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OPTIMAL ESTIMATION OF MEASUREMENT BIAS

William S. Agee, et al

National Range Operations Directorate
White Sands Missile Range, New Mexico

December 1972

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OPTIMAL ESTIMATION OF MEASUREMENT BIAS

BY

WILLIAM S. AGEE and ROBERT H. TURNER

DECEMBER 1972

MATHEMATICAL SERVICES BRANCH
ANALYSIS & COMPUTATION DIVISION
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A method is described for the optimal estimation of measurement biases in a Kalman filtering application. The method, which was originally developed by B. Friedland, is based on the decoupling of a large Kalman filter into two smaller filters. One of the smaller filters produces a state estimate which assumes that all measurement biases are zero and the other called the bias filter estimates the measurement biases. The outputs of the two smaller filters are recombined to form the optimal state estimates. Restrictions on the form of the filters which are imposed by the decoupling are discussed. Several extensions of Friedland's original method are presented. Finally, the implementation of the filters via square root filtering techniques is developed.

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TECHNICAL REPORT

No. 41

OPTIMAL ESTIMATION OF MEASUREMENT BIAS

BY

WILLIAM S. AGEE and ROBERT H. TURNER

DECEMBER 1972

MATHEMATICIAL SERVICES BRANCH
ANALYSIS & COMPUTATION DIVISION
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OPTIMAL ESTIMATION OF MEASUREMENT BIAS

INTRODUCTION. A Kalman filtering program has been developed by the authors to provide a Best Estimate of Trajectory (BET) for flight tests conducted at White Sands Missile Range (WSMR). This optimal filtering program combines measurements from radar, fixed camera, cinetheodolite, dovap, velocimeter, and accelerometer which are optimally weighted using on-line estimates of the measurement variances. One of the most important considerations in developing a BET technique is to account for inconsistencies produced by bias errors in the measurements. As a matter of fact, a BET is not very useful unless these measurement errors have been accounted for. It was for this reason that this research project began. As a result of this project, we now have a very effective technique included in the BET program to estimate the measurement bias errors.

Since our BET program uses a Kalman filter it was desirable that the bias estimation technique be developed within the framework of the Kalman filter theory. There is a natural way of including bias terms in the Kalman filter. One merely adds an additional state variable for each bias term to be considered and forms the optimal estimate of the biases in the same way as for the other states of the system. This technique is fine for cases where there are only a few bias terms to be considered. However, a typical application of our BET program has a large number of measuring instruments involved. For example, a LANCE flight test might have two radars, 28 dovap receivers, eight fixed cameras, and eight cinetheodolites. Considering only one bias error per measurement this results in 66 additional state variables to be estimated. If the dimension of dynamic state is nine, we would then have to compute filtered estimates for 75 state variables. An ordinary

Kalman filter program using a 75 dimensional state vector is computationally prohibitive at the present time. Thus, we must develop some technique, other than the straightforward method of augmenting the filter state vector, for estimating measurement bias errors within the Kalman filter framework. The research problem may be stated concisely as follows: Develop a computationally feasible method within the Kalman filter framework for estimating the biases of all measuring instruments participating in a WSMR mission.

A preliminary search of the literature reveals two possible techniques which might be applicable to the bias estimation problem. A third technique developed by the author was also considered. Finally, during the course of the research an additional technique was found. Of the four methods considered, all of which were computationally feasible, three were discarded either because of numerical difficulties or excessive errors in the bias estimates. The remaining method, which was published by B. Friedland (in Reference (1)) was developed and extensively evaluated for WSMR applications. Several extensions of Friedland's method were made so that it would be applicable to the instrument bias problem at WSMR. Also, this report employs a derivation different than Friedlands.

The evaluation of this instrument bias estimation technique was performed using simulated measurement data from a nearly ballistic trajectory. The simulated instrumentation included four fixed cameras, six cinetheodolites, one accelerometer, and two radars with doppler. Thus, there were a total of twenty-nine bias states estimated in addition to nine dynamic states of the basic filter. The technique was also tested using real data from a similar trajectory having the same instrumentation. The simulated data has bias and noise added to each of the exact measurements. Evaluation of time varying as well as constant bias were considered in the evaluation.

BIAS ESTIMATION WITH THE KALMAN FILTER. We will consider only the discrete case of the Kalman filter. The following equations describe the discrete Kalman filter as used in this report.

DYNAMIC STATE EQUATION. The model of the process we are observing is represented by the linear difference equation

$$\underset{(n\times 1)}{x(k+1)} = \underset{(n\times n)}{A(k)}\ \underset{(n\times 1)}{x(k)} + \underset{(n\times 1)}{u_x(k)} \tag{1}$$

where x_k is the state vector of the process and $u_x(k)$ is a random vector representing our uncertainties in how well the model defined by the homogeneous portion of the difference equation acually represents the system. We assume that

$$E\left[u_x(k)\right] = 0$$

and

$$E\left[u_x(k)u_x^T(\ell)\right] = Q_x(k)\delta_{k\ell}$$

OBSERVATION EQUATION. At discrete instants of time t_i we have vector valued observations z(i) available. The z(i) are assumed to be linearly related to the state by

$$\underset{(m\times 1)}{z(i)} = \underset{(m\times n)}{H(i)}\ \underset{(n\times 1)}{x(i)} + \underset{(m\times p)}{G(i)}\ \underset{(p\times 1)}{b(i)} + \underset{(m\times 1)}{v(i)} \tag{2}$$

where v(i) is a random vector of measurement errors with mean zero and covariance

$$E\left[v(i)v^T(j)\right] = R(i)\delta ij$$

b(k) is a vector of measurement biases which obey

$$b(k+1) = B(k)b(k) + u_b(k) \tag{3}$$

where $u_b(k)$ is a zero mean random vector with covariance

$$E\left[u_b(k)u_b^T(\ell)\right] = Q_b(k)\delta_{k\ell}$$

By adjoining the vector b to the vector x we form the augmented state vector y

$$y = \begin{bmatrix} x \\ b \end{bmatrix} \tag{4}$$

Then the dynamics of y are

$$y_{(k+1)} = F_{(k)}y_{(k)} + u_{(k)} \tag{5}$$

where

$$F(k) = \begin{bmatrix} \begin{matrix} A(k) \\ n\times n \end{matrix} & 0 \\ 0 & \begin{matrix} B(k) \\ p\times p \end{matrix} \end{bmatrix}$$

and

$$u(k) = \begin{bmatrix} u_x(k) \\ u_b(k) \end{bmatrix}$$

The observation equation becomes

$$z(k) = L(k)y(k) + v(k) \tag{6}$$

where

$$L(k) = \begin{bmatrix} H(k)G(k) \end{bmatrix}$$

PREDICTED STATE ESTIMATE. Let $\hat{y}(k)$ denote the optimal state estimate of the augmented system

$$\hat{y}(k) = \begin{bmatrix} \hat{x}(k) \\ \hat{b}(k) \end{bmatrix} \tag{7}$$

At t_{k+1} the predicted state estimate is

$$\hat{y}(k+1|k) = F(k)\hat{y}(k) \tag{8}$$

CORRECTED STATE ESTIMATE. The optimal estimate at t_{k+1} is defined by

$$\hat{y}(k+1) = \hat{y}(k+1|k) + W(k+1)(z(k+1) - L(k)\hat{y}(k+1|k) \tag{9}$$

where W(k+1) is the optimal gain matrix defined in terms of the posterior covariance matrix P(k+1) as

$$W(k+1) = P(k+1)L^T(k+1)R^{-1}(k+1) \tag{10}$$

The covariance of the predicted state denoted by $P(k+1|k)$ is defined by

$$P(k+1|k) = F(k)P(k)F^T(k) + Q(k) \tag{11}$$

where

$$Q(k) = \begin{bmatrix} Q_x(k) & 0 \\ 0 & Q_b(k) \end{bmatrix}$$

The posterior covariance P(k+1) may be defined as

$$P(k+1) = \left[P^{-1}(k+1|k) + L^T(k+1)R^{-1}(k+1)L(k+1)\right]^{-1} \tag{12}$$

Direct implementation on a digital computer of the augmented Kalman filter equations defined by (7) thru (12) is computationally prohibitive when the dimension of the bias vector b is large. However, by utilizing suitable restrictions on the bias dynamics, Friedland (Reference (1)) was able to decouple the augmented Kalman filter so that a dynamic state $x^*(k)$ and a bias b(k) are separately estimated and then the optimal state estimate x(k) is computed by

$$\hat{x}(k) = x^*(k) + T(k)\hat{b}(k) \tag{13}$$

where T(k) is an nxp matrix. The state estimate x^* is computed by assuming all biases to be zero. We will find that there are certain restrictions which must be placed on the form of the augmented filter in order that the decoupling be possible. One such restriction will be that Q_b, the covariance of the stochastic term in the bias equation, be zero. The development is not restricted to measurement biases; biases in assumed constants in the state dynamics may also be included.

Friedland approached the decoupling of the Kalman filter by transformation of the discrete Ricatti equation. The derivation given in this report will be based upon an examination of the conditions for which the decoupling of the state estimates specified by (13) is possible.

FILTER DECOMPOSITION. A general form for a discrete time, linear recursive filter for the augmented state vector may be written in the form of (9)

$$\hat{y}(k) = \hat{y}(k|k-1) + W(k)\Big(z(k)-H(k)\hat{x}(k|k-1)-G(k)\hat{b}(k|k-1)\Big) \tag{14}$$

where for the present we will consider $W(k)$ to be an arbitrary gain matrix. The augmented state estimate $\hat{y}(k)$ may be decomposed into estimates $\hat{x}(k)$ and $\hat{b}(k)$

$$\hat{x}(k) = \hat{x}(k|k-1) + W_x(k)\Big(z(k)-H(k)\hat{x}(k|k-1)-G(k)\hat{b}(k|k-1)\Big) \tag{15}$$

$$\hat{x}(k|k-1) = A(k-1)\hat{x}(k-1)$$

$$\hat{b}(k) = \hat{b}(k|k-1) + W_b(k)\Big(z(k)-H(k)\hat{x}(k|k-1)-G(k)\hat{b}(k|k-1)\Big) \tag{16}$$

$$\hat{b}(k|k-1) = B(k-1)\hat{b}(k-1)$$

where $W(k)$ in (14) is

$$W(k) = \begin{bmatrix} \underset{n\times m}{W_x(k)} \\ \\ \underset{p\times m}{W_b(k)} \end{bmatrix}$$

Now let $x^*(k)$ be the state estimate that would be obtained if all biases are assumed to be zero. We will call this the zero-bias estimate. The recursive estimation equations for $x^*(k)$ are

$$\begin{aligned} x^*(k) &= x^*(k|k-1) + W_*(k)(z(k) - H(k)x^*(k|k-1)) \\ x^*(k|k-1) &= A(k-1)x^*(k-1) \end{aligned} \tag{17}$$

Now we ask the question: Under what conditions can we decompose the optimal estimate $\hat{x}(k)$ as

$$\hat{x}(k) = x^*(k) + T(k)\hat{b}(k)$$

Denote the residual by $\hat{r}(k|k-1)$.

$$\hat{r}(k|k-1) = z(k) - H(k)\hat{x}(k|k-1) - G(k)\hat{b}(k|k-1) \tag{18}$$

If the decomposition (13) holds, substitution of (13) with k replaced by k-1 into (18) results in an alternative expression for $\hat{r}(k|k-1)$

$$\hat{r}(k|k-1) = z(k) - H(k)x^*(k|k-1) - S(k)\hat{b}(k-1) \tag{19}$$

where

$$S(k) = G(k)B(k-1) + H(k)A(k-1)T(k-1) \tag{20}$$

Now substituting (16), (17), (19), and (20) into (13) we find

$$\begin{aligned} \hat{x}(k) = \hat{x}(k|k-1) &+ \left[W_x^*(k) + T(k)W_b(k)\right]\hat{r}(k|k-1) \\ &+ \left[T(k)B(k) + W_x^*(k)S(k) - A(k-1)T(k-1)\right]\hat{b}(k-1) \end{aligned} \tag{21}$$

Note, if we assume that (15) and (16) define the optimal linear filter, $\hat{r}(k|k-1)$ and $\hat{b}(k-1)$ must be linearly independent random vectors. In this case the following conditions must hold in (21).

$$W_x(k) = W_x^*(k) + T(k)W_b(k) \tag{22}$$

$$T(k) = A(k-1)T(k-1)B^{-1}(k-1) - W_x^*(k)S(k)B^{-1}(k-1) \tag{23}$$

By examination of (21) we easily see that an arbitrary linear filter is also decomposable into the form specified by (13), if the above conditions hold. However, these conditions are not necessary for decomposition of a general linear filter. In addition to the conditions specified by (22), which we will call the gain condition and (23), the recursion equation for T(k), we must also have

$$\hat{x}(0) = x^*(0) + T(0)\hat{b}(0) \tag{24}$$

since we assume that the decomposition holds for all k. A natural choice for the initial conditions to satisfy (24) will in many cases be $\hat{x}(0) = x^*(0)$, $T(0) = 0$, i.e. we assume that the biases have no initial effect on the optimal estimate.

The conditions expressed by (22), (23), and (24) are the basic requirements for a general linear filter to be decomposed.

RESTRICTIONS IMPOSED BY THE GAIN CONDITION. The consequences of the gain condition stated in (22) will be examined when both the estimators for x^*_k and $x(k)$, $b(k)$ are Kalman filters. The gain for the Kalman filter giving the zero-bias estimate x^* can be written as

$$W^*_x(k) = P^*_x(k)H^T(k)R^{-1}(k) \tag{25}$$

The matrix $P^*_x(k)$ in this case (with biases actually present) may be defined as

$$P^*_x(k) = E\left[\Big(x^*(k)-x(k)+T(k)b(k)\Big)\Big(x^*(k)-x(k)+T(k)b(k)\Big)^T\right] \tag{26}$$

We will now examine the relations between the various Kalman covariance matrices $P^*_x(k)$, $P_x(k)$, and $P_b(k)$. Rewrite the decomposition equation as

$$x^*(k) - x(k) + T(k)b(k) = \left(\hat{x}(k)-x(k) + T(k)(\hat{b}(k)-b(k))\right) \tag{27}$$

Then multiplying this equation by its transpose and taking expected values we find

$$P_x^*(k) = P_x(k) + T(k)P_b(k)T^T(k) - P_{xb}(k)T^T(k) - T(k)P_{bx}(k) \tag{28}$$

where

$$P_{xb}(k) = E[(\hat{x}(k)-x(k))(\hat{b}(k)-b(k))^T],\ P_{bx}(k) = P_{xb}^T(k) \tag{29}$$

A similar result for $P^*(k|k-1)$

$$\begin{aligned} P_x^*(k|k-1) = {} & P_x(k|k-1) + A(k-1)T(k-1)P_b(k-1)T^T(k-1)A^T_{(k-1)} \\ & -P_{xb}(k|k-1)T^T(k-1)A^T(k-1)-A(k-1)T(k-1)P_{bx}(k-1) \end{aligned} \tag{30}$$

The gain for the augmented Kalman filter is

$$\begin{bmatrix} W_x(k) \\ W_b(k) \end{bmatrix} = \begin{bmatrix} P_x(k) & P_{xb}(k) \\ P_{bx}(k) & P_b(k) \end{bmatrix} \begin{bmatrix} H^T(k) \\ G^T(k) \end{bmatrix} R^{-1}(k) \tag{31}$$

Substituting the gain condition into the first component of (31)

$$\left(P_x(k)H^T(k) + P_{xb}(k)G^T(k)\right)R^{-1}(k) = W_x^*(k) + T(k)W_b(k) \tag{32}$$

Then substituting the second component of (31) for $W_b(k)$ and (25) for $W_x^*(k)$, we find the satisfaction of the gain condition requires

$$P_{xb}(k)\Big(G(k)+H(k)T(k)\Big)^T = T(k)P_b(k)\Big(G(k)+H(k)T(k)\Big)^T \tag{33}$$

Thus, the gain condition holds if we impose the requirement that

$$P_{xb}(k) = T(k)P_b(k) \tag{34}$$

We have translated the gain condition into a covariance restriction, but we must continue to examine the covariance relations to determine the meaning of (34). The relation between gain and covariance in the filter can be written as

$$P(k) = \Big(I-W(k)L(k)\Big)P(k|k-1) \tag{35}$$

or

$$\begin{bmatrix} P_x(k) & P_{xb}(k) \\ P_{bx}(k) & P_b(k) \end{bmatrix} = \begin{bmatrix} I-W_x(k)H(k) & -W_x(k)G(k) \\ -W_b(k)H(k) & I-W_b(k)G(k) \end{bmatrix} \begin{bmatrix} P_x(k|k-1) & P_{xb}(k|k-1) \\ P_{bx}(k|k-1) & P_b(k|k-1) \end{bmatrix} \tag{36}$$

Then

$$P_{xb}(k) = \Big(I-W_x(k)H(k)\Big)P_{xb}(k|k-1) - W_x(k)G(k)P_b(k|k-1) \tag{37}$$

$$P_b(k) = \Big(I-W_b(k)G(k)\Big)P_b(k|k-1) - W_b(k)H(k)P_{xb}(k|k-1) \tag{38}$$

Now suppose that the covariance condition (34) holds at k-1, $P_{xb}(k-1) = T(k-1)\,P_b(k-1)$

Then

$$P_{xb}(k|k-1) = A(k-1)T(k-1)P_b(k-1)B^T(k-1) \tag{39}$$

$$P_b(k|k-1) = B(k-1)P_b(k-1)B^T(k-1) + Q_b(k-1) \tag{40}$$

Substituting (39) and (40) in (37) and (38) and using (20) and (22) gives

$$\begin{aligned} P_{xb}(k) = {} & T(k)\Big(B(k-1)-W_b(k)S(k) \Big) P_b(k-1)B^T(k-1) \\ & - \Big[W_x^*(k)+T(k)W_b(k) \Big] G(k)Q_b(k-1) \end{aligned} \tag{41}$$

$$\begin{aligned} P_b(k) = {} & \Big(B(k-1)-W_b(k)S(k) \Big) P_b(k-1)B^T(k-1) \\ & + \Big(I-W_b(k)S(k) \Big) Q_b(k-1) \end{aligned} \tag{42}$$

Clearly, the condition

$$P_{xb}(k) = T(k)P_b(k)$$

holds if

$$Q_b(k-1) = 0$$

Thus, we cannot have a stochastic term in the dynamics of the bias states. There does not seem to be any way of removing this restriction. This restriction may lead to a divergence problem in the bias estimation, but fortunately we can treat this problem in another way which will be discussed later.

SUMMARY. The following formulas summarize the decomposition of the optimal linear filter for the process described by (1) thru (6).

ZERO-BIAS FILTER

$$X^{*}(k|k-1) = A(k-1)x^{*}(k-1)$$

$$P_x^{*}(k|k-1) = A(k-1)P^{*}(k-1)A^{T}(k-1) + Q_x(k-1)$$

$$P_x^{*}(k) = \left(P^{*-1}(k|k-1) + H(k)R^{-1}(k)H^{T}(k) \right)^{-1}$$

$$x^{*}(k) = x^{*}(k|k-1) + W_x^{*}(k)\left(Z(k)-H(k)x^{*}(k|k-1)\right)$$

$$W_x^{*}(k) = P^{*}(k)H^{T}(k)R^{-1}(k)$$

BIAS FILTER

$$S(k) = G(k)B(k-1) + H(k)T(k|k-1)$$

$$C(k) = H(k)P^{*}(k|k-1)H^{T}(k) + R(k) \tag{43}$$

$$\hat{b}(k\ k-1) = B(k-1)\hat{b}(k-1)$$

$$P_b(k|k-1) = B(k-1)P_b(k-1)B^{T}(k-1) \tag{44}$$

$$P_b(k) = \left(P_b^{-1}(k|k-1) + S^{T}(k)C^{-1}(k)S(k)\right)^{-1} \tag{45}$$

$$\hat{b}(k) = \hat{b}(k|k-1) + W_b(k)\left(Z(k)-H(k)x^{*}(k|k-1)-S(k)\hat{b}(k|k-1)\right)$$

$$W_b(k) = P_b(k)S^{T}(k)C^{-1}(k) \tag{46}$$

OPTIMAL STATE ESTIMATE

$$T(k|k-1) = A(k-1)T(k-1)$$

$$T(k) = T(k|k-1) - W_x^*(k)S(k)$$

$$\hat{x}(k) = x^*(k) + T(k)\hat{b}(k)$$

$$P_x(k) = P_x^*(k) + T(k)P_b(k)T^T(k) \tag{47}$$

The numbered equations in the summary have not been derived but are easily obtainable. Equation (44) is merely (40) with $Q_b(k)=0$, and (47) comes from (28) with $P_{xb}(k)=T(k)P_b(k)$. Equations (43, (45), and (46) can be derived by substituting the relations between covariances in (12).

EXTENSION TO THE NONLINEAR CASE. The extension of the bias estimation procedure to the case where the state dynamics are nonlinear and the observational equations are nonlinear functions of the state can be performed in any one of the ways by which the Kalman filter is usually extended to the nonlinear case. Let the discrete time nonlinear dynamics be represented by

$$x(k) = f(x(k-1),k-1) + u_x(k) \tag{48}$$

and let the nonlinear observation equations be written as

$$z(k) = h(x(k),k) + G(x(k),k)b(k) + v(k) \tag{49}$$

also we will employ the additional notation

$$H(k) = \left[\frac{\partial}{\partial x(k)}\Big(h(x(k),k)+G(x(k),k)\Big)\right]_{\hat{x}(k|k-1),\hat{b}(k|k-1)}$$

$$G(k) = G(\hat{x}(k|k-1),k)$$

$$A(k-1) = \left[\frac{\partial f(x(k-1),k-1)}{\partial x_{k-1}}\right]_{\hat{x}(k-1)}$$

Since the extended Kalman filter is not optimal we view the decomposition as applying to an arbitrary linear filter. Also, the decomposition of the extended filter is not an exact procedure as in the linear case but holds only approximately since an additional linearization is required to derive the gain condition and recursive definition of T(k).

The extended Kalman filter is

$$\hat{x}(k) = \hat{x}(k|k-1)+W_x(k)\Big(z(k)-h\Big(\hat{x}(k|k-1),k\Big)-G\Big(\hat{x}(k|k-1),k\Big)\hat{b}(k|k-1)\Big)$$

$$\hat{b}(k) = \hat{b}(k|k-1)+W_b(k)\Big(z(k)-h\Big(\hat{x}(k|k-1),k\Big)-C\Big(\hat{x}(k|k-1),k\Big)\hat{b}(k|k-1)\Big)$$

where

$$\hat{x}(k|k-1) = f\Big(\hat{x}(k-1),k-1\Big)$$

$$\hat{b}(k|k-1) = B\ (k-1)\hat{b}(k-1)$$

$$\begin{bmatrix} W_x(k) \\ W_b(k) \end{bmatrix} = \begin{bmatrix} P_x(k) & P_{xb}(k) \\ P_{bx}(k) & P_b(k) \end{bmatrix} \begin{bmatrix} H^T(k) \\ G^T(k) \end{bmatrix} R^{-1}(k)$$

$$\begin{bmatrix} P_x(k) & P_{xb}(k) \\ P_{bx}(k) & P_b(k) \end{bmatrix} = \left(\begin{bmatrix} P_x(k|k-1) & P_{xb}(k|k-1) \\ P_{bx}(k|k-1) & P_b(k|k-1) \end{bmatrix}^{-1} + \begin{bmatrix} H^T(k) \\ G^T(k) \end{bmatrix} R^{-1}(k)[H(k)G(k)] \right)^{-1}$$

$$\begin{bmatrix} P_x(k|k-1) & P_{xb}(k|k-1) \\ P_{bx}(k|k-1) & P_b(k|k-1) \end{bmatrix} = \begin{bmatrix} A(k-1) & 0 \\ 0 & B(k-1) \end{bmatrix} \begin{bmatrix} P_x(k-1) & P_{xb}(k-1) \\ P_{bx}(k-1) & P_b(k-1) \end{bmatrix} \begin{bmatrix} A^T(k-1) & 0 \\ 0 & B^T(k-1) \end{bmatrix} + \begin{bmatrix} Q_x(k-1) & 0 \\ 0 & 0 \end{bmatrix}$$

The extended Kalman filter for the zero bias case is

$$x^*(k) = x^*(k|k-1) + W_x^*(k)\left(z(k) - h(x^*(k|k-1),k) \right) \tag{50}$$

$$x^{*}(k|k-1) = f(x^{*}(k-1),k) \tag{51}$$

$$W_x^{*}(k) = P_x^{*}(k)H^{T*}(k)R^{-1}(k) \tag{52}$$

$$P_x^{*}(k) = \left[P_x^{*-1}(k|k-1)+H^{*T}(k)R^{-1}(k)H^{*}(k)\right]^{-1} \tag{53}$$

$$P_x^{*}(k|k-1) = A^{*}(k-1)P_x^{*}(k-1)A^{*T}(k-1) + Q_x(k-1) \tag{54}$$

where

$$H^{*}(k) = \left[\frac{\partial h(x,k)}{\partial x}\right]_{x^{*}(k|k-1)}$$

$$A^{*}(k-1) = \left[\frac{\partial f(x,k-1)}{\partial x}\right]_{x^{*}(k-1)}$$

The derivation of the conditions under which the decomposition approximately holds proceeds exactly as in the linear case. The additional linearizations

$$f(\hat{x}(k-1))\approx f(x^{*}(k-1),k) + A^{*}(k-1)T(k-1)\hat{b}(k-1) \tag{55}$$

$$h(\hat{x}(k|k-1),k)\approx h(x^{*}(k|k-1),k) + H^{*}(k)A^{*}(k-1)T(k-1)\hat{b}(k-1) \tag{56}$$

$$G(\hat{x}(k|k-1),k)\approx G(x^{*}(k|k-1),k) = G^{*}(k) \tag{57}$$

are required in obvious places. The details of the derivation will not be given but the results of the decomposition are summarized below.

ZERO-BIAS FILTER

$$x^*(k|k-1) = f(x^*(k|k-1),k)$$

$$P_x^*(k|k-1) = A(k-1)P^*(k-1)A^T(k-1) + Q_x(k-1)$$

$$P_x^*(k) = \left(P^{*-1}(k|k-1) + H^*(k)R^{-1}(k)H^{*T}(k) \right)^{-1}$$

$$x^*(k) = x^*(k|k-1) + W_x^*(k)\ (z(k)-h(x^*(k|k-1),k))$$

$$W_x^*(k) = P^*(k)H^{*T}(k)R^{-1}(k)$$

$$H^*(k) = \left[\frac{\partial h(x,k)}{\partial x}\right] x^*(k|k-1)$$

BIAS FILTER

$$S^*(k) = G^*(k)B(k-1) + H^*(k)T(k|k-1) \tag{58}$$

$$C^*(k) = H^*(k)P^*(k|k-1)H^{*T}(k) + R(k) \tag{59}$$

$$\hat{b}(k|k-1) = B(k-1)\hat{b}(k-1) \tag{60}$$

$$P_b(k|k-1) = B(k-1)P_b(k-1)B^T(k-1) \tag{61}$$

$$P_b(k) = (P_b^{-1}(k|k-1) + S^{*T}(k)C^{*-1}(k)S^*(k))^{-1} \tag{62}$$

$$\hat{b}(k) = \hat{b}(k|k-1)\ W_b(k)(z(k)-h(x^*(k|k-1)-S^*(k)\hat{b}(k|k-1)) \tag{63}$$

$$W_b(k) = P_b(k)S^{*T}(k)C^{*-1}(k) \tag{64}$$

APPROXIMATELY OPTIMAL STATE ESTIMATE

$$T(k|k-1) = A(k-1)T(k-1) \tag{65}$$

$$T(k) = T(k|k-1) - W_x^* S^*(k) \tag{66}$$

$$\hat{x}(k) = \quad (k) + T(k)\hat{b}(k) \tag{67}$$

$$P_x(k) = P_x^*(k) + T(k)P_b(k)T^T(k) \tag{68}$$

BIAS ESTIMATION WITH THE FADING MEMORY KALMAN FILTER. The restriction on the filter decomposition imposed by the gain condition that there be no state noise on the bias state variable, i.e. $Q_b(k)=0$ may lead to a filter divergence problem as previously noted. Thus in the bias estimation technique we need to develop some other method for devaluing the effect of old observations on the bias state estimates. The divergence problem arises from numerical errors in computation and from errors in modeling the dynamics and covariance matrices. The divergence caused by numerical errors is most easily controlled by employing some standard numerical analysis techniques and reformulating the Kalman filter in terms of the square roots of the covariance matrices, see Reference (2). We use this reformulation and also process only scalar observations in our BET program to control numerical errors. The treatment of mismodeling errors is considerably more difficult. The mismodeling errors may either be intentional, e.g., modeling a measurement bias as a single state variable when we have a much more realistic bias model available containing several state variables, or inadvertantly when we do not have a realistic model available. Intentional mismodeling may arise because we consider some terms in the bias model to be small and we do not wish to unnecessarily complicate the filter. We may also delete some more important terms from the bias model when it is obvious that the geometry of the instrumentation is not sufficient to estimate these terms.

The decomposition procedure will not permit most of the conventional methods of treating errors caused by mismodeling, e.g., we have already seen that no state noise on the bias variables is permitted. Also, the method of directly overweighting the observations is also excluded by the decomposition. Another method for devaluing the effect of old observations on the estimates is the fading memory Kalman filter which we were able to show is allowed by the decomposition. The fading memory filter weights the observations exponentially according to their age. It was first used in the Kalman framework by Fagin [5] and more recently by Tarn and Zaborsky [6], and Sorensen and Sachs [7]. Sorensen and Sachs showed that the fading factors need not be constant as previously used and also exhibited some other previously unreported properties of the fading memory Kalman filter.

The basic idea of the fading memory filter is that observations should be assigned increasingly larger variances as they become older. Let t_n be the current time at which an estimate is desired. At t_n we model the dynamics and the observations (using the augmented state vector y) by

$$y(i,n) = A(i)y(i-1,n) + u_x(i,n) \tag{69}$$

$$z(i) = L(i)y(i,n) + v(i,n) \tag{70}$$

The observation noise and state noise covariances are defined as

$$E\left[v(i,n)v^T(i,n)\right] = R(i,n) = R(i)\exp\left(\sum_{j=1}^{n-1} c_j\right) \tag{71}$$

$$E\left[u_x(i,n)u_x^T(i,n)\right] = Q_x(i,n) = Q(i)\exp\left(\sum_{j=i+1}^{n-1} c_j\right) \tag{72}$$

where Q(i) and R(i) are the usual covariance matrices. The scalars $c_j \geq 0$ are called the fading factors. From (71) and (72) we can see that the covariances assigned to the state noise and observation noise become larger as their time t_i recedes from the current time t_n. In addition, we also fade the uncertainty associated with the initial state estimate $\hat{y}(0,n)$.

$$E\left[(y(0,n)-y(0)(y(0,n)-y(0))^T\right] P(0)\exp\left(\sum_{j=1}^{n-1} c_j\right). \tag{73}$$

With the above definitions the fading memory Kalman filter estimates y(i,n) may be derived. Actually, we only consider the estimates y(n,n) and drop the double subscript. We will not derive the fading memory estimates here but will only indicate the changes from the usual Kalman filter equations. The only changes occur in the equations for computing the predicted covariance matrices which become

$$P_x(k|k-1) = a(k-1)A(k-1)P_x(k-1)A^T(k-1) + Q_x(k-1) \tag{74}$$

$$P_b(k|k-1) = a(k-1)B(k-1)P_b(k-1)B^T(k-1) \tag{75}$$

$$P_{xb}(k|k-1) = a(k-1)A(k-1)P_{xb}(k-1)B^T(k-1) \tag{76}$$

where

$$a(k-1) = e^{c_{k-1}}$$

The fading a(k) is chosen to reduce errors in the bias estimates caused by mismodeling while the state noise covariance $Q_x(k)$ is used to reduce the effects of mismodeling in the process dynamics. The fading also effects the weighting for the zero-biase state estimates $x^*(k)$, but the effect is normally small since the fading factors c_k are nearly unity in our application to instrument bias estimation. The fading memory filter, although useful in reducing errors due to mismodeling in bias estimation, is not entirely satisfactory because it does not allow for treating each bias term individually.

<u>PROBLEMS ENCOUNTERED IN BIAS ESTIMATION</u>. The major problems encounterd in estimating measurement error bias are concerned with modeling not only of the instrument biases but also modeling of the trajectory. Trajectory modeling causes problems in bias estimation since if an adequate representation of the trajectory dynamics has not been included in the Kalman filter, it will be impossible to separate the resulting trajectory estimation errors from the measurement bias estimates. For example, if a constant acceleration is assumed for the dynamics of a missile trajectory when in reality the missile is turning, estimates of accelerometer biases will be significantly effected by the trajectory estimation error caused by the constant acceleration assumption. Although the measurement bias estimates will be useless in cases where they are severely confused with the trajectory modeling errors the trajectory state estimates obtained by including the bias will often be a significant improvement over the trajectory state estimates provided by the zero-bias filter.

Another source of confusion of the bias error estimates occurs from modeling the random noise characteristics of the measurements. If the measurement noise is predominantly low frequency and the filter assumes the noise to be purely random (usually the case), the measurement bias estimate will include a significant portion of the low frequency noise components.

Another type of modeling problem occurs in modeling the measurement biases. As an example consider the following bias error model for a radar azimuth measurement.

$$\Delta A = a_0 + a_1 \tan E_0 \sin A_0 + a_2 \tan E_0 \cos A_0 + a_3 \tan E_0 + a_4 \sec E_0$$
$$+ a_5 \dot{A}_0 + a_6 \dot{A}_0 + a_7 \ddot{A}_0$$

where A_0 and E_0 are measured azimuth and elevation and ΔA is the azimuth bias error

a_0 = Zero set error

$a_1 \tan E_0 \sin A_0 + a_2 \tan E_0 \cos A_0$ = Mislevel error

$a_3 \tan E_0$ = Orthogonality error

$a_4 \sec E_0$ = Collimation error

$a_5 \dot{A}$ = Servo velocity error

$a_6 \dot{A}$ = Timing error

$a_7 \ddot{A}$ = Servo acceleration error

Many more terms could be included in the above error model but the terms listed will serve to illustrate the desired points. The first problem with the azimuth error model is obvious; the timing and servo velocity error terms have exactly the same form which implies that a_5 and a_6 cannot be separately estimated. Thus, we will only be able to estimate the sum of a_5 and a_6. Now suppose that the trajectory being estimated is approximately a level flying aircraft and that the ground range from the radar is such that the elevation angle is nearly constant over large segments of the trajectory. In this case $\tan E_0$ and $\sec E_0$ are effectively zero set error terms. In this case there is little chance of obtaining meaningful estimates of a_0, a_3, and a_4. The three terms may be lumped and a single zero set error estimated or if the a_0, a_3, and a_4 are separately estimated, the sum $\hat{a}_0+\hat{a}_3\tan E_0-\hat{a}_4\sec E_0$ may be a meaningful estimate. This example shows that each term in an error model must be well exercised along the trajectory in order to obtain meaningful estimates of the error coefficients.

Another problem in modeling measurement biases involves the large number of error coefficient terms which would be required to adequately model all measurement bias errors in some WSMR missions. Consider again the LANCE example given in the introduction where there were 66 different measurements. If an average of three error coefficients are required to model a bias error in this example, which is certainly not unrealistic, 198 bias state variables are required in the Kalman filter. Even though the technique presented in this report has greatly speeded up the computation of bias error estimates, it is doubtful that the estimation of 198 states is feasible.

To summarize the above remarks on modeling problems the following requirements must be satisfied in order to do a completely satisfactory job of bias estimation.

1. An adequate representation of the trajectory dynamics must be included in the filter.

2. A realistic model of the measurement noise including serial correlation must be available.

3. Each term in a measurement bias error model must be well exercised along the trajectory.

4. The number of bias state variables must be small enough for the computational problem to be feasible yet large enough so that no severe aliasing errors are present.

In any case careful study is required to determine what terms to include in the bias error model and to determine if the resulting bias estimatesare actually due to the error source assumed or are significantly effected by some other factors such as trajectory modeling errors, low frequency measurement noise, or missing terms in the bias error model. Although the modeling problems presented above pose some very difficult problems, they do not impose a serious limitation on the use of bias estimation, if one is willing to accept the premise that the primary purpose of bias estimation is to improve the trajectory state estimates.

SQUARE ROOT IMPLEMENTATION OF BIAS ESTIMATION. Several matrix square root formulations of the Kalman filter equations have been presented in the literature. A comprehensive survey of these methods along with an excellent bibliography is given in [8]. Until recently we used the matrix square root methods presented in [2] and [4]. Presently we employ the square root filtering methods described in [3] which are summarized below. We have found that the square root filter provides computational efficiency as well as numerical stability in the mechanization of the Kalman filter. The square root filtering equations given below consider only the processing of scalar observations. This is no restriction since either the square root method employed is easily extendable to the processing of vector observations or the problem of processing vector observations may be reduced to the processing of scalar observations.

After a time update the predicted covariance matrix $P^*(k|k-1)$ of the zero-bias filter is decomposed as

$$P^*(k|k-1) = L(k|k-1)D(k|k-1)L^T(k|k-1) \quad (77)$$

where $L^T(k|k-1)$ is a lower triangular matrix having ones along the diagonal and $D(k|k-1)$ is a diagonal matrix having positive diagonal elements. This triangular decomposition is related to the Choleski decomposition. The algorithm for decomposition is summarized in Appendix A. For further details see [9]. For each scalar observation occuring at the new time t_k an updated triangular decomposition of the covariance and an updated state estimate are computed. Let $\overset{(i)}{x^*(k)}$ denote the zero-bias state estimate after processing the ith scalar observation at t_k and let

$$\overset{(i)}{L(k)}\overset{(i)}{D(k)}\overset{(i)T}{L(k)}$$

be the triangular decomposition of the covariance matrix after the ith measurement update at t_k. The updated triangular decomposition satisfies

$$\overset{(i)}{L(k)}\overset{(i)}{D(k)}\overset{(i)T}{L(k)} = \overset{(i-1)}{L(k)}\ \overset{(i-1)}{D(k)}\ \overset{(i-1)T}{L(k)} - C_*^{(i)}y_*^{(i)}y_*^{(i)T} \quad (78)$$

where the vector $y_*^{(i)}$ and the scalar $C_*^{(i)}$ are computed from

$$y_*^{(i)} = \overset{(i-1)}{L(k)}\overset{(i-1)}{D(k)}\ u_*^{(i)} \quad (79)$$

$$C_*^{(i)} = 1/(R_i^2(k) + u_*^{(i)}D^{\overset{(i-1)}{(k)}}u_*^{(i)}) \quad (80)$$

$$u_i^* = \overset{(i-1)T}{L(k)}\ H_i^T(k) \quad (81)$$

Also $\overset{(0)}{L(k)} = L(k|k-1)$ and $\overset{(0)}{D(k)} = D(k|k-1)$.

The updated state estimate $\overset{(i)}{x^*(k)}$ is computed from

$$x^{*(i)}(k) = x^{*(i-1)}(k) + C_*^{(i)} y_*^{(i)} (Z_i(k) - H_i(k) x^{*(i-1)}(k)) \tag{82}$$

$$x^{*(0)}(k) = x^*(k|k-1).$$

An algorithm is derived in [3] for computing the $L(k)^{(i)}$ and $D(k)^{(i)}$ in (78) given $L(k)^{(i-1)}$, $D(k)^{(i-1)}$, $C_*^{(i)}$, and $y_*^{(i)}$. This algorithm is summarized in Appendix B.

The triangular decomposition

$$P_b(k) = L_b(k) D_b(k) L_b^T(k)$$

is also used for the bias filter. We will consider only the case where the elements of the bias vector are assumed to be constant. In addition to being easier to handle numerically the case of constant bias is probably the most useful in practice. The triangular decomposition for the more general case of linear bias dynamics results in slightly more numerical work. With constant bias dynamics the square root bias filter equations at a time update are

$$\hat{b}(k|k-1) = \hat{b}(k-1) \tag{83}$$

$$L_b(k|k-1) = L_b(k-1) \tag{84}$$

$$D_b(k|k-1) = D_b(k-1). \tag{85}$$

For each scalar measurement update at the new time t_k the updated triangular decomposition satisfies

$$L_b^{(i)}(k) D_b^{(i)}(k) L_b^{(i)T}(k) = L_b^{(i-1)}(k)\ D_b^{(i-1)}(k)\ L_b^{(i-1)T}(k) - C_b^{(i)}\ y_b^{(i)}\ y_b^{(i)T} \tag{86}$$

The vector $y_b^{(i)}$ and the scalar $C_b^{(i)}$ are computed from

$$y_b^{(i)} = L_b^{(i-1)}(k)\; D_b^{(i-1)}(k)\; u_b^{(i)} \tag{87}$$

$$C_b^{(i)} = 1/\left(C_i(k) + u_b^{(i)}\; D_b^{(i-1)}(k)\; u_b^{(i)}\right) \tag{88}$$

$$u_b^{(i)} = L_b^{(i-1)T}(k)\;\; s_i(k) \tag{89}$$

$$C_i(k) = R_i(k) + u_i^{*T}\, D^{(i-1)}(k)\;\; u_i^{*} \tag{90}$$

$$L_b^{(0)}(k) = L(k|\,k-1),\;\; D_b^{(0)}(k) = D(k|\,k-1)$$

The updated bias state estimate is

$$\hat{b}^{(i)}(k) = \hat{b}^{(i-1)}(k) + C_b^{(i)}\; y_b^{(i)}\; \left(r_i^{*}(k) - s_i^{T}(k)\, b^{(i-1)}(k)\right) \tag{91}$$

$$r_i^{*}(k) = Z_i(k) - H_i(k)\, x^{*(i)}(k) \tag{92}$$

$$s_i(k) = G_i(k) + H_i(k)\, T^{(i-1)}(k) \tag{93}$$

For each measurement update at t_k the combining matrix T is updated as

$$T^{(i)}(k) = T^{(i-1)}(k) - C_*^{(i)}\; y_*^{(i)}\; s_i^{T}(k) \tag{94}$$

$$T^{(0)}(k) = T(k|\,k-1)$$

BIAS FILTER REINITIALIZATION. The dimension of the bias state vector may change frequently during the execution of the filtering process. This dimension change is required when a measuring instrument begins to take observations after the initialization of the filter or when an instrument is deleted from the filtering solution because it has stopped taking observations, its bias is considered too large, or its observations are chronically inconsistent with their statistics. The decomposition of the optimal filter into a zero-bias filter and a bias filter requires that the zero-bias state estimate be orthogonal (in the usual statistical sense) to the bias state estimate. This condition must be satisfied when reinitializing the bias filter. In addition we require that there be no change in the trajectory state estimate $\hat{x}$ and the remaining bias state estimates due to reinitialization.

The orthogonality and continuity conditions are automatically met when a new measuring instrument first enters the B.E.T. program provided we assume that its initial bias estimates are uncorrelated with the zero-bias state estimate. However, when a measuring instrument is deleted from the filtering solution, considerable effort is required in reinitialization in order to meet the orthogonality and continuity conditions.

Let x_-^*, P_-^* denote the zero-bias state estimate and its covariance just prior to dropping a measurement from the filtering solution and let x_+^* and P_+^* be the same quantities after dropping the measurement and reinitializing the filter. Let b_- and b_+ be the bias state estimate before and after dropping a measurement. $\hat{b}_+$ is formed by deleting the component of $\hat{b}_-$ corresponding to the measurement being dropped. T_- and T_+ are the combining matrices before and after. T_+ has one less column than T_-. P_{b_+} and P_{b_-} are the bias covariance matrices before and after dropping a measurement. P_{b_+} is formed from P_{b_-} by deleting the row and column corresponding to the measurement being dropped. The updating equation for x^* is

$$x_+^* = x_-^* + t_i(\hat{b}_i - p^T b_+) \tag{95}$$

where t_i is the column of T_- being deleted and $\hat{b}_i$ is the bias estimate of measurement being dropped. The vector p is chosen so that x_+^* and $\hat{b}_+$ will be orthogonal. Using the triangular decompositions $P_-^* = L_- D_- L_-^T$ and $P_+^* = L_+ D_+ L_+^T$ the updating equation for P^* may be written as

$$L_+ D_+ L_+^T = L_- D_- L_-^T + (P_{b_-}(i,i) - p^T P_{b_+} p) t_i t_i^T \tag{96}$$

where $P_{b_-}(i,i)$ is the diagonal element of the row and column of P_{b_-} which are being deleted. The algorithm of Appendix B is used to compute L_+ and D_+ in the above equation. Similarly, using the triangular decompositions $P_{b_-} = L_{b_-} D_{b_-} L_{b_-}^T$ and $P_{b_+} = L_{b_+} D_{b_+} L_{b_+}^T$ delete the ith row and column of both L_{b_-} and D_{b_-}. Call the resulting matrices L_{b_-}' and D_{b_-}'. The updating equation for P_b can then be written as

$$L_{b_+} D_{b_+} L_{b_+} = L_{b_-} D_{b_-} L_{b_-} + d_i \ell_i \ell_i^T \tag{97}$$

where ℓ_i is the column deleted from L_{b_-} and d_i is the diagonal element deleted from D_{b_-}. L_{b_+} and D_{b_+} are computed using the algorithm of Appendix B. Having computed L_{b_+} and D_{b_+} the vector p which is chosen so that x_+^* and b_+ will be orthogonal is computed by solving the triangular equation

$$L_{b_+} y = \ell_i \tag{98}$$

for y and then solving the triangular equation

$$L_{b_+}^T p = D_{b_+}^{-1} y. \tag{99}$$

In order to satisfy the conditions that the state estimate be unchanged after reinitialization of the bias filter, set

$$T_+ = T_-' + t_i p^T \tag{100}$$

where $T_-^{\prime}$ is formed by deleting the ith column, t_i, from T_-. This will make $\hat{x}_+ = \hat{x}_-$. The bias estimates automatically remain unchanged.

APPENDIX A

Let P be an nxn symmetric, positive definite matrix. The following algorithm computes a lower triangular matrix L with $\ell(i,i) = 1$ and a diagonal matrix D with $d_i = d(i,i) > 0$ such that $P = LDL^T$. For further details, see [9].

$$d_1 = p(1,1)$$

$$p^*(k,1) = p(k,1) \qquad k>1$$

$$\ell(k,1) = p^*(k,1)/d_1$$

$$d_i = p(i,i) - \sum_{j=1}^{i-1} p^*(i,j)\,\ell(i,j) \qquad k>i$$

$$\ell(k,i) = p^*(k,i)/d_i$$

APPENDIX B

Let L be a lower unit triangular matrix and D be a positive diagonal matrix D such that $P = LDL^T$ is positive definite. Given a scalar c and a vector x such that $P' = P + cxx^T$ is positive definite, compute a unit lower triangular matrix L' and a positive diagonal matrix D' such that $P' = L'D'L'^T$. The following algorithm computes L' and D' given L, D, c, and x.

$$c^{(1)} = c \, , \; x^{(1)} = x$$

$$d_i' = d_i + c^{(i)} \, x_i^{(i)\,2}$$

$$x_j^{(i+1)} = x_j^{(i)} - x_i^{(i)} \, \ell(j,i) \qquad \begin{array}{l} i = 1, n \\ j = i+1, n \end{array}$$

$$\ell'(j,i) = \ell(j,i) + \frac{c^{(i)} \, x_i^{(i)}}{d_i'} \, x_j^{(i+1)}$$

$$c^{(i+1)} = c^{(i)} \left(\frac{d_i}{d_i'} \right)^2$$

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