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Gedeon Dagan

Hydronautics, Incorporated

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13. ABSTRACT

Criteria of uniformity of the linearized theory of free-surface gravity flow past submerged or floating bodies based on the thin body perturbation expansion, combining ϵ, slenderness parameter, and F, Froude number, are derived for both two- and three-dimensional flows. These criteria depend on the shape of the leading (and trailing) edge; as the shape becomes finer the linearized solution becomes valid for smaller F.

Uniform first order approximations are derived by two alternative methods: velocity straining and coordinate straining. In the first case the uniform unperturbed velocity in the free-surface condition is replaced by a variable velocity distribution. The second method leads to an apparent displacement of the most singular points of the body skeleton. In both cases the parameter ϵ appears not only in the wave amplitude, as implied by the thin body expansion, but also in the wave number function. The nonuniformity of the usual thin body expansion is, therefore, similar to that encountered in problems characterized by multiple scales.

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Notation

a_1, a_2	-	first and second order dimensionless amplitude of free waves in two-dimensional flow
A	-	velocity straining function
b_2, c_2	-	coefficients in the solution of flow past a source-sink body
B'	-	beam (half)
B	-	auxiliary function
C	-	Euler constant
D'	-	wave drag in two dimensions $(D = D'/0.5\ \rho' U'^2 L')$
f'(z')	-	complex potential in two dimensions $(f = f'/U'L')$
f'(y',z')	-	ship surface equation in three dimensions $(f = f'/L')$
$f^W(z)$	-	dimensionless free waves potential
g	-	acceleration of gravity
h'	-	submergence depth $(h = h'/L')$
Im	-	imaginary part
L'	-	reference length (generally body length)
P,Q	-	wave spectrum functions
R'	-	wave drag in three dimensions $(R = R'/0.5\ \rho' U'^2 L'^2)$
Re	-	real part
T'	-	maximum half thickness of the body in two dimensions
t(x)	-	dimensionless thickness distribution

x'	-	horizontal coordinate positive in the direction of motion of the body ($x = x'/L'$)
y'	-	upwards vertical coordinate in two dimensions; horizontal, normal to x', coordinate in three dimensions ($y = y'/L'$)
z'	-	complex variable $z' = x' + iy'$ in two dimensions, upwards vertical coordinate in three dimensions ($z = z'/L'$)
$\bar{z}$	-	complex conjugate of $z = x + iy$
u', v'	-	velocity components in x' and y' directions, respectively ($u = u'/L'$; $v = v'/L'$)
w'	-	complex velocity $w' = u' - iv'$ in two dimensions, vertical velocity component in three dimensions ($w = w'/U'$)
U'	-	velocity of uniform flow at infinity upstream
α, β	-	coordinates in the Fourier transform plane
δ	-	small angle
δz, δx	-	coordinate straining functions
ε	-	slenderness parameter ($\varepsilon = T'/L'$ in two dim., $\varepsilon = B'/L'$ for thin ships)
μ	-	artificial viscosity
$\lambda, \nu, \tau, \sigma, \rho$	-	auxiliary variables
ρ'	-	water density
ϕ'	-	velocity potential ($\phi = \phi'/U'L'$)
ψ'	-	stream function ($\psi = \psi'/U'L'$)
$\tilde{\phi}$	-	Fourier transform of ϕ
ω	-	function related to the exponential integral
$\tau(x)$	-	slope of the body profile in two dimensions.

A b s t r a c t

The linearized theory of free-surface gravity flow past submerged or floating bodies is based on a perturbation expansion of the velocity potential in the slenderness parameter ϵ, while the Froude number F is kept fixed. It is shown that although the free waves amplitude, and the associated wave resistance, tend to zero as $F \to 0$, the linearized solution is not uniform under this limit: the ratio between the second order and first order terms becomes unbounded for $F \to 0$ and fixed ϵ. This nonuniformity (called "the second Froude number paradox" in a previous work) is related to the nonlinearity of the free-surface condition. Criteria of uniformity of the thin body expansion, combining ϵ and F, are derived for both two- and three-dimensional flows. These criteria depend on the shape of the leading (and trailing) edge: as the shape becomes finer the linearized solution becomes valid for smaller F.

Uniform first order approximations are derived by two alternative methods: velocity straining and coordinate straining. In the first case the uniform unperturbed velocity in the free-surface condition is replaced by a variable velocity distribution. The second method leads to an apparent displacement of the most singular points of the body skeleton. In both cases the parameter ϵ appears not only in the wave amplitude, as implied by the thin body expansion, but also in the wave number function. The nonuniformity of the usual thin body expansion is, therefore, similar to that encountered in problems characterized by multiple scales.

INTRODUCTION

The linearized theory of free-surface gravity flow past submerged or floating bodies is based on the assumption that the body causes a small disturbance of a uniform flow. Such an approximation is incorporated in a systematic asymptotic expansion of the velocity potential by assuming that ϵ (beam/length ratio for a thin ship, draft/length ratio for a flat ship, body length submergence depth ratio in the case of deep submergence) tends to zero while the Froude number F (based on body length or submergence depth respectively) is kept fixed.

In previous works (Salvesen, 1969; Dagan, 1972a) it has been shown that it is not legitimate to let $F \to 0$, for a fixed ϵ, in the linearized solution, or in other words that the usual approximation is not uniform in F. Two "small Froude number paradoxes" have been formulated in this context (Dagan, 1972b) and ad-hoc uniformization procedures have been suggested (Ogilvie, 1968; Dagan, 1972b), leading to a quasi-linearization of the free-surface condition. It has been proved (Tuck, 1965; Salvesen, 1969; Dagan, 1972a) that the small Froude number nonuniformity is associated with the nonlinearity of the free-surface condition. In all cases detailed computations have been carried out only for two-dimensional flows.

In the present study the problem of the small Froude number nonuniformity is attacked in a different way, the results being also different of those obtained previously. For the first time the influence of the bluntness of the bow on the small F solution is discussed in detail and the analysis is extended to three-dimensional flow in general and to flow past thin ships in particular.

It is worthwhile to mention here that the problem is related mainly to three-dimensional applications, since a large class of ships operate at relatively low Froude numbers and in most such cases the usual theory of wave resistance has been found to be unsatisfactory. We begin, nevertheless, with studying the two-dimensional flow because the use of the powerful tool of analytical functions in this case permits to clarify some matters of principle much easier than in three-dimensions.

Obviously, there are various possible factors related to the discrepancy between wave resistance as measured in experiments and as predicted by the linearized theory, like viscous effects or the bow breaking wave. This should not deter us, however, from seeking a consistent solution for the wave resistance in the frame of the potential flow theory.

PART I

TWO-DIMENSIONAL FLOW PAST SUBMERGED BODIES

1. The Thin Body Expansion

We consider a steady uniform flow from infinity past a submerged body (Fig. 1). Let $z' = x' + iy'$ be a complex variable, $w' = u' - iv'$ the complex velocity, $f' = \phi' + i\psi'$ the complex potential, η' the free surface elevation, 2L' the body length, h' the submergence depth, 2T' the thickness and U' the velocity of uniform flow. First, variables are made dimensionless by referring them to L' and U', i.e., $z = z'/L'$; $f = f'/L'$; $w = w'/U'$; $h = h'/L'$; $\epsilon = T'/L'$ and $F = U'/(gL')^{1/2}$.

Under an expansion of the analytical function $f(z;\epsilon,h,F)$ in a small ϵ asymptotic series

$$f = -z + \epsilon f_1(z;h,F) + \epsilon^2 f_2(z;h,F) + \ldots \tag{1.1}$$

the following sets of equations are obtained for f_1 and f_2 from the expansion of the exact equations (Wehausen and Laitone, 1960)

$$\left.\begin{aligned} &\operatorname{Im}\left(iF^2 \frac{df_1}{dz} - f_1\right) = 0 && (1.2) \\ &\eta_1 = \psi_1 && (1.3) \end{aligned}\right\} \quad (y = 0)$$

$$f_1 \to 0 \qquad (x \to \infty \; ; \; y \to -\infty) \tag{1.4}$$

$$\psi_1 = \mp t(x) \qquad (|x| < 1 \, , \, y = -h \pm 0) \tag{1.5}$$

where $\eta = \epsilon\eta_1 + \epsilon^2\eta_2 + \ldots$ and $y = -h \pm \epsilon t(x)$ is the equation of the body profile, assumed to be symmetrical for the sake of simplicity;

$$\left.\begin{array}{ll} \mathrm{Im}\left(iF^2 \dfrac{df_2}{dz} - f_2\right) = P_2(x) = -F^2\left(\dfrac{3}{2}u_1^2 + \dfrac{1}{2}v_1^2\right) + F^4 u_1 \dfrac{\partial u_1}{\partial x} & (1.6) \\ \eta_2 = \psi_2 + \psi_1 u_1 & (1.7) \end{array}\right\} (y=0)$$

$$f_2 \rightarrow 0 \qquad (x \rightarrow \infty\ ;\ y \rightarrow -\infty) \qquad (1.8)$$

$$\psi_2 = \mp u_1 t \qquad (|x| < 1\ ,\ y = -h \pm 0) \qquad (1.9)$$

In addition, a Kutta-Jonkovsky condition has to be imposed in the case of a sharp trailing edge in order to make circulation unique.

It can be shown (Salvesen, 1969) that far behind the body the stream-function has the expressions

$$\psi_1 = \mathrm{Im}(a_1\ e^{-ix}) \qquad (x \rightarrow -\infty) \qquad (1.10)$$

$$\psi_2 = \mathrm{Im}(a_2\ e^{-ix}) + \mathrm{const} \qquad (x \rightarrow -\infty) \qquad (1.11)$$

The wave resistance is given by the following expression

$$D = \frac{1}{2F^2}\left|\epsilon a_1 + \epsilon^2 a_2\right|^2 \qquad (1.12)$$

where $D = D'/0.5\ \rho' U'^2 L'$. Hence, by expanding D

$$D = \epsilon^2 D_1 + \epsilon^3 D_2 + O(\epsilon^4) \qquad (1.13)$$

we have

$$D_1 = \frac{1}{2F^2} |a_1|^2 \quad ; \quad D_2 = \frac{1}{F^2} \mathrm{Re}(a_1 \bar{a}_2) \quad . \tag{1.14}$$

The method of determining f_1 and f_2, solutions of Eqs. (1.2) - (1.8), is well known. Let w_1^{ℓ} and w_1^{u} be the first order linearized solutions of the velocity of flow past the body or its image, respectively, in an infinite domain, i.e.,

$$w_1^{\ell}(z) = -\frac{\varepsilon}{\pi} \int_{-1-ih}^{1-ih} \tau(x_s) \frac{dz_s}{z - z_s} \tag{1.15}$$

where $z_s = x_s + iy_s$ is the coordinate of a point along the skeleton $(|x_s| < 1,\ y_s = -h)$, $\mathrm{Re}\ \tau = dt/dx$ and

$$w_1^{u}(z) = -\frac{\varepsilon}{\pi} \int_{-1+ih}^{1+ih} \bar{\tau}(x_s) \frac{d\bar{z}_s}{z - \bar{z}_s} = \bar{w}_1^{\ell}(\bar{z}) \quad . \tag{1.16}$$

Then, the solution for f_1 may be written as

$$f_1 = f_1^{\ell} + f_1^{u} - \frac{i}{\pi} \int_{-\infty}^{\infty} w_1^{u}(\sigma)\, \omega\left(\frac{z-\sigma}{F^2}\right) d\sigma \qquad (\mathrm{Im}\, z < \mathrm{Im}\, \sigma) \tag{1.17}$$

where

$$\omega(\zeta) = e^{-i\zeta} \int_{\zeta}^{\infty - io} \frac{e^{i\lambda}}{\lambda} d\lambda = \int_{0}^{\infty} \frac{e^{i\rho}}{\rho + \zeta} d\rho \tag{1.18}$$

the λ plane being cut along $\mathrm{Im}\, \lambda = 0$, $\mathrm{Re}\, \lambda > 0$, and the ρ plane is along $\mathrm{Im}(\rho+\zeta) = 0$, $\mathrm{Re}(\rho+\zeta) > 0$.

The second order solution satisfying (1.6) and regular in the lower half plane (we consider here only the free-surface second order effect and disregard (1.9), which leads to less severe effects as $F \to 0$) may be written as

$$f_2(z) = \frac{i}{\pi F^2} \int_{-\infty}^{\infty} P_2(\sigma)\,\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma \qquad (\mathrm{Im}\, z < \mathrm{Im}\, \sigma) . \qquad (1.19)$$

2. The Second Order Solution (free-surface effect)

We are going now to transform (1.19) such that f_2 will be expressed as an integral over analytical functions of σ . First, we have, by integration by parts

$$\int_{-\infty}^{\infty} u_1 \frac{\partial v_1}{\partial \sigma}\,\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma = -\frac{i}{F^2}\int_{-\infty}^{\infty}(u_1 v_1 + v_1^2)\;\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma - $$
$$- \int_{-\infty}^{\infty} \frac{u_1 v_1}{z-\sigma}\,d\sigma \quad . \qquad (1.20)$$

As far as the free waves are concerned, the last integral in (1.20) renders the well-known Stokes second order waves of amplitude $O(e^{-2h/F^2})$ Under the limit $F \to 0$ these waves are negligible, as compared to the remaining terms which are $O(e^{-h/F^2})$, and will be neglected in the sequel.

Hence, by (1.19) and (1.20) we have under these conditions

$$f_2 \cong \frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{2}v_1^2 - \frac{3}{2}u_1^2 - iu_1 v_1\right)\;\omega\left(\frac{z-\sigma}{F^2}\right)d\sigma \qquad (1.21).$$

u_1 and v_1 , which are obtained from (1.17), may be written along the real axis as follows

$$u_1(x) = -\frac{1}{2\pi F^2}\int_{-\infty+io}^{\infty+io}[w_1^u(\tau)\omega(\frac{x-\tau}{F^2}) + w_1^\ell(\tau)\bar{\omega}(\frac{x-\tau}{F^2})]d\tau \qquad (1.22)$$

$$v_1(x) = i[w_1^\ell(x)-w_1^u(x)] - \frac{i}{2\pi F^2}\int_{-\infty+io}^{\infty+io}[w_1^u(\tau)\omega(\frac{x-\tau}{F^2}) - $$
$$-w_1^\ell(\tau)\bar{\omega}(\frac{x-\tau}{F^2})]d\tau \qquad (1.23)$$

where $\bar{\omega}$ is defined, similarly to (1.18), as

$$\bar{\omega}(\zeta) = \int_0^\infty \frac{e^{-i\lambda}}{\lambda+\zeta}d\lambda \quad . \qquad (1.24)$$

Substituting (1.22) and (1.23) in (1.21) and integrating by parts we obtain f_2 (for details see Appendix I) in its final form as

$$f_2(z) = \frac{i}{\pi}\int_{-\infty}^{\infty}\{-\frac{1}{2}[w_1^\ell(\sigma) - w_1^u(\sigma)]^2 +$$
$$+\frac{1}{\pi F^2}[w_1^\ell(\sigma) + w_1^u(\sigma)]\int_{-\infty}^{\infty}w_1^u(\tau)\omega(\frac{\sigma-\tau}{F^2})d\tau\}\omega(\frac{z-\sigma}{F^2})d\sigma \qquad (1.25)$$

Besides the term containing the Stokes second order waves we have neglected in deriving (1.25) also the term related to the constant part of ψ_2 (1.11), which is associated with the DC part of $p_2(x)$, and does not contribute to the wave resistance.

3. Illustration of Results: The Source-Sink Body

(i) The complete solution

Rather than pursuing a general discussion of the second order solution under the limit $F \to 0$, we begin with a simple case which can be solved in a closed form.

We consider a closed body generated by a source at $z_\ell = 1 - ih$ and a sink at $z_t = -1 - ih$. In view of our interest in three-dimensional applications we consider only the thickness effect (the influence of circulation, generated by distributed or concentrated vorticity, may be analyzed in a similar way). This is the first order representation of a straight thin body with blunt leading and trailing edges.

Following now (1.15), (1.16), (1.17) and (1.25) we have

$$f_1 = \frac{1}{2\pi} \ln \frac{z - z_\ell}{z - z_t} + \frac{1}{\pi} \, \omega\left(\frac{z - \bar{z}_\ell}{F^2}\right) - \frac{1}{\pi} \, \omega\left(\frac{z - \bar{z}_t}{F^2}\right) \tag{1.26}$$

$$w_1^\ell = \frac{1}{2\pi} \frac{1}{z - z_\ell} - \frac{1}{2\pi} \frac{1}{z - z_t} ; \qquad w_1^u = \frac{1}{2\pi} \frac{1}{z - \bar{z}_\ell} - \frac{1}{2\pi} \frac{1}{z - \bar{z}_t} \tag{1.27}$$

$$\begin{aligned} f_2 = \frac{i}{4\pi^3} \int_{-\infty}^{\infty} \Big\{ &- \frac{1}{2} \left[\frac{1}{\sigma - z_\ell} - \frac{1}{\sigma - z_t} - \frac{1}{\sigma - \bar{z}_\ell} + \frac{1}{\sigma - \bar{z}_t}\right]^2 + \\ &+ \frac{1}{\pi F^2} \left(\frac{1}{\sigma - z_\ell} - \frac{1}{\sigma - z_t} + \frac{1}{\sigma - \bar{z}_\ell} - \frac{1}{\sigma - \bar{z}_t}\right) \int_{-\infty}^{\infty} \left(\frac{1}{\rho - \bar{z}_\ell} - \frac{1}{\rho - \bar{z}_t}\right) \times \\ &\times \omega\left(\frac{\sigma - \rho}{F^2}\right) d\rho \Big\} \, \omega\left(\frac{z - \sigma}{F^2}\right) d\sigma \ . \end{aligned} \tag{1.28}$$

The integration in (1.28) can be carried out exactly. The first term, resulting from $(w_1^{\ell} - w_1^{u})$, contributes by the residues at $\sigma = \bar{z}_\ell$ and $\sigma = \bar{z}_t$. The last terms are more intricate, but still tractable, at least for $x \to -\infty$.

We consider now the expansion of f_1 (1.26) and f_2 (1.28) for small F.

(ii) The near field solution

For $F^2 \to 0$, $\omega(z-\bar{z}_\ell/F^2)$ can be expanded in an asymptotic series for fixed z as follows

$$\omega\left(\frac{z-\bar{z}_\ell}{F^2}\right) \sim -\sum_{n=1}^{\infty} \frac{(n-1)!\, F^{2n}}{i^n (z-\bar{z}_\ell)^n} \quad . \tag{1.29}$$

This expansion is valid, however, only for $|z - \bar{z}_\ell| > \delta$ and $-\pi + \bar{\delta} < \arg(z-\bar{z}_\ell) < 0$, where δ and $\bar{\delta}$ are arbitrarily small fixed quantities (for details see the discussion of the related exponential integral function in Copson, 1965).

Substitution of (1.29) and the similar expansion of $[z-\bar{z}_t/F^2]$ into w_1, obtaining by differentiation of f_1 (1.26), yields

$$w_1 = w_1^{\ell} + w_1^{u} + O(F^2) \qquad (-\pi+\bar{\delta} < \arg(z-\bar{z}_1) < 0) \tag{1.30}$$

Hence, w_1 degenerates at zero order into the rigid wall solution, i.e., the solution of flow past the body in the presence of a rigid wall at $y = 0$. However, this limit is not uniform and in particular is not valid far behind the body, i.e., for $x \to -\infty$ and y kept fixed. For $\arg(z-\bar{z}_1) > -\pi+\bar{\delta}$ expansion (1.29) has to be supplemented by the term $2\pi i e^{-\bar{z}_\ell/F^2} e^{-iz/F^2}$ which represents precisely the trailing waves. For this reason (1.30) may be called the nearfield expansion

The rigid wall solution, and the subsequent terms of (1.30), may be obtained also by expanding first the linearized free surface condition (1.2) for $F^2 \to 0$ and solving term by term. In contrast with the previous procedure, however, (1.30) is thus obtained as a uniformly valid solution in the <u>entire z plane</u>. This difference in results, manifest in the lower half plane $y < 0$ behind the body, has been called "the first small Froude number paradox" in a previous work (Dagan, 1972b). Although the wave term is exponentially small for $y < h$, as compared to the powers of F^2 in (1.30), it is the only one which does not tend to zero for $x \to -\infty$, y = fixed and which is associated with wave resistance.

Similarly, the near field expansion of $f_2(z)$ may be obtained from (1.28) for $F^2 \to 0$ by expanding $\omega(z-\sigma/F^2)$ and computing the residue contributions at $\bar{z}_\ell$ and $\bar{z}_t$. The result is $O(F^2)$ and is a rational function of z with poles of different orders at $z = \bar{z}_\ell$ and $z = \bar{z}_t$. Hence, ϵf_1 is $O(\epsilon)$ while $\epsilon^2 f_2$ is $O(\epsilon^2 F^2)$ and the near field expansion of f is <u>uniform</u> as $F^2 \to 0$.

(iii) The free waves potential

The free waves potential is obtained in (1.26) and (1.28) by letting $x \to -\infty$. The first order solution (1.26) yields

$$f_1^w = 2i\left(e^{i\bar{z}_\ell/F^2} - e^{i\bar{z}_t/F^2}\right)e^{-iz/F^2} = -4\left(\sin \frac{1}{F^2}\right)e^{-h/F^2} e^{-iz/F^2} . \quad (1.31)$$

In the second order solution (1.28) $\omega(z-\sigma/F^2)$ is first replaced by $2\pi i\ e^{-iz/F^2} e^{i\sigma/F^2}$. Integration yields (for details see Appendix II)

$$f_2^w = -\frac{1}{F^2}\left[(b_2+ic_2)e^{i\bar{z}_\ell/F^2} + (b_2-ic_2)e^{i\bar{z}_t/F^2}\right]e^{-iz/F^2} + O(e^{-h/F^2}) \tag{1.32}$$

where $b_2 = \frac{1}{\pi}\left(\frac{1}{2} + 2C + \ln 4 + \ln \frac{1+h^2}{h^2}\right)$, $c_2 = \frac{1}{\pi}\left(\frac{\pi}{2} - \text{arc tg}\ \frac{1}{h}\right)$ and C is Euler constant. If $h \ll 1$, $c_2 \simeq 0$ and (1.32) becomes

$$f_2^w \simeq -\frac{2b_2}{F^2}\left(\cos\frac{1}{F^2}\right) e^{-h/F^2}\ e^{-iz/F^2} + O(e^{-h/F^2}) \tag{1.33}$$

Hence, the amplitude of the free waves has, by (1.10), (1.11), (1.31) and (1.32) the orders

$$\varepsilon a_1 = O(\varepsilon e^{-h/F^2}) \tag{1.34}$$

$$\varepsilon^2 a_2 = O(\varepsilon^2 e^{-h/F^2}/F^2)\quad , \tag{1.35}$$

and although for $F^2 \to 0$ and h and ε fixed both εa_1 and $\varepsilon^2 a_2$ tend to zero, their ratio becomes unbounded like ε/F^2.

This nonuniformity of the thin body expansion has been called in a previous work "the second small Froude number paradox" (Dagan, 1972b) and it has been described previously by Salvesen (1969). Eq. (1.35) shows that the usual linearized theory is valid, for the source-sink body, only if $\varepsilon/F^2 = o(1)$, i.e., for large Froude numbers based on thickness.

4. Generalization for Bodies of Different Shapes

Since for an arbitrary thickness distribution w_1^ℓ and w_1^u are represented by source distributions (1.15), (1.16) the results of the previous section may be extended to thin bodies of any shape. It is easy to ascertain that the near field solution, based on (1.17) and (1.29), has the rigid wall approximation as a leading term and is uniform in the sector $\pi - \delta < \arg(z-1-ih) < 0$ as $F^2 \to 0$.

The nonuniformity of the expansion of the free waves depends essentially on the bluntness of the leading edge (for the sake of simplicity we consider bodies of a smooth shape and assume that viscous effects ensure anyway that the trailing edge has a fine shape). The free waves, at first order, are represented by

$$f_1^w = 2e^{-iz/F^2} \int w_1^u(\sigma)\, e^{i\sigma/F^2} d\sigma \qquad (1.36)$$

which has been obtained from (1.17), the integration path circumventing the skeleton of the image of the body $-1 < \sigma - ih < 1$ in the upper half plane. For $F^2 \to 0$ the integral in (1.36) may be expanded in the usual manner, the lowest order term being provided by the highest singularity of $w_1^u(\sigma)$, at $\sigma = 1 + ih$.

We have seen that for a source-like blunt shape $a_1 = O(e^{-h/F^2})$. For an elliptical shape, (i.e. $w_1^u \sim 1/\sqrt{\sigma-1-ih}$) of the leading edge, (1.36) shows that $a_1 = O(Fe^{-h/F^2})$. Similarly, for a wedge like shape ($w_1^u \sim \ln(\sigma-1-ih)$) we obtain $a_1 = O(F^2 \ln F e^{-h/F^2})$ (see Lighthill, 1964).

To estimate the order of the amplitude of the free waves at second order we have to use the expression of f_2 (1.25), with $\omega(z-\sigma/F^2)$ replaced by $2\pi i e^{-iz/F^2} e^{i\sigma/F^2}$. The computation is facilitated by the observation, supported by the detailed solution of the previous section, that the order of the lowest term in F is determined by the term $[w_1^u(c)]^2$ in the integral of $f_2(z)$ (1.25), the other terms contributing at an equal or higher order. Hence, the order of f_2^w is determined by integrals of the type

$$e^{-iz/F^2} \int [w_1^u(\sigma)]^2 \, e^{i\sigma/F^2} \, d\sigma \quad . \tag{1.37}$$

We have, therefore, for an elliptical leading edge $a_2 = O(e^{-h/F^2})$ and for a wedge-like shape $a_2 = O(F^2 \ln^2 F \, e^{-h/F^2})$. In each case the far waves amplitude, and consequently the wave resistance, is not uniform for $F^2 \to 0$, the nonuniformity becoming, however, weaker, as the shape of the edge becomes finer. The results are collected in the following table:

TABLE 1

The shape of the ℓ.e.	The singularity of w_1^u	Order of a_1 for $F \to 0$	Order of a_2 for $F \to 0$	Order of ratio $\varepsilon^2 a_2/\varepsilon a_1$	Order of the straining $\delta \bar{z}_\ell$
⊃	$(z-1-ih)^{-1}$	e^{-h/F^2}	$e^{-h/F^2}/F^2$	ε/F^2	ε
⟩	$\sqrt{z-1-ih}$	Fe^{-h/F^2}	e^{-h/F^2}	ε/F	εF
>	$\ln(z-1-ih)$	$F^2 \ln F e^{-h/F^2}$	$e^{-h/F^2} F^2 \ln^2 F$	$\varepsilon \ln F$	$\varepsilon F^2 \ln F$

The column before the last summarizes the main findings: the quantity appearing there has to be small in order to ensure that the usual linearized thin body approximation is uniform. It is worthwhile to mention that in all the examples in which detailed computations have been carried out so far (Tuck, 1965, for a circular cylinder; Salvesen, 1969, for a hydrofoil and Dagan, 1972a for a source), the shapes were blunt.

5. Derivation of Uniform Small Froude Number Solutions

(i) *Velocity straining*

The nonuniformity of the free waves expansion is assumed to originate from the illegitimate expansion in an ε power series of an exponential of type $\exp[i\varepsilon F(z)/F^2]$ for $\varepsilon/F^2 = O(1)$. Such a term may result from the straining of the free-surface velocity due to the presence of the body. To make the idea more precise let us replace the first order free surface condition (1.2) by

$$\operatorname{Im}[iF^2(1+\varepsilon A)\frac{df}{dz} - f] = 0 \qquad (y=0) \qquad (1.38)$$

where $1+\varepsilon A$ is a strained velocity. $A(z;h,F)$ is assumed to be analytical and $A \to 0$ for $x \to \infty$. A is obviously related to w_1 and under the usual thin body expansion (1.1) it is cast in $F_2(x)$ (1.6). We now keep it in the first order equation and solve for f with boundary conditions (1.4), (1.5) (again, we do not consider the second order body effect (1.9)).

The solution is immediately obtained as follows

$$f = \epsilon[f_1^{\ell} + f_1^{u} - \frac{i}{\pi}\int_z^{\infty} \frac{B(\sigma)}{1+\epsilon A(\sigma)} \exp(-\frac{i}{F^2}\int_{\infty}^{z}\frac{d\mu}{1+\epsilon A(\mu)})d\sigma] \qquad (1.39)$$

where

$$B(\sigma) = \int_{-\infty}^{\infty} \frac{(w_1^{\ell}+w_1^{u})(1+\epsilon \operatorname{Re} A)}{\sigma - \rho} d\rho = 2i\pi\, w_1^{u}(z) + \epsilon \int_{-\infty}^{\infty} \frac{(w_1^{\ell}+w_1^{u}) \operatorname{Re} A}{\sigma - \rho} d\rho \quad (1.40)$$

For $\epsilon = o(1)$ we are generally entitled to expand in a power series the terms $B(\lambda)$ and $(1 + \epsilon A)^{-1}$ in (1.39) and (1.40). This yields

$$f = [f_1^{\ell} + f_1^{u} + 2e^{-iz/F^2} \int_z^{\infty} w_1^{u}(\sigma) e^{i\lambda/F^2} \exp(\frac{i\epsilon}{F^2}\int_{\sigma}^{z} A(\mu)d\mu)d\sigma] + O(\epsilon^2) \quad (1.41)$$

We did not expand, however, the last exponential in (1.39) since under the limit $F \to 0$ the ratio ϵ/F^2 is not necessarily small. Eq. (1.41) proves our assertion on the effect of a first order velocity straining.

The uniformization procedures of the small Froude number solution suggested in previous works are underlain by similar ideas. Ogilvie (1968) has arrived to a free-surface condition similar to (1.38) by intuitive reasoning: as the wave length of the free waves becomes small compared to the length scale of the velocity field w_1^{ℓ}, the velocity variation has to be included in a first order approximation. Moreover, it was suggested that as $F \to 0$ $A = 2(w_1^{\ell} + w_1^{u})$ for $y = 0$, since for $F \equiv 0$ w_1 degenerates into the rigid wall solution $w_1 = w_1^{\ell} + w_1^{u}$. An equation similar to (1.38) is then obtained from the exact free-surface condition by expanding with $1+\varepsilon(w_1^{\ell}+w_1^{u})$ as the basic unperturbed velocity field.

Dagan (1972b) has arrived at a similar result by using a quasi-linear equation as a model of the nonlinear free-surface condition (in both works the more ambitious task of solving the problem of a small Froude number flow past a body of finite thickness has been undertaken).

Although the arguments are plausible in principle, the assumption that $A=z(w_1^{\ell}+w_1^{u})$ is open to criticism, since it has been shown here that the degeneracy into the rigid wall solution is not uniform.

Instead, we are going to determine here the straining function $A(z)$ in a different way. We assume that for $x\to-\infty$ f in (1.41) includes the first order term f_{1w} (1.36), as well as the lowest order term in F appearing in f_{2w} (1.23). We require, therefore, that under an additional expansion for $\varepsilon A/F^2=o(1)$, (1.41) should degenerate into the thin body expansion (1.1). Expanding the exponential in (1.41) gives for $x\to-\infty$

$$f^{w}=2\varepsilon\, e^{-iz/F^2}\int_{-\infty}^{\infty} w_1^{u}(\sigma)e^{i\sigma/F^2}\,d\sigma-$$

$$-\frac{2i\varepsilon^2}{F^2}\,e^{-iz/F^2}\int_{-\infty}^{\infty}A(\sigma)d\sigma\int_{\sigma}^{\infty} w_1^{u}(\tau)e^{i\tau/F^2}\,d\tau+O(\varepsilon^2) \quad (1.42)$$

The first term in (1.41) recovers the first order solution εf_{1w} (1.36). The second term may be identified with the F lowest order term of f_2^{w} (1.25), which may be written as

$$\epsilon^2 f_2^W = \epsilon^2 e^{-iz/F^2} \int_{-\infty}^{\infty} \{[w_1^u(\sigma)]^2 e^{i\sigma/F^2} -$$

$$- \frac{4i}{F^2} [w_1^u(\sigma) + w_1^\ell(\sigma)] \int_{\sigma}^{\infty} w_1^u(\rho) e^{i\rho/F^2} d\rho\} d\sigma \qquad (1.43)$$

Identification of A such that the integrands, i.e., the lowest F term of $P_2(\sigma)$, in (1.42) and (1.43), become identical yields

$$A(z;h,F) = \frac{iF^2}{2} \frac{(w_1^u)^2 \; e^{iz/F^2}}{\int_z^{\infty} w_1^u \, e^{i\tau/F^2} d\tau} + 2(w_1^\ell + w_1^u) =$$

$$= \frac{(w_1^u)^2}{w_1^u - w_1^\ell + w_1} + 2(w_1^\ell + w_1^u) \qquad (1.44)$$

Hence, the velocity straining function A is found to be different than that suggested in previous works, which included only the last term of (1.44), but failed to take into account the singular term related to $(w_1^\ell - w_1^u)$ in (1.25). The reason is quite transparent: this last term is identically zero for $y = 0$ according to the rigid wall solution. The rigid wall solution is not uniform, however, and the singularities of f_1 (1.17) and of $f_1^\ell + f_1^u$ are different at the location of the image of the body across the free-surface.

Obviously, for $F \to 0$ and $\epsilon A/F^2 = O(1)$ the exponential in (1.41) cannot be expanded like in (1.42). By the same token, the transfer of the velocity straining factor ϵA into the right hand side of the second order free-surface condition (1.6) is not legitimate for small Froude numbers in general.

Computing the wave amplitude with the aid of (1.41) is a difficult task in the two-dimensional case and becomes extremely tedious in three-dimensions. For these reason we consider subsequently a simplified procedure for rendering the solution uniform.

(ii) Coordinate straining

We assume now that the exponential terms which cause the small Froude number nonuniformity are a result of a coordinate straining. Lighthill method (see Van Dyke, 1964) implies an infinitessimal straining of the physical plane and deriving the straining function from the equations of flow. We adopt here a modified technique applied by Van Dyke (1964) to the case of inviscid flow past airfoils: we carry out the straining in the solution, rather than in the equations, and determine the straining function from the requirement that the second order term should not be more singular than the first.

To illustrate the method we begin with the example of a source-sink body (Section I.3). The straining has as effect a virtual displacement of the images of the two singularities from $\bar{z}_\ell$, $\bar{z}_t$ to $\bar{z}_\ell + \delta \bar{z}_\ell$, $\bar{z}_t + \delta \bar{z}_t$, respectively, with $\delta \bar{z}_\ell = O(\varepsilon)$ and $\delta \bar{z}_t = O(\varepsilon)$.

The first order term of the free wave expansion becomes now by using (1.17) and (1.27) for $x \to -\infty$

$$f^W = -e^{-iz/F^2} \int_{-\infty}^{\infty} \left(\frac{1}{\sigma - \bar{z}_\ell - \delta \bar{z}_\ell} - \frac{1}{\sigma - \bar{z}_t - \delta \bar{z}_t} \right) e^{i\sigma/F^2} d\sigma \qquad (1.45)$$

For $\varepsilon = o(1)$ and F fixed we can expand in (1.45) and obtain

$$f^w = \frac{\varepsilon}{\pi} e^{-iz/F^2} \int_{-\infty}^{\infty} \left(\frac{1}{\sigma-\bar{z}_\ell} - \frac{1}{\sigma-\bar{z}_t}\right) e^{i\sigma/F^2}\, d\sigma +$$

$$\frac{\varepsilon}{\pi} e^{-iz/F} \int_{-\infty}^{\infty} \left[\frac{\delta\bar{z}_\ell}{(\sigma-\bar{z}_\ell)^2} - \frac{\delta\bar{z}_t}{(\sigma-\bar{z}_t)^2}\right] e^{i\sigma/F^2}\, d\sigma =$$

$$= f_1^w - \frac{2\varepsilon}{F^2} e^{-iz/F^2} (\delta\bar{z}_\ell\, e^{i\bar{z}_\ell/F^2} - \delta\bar{z}_t\, e^{i\bar{z}_t/F^2}) \qquad (1.46)$$

Hence, the first term of (1.46) recovers f_{1w} (1.31). Consequently, the second order term of the free waves potential will be made up this time from f_2^w (1.32) plus the last term of (1.46), provided that the straining is of order ε .

We determine now $\delta\bar{z}_\ell$ and $\delta\bar{z}_t$ from the requirement that the term of order ε/F^2 in the amplitude of the free waves, which is the origin of the small F nonuniformity, should vanish in the solution, separately for the source and the sink. We thus obtain

$$\delta\bar{z}_\ell = -\frac{\varepsilon}{2\pi}(b_2 + ic_2) \qquad (1.47)$$

$$\delta\bar{z}_t = \frac{\varepsilon}{2\pi}(b_2 - ic_2) \qquad (1.48)$$

where b_2, c_2 are given in (1.32).

The uniform first order solution, valid for $\varepsilon/F^2 = O(1)$, is easily derived from (1.45)

$$f^w = 2i\, e^{-iz/F^2} [e^{i(\bar{z}_\ell+\delta\bar{z}_\ell)/F^2} - e^{i(\bar{z}_t + \delta\bar{z}_t)/F^2}] \qquad (1.49)$$

By using (1.47) and (1.48) we finally obtain from (1.49)

$$f^w = -4e^{-[h-(c_2\varepsilon/2\pi)]/F^2} \sin \frac{1-(\varepsilon b_2/2\pi)}{F^2} e^{-iz/F^2} \qquad (1.50)$$

The coefficient c_2 (1.32) is associated with the second order interaction between the source and the sink; it has the effect of diminishing the effective submergence depth of the body (when $h \to 0$ this term vanishes). b_2 is associated with the nonlinear effects of the leading and trailing edges singularities upon themselves. It manifests in an apparent change of the body length and consequently in a shift of the curve of the amplitude (and wave resistance) as function of F.

Again, (1.50) shows that the small Froude number nonuniformity is a result of an illegitimate expansion of the exponential and trigonometric functions in (1.50) in an ε power series for $\varepsilon/F^2 = O(1)$. In other words, when the straining becomes of the order of the wave length, it has to be maintained in the first order approximation.

We are going now to generalize the procedure for a body of arbitrary thickness distribution $\tau(x_s) = dt/dx$. The straining has now as effect a continuous infinitessimal displacement of the image of the body skeleton from $\bar{z}_s$ to $\bar{z}_s + \delta\bar{z}_s$. The first order solution ((1.16) and (1.17)) becomes

$$f^W = -\frac{2\varepsilon}{\pi} e^{-iz/F^2} \int_{-\infty}^{\infty} d\sigma \, e^{i\sigma/F^2} \int_{-1}^{1} dx_s \frac{\tau(x_s)}{\sigma - \bar{z}_s - \delta\bar{z}_s} \tag{1.51}$$

where $\bar{z}_s = x_s + ih$. The straining has been taken into account only in the denominator of (1.51), because only the residues at $\sigma - \bar{z}_s - \delta\bar{z}_s = 0$ are contributing to the lowest order terms in F. An infinitessimal change of the limits of integration or of $\tau(x_s)$ in (1.15) yields higher order terms.

For $\delta\bar{z}_s = O(\varepsilon)$ and F fixed the integrand in (1.51) may be expanded as follows

$$f^w = -\frac{2\varepsilon}{\pi} e^{-iz/F^2} \int_{-\infty}^{\infty} d\sigma \int_{-1}^{1} dx_s \frac{\tau}{\sigma - \bar{z}_s} e^{i\sigma/F^2} -$$

$$-\frac{2\varepsilon}{\pi} e^{-iz/F^2} \int_{-\infty}^{\infty} d\sigma \int_{-1}^{1} dx_s \frac{\tau\delta\bar{z}_s}{(\sigma - \bar{z}_s)^2} = \tag{1.52}$$

$$= -4i\varepsilon\, e^{-iz/F^2} \int_{-1}^{1} \tau(x_s) e^{i\bar{z}_s/F^2} dx_s + \frac{4\varepsilon}{F^2} e^{-iz/F^2} \int_{-1}^{1} \tau(x_s)\delta\bar{z}_s\, e^{i\bar{z}_s/F^2} dx_s$$

The first term in (1.52) is precisely f_{1w} (1.36); the unknown straining function $\delta\bar{z}_s$ is now determined from the requirement that the second term of (1.52) should cancel the lowest F term of f_{2w} (1.43), i.e.,

$$\int_{-1}^{1} \tau(x_s)\delta\bar{z}_s\, e^{i\bar{z}_s/F^2} dx_s =$$

$$= \frac{\varepsilon F^2}{2} \int_{-\infty}^{\infty} [- \frac{(w_1^u)^2}{2} + \frac{1}{\pi F^2} (w_1^\ell + w_1^u) \int_{-\infty}^{\infty} w_1^u\, \omega(\frac{\sigma-\rho}{F^2})\, d\rho] e^{i\sigma/F^2} d\sigma \tag{1.53}$$

To determine $\delta\bar{z}_s$ in a simple way, advantage is taken of the fact that the lowest F terms in (1.53) are associated with the singularities of the edges (an intermediate point of discontinuity can be easily accounted). What matters, therefore, is $\delta\bar{z}_\ell$ and $\delta\bar{z}_t$. Any continuous straining between the edges is acceptable as far as the most singular terms are concerned. Assuming, for the sake of simplicity, a linear straining we have

$$\delta\bar{z}_s = \frac{\delta\bar{z}_\ell - \delta\bar{z}_t}{2} x_s + \frac{\delta\bar{z}_\ell + \delta\bar{z}_t}{2} . \tag{1.54}$$

Substitution of (1.54) into the left hand side of (1.53) yields for the lowest F term

$$(\delta \bar{z}_\ell + \delta \bar{z}_t) \int_{-1}^{1} \tau(x_s)\, e^{i\bar{z}_s/F^2} dx_s \; . \qquad (1.55)$$

Equating the lowest F terms resulting from the integration in the r.h.s. of (1.54) and from (1.55) renders in an unique manner $\delta \bar{z}_\ell + \delta \bar{z}_t$. An additional relationship is obtained from the requirement of separate cancelation of the leading and trailing edges waves (obviously, for a fine shape of the trailing edge $\delta \bar{z}_t = 0$).

The estimates of Section I.4 permit to evaluate the order of the straining $\delta \bar{z}_\ell$ for different types of leading edge singularities. The results are given in the last column of Table 1. The straining becomes weak as the shape of the leading edge becomes fine and it is F dependent, excepting the source-like case.

6. Conclusions

It has been shown that the slenderness small parameter ε appears in the expression of the potential of the free waves generated by a submerged body not only in the amplitude, but also as the ratio ε/F^2 in the wave number. Like in other problems characterized by two scales (Cole, 1968) a power expansion in ε does not render a uniform solution unless $\varepsilon/F^2 = o(1)$; this last estimate has been sharpened and shown to depend on the nature of the leading (and trailing) edge singularity.

Two procedures of rendering the small Froude number solution uniform have been suggested: free-surface velocity straining and coordinate straining. The first procedure has the advantage of making uniform the second order pressure term, whose integration yields the amplitude of the free waves; for this reason the velocity straining is easily expressed with the aid of the first order solution. Computing the uniform solution is, however, extremely difficult. The coordinate straining ensures the uniformity of the expansion of the free wave amplitude (and the wave resistance) and the computation of the straining factors is more difficult than in the first case. Once determined, however, they provide immediately the uniform first order solution.

Two problems have not been touched: (i) the second order body effect, and (ii) circulation. As for (i) it has been shown previously (Dagan, 1972a) that the body correction is uniform as $F \to 0$ (e.g., for a source $\varepsilon^2 a_2 = O(\varepsilon^2 e^{-3h/F^2})$. Circulation may have an important influence on the wave amplitude (Salvesen, 1969). It can be treated similarly to the thickness with no problems of principle. We have purposely considered the effect of thickness solely because the two-dimensional solution serves here only as a case study for the three-dimensional flow problem. In applications, two-dimensional flows are generally at high Froude numbers.

PART II

THREE-DIMENSIONAL FLOWS

1. The Thin Body Expansion and Fourier Transforms

Let now z be a vertical coordinate, while the axis x and y lie in the horizontal plane of the unperturbed free surface. ϕ is the dimensionless velocity potential which is expanded in a thin body approximation as follows

$$\phi(x,y,z;\epsilon,F,\ldots) = -x+\epsilon\phi_1(x,y,z;F,\ldots)+\epsilon^2\phi_2(x,y,z;F,\ldots)+\ldots \quad (2.1)$$

where $F = U'/(gL')^{1/2}$ is again the length Froude number.

The free surface conditions satisfied by the harmonic functions ϕ_1 and ϕ_2, given here for the sake of completeness, are

$$F \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial\phi_1}{\partial z} - \mu \frac{\partial\phi_1}{\partial x} = 0 \qquad (z=0) \qquad (2.2)$$

$$F^2 \frac{\partial^2\phi_2}{\partial x^2} + \frac{\partial\phi_2}{\partial z} - \mu \frac{\partial\phi_2}{\partial x} = P_2(x,y) =$$

$$= -F^2(3u_1 \frac{\partial u_1}{\partial x} + 2v_1 \frac{\partial v_1}{\partial x} + u_1 \frac{\partial v_1}{\partial y} + 2w_1 \frac{\partial w_1}{\partial x} - F^2 u_1 \frac{\partial^2 u_1}{\partial x\partial z}) \qquad (z=0)\ (2.3)$$

where u_1, v_1, w_1 are the first order x,y,z velocity components and $\mu \to 0$ represents the "artificial viscosity" added in order to satisfy the radiation condition (we introduce it rather for an easy account of the integration paths in the complex plane).

The body boundary conditions differ depending on whether one considers slender or thin, submerged or surface piercing bodies. For this reason they will be formulated separately in the subsequent examples.

Again, like in the two-dimensional case, we consider here the part of the second order solution which is regular in $z < 0$ and satisfies (2.3), and do not investigate the second order body correction as well as the line integral correction, since the nonlinearity of the free surface condition is the most severe as $F \to 0$.

We summarize now the different Fourier transforms to be used in the sequel: with $\tilde{\phi}$ denoting the Fourier transform of ϕ with respect to x,y, we have

$$\tilde{\phi}(\alpha,\beta,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \; \phi(x,y,z) e^{i(\alpha x+\beta y)/F^2} \tag{2.4}$$

$$\phi(x,y,z) = \frac{1}{2\pi F^4} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \tilde{\phi}(\alpha,\beta,z) e^{-i(\alpha x+\beta y)/F^2} \tag{2.5}$$

where α,β are coordinates in the transform plane. We shall also use the convolution transform which may be written as

$$\widetilde{\phi(x,y)\psi(x,y)} = \frac{1}{2\pi F^4} \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\tau \; \tilde{\phi}(\nu,\tau)\tilde{\psi}(\alpha-\nu,\beta-\tau) \tag{2.6}$$

where ϕ,ψ are arbitrary functions which have FT (Fourier transform).

Let now ϕ_1^ℓ represent the first order potential of flow past the body in an infinite domain and ϕ_1^u the same for the flow past the image of the body across the plane $z = 0$. Then, the well known (Wehausen and Laitone, 1960) first order solution ϕ_1 satisfying Laplace equation and (2.2) becomes

$$\phi_1 = \phi_1^\ell - \phi_1^u + \phi_1^r \tag{2.7}$$

where

$$\tilde{\phi}_1^r = -2F^2 \frac{\tilde{w}_1^u(\alpha,\beta,0)e^{\rho z/F^2}}{\alpha^2 - \rho - i\mu\alpha} \qquad (z < 0) \;, \tag{2.8}$$

$w_1^u = \partial\phi_1^u/\partial z$ and $\rho = \sqrt{\alpha^2 + \beta^2}$. By (2.5) we have

$$\phi_1^r = -\frac{1}{\pi F^2} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \, \frac{\tilde{w}_1^u(\alpha,\beta,0)}{\alpha^2 - \rho - i\mu\alpha} \, e^{-i(\alpha x + \beta y + i\rho z)/F^2} \tag{2.9}$$

being understood that $\mu \to 0$ in the final expressions (ϕ_1^r is regular for $z < 0$ and $\phi_1^r \to 0$ for $z \to -\infty$).

The FT of the second order solution, regular for $z < 0$ and satisfying (2.3), is similarly given by

$$\tilde{\phi}_2(\alpha,\beta,z) = -\frac{F^2 \tilde{P}_2(\alpha,\beta) \; e^{\rho z/F^2}}{\alpha^2 - \rho - i\mu\alpha} \tag{2.10}$$

From (2.3) and (2.6) $\tilde{P}_2(\alpha,\beta)$ may be written after a few integrations by parts as follows

$$\tilde{P}_2(\alpha,\beta) = -\frac{i}{2\pi F^8} \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\tau \; \{[3(\alpha-\nu)\nu^2 + 2(\beta-\tau)\nu\tau + (\alpha-\nu)\tau^2 -$$

$$- 2(\alpha-\nu)^2\nu^3 - (\alpha-\nu)\nu^4]\tilde{\phi}_1^r(\nu,\tau,0)\tilde{\phi}_1^r(\alpha-\nu,\beta-\tau,0)\} \tag{2.11}$$

It seems that all the terms of $P_2(x,y)$ (2.3) contribute at the same F order in the expression of $\tilde{P}_2(\alpha,\beta)$. It is sufficient, therefore, to single out one of these terms in order to establish the asymptotic behavior of ϕ_2 as $F \to 0$. Like in the case of two-dimensional flow (Sec. I5), it is convenient to select the term $w_1(\partial w_1/\partial x)$ and to write

$$P_2(x,y,F) \sim -2F^2 w_1 \frac{\partial w_1}{\partial x} \qquad (z=0) \qquad (2.12)$$

being understood that (2.12) expresses the asymptotic dependence of P_2 on the variable F.

Finally, it is sometimes convenient to operate with polar coordinates in the transform plane, $\alpha = \rho \cos\theta$ and $\beta = \rho \sin\theta$. Then, if the potential $\phi^w = \epsilon\phi_1^w + \epsilon^2\phi_2^w$ of the far free waves is written in the following form

$$\phi^w = \frac{1}{\pi} \mathrm{Re} \int_{-\pi/2}^{-\pi/2} d\theta \, [Q(\theta) + iP(\theta)] e^{-i(x \sec\theta + y \sin\theta \sec^2\theta + iz \sec^2\theta)/F^2} \sec^2\theta \quad (x \to -\infty) \qquad (2.13)$$

the dimensionless wave resistance is given by (Maruo, 1966)

$$R = \frac{R'}{0.5\rho' U'^2 L'^2} = \frac{1}{\pi} \int_{-\pi/2}^{-\pi/2} [Q^2(\theta) + P^2(\theta)] \sec^3\theta \, d\theta \qquad (2.14)$$

where ρ' is the fluid density. The amplitude functions P and Q result from the thin body approximations ϕ_1 and ϕ_2 and can be written as

$$P = \epsilon P_1 + \epsilon^2 P_2$$
$$Q = \epsilon Q_1 + \epsilon^2 Q_2 \qquad (2.15)$$

which leads by substitution in (2.14) to

$$R = \varepsilon^2 R_1 + \varepsilon^3 R_2 + \ldots \quad . \tag{2.16}$$

2. Small Froude Number Solution for An Isolated Source

(i) First order solution

Like in the two-dimensional case we begin the study of the limit $F \to 0$ of the thin body solution with the example of an isolated source, because of its simplicity and because the source is the fundamental singularity underlying any slender or thin body solution.

The potentials of flow in infinite domains are

$$\phi_{1s}^{\ell} = -\frac{1}{4\pi} \frac{1}{[x^2 + y^2 + (z+h)^2]^{1/2}} \tag{2.17}$$

$$\phi_{1s}^{u} = -\frac{1}{4\pi} \frac{1}{[x^2 + y^2 + (z-h)^2]^{1/2}} \quad . \tag{2.18}$$

The source is located at $x = y = 0$, $z = -h$ (the reference length L' is left unspecified). In the frame of the first order approximation $\phi = -x + \varepsilon\phi_1$ represents the flow past a slender body of revolution with a blunt nose and of semi-infinite length $(-\infty < x < 0)$. The small parameter ε is equal to $\pi(r'/L')^2$, where r' is the radius of the body circular cross-section.

The expression of ϕ^r_{1s} (2.9) is well known (see, Wehausen and Laitone, 1960)

$$\phi^r_{1s} = \frac{1}{4\pi^2 F^2} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} \frac{e^{-i[\alpha x+\beta y+i\rho(z-h)]/F^2}}{\alpha^2 - \rho - i\mu\alpha}$$

$$= \frac{1}{4\pi^2 F^2} \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} \frac{e^{-i\rho[x\cos\theta+y\sin\theta+i(z-h)]/F^2}}{\rho - \sec^2\theta - i\mu\sec\theta} \sec^2\theta \, d\rho$$

$$(z < h, \quad \mu \to 0) \qquad (2.19)$$

Like in the two-dimensional case, $\phi_1 = \phi_1^\ell - \phi_1^u + \phi_1^r$ can be expanded as $F \to 0$ in a near field solution. For this purpose let write again ϕ^r_{1s} (2.19) as follows

$$\phi^r_{1s} = \frac{1}{2\pi^2 F^2} \int_0^{\infty} \cos(\beta y/F^2) d\beta \int_{-\infty}^{\infty} \frac{e^{-i[\alpha x+i\rho(z-h)]/F^2}}{\alpha^2 - \rho - i\mu\alpha} d\alpha. \quad (2.20)$$

We consider now the complex α plane. The function $\rho = \sqrt{\alpha^2 + \beta^2}$ has its branch lines depicted in Fig. 2 with the values of the argument selected such that for α, β reals ρ is real and positive, such that the condition $\phi^r_{1s} \to 0$ for $z \to -\infty$ is identically satisfied.

The poles in (2.20) are $\alpha = \pm\alpha_p = \{\frac{1}{2}[1 + (1+4\beta^2)^{1/2}]\}^{1/2}$ and have to be circumvented from below. We can now swing the integration path from the real axis in the α plane to the branch cuts. The result is easily found to be made up of the so called local disturbance $\phi_{1s}^{r\,loc}$ and the free waves term ϕ_{1s}^{w}

$$\phi_{1s}^{r\,loc} = -\frac{1}{\pi^2 F^2}\int_0^\infty e^{-\beta|x|/F^2}\cos(\beta y/F^2)\,d\beta \times$$

$$\times \int_0^\infty \frac{e^{-s|x|/F^2}\{(\beta+s)^2\sin[\gamma(z-h)/F^2] + \gamma\cos[\gamma(z-h)/F^2]\}}{(\beta+s)^4 + \gamma^2}\,ds \tag{2.21}$$

where $\gamma = \sqrt{s^2 + 2\beta s}$, and

$$\phi_{1s}^{w} = \frac{2}{\pi F^2}\int_0^\infty \frac{\sin(\alpha_p x/F^2)\cos(\beta y/F^2)e^{\alpha_p^2(z-h)/F^2}}{2\alpha_p - \alpha_p^{-1}}\,d\beta =$$

$$= \frac{1}{\pi F^2}\int_{-\pi/2}^{\pi/2} \sin[(x\cos\theta + y\sin\theta\sec^2\theta)/F^2]\,e^{(z-h)\sec^2\theta/F^2}\sec^2\theta\,d\theta$$

$$(x < 0) \tag{2.22}$$

Like in the two dimensional case $\phi_1^{r\,loc}$ is regular for $z < h$ and tends to zero algebraically for $|x| \to \infty$. After a change of variables it can be expanded uniformly in an F^2 power series for $F \to 0$ and fixed x,y,z (with $x^2 + y^2 + (z-h)^2 > 0$); as a result $\phi_{1s}^{r\,loc} = 2\phi_{1s}^{u} + O(F^2)$.

The free wave potential ϕ_{1s}^{w} is different of zero only for $x < 0$. Because of the exponential term, it is negligible with respect to $\phi_{1s}^{r\,loc}$ excepting in an arbitrarily thin wedge $z-h = \delta x$ ($x < 0$, δ arbitrarily small). Again, like in the two-dimensional case $\phi_{1s}^{r\,loc}$ is not an uniform approximation of ϕ_{1s}^{r} and for $x \to -\infty$ ϕ_{1s}^{w} dominates the solution.

Summarizing, as $F \to 0$ $\phi_{1s} = \phi_{1s}^{\ell} + \phi_{1s}^{u} + O(F^2)$, where $\phi_{1s}^{\ell} + \phi_{1s}^{u}$ is the rigid wall solution, but the limit is not uniform for $z-h = \delta x$ $(x < 0)$, where it has to be supplemented by ϕ_{1s}^{w}.

(ii) Second order solution (the free waves potential)

In contrast with the two-dimensional case, now it is not possible to obtain the solution in a closed form. It is relatively easy, however, to evaluate the order of ϕ_2 for $F \to 0$ by retaining only part of the terms of P_2 (2.12), as it has been done in Section I4. Like in two-dimensions w_1 may be written as (see (2.7))

$$w_{1s} = w_{1s}^{\ell} - w_{1s}^{u} + w_{1s}^{r} \tag{2.23}$$

where in the case of a source (2.17-2.18)

$$w_{1s}^{\ell} = \frac{1}{4\pi} \frac{z+h}{[x^2+y^2+(z+h)^2]^{3/2}} \tag{2.24}$$

$$w_{1s}^{u} = \frac{1}{4\pi} \frac{z-h}{[x^2+y^2+(z-h)^2]^{3/2}} \tag{2.25}$$

and $w_{1s}^{r} = \partial\phi_{1s}^{r}/\partial z$ (2.20). After substituting w_1 in $P_2(x,y)$ (2.22) it can be shown that the term $-2F^2 w_{1s}^{u}(\partial w_{1s}^{u}/\partial x)$ contributes to the lowest order term of ϕ_2 as $F \to 0$. Hence, by (2.22) and (2.25),

$$\tilde{P}_2(\alpha,\beta) \sim -2F^2 w^u_{1s} \widetilde{\frac{\partial w^u_{1s}}{\partial x}} = 4i\alpha\widetilde{(w^u_{1s})^2} \qquad (z=0) \tag{2.26}$$

$$\tilde{\phi}_2(\alpha,\beta) \sim \frac{-4i\alpha\, F^2\, \widetilde{[w^u_{1s}(\alpha,\beta)]^2}}{\alpha^2 - \rho - i\mu\alpha} = \frac{4iF^2\, \widetilde{[w^u_{1s}(\rho,\theta)]^2} \sec\theta}{\rho - \sec^2\theta - i\mu\sec\theta} \tag{2.27}$$

The Fourier transform of $(w^u_{1s})^2$ is easily found in polar ρ,θ coordinates. From (2.25) we have

$$[w^u_{1s}(x,y,0)]^2 = \frac{1}{16\pi^2}\frac{h^2}{(x^2+y^2+h^2)^3} \tag{2.28}$$

and by (2.4)

$$\widetilde{(w^u_{1s})^2} = \frac{h^2}{32\pi^3}\int_{-\pi}^{\pi} d\lambda \int_0^\infty r\, \frac{e^{i\rho r\cos(\lambda-\theta)/F^2}}{(r^2+h^2)^2}\, dr \tag{2.29}$$

where $x = r\cos\lambda$, $y = r\sin\lambda$. Integration in (2.29) (Gradshteyn & Ryzhik, 1965) yields

$$\widetilde{(w^u_{1s})^2} = \frac{h^2}{16\pi^2}\int_0^\infty J_o\left(\frac{\rho r}{F^2}\right)\frac{r\,dr}{(r^2+h^2)^3} = \frac{\rho^2}{128\pi^2F^4} K_{-2}\left(\frac{\rho h}{F^2}\right) \tag{2.30}$$

where J_o and K_{-2} are Bessel functions of the first kind and of the second kind (modified), respectively. By using (2.27), (2.30) and (2.5) we obtain

$$\phi_{2s} \sim -\frac{i}{64\pi^3F^6}\int_{-\pi}^{\pi} d\theta \int_0^\infty d\rho\, \frac{\rho^3 K_{-2}(\rho h/F^2) e^{-i\rho(x\cos\theta + y\sin\theta + iz)/F^2}\sec\theta}{\rho - \sec^2\theta - i\mu\sec\theta} \tag{2.31}$$

Like in the two-dimensional case it can be shown that the near field expansion of (2.31) yields a uniform F^2 power series expansion of ϕ_s .

The free waves term of ϕ_{2s} (2.31) may be written as

$$\phi_{2s}^{w} \sim \frac{1}{16\pi^2F^6} \int_{-\pi/2}^{\pi/2} \sec^6\theta K_{-2}(h\sec^2\theta/F^2)\cos[(x\sec\theta + y\sec\theta\sec^2\theta)/F^2] \times$$
$$\times\, e^{z\sec^2\theta/F^2} \sec\theta\, d\theta \qquad (2.32)$$

As $F \to 0$, $K_{-2}(h\sec^2\theta/F^2)$ can be expanded in an asymptotic series which yields for the leading term of ϕ_{2s}^{w}

$$\phi_{2s}^{w} \sim \frac{1}{F^5} \int_{-\pi/2}^{\pi/2} \sec^6\theta\, e^{(z-h)\sec^2\theta/F^2} \cos[(x\sec\theta + y\sin\theta\sec^2\theta)/F^2]d\theta + O(\frac{1}{F^3}) \,. \qquad (2.33)$$

(iii) Discussion of results

We are now in a position to discuss the final, and most important, topic of this section, namely that of the uniformity of the free waves expansion $\phi_s^w = \varepsilon\phi_{1s}^w + \varepsilon^2\phi_{2s}^w$ under the small Froude number limit. Eqs. (2.22) and (2.33) show that the thin body solution is not uniform unless $\varepsilon/F^3 = o(1)$; the latter condition originates from the comparison of the amplitude functions $P_1(\theta)$ and $Q_2(\theta)$ in (2.22) and (2.33), respectively. The same condition ensures the convergence of the coefficient of wave resistance (2.16). Taking the submergence depth as reference length, we can write in term of variables with dimensions

$$\frac{\varepsilon}{F^3} \sim \frac{r'^2g^2}{U'^4} \frac{U'}{(gh')^{1/2}} \qquad (2.34)$$

where r' is the radius of the body generated by the source. This ratio has to be compared with the criterion $T'g/U'^2 = o(1)$ which ensures the uniformity of the thin body solution in two dimensions.

We can continue and analyze, like in Section I4, uniformity criteria of the thin body expansion for $F \to 0$ in the case of bodies of revolution with milder singularities at the nose (and the tail). From the applications point of view, however, it is of interest to focus the analysis on the case of thin ships rather than on that of submerged slender bodies.

3. Small Froude Number Solution for Thin Ships

(i) *General. First order solution*

Let $y = \pm \varepsilon f(x,z)$ be the equation of the surface of the ship, where $\varepsilon = B'/L'$ is the beam length ratio. The first order velocity potential ϕ_1 is obtained by integration, over the area of the center plane S, of the potential of an isolated source ϕ_{1s}

$$\phi_1 = -2 \int\int_S \frac{\partial f(\bar{x},\bar{z})}{\partial \bar{x}} \phi_{1s}(x-\bar{x},y,z-\bar{z}) \, dxdz \tag{2.35}$$

where $\phi_{1s} = \phi_{1s}^{\ell} - \phi_{1s}^{u} + \phi_{1s}^{r}$ is given in (2.17), (2.18) and (2.19) with $\bar{z}$ replacing $-h$. For the sake of simplicity we consider a symmetrical ship solely.

The essential difference, in the present context, between ϕ_1 (2.35) and ϕ_1 in all the other cases (of submerged bodies) considered in the preceding sections, stems from the fact that the body is now piercing the free-surface. Consequently, the exponential factor e^{-h/F^2} and $e^{-h \sec^2\theta/F^2}$, present in the two- and three-dimensional solutions, respectively, does not appear anymore in (2.35) for the waterline $(\bar{z} = 0)$ singularities. For $F \to 0$ and h fixed, the exponential factor ensured previously the rapid decay of the free waves amplitude and of the wave resistance coefficient no matter how blunt the shape was. It also ensured the separation of the near field expansion and of the free waves potential.

In the case of a thin ship the shape has to be sufficiently fine in order to ensure the finiteness of the velocity potential. It can be shown, for instance, that for a source-like bow shape the second order potential is not integrable. Since usual shapes are far from being so blunt, we shall limit the discussion here to wedge like bodies, i.e., with singularities associated with finite entrance and shoulder angles at most (the case of an elliptical bow is also less interesting).

Further simplifications of the analysis are achieved if we take into consideration the well known asymptotic properties of ϕ_1 (2.35) as $F \to 0$ (see, for instance, Lunde, 1963): (i) the dominant contribution originates from the shape at the waterline $f(\bar{x},0)$, and (ii) from the singularities of $f(\bar{x},0)$, i.e., from the points of discontinuity of $\partial f(\bar{x},0)/\partial\bar{x}$. For this reason we consider the simple example of flow past a wedge-shape cylindrical bow, i.e.,

$$\begin{aligned} f(\bar{x},\bar{z}) &= -\bar{x} \qquad && (-1 < \bar{x} < 0,\ -h < \bar{z} < 0) \\ f(\bar{x},\bar{z}) &= 1 \qquad && (\bar{x} < -1,\ -h < \bar{z} < 0) \end{aligned} \tag{2.36}$$

the reference length being the forebody length. f (2.36) incorporates the essential features of any smooth shape between $\bar{x} = 0$ and $\bar{x} = -1$ with the same angle discontinuities. The influence of the stern is also disregarded because as will be shown later, the most singular terms are associated with the interaction between the singularity at $\tilde{x} = 0$ (or $\tilde{x} = -1$) with itself and not with the interaction between the different singularities.

From (2.35), (2.19) and (2.36) we immediately obtain for $\phi_1^r = \phi_1 - \phi_1^\ell + \phi_1^u$

$$\phi_1^r = -\frac{iF^2}{2\pi^2} \int_{-\infty}^{\infty} d\alpha \int_{-\infty}^{\infty} d\beta \, \frac{(1-e^{-\rho h/F^2})(1-e^{-i\alpha/F^2})}{\alpha\rho(\alpha^2-\rho-i\mu\alpha)} \, e^{-i(\alpha x+\beta y+i\rho z)/F^2} =$$

$$= -\frac{iF^2}{2\pi^2} \int_{-\pi}^{\pi} d\theta \int_{0}^{\infty} d\pi \, \frac{(1-e^{-\rho h/F^2})(1-e^{-i\rho\cos\theta/F^2})}{\rho^2(\rho-\sec^2\theta-i\mu\sec\theta)} \times$$

$$\times \, e^{-i\rho(x\cos\theta+y\sin\theta+iz)/F^2} \sec^3\theta \qquad (2.37)$$

The near field expansion, i.e., the local disturbance $\phi_1^{r\,loc}$ may be found like in (2.21) and again $\phi_1^{r\,loc} = 2\phi_1^u + O(F^2)$. The free waves potential is found from (2.37) as follows

$$\phi_1^w = \frac{2F^2}{\pi} \operatorname{Re} \int_{-\pi/2}^{\pi/2} (1-e^{-h\sec^2\theta/F^2})(1-e^{i\sec\theta/F^2}) \times$$

$$\times \, e^{i(x\cos\theta+y\sin\theta-iz)/F^2} \cos\theta \, d\theta \qquad (2.38)$$

As $F \to 0$ a lowest order term in (2.38) is obtained, for instance, from integration over $e^{i(x\cos\theta+y\sin\theta-iz)/F^2}\cos\theta$, i.e., from the potential of a wedge of infinite draft and of constant aperture angle, which leads to $\phi_1^w = O(F^3)$ for fixed x,y $(x^2+y^2 \neq 0)$ and $z=0$. By (2.14) the coefficient of wave resistance derived from this term is, at first order, $R = O(\varepsilon^2F^4)$. Hence, for $z=0$ ϕ_1^w is no more exponentially small, as $F \to 0$, in comparison with the near field solution; moreover, for

$(x^2+y^2)/F^2 = O(1)$ and $z = 0$ it becomes of the same order as the second term of the near field solution, i.e., $O(F^2)$. For any fixed $z < 0$ the decay is, however exponential.

(ii) Second order solution (the free waves potential)

To estimate the order of ϕ_2^w we again retain, like in Section II2(ii), only w_1^u in the expression of w_1 (2.23). For f given by (2.31) we immediately obtain

$$w_1^u = -\frac{1}{2\pi}\int_{-1}^{0} d\bar{x}\int_{-h}^{0} d\bar{z}\,\frac{\partial}{\partial\bar{z}}\left\{\frac{1}{[(x-\bar{x})^2+y^2+(z+\bar{z})^2]^{1/2}}\right. =$$

$$= -\frac{1}{2\pi}\ln\frac{[(x^2+y^2)^{1/2}-x]\,\{[(x+1)^2+y^2+h^2]^{1/2}-x-1\}}{\{[(x+1)^2+y^2]^{1/2}-x-1\}[(x^2+y^2+h^2)^{1/2}-x]} \quad (2.39)$$

Based on the results of the preceding section we can retain in (2.39) only the term originating from the singularity at $x = 0$, for an infinite draft, in order to estimate the term of lowest order in F, i.e.,

$$w_1^u \sim \frac{1}{2\pi}\ln\,[(x^2+y^2)^{1/2}-x] \quad (2.40)$$

P_2 and $\tilde{\phi}_2$ have the expressions (2.12) and (2.10), respectively. The FT of $w_1^u(\partial w_1^u/\partial x)$ has the following estimate

$$\widetilde{\left(w_1^u\frac{\partial w_1^u}{\partial x}\right)} \sim \frac{1}{8\pi^3}\int_{-\infty}^{\infty} dy\; e^{i\beta y/F^2}\int_{-\infty}^{\infty}\frac{\ln[(x^2+y^2)^{1/2}-x]}{(x^2+y^2)^{1/2}}\,e^{i\alpha x/F^2}dx =$$

$$= \frac{1}{8\pi^3}\int_{-\pi}^{\pi} d\lambda\int_{0}^{\infty}[\ln r(1-\cos\lambda)]e^{ir\rho\cos(\lambda-\theta)/F^2}dr \quad (2.41)$$

where the last expression in (2.41) has been obtained by substituting polar coordinates in the x,y and α,β planes. The order of magnitude of (2.41) may be estimated from the ln r term

$$\overline{\left(w_1^u \frac{\partial w_1^u}{\partial x}\right)} \sim \frac{1}{8\pi^3} \int_{-\pi}^{\pi} d\lambda \int_0^{\infty} (\ln r)\, e^{ir\rho\cos(\lambda-\theta)/F^2}\, dr =$$

$$= -\frac{1}{4\pi^2}\frac{F^2}{\rho}\left(C + \ln r + \ln\frac{\rho}{F^2}\right) \tag{2.42}$$

where C is the Euler constant (Gradshteyn & Rhyzhik, 1965). By (2.42) and (2.10) we have

$$\tilde{\phi}_2 \sim \frac{F^6(C + \ln r + \ln\rho - 2\ln F)}{\rho^2(\rho - \sec^2\theta - i\mu\sec\theta)} \sec^2\theta\, e^{\rho z/F^2} \tag{2.43}$$

Finally, the estimate of the lowest order term of ϕ_2^w becomes (2.5)

$$\phi_2^w \sim F^2 \ln F\, \mathrm{Im} \int_{-\pi}^{\pi} d\theta\, e^{-i(x\sec\theta + y\sin\theta\sec^2\theta + iz\sec^2\theta)/F^2} \tag{2.44}$$

(iii) *Discussion of results*

Inspection of $\varepsilon\phi_1^w$ (2.38) and $\varepsilon^2\phi_2^w$ (2.44) shows that the amplitude functions are $O(\varepsilon F^2)$ and $O(\varepsilon^2 F^2 \ln F)$, respectively. Hence, the expansion of the free waves potential is not uniform for fixed ε and $F \to 0$, although the singularity is much milder than that corresponding to a blunt submerged body. To ensure the validity of the thin body solution the condition $|\varepsilon \ln F| = o(1)$ must be satisfied, i.e., $|B'\ln[U'/(gL')^{1/2}]/L'| \ll 1$, where L' is the forebody length (it is worthwhile to mention that this criterion is not different of that valid for two-dimensional flow past a body of similar shape, Table 1). This criterion is only marginally satisfied by typical slow commercial ships. It is emphasized, however, that this is an asymptotic estimate and the actual ratio between the waves amplitude may be evaluated only from detailed computations of ϕ_1^w and ϕ_2^w. Obviously, the above condition applies also to the expansion of the coefficient of wave resistance (2.16).

Again, similarly to the two-dimensional flow, blunter bow shapes will impose more stringent criteria of validity of the thin body solution for small F.

4. Derivation of Uniform Small Froude Number Solutions

(i) *Velocity straining*

The argument is similar to that given in Section I5: we assume that the presence of the body causes a free-surface velocity straining which has to be incorporated in the first order solution since it causes a change of the wave number and not only of the wave amplitude.

If we assume that all the terms of P_2 (2.3), (2.11) contribute to the F lowest order terms of ϕ_2^w, the simple straining of the horizontal uniform velocity of Section I5, is not sufficient. The generalized straining suggested by the free surface conditions yields

$$F^2(1+3\epsilon u_1)\frac{\partial^2\phi}{\partial x^2} + F^2\epsilon\left(2v_1\frac{\partial^2\phi}{\partial x\partial y} + u_1\frac{\partial^2\phi}{\partial y^2} + 2w_1\frac{\partial^2\phi}{\partial x\partial z} - F^2u_1\frac{\partial^3\phi}{\partial x^2\partial z}\right) +$$

$$+\frac{\partial\phi}{\partial z} - \mu\frac{\partial\phi}{\partial x} = 0 \qquad (z=0,\quad \mu\to 0) \qquad (2.45)$$

(2.45) is supposed to replace both (2.2) and (2.3); we are going to prove that only for $\epsilon = o(1)$ and F fixed it does indeed separate into these two equations.

Keeping the body boundary condition in its first order version, we write, like in (2.7)

$$\phi = \phi_1^\ell - \phi_1^u + \phi^r$$

where ϕ^r is harmonic and regular for $z < 0$. Substituting (2.7) into (2.45) and using the FT relationships (2.4) and (2.6) we obtain from (2.45)

$$\tilde{\phi}^r(\alpha,\beta;F) - \frac{i\epsilon}{2\pi F^6(\alpha^2-\rho-i\mu\alpha)} \int_{-\infty}^{\infty} d\nu \int_{-\infty}^{\infty} d\tau\, K(\alpha,\beta,\nu,\tau;F)\, \tilde{\phi}^r(\nu,\tau;F) =$$

$$= \frac{2F^2\tilde{w}_1^u(\alpha,\beta)}{\alpha^2-\rho-i\mu\alpha} \qquad (2.46)$$

where

$$K = [3(\alpha-\nu)\alpha^2 + 2(\beta-\tau)\alpha\beta - 2(\alpha-\nu)^2\alpha^3 - (\alpha-\nu)\alpha^4]\tilde{\phi}_1^r(\alpha-\nu,\beta-\tau) \quad . \qquad (2.47)$$

$\tilde{\phi}^r(\alpha,\beta;F)\big|_{z=0}$ satisfies, therefore, the Fredholm integral equation (2.40). This equation degenerates precisely into the FT of the thin body free surface conditions (2.2) and (2.3) if the integral equation is solved by successive approximations. This is legitimate, however, only for sufficiently small values of the combination between ϵ and F which multiplies the second term of (2.46); this combination depends on the order of K, which in turn depends on the order of $\tilde{\phi}_1^r$. A sufficient condition is, however, $\epsilon = o(1)$ $F = O(1)$.

In those cases in which F is so small that the usual thin body expansion is not uniform, φ^r, solution of (2.46) is presumably a valid approximation of the exact solution for small and F.

Solving (2.46) is an extremely difficult task. We consider, therefore, the method of coordinate straining as an alternative simplified approach.

(ii) Coordinate straining

Again, the argument is the same as in the case of two-dimensional flow: it is assumed that the nonlinear free surface effect manifests in a straining of the coordinates which results in a virtual displacement of the system of singularities representing the image of the body.

We begin the discussion of the method with the case of thin ships. The first order free waves potential may be written as follows (2.22), (2.35)

$$\phi_1^w = \mathrm{Im}\{\frac{2}{\pi F^2} \iint_S \frac{\partial f(\bar{x},\bar{z})}{\partial \bar{x}} d\bar{x}d\bar{z} \int_{-\pi/2}^{\pi/2} \exp\{i[(x-\bar{x})\sec\theta + y\sin\theta\sec^2\theta -$$

$$- i(z+\bar{z})\sec^2\theta]/F^2\}\sec^2\theta\, d\theta\} \quad . \quad (2.48)$$

Like in (1.49) we may assume now that in the exponential function of (2.48) $\bar{x}$ and $\bar{z}$ are replaced by $\bar{x} + \delta\bar{x}$ and $\bar{z} + \delta\bar{z}$, respectively, where $\delta\bar{x}$, $\delta\bar{z}$ are straining functions of order ϵ which depend on $\bar{x}$, $\bar{z}$ and F. Such a general and complicated straining is, however, unnecessary; from the discussion of Sec. II3 it is seen that in order to ensure uniformity for $F \to 0$ it is enough to consider the displacement of the most singular points of $\partial f(\bar{x},0)/\partial\bar{x}$, i.e., in the case of finite jumps of $\partial f/\partial\bar{x}$, the points of abrupt angle changes at the water-line. Obviously, for $\bar{z}=0$, $\delta\bar{z}=0$ and the straining is horizontal solely. Along these lines let us consider, for instance, the singularity associated with the bow for a cylindrical ship of finite draft. First, before straining, ϕ_1^w becomes by integration by parts

$$\phi_{1b}^w = \mathrm{Im}\,\frac{2iF^2}{\pi}\,\frac{\partial f(\bar{x}_b,0)}{\partial\bar{x}}\int_{-\pi/2}^{\pi/2}\exp\{i[(x-\bar{x}_b)\sec\theta + y\sin\theta\sec^2\theta - iz\sec^2\theta]/F^2\}\cos\theta\,d\theta \qquad (2.49)$$

where $\bar{x}_b$ is the abscissa of the bow for $\bar{z}=0$; for the sake of simplicity we may take (like in Sec. II3) $\partial f(\bar{x}_b,0)/\partial\bar{x} = -1$ which is tantamount to defining ε as the tangent of the entrance angle. Assuming now that the free surface nonlinear effect leads to a straining $\delta\bar{x}_b$, ϕ_{1b}^w becomes

$$\phi_{1b}^w = -\mathrm{Im}\,\frac{2iF^2}{\pi}\int_{-\pi/2}^{\pi/2}\exp\{i[(x-\bar{x}_b-\delta\bar{x}_b)\sec\theta + y\sin\theta\sec^2\theta - iz\sec^2\theta]/F^2\}\cos\theta\,d\theta \qquad (2.50)$$

Like in Sec. I5 $\delta\bar{x}_b = O(\varepsilon)$ is sought by requiring that for fixed F and $\varepsilon = o(1)$, the thin body expansion $\phi_b^w = \varepsilon\phi_{1b}^w + \varepsilon^2\phi_{2b}^w$ of the free waves potential remains uniform as $F\to 0$, i.e., the second order approximation is not more singular than the first.

The expansion of ϕ_{1b}^w (2.50) for $(x-\bar{x}_b)^2 + z^2 \neq 0$ yields

$$\phi_{1b}^w = \phi_{1b}^w + \mathrm{Im}\,\frac{2\delta\bar{x}_b}{\pi}\int_{-\pi/2}^{\pi/2}\exp\{i[(x-\bar{x}_b)\sec\theta + y\sin\theta\sec^2\theta - iz\sec^2\theta]/F^2\}d\theta \qquad (2.51)$$

$\delta\bar{x}_b$ has now to be determined such that the last term of (2.51) should cancel the lowest F order term of ϕ^w_{2b}. It turns out that this is possible for the estimate of ϕ^w_{2b} given in (2.44). As result $\delta\bar{x}_b = O(\epsilon F^2 \ln F)$ which is the same as in the two-dimensional case (Table 1). $\delta\bar{x}_b$ can be substituted in ϕ^w_{1b} which becomes a valid approximation even if $\epsilon \ln F = O(1)$. Obviously, the nonuniformity of the thin body usual approximation in the latter case is a result of the expansion of the wave number function in (2.50) in an ϵ power series.

The expression of the free waves $\bar{\phi}^w_{1b}$ (2.50) differs from ϕ^w_{1b} (2.49) only by a change of phase resulting from an apparent displacement of the bow, which is velocity dependent. The same is true for the curve representing the coefficient of wave resistance as function of F. This result is, at least in principle, in agreement with experimental findings.

If there are other points of slope discontinuity, additional straining factors have to be incorporated in ϕ^w_1. If the procedure is applied to a submerged body there is an additional freedom in selecting a vertical straining; the choice between horizontal and vertical strainings depends on whether we have to cancel P or Q type functions in the expression of R (2.14). In the case of a source-like blunt nose (Sec. II2) the straining is of order ϵ/F as compared to the order ϵ in two-dimensional flows (Table 1).

Since our estimate of the lowest order term of ϕ^w_2 (2.44) was based only on part of the expression of $P_2(x,y)$ (2.3), it is not sure that the complete wave spectrum functions $P_2(\theta)$ and $Q_2(\theta)$ of ϕ^w_2 are the same as in (2.44). If they are different the coordinate straining may be successful only if we assume that $\delta\bar{x}_b$ is a function of θ, which makes the straining less meaningful then in the case of two-dimensional flow. This question is, however, left open at present.

5. Conclusions

The results obtained in the study of the small Froude number limit of the thin body expansion in the case of three-dimensional flows do not differ in principle from those pertaining to two-dimensional flows; the computations, however, are much more tedious. Again, the nonuniform behavior of the expansion of the potential of the free waves is apparently related to the presence of the small parameter ε in the wave number of the spectrum function, and not only in the amplitude. It is worthwhile to mention that the nonlinear effects considered here are associated with terms of the second order pressure which result from the local disturbance of the free-surface. The interaction between the first and second order free waves yields terms which are of higher order as $F \to 0$.

A relatively simple method of rendering uniform the expansion of the free waves potential suggested here is the coordinate straining. It results in an apparent horizontal displacement of the singularities of the water-line contour in the case of thin ships. This displacement is of the order of the beam and its dependance on the Froude number is related to the nature of the singularity. The actual value of the straining factors has to be computed numerically for each particular case. The influence of the straining becomes appreciable for blunt bodies moving at low speeds.

APPENDIX I

Derivation of $f_2(z)$ (1.25)

The expression of f_2 (1.21) becomes, by substitution of u_1 (1.22) and v_1 (1.23)

$$f_2(z) = \frac{i}{\pi} \int_{-\infty}^{\infty} d\sigma\, \omega\left(\frac{z-\sigma}{F^2}\right) \{ -\frac{1}{2} [w_1^{\ell}(\sigma) - w_1^{u}(\sigma)]^2 - \frac{1}{\pi F^2} [w_1^{\ell}(\sigma) - w_1(\sigma)] \times$$

$$\times [\int_{-\infty}^{\infty} d\tau\, w_1^{\ell}(\tau)\, \bar{\omega}\left(\frac{\sigma-\tau}{F^2}\right)] - \frac{1}{4\pi^2 F^4} \int_{-\infty}^{\infty} d\tau_1 \int_{-\infty}^{\infty} d\tau_2 [w_1^{u}(\tau_1) w_1^{u}(\tau_2)\, \omega\left(\frac{\sigma-\tau_1}{F^2}\right) \omega\left(\frac{\sigma-\tau_2}{F^2}\right) +$$

$$+ 2 w_1^{\ell}(\tau_1) w_1^{u}(\tau_2)\, \bar{\omega}\left(\frac{\sigma-\tau_1}{F^2}\right) \omega\left(\frac{\sigma-\tau_2}{F^2}\right) + 3 w_1^{\ell}(\tau_1) w_1^{\ell}(\tau_2)\, \bar{\omega}\left(\frac{\sigma-\tau_1}{F^2}\right) \bar{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)]\} \qquad (A.1)$$

By integration by parts it can be shown that

$$\int_{-\infty}^{\infty} \omega\left(\frac{\sigma-\tau_1}{F^2}\right) \omega\left(\frac{\sigma-\tau_2}{F^2}\right) \left(\frac{z-\sigma}{F^2}\right) d\sigma = iF^2 \int_{-\infty}^{\infty} \frac{1}{\sigma-\tau_1}\, \omega\left(\frac{\sigma-\tau_1}{F^2}\right) + \frac{1}{\sigma-\tau_2}\, \omega\left(\frac{\sigma-\tau_2}{F^2}\right)]\, \omega\left(\frac{z-\sigma}{F^2}\right) d\sigma \qquad (A.2)$$

$$\int_{-\infty}^{\infty} \omega\left(\frac{\sigma-\tau_1}{F^2}\right) \left(\frac{\sigma-\tau_2}{F^2}\right) \left(\frac{z-\sigma}{F^2}\right) d\sigma = -iF^2 \int_{-\infty}^{\infty} \frac{1}{\sigma-\tau_1}\, \omega\left(\frac{\sigma-\tau_1}{F^2}\right) + \frac{1}{\sigma-\tau_2}\, \bar{\omega}\left(\frac{\sigma-\tau_2}{F^2}\right)]\, \omega\left(\frac{z-\sigma}{F^2}\right) d\sigma \qquad (A.3)$$

By residues we also have

$$\int_{-\infty}^{\infty} \frac{w_1^{u}(\tau_2)}{\sigma - \tau_2}\, d\tau_2 = 2\pi i\, w_1^{u}(\sigma) \qquad \int_{-\infty}^{\infty} \frac{w_1^{\ell}(\tau_1)}{\sigma - \tau_1}\, d\tau_1 = -2\pi i\, w_1^{\ell}(\sigma) \qquad (A.4)$$

Substituting (A.2), (A.3) and (A.4) in (A.1) results in the final expression of $f_2(z)$ (1.25).

APPENDIX II

Derivation of $f_2^w(z)$ (1.32)

The complete expression of f_2 is given in (1.28). By residues we obtain for the last integral of (1.28)

$$\int_{-\infty}^{\infty} \left(\frac{1}{\rho-\bar{z}_\ell} - \frac{1}{\rho-\bar{z}_t}\right) \omega\left(\frac{\sigma-\rho}{F^2}\right) d\rho = 2\pi i \left[\omega\left(\frac{\sigma-\bar{z}_\ell}{F^2}\right) - \omega\left(\frac{\sigma-\bar{z}_t}{F^2}\right)\right] \tag{A.5}$$

We represent now ω like in (1.18)

$$\omega\left(\frac{\sigma-\bar{z}_\ell}{F^2}\right) = \int_0^{\infty} \frac{e^{i\lambda/F^2}}{\lambda+\sigma-\bar{z}_\ell}\, d\lambda \tag{A.6}$$

and similarly $\omega(\sigma-\bar{z}_t/F^2)$. To derive the expression of the far free waves we replace $\omega(z-\sigma/F^2)$ by $2\pi i\ e^{-iz/F^2}\ e^{i\sigma/F^2}$. With these transformations the last term of f_2 becomes

$$-\frac{i}{\pi^2 F^2}\int_0^{\infty} e^{i\lambda/F^2} \int_{-\infty}^{\infty} \left(\frac{1}{\sigma-z_\ell} - \frac{1}{\sigma-z_t} + \frac{1}{\sigma-\bar{z}_\ell} - \frac{1}{\sigma-\bar{z}_t}\right)\left(\frac{1}{\lambda+\sigma-\bar{z}_\ell} - \frac{1}{\lambda+\sigma-\bar{z}_t}\right) e^{i\sigma/F^2}\, d\sigma \tag{A.7}$$

Again, by using the residue theorem in the last integral, (A.7) becomes

$$\frac{2}{\pi F^2}\Big\{e^{i\bar{z}_\ell/F^2} \int_0^{\infty} \left[\left(-\frac{1}{\lambda-2ih} + \frac{1}{\lambda-2-2ih} - \frac{1}{\lambda} + \frac{1}{\lambda-2}\right) + \frac{e^{i\lambda/F^2}}{\lambda}\right] d\lambda -$$

$$- e^{-\bar{z}_t/F^2} \int_0^{\infty} \frac{e^{i\lambda/F^2}}{\lambda-2}\, d\lambda\Big\} \tag{A.8}$$

where $h = \mathrm{Im}\,\bar{z}_\ell = \mathrm{Im}\,\bar{z}_t$ and $\mathrm{Re}\,z_\ell = 1$, $\mathrm{Re}\,z_t = -1$.

Integrating in (A.8) and adding the residues of the first term of f_2(1.28) leads to the final expression of f_2^w (1.32).

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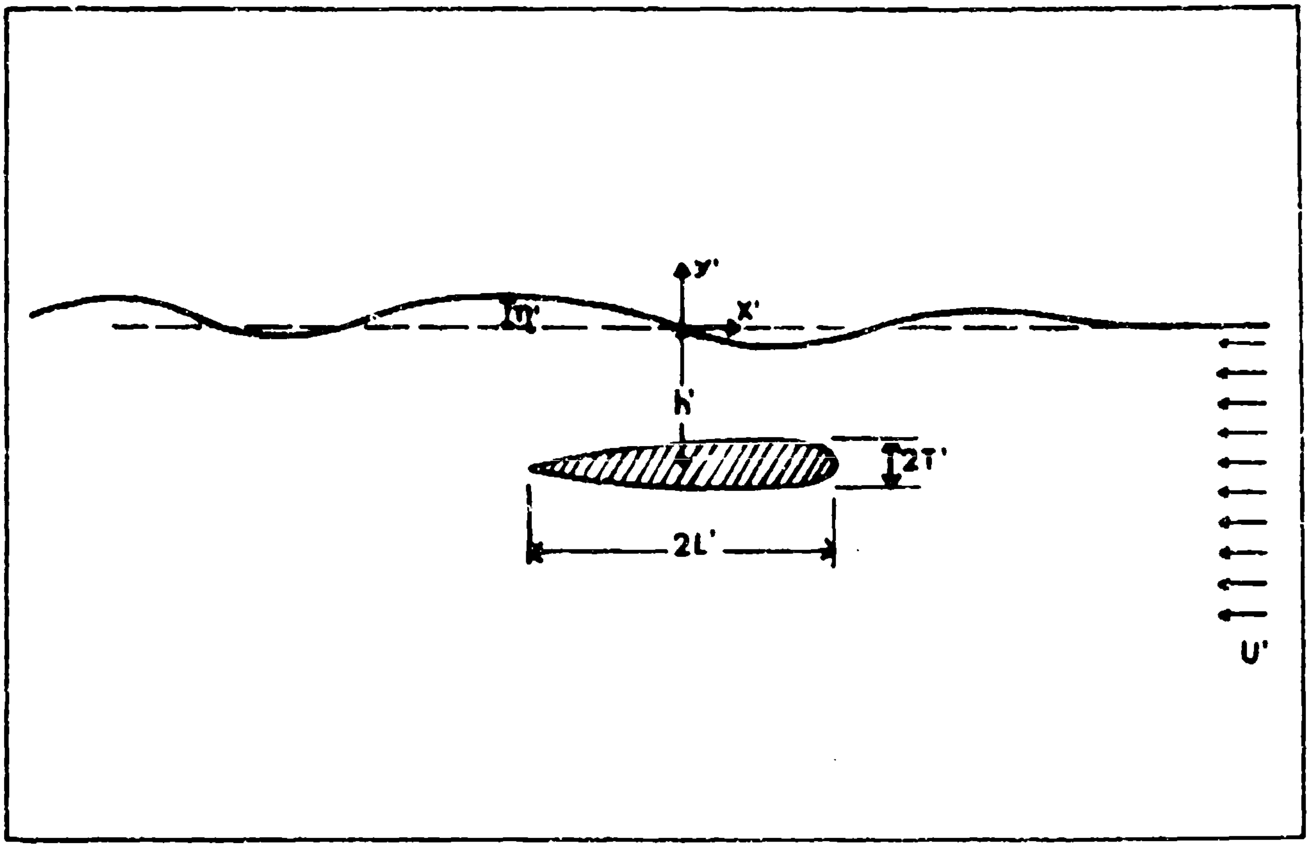

FIGURE 1 - TWO-DIMENSIONAL FLOW PAST A SUBMERGED BODY