBUTION...2-PARAMETER   MAXIMUM LIKELIHOOD ESTIMATES   ALPHA = 30.714

BETA = 37.984



```
                         RESULTS FROM PROGRAM RELIAB

SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL        MARCH 1973

THIS PROGRAM DETERMINES POINT ESTIMATES AND LOWER CONFIDENCE LIMITS
OF SYSTEM, SUBSYSTEM AND ELEMENT RELIABILITIES FROM ELEMENT DATA

EXPECTED OR SPECIFIED SYSTEM LIFE  =  2000.0000000000
NUMBER OF MISSIONS OVER SYSTEM LIFE=  10
MISSION LENGTH =   200.0000000000
SPECIFIED CONFIDENCE LEVEL =       0.7000000000
NUMBER OF SUBSYSTEMS =  3
NUMBER OF ELEMENTS FOR WHICH DATA IS SUPPLIED =   5

SUBSYSTEM...SAMPLE PROBLEM SUBSYSTEM A
         NUMBER OF ELEMENTS FOR THIS SUBSYSTEM =  3

SUBSYSTEM...SAMPLE PROBLEM SUBSYSTEM B
         NUMBER OF ELEMENTS FOR THIS SUBSYSTEM =  1

SUBSYSTEM...SAMPLE PROBLEM SUBSYSTEM C
         NUMBER OF ELEMENTS FOR THIS SUBSYSTEM =  1
```

```
    RESULTS OF COMPUTATIONS FOR ELEMENT ID NUMBER         100000001
     SAMPLE PROBLEM ELEMENT A1

NUMBER OF FAILURES =     9
NUMBER OF SUSPENSIONS =   0
TOTAL TEST TIME =         10500.0

SAMPLE OF FAILURE TIMES
           850.     SPACE FOR FAILURE DESCRIPTION
           900.     SPACE FOR FAILURE DESCRIPTION
          1000.     SPACE FOR FAILURE DESCRIPTION
          1100.     SPACE FOR FAILURE DESCRIPTION
          1200.     SPACE FOR FAILURE DESCRIPTION
          1300.     SPACE FOR FAILURE DESCRIPTION
          1300.     SPACE FOR FAILURE DESCRIPTION
          1350.     SPACE FOR FAILURE DESCRIPTION
          1500.     SPACE FOR FAILURE DESCRIPTION

ARITHEMATIC AVERAGE OF FAILURE TIMES =     1166.6666667
ARITHEMATIC STANDARD DEVIATION OF FAILURE TIMES =      219.3741097

THE THEORETICAL DISTRIBUTION OF FAILURE TIMES ASSUMED FOR THIS COMPONENT IS THE EXPONENTIAL

MAXIMUM LIKELIHOOD RESULTS FROM SUBROUTINE MLPMSD

SUMMARY OF MAXIMUM LIKELIHOOD ESTIMATES

SCALE PARAMETER =      1166.6666666667
AVERAGE =        1166.6666666667
STANDARD DEVIATION =     1166.6666666667
```

```
                    ELEMENT ID NUMBER        100000001

     SAMPLE PROBLEM ELEMENT A1

RESULTS FROM SUBROUTINE INTREL

DISTRIBUTION CODE =  1
SCALE PARAMETER =    1166.6665039
SHAPE PARAMETER =       0.0
LOCATION PARAMETER =       0.0
MAXIMUM TIME =    2000.0000000
NUMBER OF TIME INTERVALS =    0
NUMBER OF MISSIONS =    10

COMPUTED RESULTS

AVERAGE INTERVAL RELIABILITY = 0.8424605

INTERVAL RELIABILITY FOR EVERY MISSION
       0.8424605     0.8424605     0.8424605     0.8424605     0.8424605
       0.8424605     0.8424605     0.8424605     0.8424605     0.8424605

COMPONENT CONFIDENCE LIMIT ON RELIABILITY IS PRESENTLY
DETERMINED ASSUMING CONSTANT FAILURE RATE

SPECIFIED CONFIDENCE LEVEL =    0.7000000000
LOWER CONFIDENCED RELIABILITY =    0.8050100940
```

```
    RESULTS OF COMPUTATIONS FOR ELEMENT ID NUMBER        100000002
     SAMPLE PROBLEM ELEMENT A2

NUMBER OF FAILURES =    1
NUMBER OF SUSPENSIONS =    1
TOTAL TEST TIME =         9500.00

SAMPLE OF FAILURE TIMES
          1500.     SPACE FOR FAILURE DESCRIPTION

SAMPLE OF SUSPENSION TIMES
          8000.        SPACE FOR SUSPENSION DESCRIPTION

ARITHEMATIC AVERAGE OF FAILURE TIMES =    1500.0000000

THE THEORETICAL DISTRIBUTION OF FAILURE TIMES ASSUMED FOR THIS COMPONENT IS THE
EXPONENTIAL

MAXIMUM LIKELIHOOD RESULTS FROM SUBROUTINE MLPMSD

SUMMARY OF MAXIMUM LIKELIHOOD ESTIMATES

SCALE PARAMETER =     9500.0000000000
AVERAGE =     9500.0000000000
STANDARD DEVIATION =     9500.0000000000
```

```
                    ELEMENT ID NUMBER          100000002
     SAMPLE PROBLEM ELEMENT A2

RESULTS FROM SUBROUTINE INTREL

DISTRIBUTION CODE =  1
SCALE PARAMETER =    9500.0000000
SHAPE PARAMETER =       0.0
LOCATION PARAMETER =       0.0
MAXIMUM TIME =    2000.0000000
NUMBER OF TIME INTERVALS =    0
NUMBER OF MISSIONS =    10

COMPUTED RESULTS

AVERAGE INTERVAL RELIABILITY = 0.9791675

INTERVAL RELIABILITY FOR EVERY MISSION
         0.9791675     0.9791675      0.9791675      0.9791675      0.9791675
         0.9791675     0.9791675      0.9791675      0.9791675      0.9791675

COMPONENT CONFIDENCE LIMIT ON RELIABILITY IS PRESENTLY
DETERMINED ASSUMING CONSTANT FAILURE RATE

SPECIFIED CONFIDENCE LEVEL =    0.7000000000
LOWER CONFIDENCED RELIABILITY =    0.9499443016
```

```
    RESULTS OF COMPUTATIONS FOR ELEMENT ID NUMBER        100000003
     SAMPLE PROBLEM ELEMENT A3

NUMBER OF FAILURES =     0
NUMBER OF SUSPENSIONS =     1
TOTAL TEST TIME =          9000.00

SAMPLE OF SUSPENSION TIMES
          9000.     SPACE FOR SUSPENSION DESCRIPTION

THE THEORETICAL DISTRIBUTION OF FAILURE TIMES ASSUMED FOR THIS COMPONENT IS THE
EXPONENTIAL

MAXIMUM LIKELIHOOD RESULTS FROM SUBROUTINE MLPMSD

NO FAILURE DATA INDICATES INFINITE POINT ESTIMATE OF MEAN
```

```
                    ELEMENT ID NUMBER        100000003
    SAMPLE PROBLEM ELEMENT A3

RESULTS FROM SUBROUTINE INTREL

DISTRIBUTION CODE =  1
SCALE PARAMETER = ****************
SHAPE PARAMETER =      0.0
LOCATION PARAMETER =      0.0
MAXIMUM TIME =    2000.0000000
NUMBER OF TIME INTERVALS =     0
NUMBER OF MISSIONS =    10

COMPUTED RESULTS

AVERAGE INTERVAL RELIABILITY  = 1.0000000

INTERVAL RELIABILITY FOR EVERY MISSION
      1.0000000       1.0000000       1.0000000       1.0000000       1.0000000
      1.0000000       1.0000000       1.0000000       1.0000000       1.0000000

COMPONENT CONFIDENCE LIMIT ON RELIABILITY IS PRESENTLY
DETERMINED ASSUMING CONSTANT FAILURE RATE

SPECIFIED CONFIDENCE LEVEL =      0.7000000000
LOWER CONFIDENCED RELIABILITY =     0.9735997829
```

```
RESULTS OF COMPUTATIONS FOR ELEMENT ID NUMBER        200000001
 SAMPLE PROBLEM ELEMENT B1

NUMBER OF FAILURES =      9
NUMBER OF SUSPENSIONS =     0
TOTAL TEST TIME =        10500.00

SAMPLE OF FAILURE TIMES
            850.     SPACE FOR FAILURE DESCRIPTION
            900.     SPACE FOR FAILURE DESCRIPTION
           1000.     SPACE FOR FAILURE DESCRIPTION
           1100.     SPACE FOR FAILURE DESCRIPTION
           1200.     SPACE FOR FAILURE DESCRIPTION
           1300.     SPACE FOR FAILURE DESCRIPTION
           1300.     SPACE FOR FAILURE DESCRIPTION
           1350.     SPACE FOR FAILURE DESCRIPTION
           1500.     SPACE FOR FAILURE DESCRIPTION

ARITHEMATIC AVERAGE OF FAILURE TIMES =     1166.6666667
ARITHEMATIC STANDARD DEVIATION OF FAILURE TIMES =     219.3741007

THE THEORETICAL DISTRIBUTION OF FAILURE TIMES ASSUMED FOR THIS COMPONENT IS THE
TWO PARAMETER WEIBULL

MAXIMUM LIKELIHOOD RESULTS FROM SUBROUTINE MLPMSD

CONVERGENCE HAS BEEN ACHIEVED

SUMMARY OF MAXIMUM LIKELIHOOD ESTIMATES

SCALE PARAMETER =      1253.3223250387
SHAPE PARAMETER =         6.5319380785
AVERAGE =      1168.1272794997
STANDARD DEVIATION =      209.2752304140
```

```
                     ELEMENT ID NUMBER         200000001
    SAMPLE PROBLEM ELEMENT B1

RESULTS FROM SUBROUTINE INTREL

DISTRIBUTION CODE =  2
SCALE PARAMETER  =    1253.3222656
SHAPE PARAMETER  =       6.5319376
LOCATION PARAMETER         0.0
MAXIMUM TIME  =    2000.0000000
NUMBER OF TIME INTERVALS =  20
NUMBER OF MISSIONS =      10

TIME
          0.0          100.00000       200.00000       300.00000       400.00000
        500.00000      600.00000       700.00000       800.00000       900.00000
       1000.00000     1100.00000      1200.00000      1300.00000      1400.00000
       1500.00000     1600.00000      1700.00000      1800.00000      1900.00000
       2000.00000

RENEWAL RATE
       0.0            0.00000000      0.00000020      0.00000191      0.00000939
       0.00003222     0.00008785      0.00020321      0.00041238      0.00074388
       0.00118923     0.00165421      0.00193338      0.00180027      0.00124383
       0.00059645     0.00022871      0.00017360      0.00026958      0.00043246
       0.00064550

COMPUTED RESULTS

AVERAGE INTERVAL RELIABILITY =  0.8868418

INTERVAL RELIABILITY FOR EVERY MISSION
       0.9999927      0.9994249       0.9924625       0.9562320       0.8474288
       0.6753514      0.6540568       0.8713890       0.9602432       0.9118366
```

(Continued)

(Continued)

```
COMPONENT CONFIDENCE LIMIT ON RELIABILITY IS PRESENTLY
DETERMINED ASSUMING CONSTANT FAILURE RATE

SPECIFIED CONFIDENCE LIVEL =     0.7000000000
LOWER CONFIDENCED RELIATILITY =     0.8050100940
```

```
    RESULTS OF COMPUTATIONS FOR ELEMENT ID NUMBER        300000001
     SAMPLE PROBLEM ELEMENT C1

NUMBER OF FAILURES =     9
NUMBER OF SUSPENSIONS =    0
TOTAL TEST TIME =        10500.00

SAMPLE OF FAILURE TIMES
            850.     SPACE FOR FAILURE DESCRIPTION
            900.     SPACE FOR FAILURE DESCRIPTION
           1000.     SPACE FOR FAILURE DESCRIPTION
           1100.     SPACE FOR FAILURE DESCRIPTION
           1200.     SPACE FOR FAILURE DESCRIPTION
           1300.     SPACE FOR FAILURE DESCRIPTION
           1300.     SPACE FOR FAILURE DESCRIPTION
           1350.     SPACE FOR FAILURE DESCRIPTION
           1500.     SPACE FOR FAILURE DESCRIPTION

ARITHEMATIC AVERAGE OF FAILURE TIMES =     1166.6666667
ARITHEMATIC STANDARD DEVIATION OF FAILURE TIMES =      219.3741097

THE THEORETICAL DISTRIBUTION OF FAILURE TIMES ASSUMED FOR THIS COMPONENT IS THE
TWO PARAMETER LOGNORMAL

MAXIMUM LIKELIHOOD RESULTS FROM SUBROUTINE MLPMSD

CONVERGENCE HAS BEEN ACHIEVED

SUMMARY OF MAXIMUM LIKELIHOOD ESTIMATES

SCALE PARAMETER =        7.0455385748
SHAPE PARAMETER =        0.1829129466
AVERAGE =     1167.0881299857
STANDARD DEVIATION =      215.2736078302
```

```
                    ELEMENT ID NUMBER        300000001
   SAMPLE PROBLEM ELEMENT C1

RESULTS FROM SUBROUTINE INTREL

DISTRIBUTION CODE =  4
SCALE PARAMETER  =        7.0455379
SHAPE PARAMETER  =        0.1829129
LOCATION PARAMETER =          0.0
MAXIMUM TIME =   2000.0000000
NUMBER OF TIME INTERVALS =   20
NUMBER OF MISSIONS =     10

TIME
          0.0          100.00000        200.00000        300.00000        400.00000
      500.00000        600.00000        700.00000        800.00000        900.00000
     1000.00000       1100.00000       1200.00000       1300.00000       1400.00000
     1500.00000       1600.00000       1700.00000       1800.00000       1900.00000
     2000.00000

RENEWAL RATE
     0.0              0.00000000       0.00000000       0.00000000       0.00000000
     0.00000014       0.00000676       0.00008068       0.00038913       0.00100167
     0.00164231       0.00193004       0.00176446       0.00133054       0.00086481
     0.00050601       0.00029552       0.00023493       0.00032048       0.00053527
     0.00082720

COMPUTED RESULTS

AVERAGE INTERVAL RELIABILITY  = 0.8870960

INTERVAL RELIABILITY FOR EVERY MISSION
     1.0000000        1.0000000        0.9997554        0.9760462        0.7987295
     0.6291037        0.7349525        0.8938541        0.9481429        0.8903761    (Continued)
```

(Continued)

```
COMPONENT CONFIDENCE LIMIT ON RELIABILITY IS PRESENTLY
DETERMINED ASSUMING CONSTANT FAILURE RATE

SPECIFIED CONFIDENCE LEVEL =     0.7000000000
LOWER CONFIDENCED RELIABILITY  =    0.8050100940
```

SUMMARY OF ELEMENT RELIABILITY COMPUTATIONS

ELEMENT ID NUMBER	NUMBER OF FAILURES	NUMBER OF SUSPENSIONS	TOTAL TEST TIME	DISTRIBUTION CODE	MEAN TIME BETWEEN FAILURE
100000001	9	0	10500.0	1	1166.7
100000002	1	1	9500.0	1	9500.0
100000003	0	1	9000.0	1	*********
200000001	9	0	10500.0	2	1168.1
300000001	9	0	10500.0	4	1167.1

ELEMENT ID NUMBER	POINT ESTIMATE AVG. RELIABILITY	LOWER CONFIDENCED AVG. RELIABILITY (CONSTANT F.R.)
100000001	0.8424605	0.8050101
100000002	0.9791675	0.9499443
100000003	1.0000000	0.9735998
200000001	0.8868418	0.8050101
300000001	0.8870960	0.8050101

SPECIFIED CONFIDENCE LEVEL = 0.70000000
MISSION TIME = 200.0

DESCRIPTION OF ELEMENTS

SUBSYSTEM...SAMPLE PROBLEM SUBSYSTEM A
100000001 SAMPLE PROBLEM ELEMENT A1
100000002 SAMPLE PROBLEM ELEMENT A2
100000003 SAMPLE PROBLEM ELEMENT A3

(Continued)

(continued)

```
SUBSYSTEM...SAMPLE PROBLEM SUBSYSTEM B
          200000001      SAMPLE PROBLEM ELEMENT B1

SUBSYSTEM...SAMPLE PROBLEM SUBSYSTEM C
          300000001      SAMPLE PROBLEM ELEMENT C1
```

```
                 RESULTS OF SUBSYSTEM RELIABILITY COMPUTATIONS

CONFIDENCED RELIABILITY FOR SUBSYSTEM NUMBER 1
                 POINT ESTIMATE OF MISSION RELIABILITY  =   0.8249098636

                 SUBSYSTEM CONFIDENCED RELIABILITY IS CURRENTLY
                 DETERMINED ASSUMING CONSTANT FAILURE RATE

                 SPECIFIED CONFIDENCE LEVEL  =   0.7000000000
                 LOWER CONFIDENCED MISSION RELIABILITY  =   0.7798356233


CONFIDENCED RELIABILITY FOR SUBSYSTEM NUMBER 2

                 POINT ESTIMATE OF MISSION  RELIABILITY  =   0.8868417740

                 SUBSYSTEM CONFIDENCED RELIABILITY IS CURRENTLY
                 DETERMINED ASSUMING CONSTANT FAILURE RATE

                 SPECIFIED CONFIDENCE LEVEL  =   0.7000000000
                 LOWER CONFIDENCED MISSION RELIABILITY  =   0.8050101907


CONFIDENCED RELIABILITY FOR SUBSYSTEM NUMBER 3

                 POINT ESTIMATE OF MISSION RELIABILITY  =   0.8870960474

                 SUBSYSTEM CONFIDENCED RELIABILITY IS CURRENTLY
                 DETERMINED ASSUMING CONSTANT FAILURE RATE

                 SPECIFIED CONFIDENCE LEVEL  =   0.7000000000
                 LOWER CONFIDENCED MISSION RELIABILITY  =   0.8050101907
```

SUMMARY OF SUBSYSTEM RELIABILITY COMPUTATIONS

SUBSYSTEM NUMBER	NUMBER OF ELEMENTS	POINT ESTIMATE AVG. RELIABILITY	LOWER CONFIDENCED AVG. RELIABILITY (CONSTANT F.R.)
1	3	0.8249099	0.7798356
2	1	0.8868418	0.8050102
3	1	0.8870960	0.8050102

SPECIFIED CONFIDENCE LEVEL = 0.7000000
MISSION TIME = 200.0

DESCRIPTION OF SUBSYSTEMS
SUBSYSTEM NUMBER 1...SAMPLE PROBLEM SUBSYSTEM A
SUBSYSTEM NUMBER 2...SAMPLE PROBLEM SUBSYSTEM B
SUBSYSTEM NUMBER 3...SAMPLE PROBLEM SUBSYSTEM C

```
SAMPLE PROBLEM TO DEMONSTRATE RELIABILITY DATA ANALYSIS MODEL  MARCH 1973
CONFIDENCED RELIABILITY FOR TOTAL SYSTEM

            POINT ESTIMATE OF MISSION RELIABILITY =  0.6489680001

            TOTAL SYSTEM CONFIDENCED RELIABILITY IS CURRENTLY
            DETERMINED ASSUMING CONSTANT FAILURE RATE

            SPECIFIED CONFIDENCE LEVEL =  0.7000000000
            LOWER CONFIDENCED MISSION RELIABILITY  =  0.5423481333
```