AD-757 872

RAPID ENGINEERING CALCULATION OF TWO-DIMENSIONAL TURBULENT SKIN FRICTION

Frank M. White, et al

Rhode Island University

Prepared for:

Air Force Flight Dynamics Laboratory

November 1972

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AFFDL-TR-72-136
Supplement 1

RAPID ENGINEERING CALCULATION OF TWO-DIMENSIONAL TURBULENT SKIN FRICTION

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TECHNICAL REPORT AFFDL-TR-72-136, SUP. 1

NOVEMBER 1972

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FOREWORD

The final technical report and this Supplement were prepared by F.M. White, R.C. Lessmann, and G.H. Christoph of the Department of Mechanical Engineering and Applied Mechanics of the University of Rhode Island under Contract F33615-71-C-1585, "Analysis of the Turbulent Boundary Layer in Axisymmetric and Three-Dimensional Flows."

The contract was initiated under Project No. 1426, "Experimental Simulation of Flight Mechanics," Task No. 142604, "Theory of Dynamic Simulation of Flight Environment." The work was administered by the Air Force Flight Dynamics Laboratory. Wright-Patterson Air Force Base, Ohio, Dr. James T. Van Kuren (FX), Project Engineer.

The work was accomplished during the period 1 June 1971 through 30 June 1972.

The report was submitted by the authors in July 1972.

This Supplement contains an outline of the theoretical method for engineering use.

This Supplement has been reviewed and is approved.

Philip P. Antonatos

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UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1 ORIGINATING ACTIVITY (Corporate author)
Division of Engineering Research &Development
University of Rhode Island
Kingston, Rhode Island 02881

2a REPORT SECURITY CLASSIFICATION
Unclassified

2b GROUP
N/A

3 REPORT TITLE
RAPID ENGINEERING CALCULATION OF TWO-DIMENSIONAL TURBULENT SKIN FRICTION

4 DESCRIPTIVE NOTES (Type of report and inclusive dates)
Supplement - Design Report - August 1972

5 AUTHOR(S) (Last name, first name, initial)
White, Frank M.
Christoph, George H.

6 REPORT DATE
November 1972

7a TOTAL NO. OF PAGES
26 33

7b NO. OF REFS
12

8a CONTRACT OR GRANT NO
F33615-71-C-1585

b PROJECT NO 1426

c Task 04

d Work Unit 17

9a ORIGINATOR'S REPORT NUMBER(S)

9b OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
AFFDL-TR-72-136, Sup. 1

10 AVAILABILITY/LIMITATION NOTICES
Approved for public release, distribution unlimited.

11 SUPPLEMENTARY NOTES This is a supplement to AFFDL-TR-72-136 "Simplified Approach to the Analysis of Turbulent Boundary Layers in 2&3 Dimensions."

12 SPONSORING MILITARY ACTIVITY
Department of the Air Force
AF Flight Dynamics Laboratory (AFSC)
Wright-Patterson AFB, Ohio 45433

13 ABSTRACT

It is the purpose of this report to outline the complete details for engineering use of White's two-dimensional theory of turbulent skin friction under arbitrary compressible flow conditions. This new method is not only the simplest but also the most accurate computational scheme in the literature. This procedure is recommended for general use by engineering designers. The method concerns skin friction only. This report presents methods for hand or digital computer computation. Several examples are presented.

DD FORM 1 JAN 64 1473

UNCLASSIFIED
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14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Aerodynamic drag						
Aerodynamic heating						
Boundary Layer flow						
Compressible flow						
Law of the wall						
Skin friction						
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*U.S. Government Printing Office: 1973 — 759-491/369

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Security Classification

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I INTRODUCTION

The authors have been engaged for several years in the development of a new method of analysis of turbulent skin friction under fairly arbitrary conditions of freestream and wall parameters. Early efforts [1,2,3] were devoted to two-dimensional skin friction calculations, while later papers consider axisymmetric skin friction [4,5], three-dimensional skin friction [5], wall heat transfer [5], and wall roughness and transpiration conditions [6]. It is the purpose of this report to outline the complete details for engineering use of our two-dimensional theory [3] of turbulent skin friction under arbitrary compressible flow conditions. It appears to the authors that this new method is not only the simplest but also the most accurate computational scheme in the literature. Therefore we feel justified in recommending this procedure for general use by engineering designers.

This method concerns skin friction only. No attempt is made to estimate the value of any "traditional" integral parameter such as a shape factor or a momentum thickness. Such parameters could easily be added to the present analysis, but we adamantly discourage their use. In our opinion, shape factors and integral thicknesses are of little significance and serve mainly to divert the attention of the engineer from his basic problem, the turbulent skin friction. This theme, admittedly biased, dominates all our work [1-6].

II OUTLINE OF THE BASIC THEORY

In reference [3], the use of a compressible law-of-the-wall and a Crocco temperature approximation are shown to lead to closure of the boundary layer continuity and momentum equations for two-dimensional compressible turbulent flow. The variables which arise quite naturally from this analysis are the skin friction variable:

$$\lambda = (2/C_f)^{1/2} , \quad C_f = 2\tau_w/\rho_e U_e^2 , \tag{1}$$

which varies with the dimensionless coordinate in the freestream direction:

$$x^* = x/L \tag{2}$$

A basic parameter in the theory is the dimensionless freestream velocity distribution:

$$V = U_e(x)/U_o = V(x^*) \tag{3}$$

The quantities L and U_o are reference constants; they may be taken equal to unity if one is prepared to handle the resulting "unit Reynolds numbers" and "unit lengths" in the basic equation. In our own work, we commonly take L equal to the body length (so that x^* varies from zero to unity) and take U_o equal to the initial velocity $U_e(x=0)$. We will illustrate both these approaches in what follows, since it is our experience that improper use of "x^*" and "R_L" in the theory is the main cause of erroneous results when the theory is used.

The analysis not only requires accurate knowledge of $V(x^*)$ but also its first and second derivatives, V' and V''. Therefore it is essential that an accurate curve-fit or other smooth formula be provided for the distribution $V(x^*)$. Failure to provide a smooth curve-fit is the second most common source of error in the theory. We shall also illustrate this problem with examples. Note that there are no limitations upon $U_e(x)$, which may be subsonic, transonic, supersonic, or hypersonic. The theory is valid for all.

The second important running parameter in the theory is a sort of "stretched" Reynolds number, R^*:

$$R^* = R_L/(1/V)' \quad , \tag{4}$$

$$\text{where} \quad R_L = (U_o L/\nu_e)(\mu_e/\mu_w)(T_e/T_w)^{1/2}.$$

The prime indicates differentiation with respect to x^*. The Reynolds number R_L is a constant for low speed adiabatic flow [ref. 1] but may vary with x^* at high speeds due to wall and freestream temperature variations. Finally, the freestream Mach number variation $M_e(x^*)$ and the wall temperature ratio $T_w(x)/T_e(x)$ are combined into a single parameter $A(x^*)$, first used by van Driest [7] in a flat plate analysis:

$$A = \sqrt{(T_{aw}/T_e - 1)}/[\sin^{-1}(a/c) + \sin^{-1}(b/c)] \quad , \tag{5}$$

where T_{aw} is the adiabatic wall temperature and a/c, b/c vary only with Mach number and temperature ratio:

$$a = T_{aw} + T_w - 2T_e \; ; \quad b = T_{aw} - T_w \; ; \quad c^2 = (T_{aw}+T_w)^2 - 4T_wT_e . \qquad (6)$$

Numerical values of this parameter A are shown in Figure 1. This figure may be used to pick off values of A when making a hand computation with the theory. Otherwise, when using the computer program in this report, computation of A from (5) and (6) is already programmed. Note that A varies only slowly with Mach number and temperature ratio. Therefore, an accurate curve-fit expression for $M_e(x^*)$ and (T_w/T_e) is not needed; any reasonable approximation will do.

All of these parameters arise in the basic differential equation for computing $\lambda(x^*)$ from an arbitrary (known) distribution of V, M_e, and T_w/T_e. The equation is:

$$\frac{d\lambda}{dx^*} \doteq \frac{\frac{(1/V)'}{(1/V)}\,[1 + 9\,A^{-2}\,g^*\,R^{*0.07}] + \frac{(1/V)''}{(1/V)'}[3\,A^2\,g^*\,R^{*0.07}]}{0.16\,f^*\,A^3} \qquad (7)$$

For low speed flow, $A \doteq 1.0$ and Eq.(7) reduces to the incompressible analysis of reference [1]. The functions f^* and g^* arise from the integral coefficients of [1,3] and vary only with the quantity (λ/λ_{max}). They are plotted in Figure 2, which is the second figure required for making a hand computation from the theory. Finally, the quantity "λ_{max}" is the value of λ at which boundary layer separation

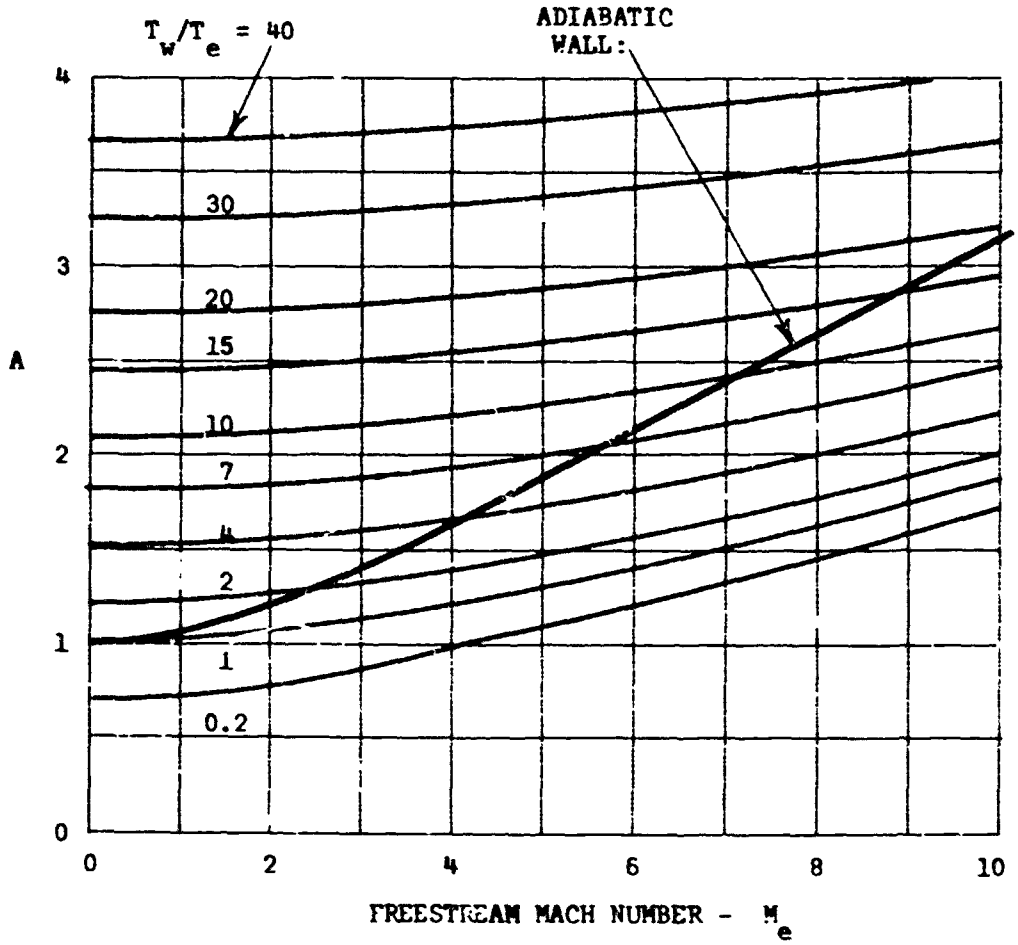

Figure 1. The Van Driest Parameter A, from Equations (5,6).

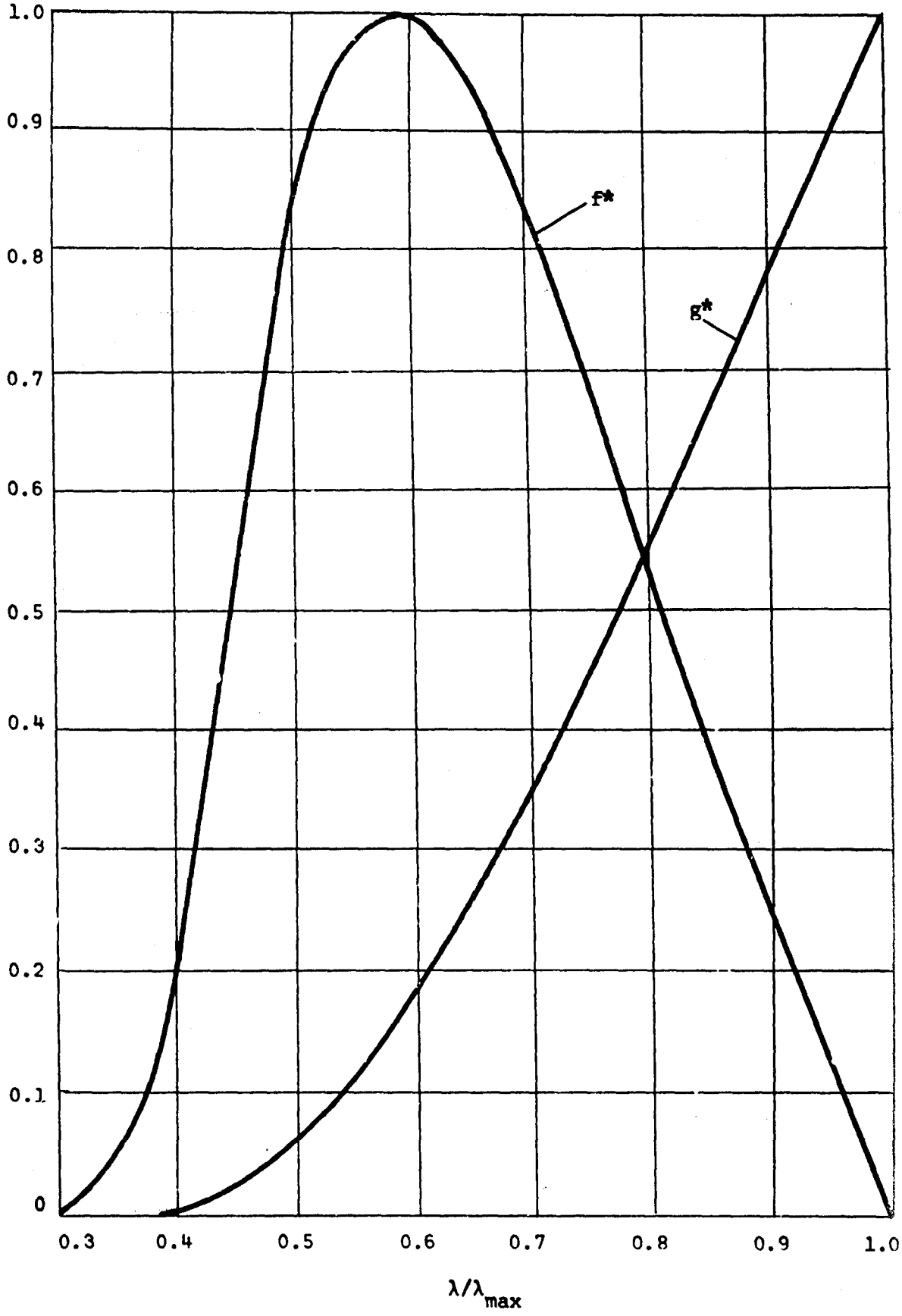

Figure 2. The Functions f* and g* for Use with Equation (7).

occurs. This is correlated in [1,3] with R^* and A, as follows:

$$\lambda_{max} \doteq 8.7\ A\ \log_{10}(R^*) \tag{8}$$

As this point, f^* approaches zero (Figure 2) and thus the derivative $(d\lambda/dx^*)$ becomes infinite and $C_f = \lambda/\lambda^2$ approaches zero. This of course is the definition of the separation point. The theory at present cannot be used beyond the separation point.

Numerical values of f^* and g^*, taken from [1], are given in Table 1. In the computer program which follows, those values are curve-fit with the following expressions which give excellent accuracy in the range $0.36 \leq \lambda/\lambda_{max} \leq 1.0$:

$$f^* \doteq (2.434\ Z + 1.443\ Z^2)\ \exp[-\ 44.0\ Z^6] \quad ;$$

$$g^* \doteq 1 - 2.3\ Z + 1.76\ Z^3 \quad , \tag{9}$$

$$\text{where} \quad Z = 1 - (\lambda/\lambda_{max}).$$

If (λ/λ_{max}) is less than 0.36, then $f^* \doteq g^* \doteq 0$, and Eq.(7) is not necessary. The same is true if R^* is negative, which corresponds to a favorable pressure gradient. In these cases, the following simplified differential equation is used:

$(\lambda < 0.36\ \lambda_{max}$ or $R^* < 0)$:

$$\frac{d\lambda}{dx^*} \doteq \frac{1}{8} R_L\ V \exp(-0.48\ \lambda/A) - 5.5\ V'/V. \tag{10}$$

TABLE 1

THE FUNCTIONS f* AND g*
FOR USE WITH EQUATION (7)

λ/λ_{max}	g*	f*
< 0.36:	Use Equation (10) instead.....	
0.36	0.00237	0.07575
0.38	0.00436	0.13522
0.40	0.00774	0.22410
0.42	0.01310	0.34015
0.44	0.02098	0.47212
0.46	0.03176	0.60479
0.48	0.04562	0.72521
0.50	0.06255	0.82546
0.52	0.08242	0.90229
0.54	0.10503	0.95566
0.56	0.13018	0.98735
0.58	0.15763	0.99997
0.60	0.18715	0.99642
0.62	0.21853	0.97944
0.64	0.25159	0.95155
0.66	0.28615	0.91498
0.68	0.32204	0.87164
0.70	0.35914	0.82313
0.72	0.39732	0.77079
0.74	0.43646	0.71574
0.76	0.47646	0.65887
0.78	0.51725	0.60092
0.80	0.55873	0.54250
0.82	0.60085	0.48407
0.84	0.64353	0.42601
0.86	0.68673	0.36864
0.88	0.73040	0.31219
0.90	0.77449	0.25682
0.92	0.81895	0.20268
0.94	0.86377	0.14986
0.96	0.90890	0.09844
0.98	0.95431	0.04846
1.00	1.00000	0.00000

III HAND CALCULATION

The basic relation, Eq.(7), is quite suitable for hand calculation and its integration may be accomplished either numerically or graphically. The following steps outline a typical portion of the calculation:

1) given the value of λ at some intermediate position x^*.

2) Establish numerical values, from given input information, for V, $(1/V)'$, $(1/V)''$, M_e and (T_w/T_e) at that x^*.

3) Compute R_L from Eq.(4) and then compute R^* from (4) also.

4) For the given M_e and (T_w/T_e), read A from Figure 1.

5) Compute λ_{max} from Eq.(8) and hence compute λ/λ_{max}.

6) Enter Figure 2 or Table 1 at (λ/λ_{max}) and read f^* and g^*.

7) Using A, f^*, g^* and R^* from above, compute the local rate of change $(d\lambda/dx^*)$ from the basic relation, Eq.(7). You are now ready to "integrate" forward to the next position $(x^* + \Delta x^*)$ using a numerical formula such as $\lambda(x^*+\Delta x^*) \doteq \lambda(x^*) + \Delta x^* \lambda'(x^*+\Delta x^*)$. Alternately, the graphical "method of isoclines" is very convenient, since we are dealing only with a first order differential equation.

<u>Numerical Example:</u>

At the initial position of the supersonic relaxing flow experiment of F. W. Zwarts (Case 1 in what follows), we are given the following information:

$C_f = 0.000894$

$X = 0.75$ inches

$U_e = U_o = 2205$ ft/sec

$\nu_e = 0.00127$ ft^2/sec

$V = 1.0$

$(1/V)' \doteq 0.008$ } per <u>inch</u> (L = 1")

$(1/V)'' \doteq 0.003$

$M_e = 4.02$

Adiabatic Wall

This is sufficient input to proceed forward with Eq.(7), as follows:

Step 1. Establish (T_w/T_e) from an adiabatic wall formula with, say, a recovery factor of 0.89 for turbulent flow:

$$T_w/T_e \doteq 1 + 0.89(0.2)\, M_e^2 = 3.88$$

Also estimate the viscosity ratio from a power-law formula for air:

$$\mu_w/\mu_e \doteq (T_w/T_e)^{0.67} = 2.47$$

Step 2. Compute the initial value of λ from the skin friction:

$$\lambda = (2/C_f)^{1/2} \doteq 47.3$$

Step 3. Compute R_L and R^* from Eq.(4): use L = 1 inch for convenience.

$$R_L = [2205(1/12)/0.000127]/(3.88)^{1/2}/(2.47) = 3.56 \times 10^5.$$

$$R^* = R_L/(1/V)' = 1.25 \times 10^8$$

Step 4. For $M_e = 4.02$ and $(T_w/T_e) = 3.88$ (adiabatic), read $A \doteq 1.63$ from Figure 1.

Step 5. From Eq.(8), compute $\lambda_{max} = 8.7(1.63)\log_{10}(1.25\times10^8) = 115.$ Hence $\lambda/\lambda_{max} = (47.3)/115. \doteq 0.412.$

Step 6. From Figure 2 or Table 1, at 0.412, read $f^* \doteq 0.294$ and $g^* \doteq 0.011$. For convenience, compute $[3g^* R^{*0.07}] \doteq 0.131$.

Step 7. Evaluate the derivative from Eq.(7):

$$\frac{d\lambda}{dx^*} = \frac{\frac{0.008}{(1.0)}[1 + 3(0.131)/1.63^2] + \frac{0.003}{0.008}[0.131(1.63)^2]}{0.16\ (0.294)\ (1.63)^3}$$

$$= \frac{0.0091 + 0.121}{0.204} = 0.64$$

This is a rather small rate of change, corresponding to a modest adverse pressure gradient. We may integrate forward about three inches, which gives about a 4% change in λ. In no case should we try to extrapolate forward more than about a 3%-5% change in λ, and in this way we will avoid serious numerical error. Here we estimate:

$$\lambda(3.75") \doteq \lambda(0.75") + (3.0") \frac{d\lambda}{dx^*}(0.75")$$

$$= 47.3 + (3.0)(0.64) = 49.22$$

$$C_f(3.75") = 2/(49.22)^2 = \underline{\underline{0.0008256}}$$

If M_e, V, etc. are known at 3.75", we are now in a position to evaluate $(d\lambda/dx^*)$ at this new station and proceed forward again. If this derivative is markedly different from the previous one of 0.64, we should backtrack and take an "average" slope over the previous three inches to estimate a more accurate $\lambda(3.75")$. Note that we are using Δx^* in inches here, since we chose L = one inch. If we had chosen L = 18" (the total length of Zwarts' model), then both R_L and (1/V)' would be eighteen times larger and hence R^* would still be exactly the same (1.25×10^8), leading to the same values of f^* and g^*. However, (1/V)'' would be $(18)^2$ larger and the result would be that $(d\lambda/dx^*)$ is eighteen times larger. Meanwhile, Δx^* is eighteen times smaller, so that the value of λ at 3.75"would be still the same (49.22). Thus the choice of L does not affect the result.

IV DIGITAL COMPUTER CALCULATION

The basic theory, Equation (7), has also been programmed for use on a digital computer. The suggested FORTRAN program is on the next page, followed by a subroutine (RUNGE) which implements the required numerical integration by a Runge-Kutta procedure.

The user must provide a smooth curve-fit to the dimensionless freestream velocity distribution $V(x^*)$. The suggestion here is for a fourth order polynomial:

$$V \doteq a_o + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$$

The first READ statement calls for the user's coefficients (a_o, a_1, a_2, a_3, a_4). If some other expression is used (exponential, sine wave, etc), the user must himself modify the three statements following statement ten, which compute V, VP, and VPP. Immediately following these three are three more statements which invert and compute (1/V)'', (1/V)', and (1/V). These should <u>not</u> be changed.

The second READ statement calls for initial data appropriate to the particular run:

$$U_o L/\nu_e \quad , \quad C_{f_o} \quad , \quad \Delta X^*, \quad X^*_o \quad , \quad X^*_{max}$$

The program will increment itself in steps of ΔX^* (which is called H in the program) and will stop when XMAX is reached.

At statement 21, the user must provide an analytical estimate of the freestream Mach number distribution $M_e(x^*)$. These need not be a

FORTRAN Program for Solving Equation (7)

```
      DIMENSION Y(10), F(10)
C  READ IN POLYNOMIAL COEFFICIENTS FOR V(X*) AND ALSO INITIAL DATA.
    7 READ(5,*)AZ,A1,A2,A3,A4
      READ(5,*)RZERO,CFZERO,H,X,XMAX
      Y(1) = SQRT(2./CFZERO)
      M = 0
    8 IF(X-XMAX) 6,6,7
    6 CALL RUNGE(1,Y,F,X,H,M,K)
      GO TO (10,20), K
C  NOW COMPUTE LOCAL V, V', V", PROGRAM COMPUTES 1/V,(1/V)',(1/V)".
C  BE CAREFUL, THOUGH: NOTICE FOR ZWARTS DATA WE USE (X-0.75), NOT X.
   10 Z = X - 0.75
      V = AZ + A1*Z + A2*Z**2 + A3*Z**3 + A4*Z**4
      VP = A1 + 2.*A2*Z + 3.*A3*Z**2 + 4.*A4*Z**3
      VPP = 2.*A2 + 6.*A3*Z + 12.*A4*Z**2
      VPP = (2.*VP*VP/V - VPP)/V/V
      VP = -VP/V/V
      V = 1./V
C  NOW COMPUTE THE LOCAL MACH NUMBER. AN APPROXIMATION IS O.K.
      IF(X-6.75)21,22,22
   21 CM = 4.125 - X/6.
      GO TO 23
   22 CM = 3.0
C  NOW COMPUTE ADIABATIC WALL AND LOCAL WALL TEMPERATURE, THEN "A".
   23 TAW = 1. + 0.89*0.2*CM**2
      TW = TAW
      A = TW + TAW - 2.
      B = TAW - TW
      C = SQRT((TAW + TW)**2 - 4.*TW)
      A = SQRT(TAW-1.)/(ASIN(A/C) + ASIN(B/C))
C  TEST TO SEE IF EQ.(10)SHOULD BE USED, OTHERWISE USE EQ.(7).
      IF(VP)41,41,42
   41 F(1) = RL/V/EXP(.40*Y(1)/A)/8. + 5.5*VP/V
      GO TO 6
   42 RSTAR = RZERO/SQRT(TW)/TW**0.67/VP
      Q = Y(1)/8.7/A/ALOG10(RSTAR)
      IF(Q-0.4)41,41,50
   50 IF(Q-1.)51,52,52
   52 WRITE(6,606)X
  606 FORMAT(' SEPARATION HAS OCCURRED AT X = ',F12.4)
      GO TO 7
   51 Z = 1. - Q
      GS = 1. - 2.3*Z + 1.76*Z**3
      FS = (2.424*Z + 1.443*Z**2)*EXP(-44.*Z**6)
      GR = 3.*GS*RSTAR**0.07
      Q = VP*(1. + 3.*GR/A/A)/V + VPP*GR*A*A/VP
      F(1) = Q/.16/FS/A**3
      GO TO 6
   20 CF = 2./Y(1)**2
      WRITE(6,707) X, Y(1), CF, V, CM
  707 FORMAT(2F10.4,2F14.8,F10.4)
      GO TO 8
      END
```

```
      SUBROUTINE RUNGE(N,Y,F,X,H,M,K)
C     THIS ROUTINE PERFORMS RUNGE-KUTTA CALCULATION BY GILLS METHOD
      DIMENSION Y(10), F(10), Q(10)
      M = M + 1
      GO TO (1,4,5,3,7) , M
    1 DO 2 I = 1, N
    2 Q(I) = 0
      A = .5
      GO TO 9
    3 A = 1.707107
C     IF YOU NEED MORE ACCURACY, USE  A = 1.7071067811865475244
    4 X = X + .5*H
    5 DO 6 I = 1, N
      Y(I) = Y(I) + A*(F(I)*H - Q(I))
    6 Q(I) = 2.*A*H*F(I) + (1. - 3.*A)*Q(I)
      A = .2928932
C     IF YOU NEED MORE ACCURACY, SET A = .29283218813452475 6
      GO TO 9
    7 DO 8 I = 1,N
    8 Y(I) = Y(I) + H*F(I)/6. - Q(I)/3.
      M = 0
      K = 2
      GO TO 10
    9 K = 1
   10 RETURN
      END
```

smooth curve-fit, since the expression will not be differentiated and is used only to compute van Driest's parameter A from Equation (5).

Finally, just below statement number 23, the local wall temperature must be estimated. In the example shown, it is simply set equal to the local adiabatic wall temperature (the conditions of Zwarts' data). Again, any reasonable estimate will do, because T_w is used only to compute van Driest's parameter A, which Figure 1 shows is only slowly varying with both T_w and M_e.

All other procedures in the theory are carried out by the computer program, including testing for boundary layer separation and printing out the separation point if λ exceeds λ_{max}.

Example 1. THE SUPERSONIC RELAXING FLOW OF ZWARTS (UNPUBLISHED)

In unpublished work, Dr. F. W. Zwarts of McGill University measured a two-dimensional supersonic flow which decelerated sharply from Mach-4 to Mach-3 in about eight inches and thereafter remained nearly constant at Mach-3. This flow was used as an example to test the finite difference calculations of Bradshaw (8). It is a difficult test for any method. The freestream velocity $V(x)$ and Mach number $M_e(x)$ are shown in Figure 3. The velocity is fit by the solid line to a least-squares fourth order polynomial: (U_o = 2204. ft/sec)

$$V = 1.0013 - 0.00051\,x - 0.003885\,x^2 + 0.0004466\,x^3 - 0.0000139\,x^4 \tag{11}$$

The smoothness of the fit is seen to be quite adequate, and one can take confidence in a reasonably smooth first and second derivative. The Mach

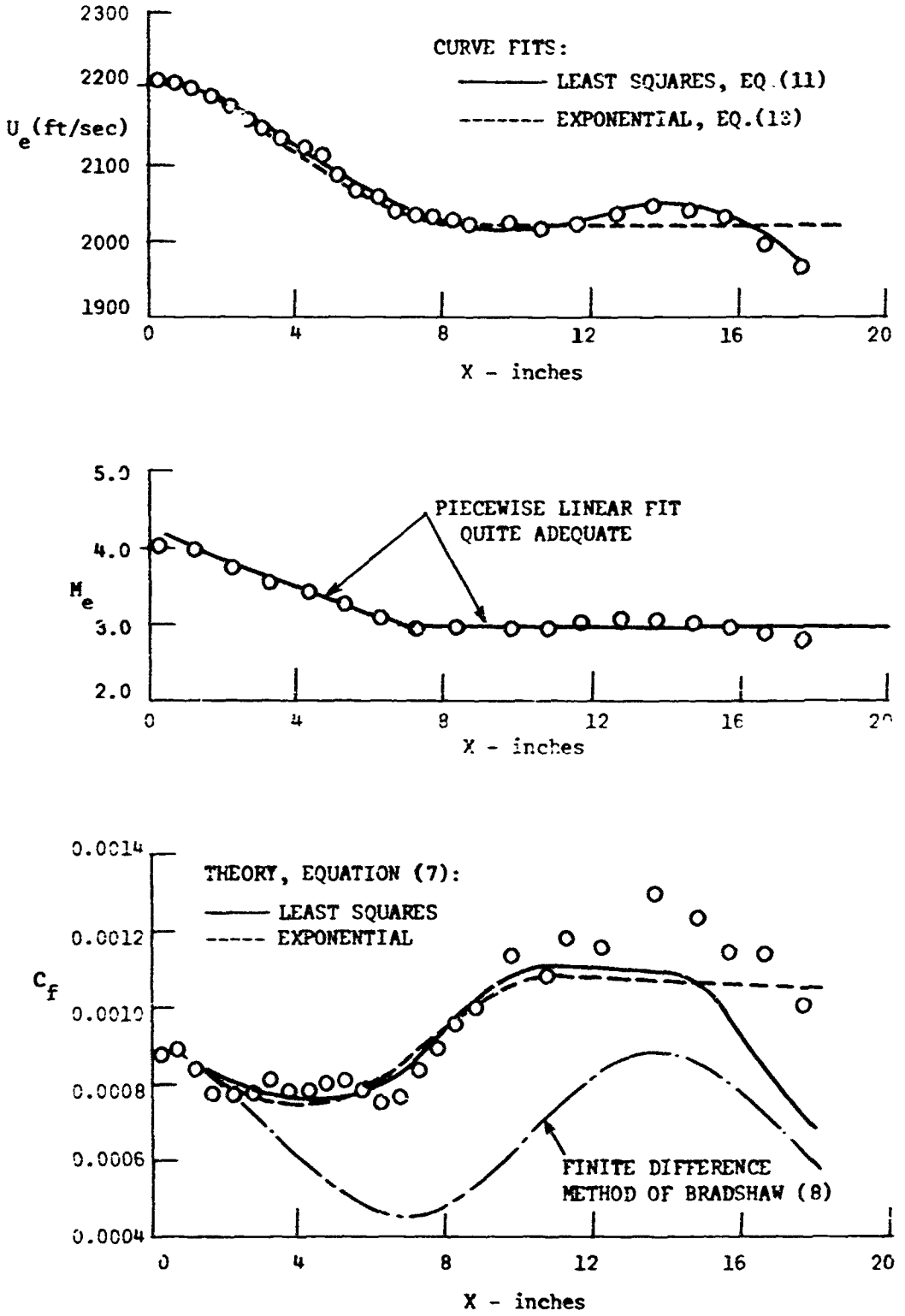

Figure 3. COMPARISON OF THEORY WITH THE SUPERSONIC RELAXING FLOW EXPERIMENT OF ZWARTS (UNPUBLISHED).

number is fit adequately by two linear pieces:

$$x \leq 6.75'': \quad M_e \doteq 4.125 - x/6$$
$$x > 6.75'': \quad M_e \doteq 3.0 \tag{12}$$

Note that x is in inches in both Eqs.(11) and (12), that is, L = one inch.

Also shown on Figure 3 is an exponential approximation to $M_e(x)$:

$$V \doteq 0.9165 + 0.0835 \exp[-0.03\, x - 0.05\, x^2] \tag{13}$$

This is plotted as a dotted line. The fit is excellent up to x = 12 inches, after which the curvature in the data - which may be an artifact - is not fit by the exponential curve.

The initial data to be read into the program are as follows:

$R_c = U_o L/\nu_e = (2204)(1/12)/(0.000106) = 1.73 \times 10^6.$

$C_{f_o} = 0.000894$ (from Zwarts' data).

$x_o = 0.75$ inches.

$h = \Delta x = 0.5$ inches.

$x_{max} = 18$ inches.

The choice $\Delta x = 0.5''$ is an engineering judgment designed to break the interval 0 to L down into about forty steps. Doubling this, to $\Delta x = 1''$, will cause a numerical error of about one per cent in the computed skin friction. The computed results from the program are also shown in Figure 3. Both the polynomial and exponential fits give theoretical C_f in reasonable agreement with the Preston tube data. Both are far more accurate than the finite-difference calculations of Bradshaw and Ferriss (8), also shown, which fall thirty to forty per cent low.

Example 2. THE UPPER SURFACE OF AN NACA 64A210 AIRFOIL

Mann and Whitten (9) measured skin friction with a Stanton tube and a Preston tube on the upper surface of an NACA 64A210 airfoil in high subsonic flow, $M_\infty = 0.7$. The measured velocity distribution for zero angle of attack is shown in Figure 4. An excellent fit is provided by the solid line, a fourth order least squares polynomial:

$$V = 0.8664 + 2.6494x^* - 5.6028x^{*2} + 4.1633x^{*3} - 1.2079x^{*4} \tag{14}$$

Since the flow was subsonic, the static temperature was nearly constant and thus the freestream Mach number was nearly proportional to V:

$$M_e \doteq 0.70\ V \tag{15}$$

For initial data, Mann and Whitten suggest that transition to turbulence occurred at about $x^* = 0.1$, with C_f somewhere between 0.003 and 0.004:

$$\begin{aligned} R_o &= 9.55 \times 10^6 \quad ; & x_o^* &= 0.1 \\ & & h &= \Delta x^* = 0.05 \\ C_{f_o} &= 0.003 \quad \text{or} \quad 0.004 \quad ; & x_{max}^* &= 1.0 \\ T_w &\doteq T_{aw} \end{aligned} \tag{16}$$

These values were run with the computer program of this report. The results are shown in Figure 4. Both runs are in reasonable agreement with the data. The dotted curve ($C_{f_o} = 0.003$) shows separation at the trailing edge, which was probably the case. Mann and Whitten themselves demonstrated good agreement with the Karman integral theory of Sasman and Cresci (10), and also with a finite-difference procedure by Cebeci et al (11).

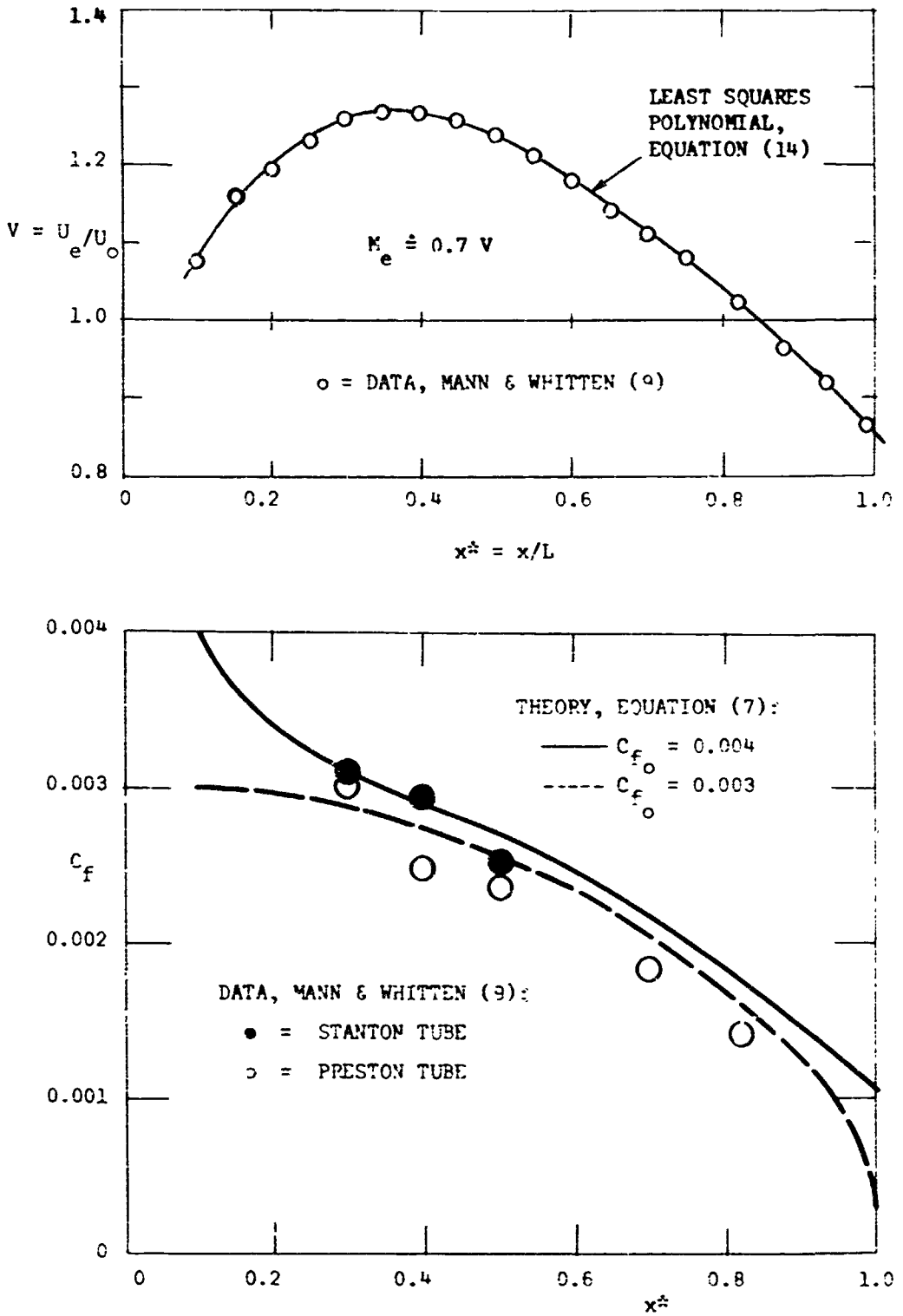

Figure 4. COMPARISON OF THEORY WITH DATA ON THE UPPER SURFACE OF AN NACA 64A210 AIRFOIL AT $M_\infty = 0.7$ (REF. 9).

Example 3. SUPERSONIC FLOW PAST A CURVED COMPRESSION RAMP

Sturek and Danberg (12) placed a faired curved ramp on the floor of a supersonic wind tunnel and generated a smooth decrease in freestream Mach number from 3.5 to 2.9 in a distance of ten inches. The freestream velocity data are shown in Figure 5. They are fit quite well by the solid curve, which is a least squares polynomial:

$$V = 1.1132 - 0.04396\,x + 0.005943\,x^2 - 0.0003263\,x^3 + 0.00000592\,x^4 \quad , \tag{17}$$

where x is in inches. The derivatives of this fit may not be reliable, since there are only eight data points to be fit and none are in the important region where the ramp begins. The step size and range are:

$$x_o = 6 \text{ inches}, \quad h = \Delta x = 0.5 \text{ inches}, \quad x_{max} = 22 \text{ inches}. \tag{18}$$

Skin friction data are shown for two different tunnel stagnation pressures, which correspond to two different Reynolds numbers and initial values:

p_o - psia	$U_o(1")/\nu_e$	C_{f_o}	
38.7	358,000.	0.001085	$\} \; T \doteq T_{aw}$
56.0	538,000.	0.00103	

These were run on the computer program, and the results are shown in Figure 5. It is seen that the theory is in excellent agreement with the skin friction data. No other theory has been applied to this data, and in fact most other theories, both Karman integral and finite difference type, are invalid for this experiment because of the surface curvature, which causes large pressure gradients normal to the wall. (8,10) The present theory is insensitive to normal pressure gradients.

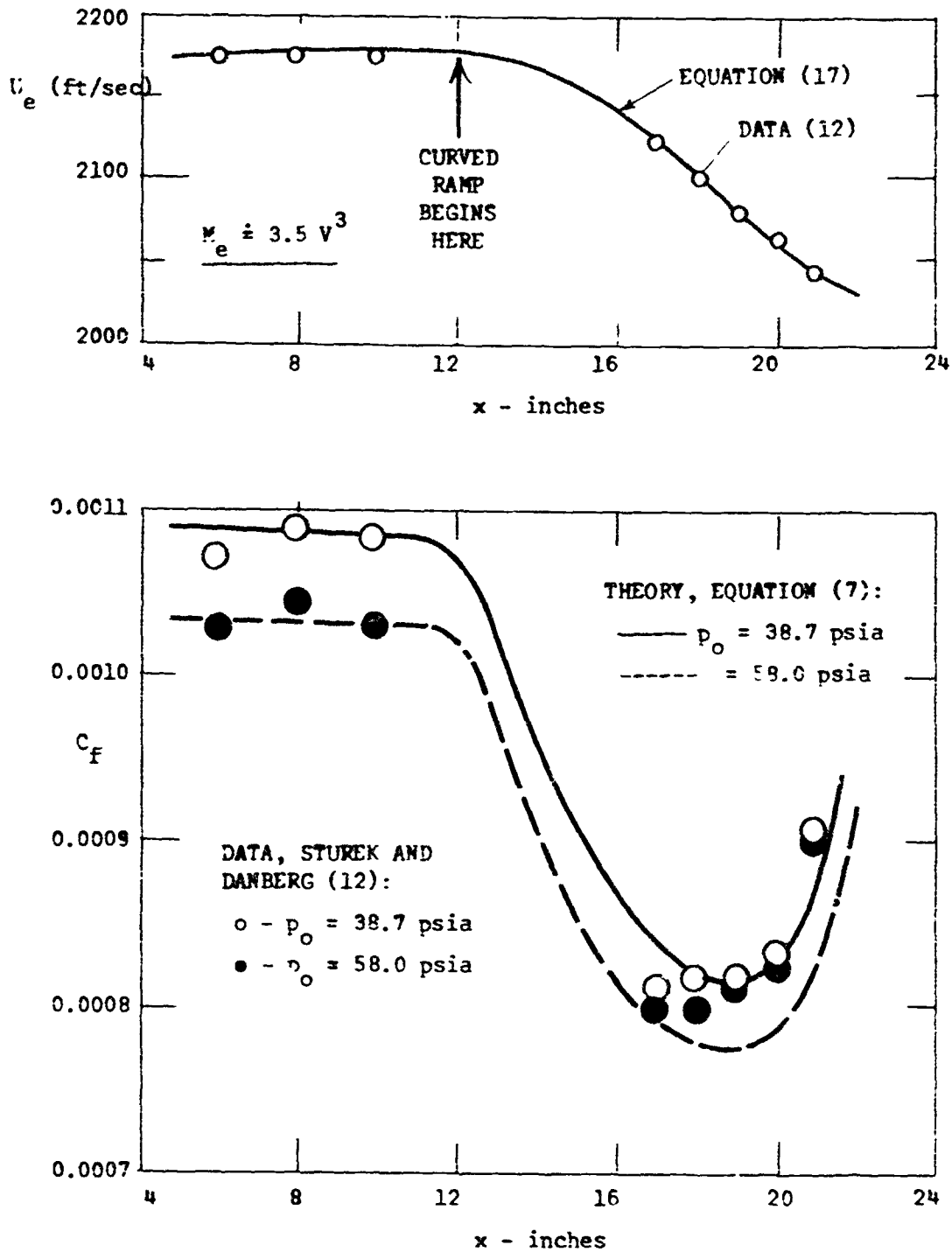

Figure 5. COMPARISON OF THEORY WITH THE SUPERSONIC CURVED RAMP EXPERIMENT OF STUREK AND DANBERG (12).

Example FLOW PAST A FLAT PLATE

For flow at zero incidence past a flat plate, $\alpha = V' = V'' = 0$, and the theory reduces to Equation (10), which further reduces to:

$$\frac{d\lambda}{dx^*} = \frac{1}{8} R_L \exp(-0.48\,\lambda/A) \quad , \text{ with } \lambda = \lambda_o \text{ at } x = x_o . \qquad (19)$$

The variables are separable and integrable, and an exact solution is possible:

$$C_f = 0.455\, A^{-2}\, \ln^{-2}[\exp(0.48\lambda_o/A) + \frac{0.06}{A}(R_x - R_{x_o})(\frac{T_e}{T_w})^{\frac{1}{2}} (\frac{\mu_e}{\mu_w})\ ] , \qquad (20)$$

where $R_x = U_e x/\nu_e$ is the local Reynolds number. This formula may be used to proceed from a position $C_{f_o}(R_{x_o})$ downstream to a new value $C_f(R_x)$. If the plate is turbulent from the leading edge, $\lambda_o = R_{x_o} = 0$, and Eq.(20) reduces to the flat plate formula of reference 3:

$$C_f(\text{flat plate}) = 0.455\, A^{-2}\, \ln^{-2}[\frac{0.06}{A} R_x (T_e/T_w)^{1/2} (\mu_e/\mu_w)] \qquad (21)$$

When compared in reference 3 with almost all available data on turbulent flat plate skin friction (657 reported values), Eq.(21) was shown to be the most accurate formula ever derived in terms of mean absolute error. It is more accurate than the tentative formula we first suggested (in ref. 2) and is a consequence of a better correlation of the integral functions G and H of that reference.

Some numerical values from Eq.(21) for adiabatic walls are given in Table 2, and a plot showing the effect of wall temperature is given in Figure 6. Note that there is a substantial effect of Reynolds number in this plot, so that the common practice in the literature of plotting all data on a single figure of this type can be very misleading.

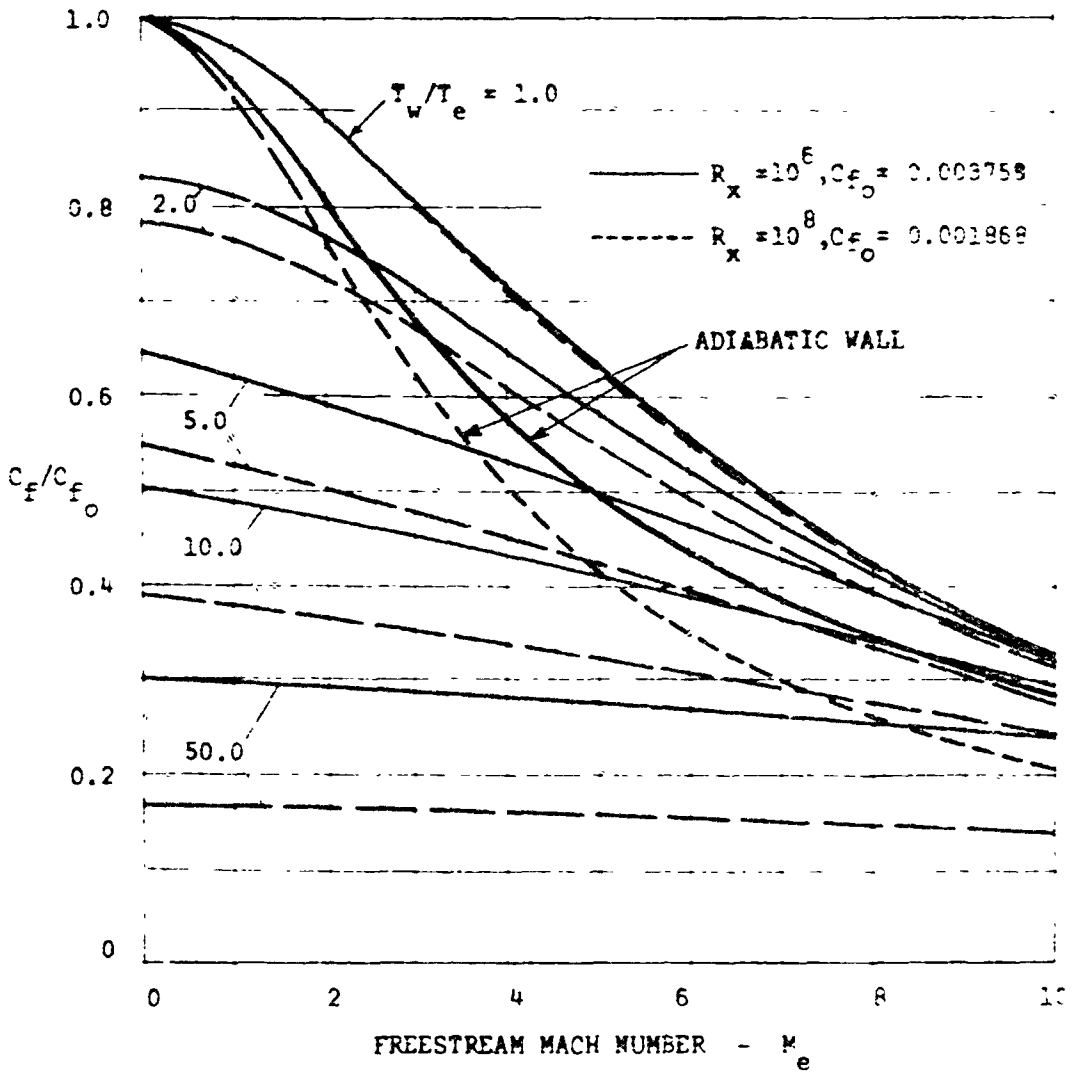

Figure 6. SKIN FRICTION ON A FLAT PLATE AT VARIOUS MACH NUMBERS, WALL TEMPERATURES, AND REYNOLDS NUMBERS, FROM EQUATION (21).

Table 2

TURBULENT SKIN FRICTION ON ADIABATIC FLAT PLATES, EQ. (21)

MACH NUMBER	LOCAL REYNOLDS NUMBER, R_x				
	10^5	10^6	10^7	10^8	10^9
0	0.006012	0.003759	0.002570	0.001868	0.001419
1	0.005703	0.003522	0.002390	0.001727	0.001306
2	0.005049	0.003025	0.002012	0.001435	0.001074
4	0.003892	0.002146	0.001357	0.000935	0.000683
6	0.003225	0.001620	0.000972	0.000647	0.000462
8	0.002868	0.001308	0.000745	0.000481	0.000335
10	0.002698	0.001110	0.000601	0.000376	0.000258

Three other examples of the use of this new theory to predict turbulent skin friction, particularly in supersonic flows, are given in references 3 and 8.

V. CONCLUSIONS

It has been shown that one can compute turbulent skin friction in two-dimensional flows with arbitrary conditions, using Eq.(7) of this report, which has been programmed for ready use with a digital computer. The user must only supply: 1) an initial value of skin friction; 2) a smooth curve fit to the freestream velocity distribution; and 3) reasonable estimates of the freestream Mach number and wall temperature distributions. The authors believe this theory is the most accurate in the literature.

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