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PROPAGATION OF ELECTROMAGNETIC WAVES IN MEDIA WHICH VARY SLOWLY WITH POSITION AND TIME

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13. ABSTRACT

Using WKB methods we have considered the propagation of electromagnetic waves in isotropic lossless media which vary slowly with both position and time. It is found that in such media the meaning of various quantities, such as the group velocity, must be reinterpreted. The theory is applied to study propagation in space-time, varying dielectrics, and plasmas.

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Propagation of electromagnetic waves in media which vary slowly with position and time

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Using WKB methods we have considered the propagation of electromagnetic waves in isotropic lossless media which vary slowly with both position and time. It is found that in such media the meaning of various quantities, such as the group velocity, must be reinterpreted. The theory is applied to study propagation in space-time, varying dielectrics, and plasmas.

1. INTRODUCTION

A difficult problem to consider conceptually is wave propagation in media which vary with both position and time. In such media our standard concepts of frequency, wavenumber, and group velocity no longer apply. That is, we find that frequency can be defined only as the time derivative of the phase function and that the quantity so defined is a function of both the position and the time at which the observation is made. We also find that the group velocity no longer retains its conventional meaning. In spatially-homogeneous, time-invariant media the group velocity is interpreted as the velocity at which wave packets centered around some wavevector $\mathbf{k}_0$ propagate [*Jeffreys and Jeffreys*, 1962]. In space-time varying media this is no longer true. In fact, as we shall see, values of ω and $\mathbf{k}$ do not propagate with the group velocity $\nabla_k\omega$; it is rather different quantities (which are functions of ω and $\mathbf{k}$) which are propagated at this velocity. The same conclusions hold true for energy flow (i.e., the energy flux does not propagate with the group velocity).

In this paper we will study the propagation of electromagnetic waves in lossless media which vary slowly with position and time. We will therefore employ the four-dimensional WKB (Wentzel-Kramers-Brillouin) method. The WKB method was first applied in three dimensions by *Sommerfeld and Runge* [1911]. That is, in media in which the properties depended on position $\mathbf{x}$ but not on time, Sommerfeld and Runge considered solutions of the form $\exp(i\omega t - \int \mathbf{k}\cdot d\mathbf{x})$. As a consequence of the fact that the phase function must be uniquely defined, Sommerfeld and Runge then concluded that $\nabla\times\mathbf{k} = 0$ was required. This is the original version of the Sommerfeld-Runge law. This result can be extended to the four-dimensional case, as has been done by *Whitham* [1960] and *Poeverlein* [1962]. That is, in space-time varying media the existence of a uniquely defined wave function, $\exp[i \int (\omega\, dt - \mathbf{k}\cdot d\mathbf{x})]$ requires that $\nabla\omega - \partial\mathbf{k}/\partial t = 0$. This is the four-dimensional Sommerfeld-Runge law.

In the present paper we will first give an elementary derivation of the four-dimensional Sommerfeld-Runge law. We will then examine its implications, and finally indicate its use in studying electromagnetic wave propagation in isotropic, lossless media which vary slowly with position and time.

2. GENERAL THEORY

2.1. *Discussion of the Generalized Sommerfeld-Runge Law*

In a medium which varies slowly (e.g., in a dielectric the conditions for slow variation are that $k^2 \gg \nabla\cdot\mathbf{k}$, $\omega^2 \gg \partial\omega/\partial t$, $\omega \gg \partial/\partial t(\ln\epsilon)$, and $k \gg (1/\epsilon)|\nabla\epsilon|$, where ϵ is the permittivity of the dielectric) with both position and time, the WKB approximation for the electric field strength can be written as

$$\mathbf{E}(\mathbf{x}, t) = \mathbf{e}_0(\mathbf{x}, t) \exp\left[i \int_L (\mathbf{k}\cdot d\mathbf{x} - \omega\, dt)\right] \qquad (1)$$

where $\mathbf{e}_0(\mathbf{x}, t)$ varys slowly with position and time compared with the exponent, and L is a line integral in four space between some initial point $(\mathbf{x}_0, t_0)$ and the observation point $(\mathbf{x}, t)$. The functions ω and $\mathbf{k}$ are generally related through a dispersion relation of the form

$$\omega = W(k, \mathbf{x}, t) \qquad (2)$$

or

$$k = K(\omega, \mathbf{x}, t) \tag{3}$$

where $k = |\mathbf{k}|$. For example, in a plasma we have $k = c^{-1}[\omega^2 - \omega_p^2(\mathbf{x}, t)]^{1/2}$, where c is the speed of light, and ω_p is the electron plasma frequency. Now since the exponent in (1) represents a phase function, its value must be unique. That is, the line integral L should be independent of the path, which implies that the integrand is a perfect differential. Therefore, there must exist some function ϕ, such that $\mathbf{k} = \nabla\phi$ and $\omega = -\partial\phi/\partial t$. As a consequence of these relations we then have

$$\partial\mathbf{k}/\partial t + \nabla\omega = 0 \tag{4}$$

Equation 4 is the four-dimensional version of the Sommerfeld-Runge law. This result can alternately be derived by employing results on wave conservation, as has been done by *Lighthill and Whitham* [1955]. Equation 4 has also been considered by *Whitham* [1960], *Poeverlein* [1962], and *Landau and Lifschitz* [1959].

2.2. *Discussion of the Properties of Equation 4*

2.2.1. *Generalized group velocity.* To discuss the properties of (4) in a medium which varies with both position and time, let us substitute (2) into (4). Using the fact that $\nabla \times \mathbf{k} = 0$, we get

$$\partial\mathbf{k}/\partial t + (\mathbf{V}\cdot\nabla)\mathbf{k} = -(\nabla W)_{k,t} \tag{5}$$

where

$$\mathbf{V} = \nabla_k W = \left(\frac{\partial W}{\partial \mathbf{k}}\right)_{x,t} \tag{6}$$

The quantity $\mathbf{V}$ in (6) can be interpreted as a generalized group velocity as we shall see in the following discussion. Upon defining $d/dt = \partial/\partial t + \mathbf{V}\cdot\nabla$ equation 5 can be rewritten as

$$d\mathbf{k}/dt = -(\nabla W)_{k,t} \tag{7}$$

From (7) we see that if W does not depend explicitly on position, then $d\mathbf{k}/dt = 0$. This means that if one moves along the ray with the velocity $\mathbf{V}$, he will observe constant values of $\mathbf{k}$ (i.e., the observer moving with $\mathbf{V}$, measures constant values of wavelength). Therefore, if W is a function of k and t, but does not depend explicitly on $\mathbf{x}$, then $\mathbf{k}$ is a constant of the motion, so that wave packets sharply centered around some wavenumber $\mathbf{k}_0$ will be propagated with $\mathbf{V} = \partial W/\partial\mathbf{k}$ evaluated at $\mathbf{k} = \mathbf{k}_0$.

To examine the other limit when the properties of the medium vary with position but not with time, it is convenient to use (3) in (4). For this case we obtain (assuming the medium is isotropic)

$$\partial\omega/\partial t + \mathbf{V}\cdot\nabla\omega = -(\partial\omega/\partial k)_{x,t}\left(\frac{\partial K}{\partial t}\right)_{\omega,x} \tag{8}$$

From (8) it is clear that if K depends on ω and $\mathbf{x}$, but does not explicitly depend on time, then

$$d\omega/dt = 0 \tag{9}$$

This means that the observer moving along the ray with the velocity $\mathbf{V}$ will observe constant values of ω (i.e., constant wave period). Snell's law follows immediately from (9). Therefore wave packets sharply centered about some frequency ω_0 will propagate with the group velocity $\mathbf{V}$ (evaluated at $\omega = \omega_0$).

For the general case when the properties of the medium depend on both position and time, neither ω nor $\mathbf{k}$ will be invariant as one moves along the ray with the group velocity $\mathbf{V}$. (However, in the next subsection we do show that there are velocities $\mathbf{V}^{(k)} \neq \mathbf{V}$ and $\mathbf{V}^{(\omega)} \neq \mathbf{V}$ with which values of $\mathbf{k}$ and ω are propagated in space-time varying media). In this case other quantities will be invariants of the motion. Consider each scalar component of (5). We have

$$\partial k_1/\partial t + (\mathbf{V}\cdot\nabla)k_1 = -\frac{\partial W}{\partial x_1} \tag{10}$$

To solve (10) we consider the subsidiary set

$$dt = dx_1/V_1 = dx_2/V_2 = dx_3/V_3 = -dk_1/(\partial W/\partial x_1) \tag{11}$$

Let us denote the particular integrals of (11) by $f(k_1, \mathbf{x}, t) = C_1$, $g(k_1, \mathbf{x}, t) = C_2$, $h(k_1, \mathbf{x}, t) = C_3$, $\psi(k_1, \mathbf{x}, t) = C_4$. Then it can be shown that [*Sneddon*, 1957] (a) the general solution of (10) is given by

$$C_1 = \Phi(C_2, C_3, C_4) \tag{12}$$

where Φ is an arbitrary function, determined by the boundary conditions imposed and (b) the invariants of motion, for an observer moving with the velocity $\mathbf{V}$ given by (6), are C_1, C_2, C_3, and C_4. That is, each C_i satisfies

$$dC_i/dt = (\partial/\partial t + \mathbf{V}\cdot\nabla)C_i = 0 \tag{13}$$

Therefore, for media which vary with both position and time, the generalized group velocity defined in (6) is the velocity with which the quantities C_i are propagated. It is only in the limit of spatially homogeneous, time-invariant media that some of the constants C_i can be identified with ω and $\mathbf{k}$

In section 3.1 we will calculate the constants of motion for several examples of media which vary with position and time.

2.2.2. *The angle between* **V** *and* **k**. In this subsection we will demonstrate that in isotropic lossless media the normal, $\hat{\mathbf{k}} = (\mathbf{k}/k)$, to the phase surface $\phi(\mathbf{x}, t)$ lies in the same direction as the group velocity **V**. We will also calculate the angle between $\hat{\mathbf{k}}$ and **V** for the case of a simple anisotropy.

Let us consider (3) for the case when the dispersion relation can be written as

$$k = K(\omega, \theta, \mathbf{x}, t) \tag{14}$$

where θ is the angle which the vector **k** makes with the z axis. (This corresponds to the case in which the dispersion relation is given by the Appelton-Hartree formula). Using (14) in (5), we find, upon differentiating implicitly, that the group velocity is

$$\mathbf{V} = (\partial\omega/\partial k)[\hat{\mathbf{k}} - (\hat{\theta}/k)(\partial K/\partial\theta)] \tag{15}$$

Upon taking the dot and cross products of **k** with (15) we obtain for the angle γ between **k** and **V**

$$|\tan\gamma| = |\mathbf{k}\times\mathbf{V}|/\mathbf{k}\cdot\mathbf{V} = k^{-1}\frac{\partial K}{\partial\theta} = k^{-1}\frac{\partial k}{\partial\theta} \tag{16}$$

For isotropic media (k independent of θ) the angle $\gamma = 0$. Equation 16 is a well-known result from the study of Whistlers [*Holt and Haskell*, 1965; *Kelso*, 1964]. We have shown here that the result also applies to media which vary slowly with both position and time, provided we understand that **V** is not the velocity with which values of **k** and ω are propagated. One final result of interest is to use (16) and (15) to calculate the magnitude of the group velocity. We get

$$|V| = [(\partial k/\partial\omega)\cos\gamma]^{-1} \tag{17}$$

2.2.3. *The equation of motion of* $\hat{\mathbf{k}}$. In this subsection we shall demonstrate that the classical equation presented by *Landau and Lifschitz* [1959] for the motion of the normal to $\phi(\mathbf{x}, t)$, can be generalized to include space-time varying media. For isotropic media we write, using $\mathbf{k} = k\hat{\mathbf{k}}$ in (5)

$$k(d\hat{\mathbf{k}}/dt) + \hat{\mathbf{k}}(dk/dt) = -(\nabla W)_{k,t} \tag{18}$$

We next use the dispersion relation of (3) in the second term on the left-hand side of (18). We obtain, after regrouping terms

$$k(d\hat{\mathbf{k}}/dt) + \hat{\mathbf{k}}[V^{-1}(d\omega/dt) + (\partial K/\partial t)_{\omega,\mathbf{x}} + \mathbf{V}\cdot(\nabla K)_{\omega,t}] = -(\nabla W)_{k,t} \tag{19}$$

where $V = |\mathbf{V}| = (\partial k/\partial\omega)^{-1}$. Now substitute (8) for $d\omega/dt$ in (19). The result is

$$k(d\hat{\mathbf{k}}/dt) + V\hat{\mathbf{k}}\hat{\mathbf{k}}\cdot(\nabla K)_{\omega,t} = -(\nabla W)_{\mathbf{k},t} \tag{20}$$

Equation 20 is valid for an arbitrary lossless isotropic medium. To obtain the analog of the ray-normal equation of *Landau and Lifschitz* [1959], we next specialize (20) to dielectrics. For this case $K = \omega/v$ where $v = V =$ phase velocity. In this limit (20) becomes

$$d\hat{\mathbf{k}}/dt - \hat{\mathbf{k}}(\hat{\mathbf{k}}\cdot\nabla v) = -\nabla v(\mathbf{x}, t) \tag{21}$$

Equation 21 determines the motion of the ray normal in dielectrics which vary slowly with both position and time. As demonstrated by Landau and Lifschitz, who obtained the same equation for the case when v varies with position only, (21) predicts a bending of the rays toward the region where v is smaller.

We also note from (20) and (21) that if v is independent of position but does depend on time, we have

$$d\hat{\mathbf{k}}/dt = 0 \tag{22}$$

Therefore, as expected, the ray does not change its direction of propagation in media which vary only with time.

2.2.4. *Temporal discontinuities*. It is often desirable to know the behavior of ω and **k** when the properties of the medium are suddenly altered. For example, suppose we have a dielectric in which $\epsilon = \epsilon_1(\mathbf{x})$ for $t < t_1$, and $\epsilon = \epsilon_2(\mathbf{x})$ for $t > t_1$. To study the behavior of ω and **k** when temporal discontinuities occur let us integrate (4) from $t_1 - \delta$ to $t_1 + \delta$ (note that (4) is not strictly valid for $\delta = 0$). We obtain

$$\mathbf{k}(\mathbf{x}, t_1 + \delta) - \mathbf{k}(\mathbf{x}, t_1 - \delta) = -\int_{t_1-\delta}^{t_1+\delta} (\nabla\omega)\,dt \tag{23}$$

In the limit as $\delta \to 0$ the right-hand side of (23) vanishes, unless $\nabla\omega$ has a delta-function behavior; therefore,

$$\mathbf{k}(\mathbf{x}, t_1 + \delta) = \mathbf{k}(\mathbf{x}, t_1 - \delta) \tag{24}$$

Since (24) implies that both the magnitude and direction of **k** cannot change instantaneously then from (3) we may write

$$K_1[\omega(\mathbf{x}, t_1 + \delta), \mathbf{x}, t_1 + \delta] = K_2[\omega(\mathbf{x}, t_1 - \delta), \mathbf{x}, t_1 - \delta] \tag{25}$$

For a dielectric, in which $K = \pm\omega(\mathbf{x}, t)[\mu_0\epsilon(\mathbf{x}, t)]^{1/2}$, (25) yields

$$\omega(\mathbf{x}, t + \delta) = \pm[\epsilon_1(\mathbf{x})/\epsilon_2(\mathbf{x})]^{1/2}\omega(\mathbf{x}, t - \delta) \tag{26}$$

The positive sign in (26) is appropriate for the wave traveling along **k**, while the negative sign is appropriate for the reflected component which travels along $-\mathbf{k}$. This latter component is negligible in the WKB limit. In the limit of a spatially homogeneous dielectric, (26) reduces to the previous result of *Morgenthaler* [1958].

2.3. *Velocity of Propagation of Values of* k *and* ω

One of the points we have made is that in a space-time varying medium neither ω nor $\mathbf{k}$ is an invariant as one moves with the group velocity $\mathbf{V} = \nabla_k \omega$. However, it is possible to define a new velocity $\mathbf{V}^{(k)}$ such that $\mathbf{k}$ will be constant when the observer moves with this velocity. To obtain $\mathbf{V}^{(k)}$, let us suppose that the solution of (5) is given by $\mathbf{k}(\mathbf{x}, t)$. Then we can define a velocity $\mathbf{V}'$ through the equation

$$(\mathbf{V}'\cdot\nabla)\mathbf{k} = (\nabla W)_{\mathbf{k},t} \tag{27}$$

If this were done, (5) could be rewritten as

$$\partial \mathbf{k}/\partial t + [(\mathbf{V} + \mathbf{V}')\cdot\nabla]\mathbf{k} = 0 \tag{28}$$

from which we immediately identify

$$\mathbf{V}^{(k)} = \mathbf{V} + \mathbf{V}' \tag{29}$$

Equation 27 can be solved for $\mathbf{V}'$ by standard matrix methods. It is interesting to consider the one dimensional limit (i.e., $\partial/\partial x = \partial/\partial y = 0$). Then (27) becomes

$$V'(z, t)\left(\frac{\partial k}{\partial z}\right)_t = (\partial W/\partial z)_{k,t} \tag{30}$$

which is readily solved for V'. Using this result, along with (6), in (29) then gives

$$V^{(k)} = \left(\frac{\partial W}{\partial k}\right)_{z,t} + \left(\frac{\partial W}{\partial z}\right)_{k,t} \Big/ \left(\frac{\partial k}{\partial z}\right)_t \tag{31}$$

Therefore, we have demonstrated it is possible to define a velocity $V^{(k)}$ such that the observer moving with this velocity sees constant values of k. A similar argument holds for ω.

2.4. *Approximate Solution of Equation 8*

In many instances it is difficult to obtain exact solutions of (5) and (8) for space-time varying media. This is especially true when dispersion is present. However, when the frequency shift in propagating through the medium is small compared with the transmitter frequency, it is possible to solve (8) by iteration. Let us consider the limit when

$$\omega^{-1} \int (\partial K/\partial t)\, ds \ll 1 \tag{32}$$

where, as before, $k = K(\omega, \mathbf{x}, t)$ and $ds = \hat{\mathbf{k}}\cdot d\mathbf{x}$. The integral is along the ray path in the medium between the transmitter and the observer. For the special case of a dielectric, this requires that, in addition to the condition that the medium vary slowly, the path length in the medium cannot be so large that $\int K\, ds \gg 1$. When (32) is satisfied, we may neglect the right-hand side of (8), and conclude as a lowest-order approximation that ω is a constant of the motion. Furthermore, if (as is usually the case) we specify that $\omega = \omega_0$ for all time at the position of the transmitter, then $\omega \simeq \omega_0$ for all $\mathbf{x}$ and t (for which (32) is still satisfied). Using $\omega \simeq \omega_0$ then gives

$$|\nabla\phi| = i(\omega_0/c)n(\omega_0, \mathbf{x}, t) \tag{33}$$

where $n(\omega, \mathbf{x}, t)$ is the index of refraction, defined by $|\mathbf{k}| = (\omega/c)n$. Equation 33 is, of course, the standard eikonal used in ray optics. Similarly, for ϕ we have, substituting $|\mathbf{k}| = (\omega_0/c)n(\omega_0, \mathbf{x}, t)$,

$$\phi \simeq i\left[\omega_0/c \int n(\omega_0, \mathbf{x}, t)\hat{\mathbf{k}}\cdot d\mathbf{x} - \omega_0 t\right] \tag{34}$$

where, as before, $\hat{\mathbf{k}}$ is the unit vector normal to the phase surface, and is determined by solving (33). Finally, since $\omega = -\partial\phi/\partial t$ we have for the first iteration to the instantaneous frequency

$$\omega \simeq \omega_0 - (\omega_0/c) \int \{[\partial n(\omega_0, \mathbf{x}, t)]/\partial t\}\, ds \tag{35}$$

where $ds = \hat{\mathbf{k}}\cdot d\mathbf{x} =$ path length along the ray. Equation 35 is the result used by ionospheric researchers in studying the Doppler shift through an ionospheric region which varies slowly with path position and time [*Weekes*, 1958; *Kelso*, 1960, 1964; *Ginzburg*, 1964; *Bennett*, 1967]. It is evident, from (32) and (35) that (35) is valid only when the Doppler shift is small compared with the transmitter frequency. For most problems of propagation through the earth's ionosphere, (35) is an adequate approximation for the instantaneous frequency. However, there are laboratory plasmas and some planetary atmospheres (e.g., Jupiter) where (35) may not be a good approximation. In addition, the constraint of (32) may not hold in many space-time varying dielectrics. In the next section, we will study the exact solutions of (5) and (8) in some dielectric materials.

3. APPLICATION TO ISOTROPIC, LOSSLESS DIELECTRICS

3.1. *Calculation of the Invariants,* ω, *and* k *in Dielectrics*

We will now use the results of section 2 to study the propagation in lossless dielectrics with permittivity varying slowly with position and time. To simplify the problem we will also assume that the propagation is in the same direction as $\nabla\epsilon$, which we choose to be along the z axis in a rectangular coordinate system. For this case (8) becomes

$$\partial\omega/\partial t + \beta^{-1}(\partial\omega/\partial z) = -\omega(\partial/\partial t)(\ln \beta) \qquad (36)$$

where $\beta = (\mu_0\epsilon)^{1/2}$. From the theory presented in (11) and (12) we know that the solution of (36) will have the form $C_1 = \Phi(C_2)$ where $C_1 = f(\omega, z, t)$ and $C_2 = g(\omega, z, t)$ are the particular integrals of

$$dt = dz/\beta^{-1} = -d\omega/\{\omega[\partial(\ln \beta)/\partial t]\} \qquad (37)$$

We discuss the solutions of (37), below, for several special cases.

3.1.1. *β separable.* When β is separable we may write $\beta = \beta_1(z)\,\beta_2(t)$. In this case, upon combining the first and third members in (37) we get

$$\omega\beta_2(t) = C_1 \qquad (38)$$

Therefore, the observer moving with the velocity $V = \beta^{-1}$ finds that $\omega\beta_2(t)$ is an invariant. Similarly combining the first and second members of (37) we get

$$\int^z \beta_1(z')\,dz - \int^t dt'/[\beta_2(t')] = C_2 \qquad (39)$$

so that the general solution of (36) when β is separable is

$$\omega(z, t) = [\beta_2(t)]^{-1}\Phi\left[\int^z \beta_1(z')\,dz - \int^t dt'/\beta_2(t)\right] \qquad (40)$$

where the arbitrary function Φ is determined by specifying boundary conditions on ω along any curve in the $z - t$ plane. For example, if β_2 did not depend on time, and we specified $\omega = \omega_0$ at $z = 0$ for all t, then Φ = constant, and therefore ω = constant, as would be expected.

3.1.2. *Taylor expansion of β.* The situation when β is separable, studied above, does not usually occur in practical situations. However, there are often problems in which it is appropriate to expand $\beta(z, t)$ in Taylor series in either z or t. For example, we can consider the case in which we desire to study the propagation only over the time interval $t_1 \le t \le t_2$. In that interval we may expand β in Taylor series in t as

$$\beta(z, t) = \beta_0(z) + t\beta_1(z) + \cdots \qquad (41)$$

For the variation of (41) the constants C_1 and C_2 are

$$\omega \exp [S(z)] = C_1 \qquad (42)$$

$$t \exp [-S(z)] = \int^z dz'\,\beta_0(z') \exp [-S(z')] = C_2 \qquad (43)$$

where $S(z) = \int^z \beta_1(z)\,dz$. Upon using (42) and (43) in the general solution $C_1 = \Phi(C_2)$ we get

$$\omega(z, t) = \exp [-S(z)]\Phi \cdot\left\{t \exp [-S(z)] - \int^z dz'\beta_0(z') \exp [-S(z')]\right\} \qquad (44)$$

The wavenumber $k(z, t)$ is related to ω by $k = \omega\beta$. We can determine the arbitrary function $\Phi(\cdots)$ by specifying boundary conditions on ω. For example, suppose we specify that $\omega = \omega_0$, for all t, at $z = 0$. (This condition is appropriate for the case of a plane wave of frequency ω_0 transmitted into a space-time varying half-space). This requires that, in (44), the function Φ = constant. We therefore obtain

$$\omega = \omega_0 \exp \left[-\int_0^z \beta_1(z')\,dz'\right] \qquad (45)$$

In the limiting case when β_0 and β_1 are independent of z this result can be readily shown to be identical with the previous result of *Morgenthaler* [1958].

It is also possible to obtain the constants of motion for other variations in β. For example, suppose β is a function of $z - v_0 t$. Then it can be shown that $\omega(z, t)\,[1 - v_0\,\beta(z - v_0 t)]$ is an invariant of motion, except in the limit when v_0 approaches the phase velocity in the unmodulated medium. (This is known as the sonic region and has been discussed in detail by *Hessel and Oliner* [1961]).

3.2. *Time of Transit in Space-Time Varying Dielectrics*

Since the phase and group velocities in a space-time varying medium are functions of position and time, it is not immediately evident how long it would take for a disturbance to travel a distance L along a ray. To study this problem let us consider the motion of the point at which the phase $\phi = 0$. In particular, let us suppose $\phi = 0$ for $t = t_0$ at some point $\mathbf{x}_0$ on a given ray. Then the time t_L at which $\phi = 0$ will reach another point $\mathbf{x}_L$ along the ray is a solution of

$$\int_{(\mathbf{x}_0, t_0)}^{(\mathbf{x}_L, t_L)} (\mathbf{k}\cdot d\mathbf{x} - \omega\,dt) = 0 \qquad (46)$$

To study the solution of (46) let us specialize to the case when $\mathbf{k}$ and $\nabla\epsilon$ lie along the z axis. If we assume that the initial point $\mathbf{x}_o$ is $z = 0$ and the observation point is at $z = L$ we can rewrite (46) as

$$\int_{(0,t_o)}^{(L,t_L)} (k\,dz - \omega\,dt) = 0 \qquad (47)$$

Since the line integral in (47) is independent of the path, it can be taken along any curve joining $(0, t_o)$ to (L, t_L) in the z-t plane. For instance, we shall find it convenient to write

$$\int_0^L k(z, 0)\,dz = \int_{t_o}^{t_L} \omega(L, t)\,dt = 0 \qquad (48)$$

Equation 48 is an integral equation to be solved for the transit time $(t_L - t_o)$. To understand the meaning of (48) let us first suppose that $\beta(z, t)$ is independent of t, and that at $\omega(z = 0)$ is specified to be equal to ω_0 for all t. Then since $\omega(z = 0) = \omega(z = L) = \omega_0$, (48) becomes the obvious result

$$t_L - t_o = \int_0^L dz/V(z) \qquad (49)$$

where $V(z) = [\mu_0\,\epsilon(z)]^{1/2}$. In the other limit when $\beta(z, t)$ is independent of z, we get (assuming $k(t = t_o) = k_o$ for all z) that t_L is a solution of

$$L = \int_{t_o}^{t_L} V(t')\,dt' \qquad (50)$$

which is the result obtained previously [*Fante*, 1971].

As an example of the application of (47) to dielectrics which vary with both position and time, let us consider the case when β is given by (41), along with the boundary condition that $\omega = \omega_0$, at $z = 0$, for all t. Using (45) in (48) then gives for the transit time

$$\Delta t = t_L - t_o = \exp[S(L)] \int_0^L \beta_o(z') \exp[-S(z')]\,dz' \qquad (51)$$

where

$$S(z) = \int_0^z \beta_1(\xi)\,d\xi$$

3.3. *Discussion of Transmission Through a Dielectric Slab*

The results of sections 3.1 and 3.2 can be applied to consider the transmission of plane waves through a lossless dielectric slab of thickness L in which the permittivity varies slowly with space and time. Let us denote the solutions in the slab by

$$E = e_o(z, t; z_o, t_o) \exp\left[i \int (\omega\,dt \pm k\,dz)\right] \qquad (52)$$

where ω and k are the appropriate solutions of (36) and $e_o(z, t)$ is the appropriate solution of (69). For example, if $\beta(z, t) = \beta_1(z)\,\beta_2(t)$ then

$$e_o(z, t; z_o, t_o) = e_o(0)\{[\beta_1(z_o)]/[\beta_1(z)]\}^{1/2}\{[\beta_2(t_o)]/[\beta_2(t)]\}^{3/2} \qquad (53)$$

Now suppose the slab occupies the region $0 \le z \le L$, and a plane wave $E_1 = \exp[i\omega_0(t - z/c)]$ is normally incident upon the slab from the region $z < 0$. Then the field transmitted through the slab at time t_o, into the region $z > L$, will consist of a number of components. First, there will be a wave which (from $z < 0$) crosses the $z = 0$ boundary at time τ_1 and is transmitted directly through the dielectric, arriving at $z = L^+$ at the time t_o. Next, there is a wave which crosses the $z = 0$ boundary at time τ_3, and arrives at $z = L^+$ at the time t_o after being internally reflected at time τ_2 by the $z = L$ boundary and at time τ_1 by the $z = 0$ boundary. Next, there is a wave which (from $z < 0$) crosses the $z = 0$ boundary at time τ_5 and arrives at $z = L^+$ at the time t_o, after being reflected twice at the $z = 0$ boundary (at times τ_1 and τ_3) and twice by the $z = L$ boundary (at times τ_2 and τ_4), etc. Let us define $R(0, \tau)$ as the internal reflection coefficient at $z = 0^+$ boundary at time τ, $R(L, \tau)$ as the internal reflection coefficient at $z = L^-$ at time τ (for the case in which the medium is spatially homogeneous $R(0, \tau) = R(L, \tau) = \{[\epsilon(\tau)/\epsilon_0]^{1/2} - 1\}\{[\epsilon(\tau)/\epsilon_0]^{1/2} + 1\}^{-1}$), $T(\tau)$ as the transmission coefficient, from $z = 0^-$ to $z = 0^+$ at time τ, and $\hat{T}(\tau)$ as the transmission coefficient from $z = L^-$ to $z = L^+$. Then, the transmitted field at $z = L^+$ can be written as

$$\begin{aligned} E(z = L^+, t = t_o) = \hat{T}(t_o)[&T(\tau_1)A(\tau_1)\exp(i\omega_0\tau_1) \\ &+ T(\tau_3)A(\tau_3)R(0, \tau_1)R(L, \tau_2)\exp(i\omega_0\tau_3) \\ &+ T(\tau_5)A(\tau_5)R(0, \tau_1)R(L, \tau_2)R(0, \tau_3)R(L, \tau_4) \\ &\cdot \exp(i\omega_0\tau_5) + \cdots] \end{aligned} \qquad (54)$$

where $A(\tau) = e_o(z = L^+, t = t_o; z_o = 0, t_o = \tau)$. From (54) it is clear that the nature of the transmitted field will be known once the times $\tau_1, \tau_2, \tau_3, \cdots$ have been determined. Extending the discussion of section 3.2 we see that these are solutions of the equations

$$\int_0^L k(z, t_o)\,dz = \int_{\tau_1}^{t_o} \omega(0, t')\,dt' \qquad (55)$$

$$-\int_L^0 k(z, \tau_1)\,dz = \int_{\tau_2}^{\tau_1} \omega(L, t')\,dt' \qquad (56)$$

$$\int_0^L k(z, \tau_2)\,dz = \int_{\tau_3}^{\tau_2} \omega(0, t')\,dt' \qquad (57)$$

Therefore τ_1, τ_2, and τ_3 could be obtained by first solving (55) for τ_1. Then, using the solution of (55) in (56), the resulting equation could be solved for τ_2.

Equation 54 can also be applied to the case of transmission into an infinite half-space. We can obtain this limit by setting $\hat{T}(t_*) = 1$ and $R(L, \tau) = 0$. We therefore obtain for the field at a point z_* at time t_*

$$E(z_*, t_*) = T(\tau_1) A(\tau_1) \exp (i\omega_0 \tau_1) \tag{58}$$

where τ_1 is the solution of the equation

$$\int_0^{z_*} k(z', t_*)\, dz' = \int_{\tau_1}^{t_*} \omega(0, t')\, dt' \tag{59}$$

When $\beta(z, t)$ is given by (41), we can use the results of section 3.2 to express τ_1 as

$$\tau_1 = t_* - \exp [S(z_*)] \int_0^{z_*} \beta_0(z') \exp [-S(z')]\, dz' \tag{60}$$

Once τ_1 is known, $T(\tau_1)$ can be obtained by applying the usual WKB methods to the profile $\epsilon(z) = \mu_0^{-1} [\beta_0(z) + \tau_1 \beta_1(z)]^2$.

As an example of the application of the more general result of (54) let us consider the case in which the dielectric slab is spatially homogeneous. Then (54) becomes

$$\begin{aligned} E(z = L^+, t = t_*) = \hat{T}(t_*)[T(\tau_1) A(\tau_1) \exp (i\omega_0 \tau_1) \\ + T(\tau_3) A(\tau_3) R(\tau_1) R(\tau_2) \exp (i\omega_0 \tau_3) \\ + T(\tau_5) A(\tau_5) R(\tau_1) R(\tau_2) R(\tau_3) R(\tau_4) \\ \cdot \exp (i\omega_0 \tau_5) + \cdots] \end{aligned} \tag{61}$$

where

$$A(\tau) = \{[\epsilon(\tau)]/[\epsilon(t_*)]\}^{3/4} \tag{62}$$

and τ_m is the solution of

$$\int_{\tau_m}^{t_*} v(t')\, dt' = mL \qquad m = 1, 2, 3, \cdots \tag{63}$$

To examine the various frequency components present in the transmitted wave of (61) we can Taylor-expand the functions $\tau_m(t)$. That is

$$\tau_m(t) = \tau_m^* + (t - t_*)(\partial \tau_m/\partial t)_{t_*} + \cdots \tag{64}$$

where

$$(\partial \tau_m/\partial t)_{t_*} = [v(t_*)]/[v(\tau_m)] = \{[\epsilon(\tau_m)]/[\epsilon(t_*)]\}^{1/2} \tag{65}$$

Using this result we may rewrite (61) as:

$$E(z = L^+, t) = \sum_l B_l \exp [i\Omega_l(t - t_*)] \tag{66}$$

where

$$B_l = T(\tau_{2l-1}) \hat{T}(t_*) \left[\frac{\epsilon(\tau_{2l-1})}{\epsilon(t_*)}\right]^{3/4} \cdot \exp (i\omega_0 \tau_{2l-1}^*) \prod_{j=0}^{2(l-1)} R(\tau_j)$$

$$\Omega_l = \omega_0 \left[\frac{v(t_*)}{v(\tau_{2l-1})}\right] = \omega_0 \left[\frac{\epsilon(\tau_{2l-1})}{\epsilon(t_*)}\right]^{1/2} \tag{67}$$

From (66) we see that the transmitted signal consists of components at the instantaneous frequencies $\Omega_1, \Omega_2, \Omega_3, \Omega_4, \cdots$. This interpretation assumes that B_l is a slowly varying function of time in comparison with $\exp (i\Omega_l t)$, which is the case in the WKB approximation. The importance of the frequency components at $\Omega_2, \Omega_3, \Omega_4, \cdots$ in comparison to that at Ω_1 will depend on the amplitude of the reflection coefficient R. For $R \ll 1$ only Ω_1 will be significant, but for R near unity this conclusion is clearly not true.

Therefore, we see that, because of the spatial boundaries, the transmitted signal has components at $\Omega_1, \Omega_2, \Omega_3, \cdots$ and not just at Ω_1 as found by *Morgenthaler* [1958], who did not account for boundary effects. For relative permittivities near unity the components at $\Omega_2, \Omega_3, \cdots$ will be negligible compared with those at Ω_1. However, for large relative permittivities the higher-order frequency components will be significant.

4. RAY TRACING METHODS

In the general case when the direction of propagation is not along $\nabla \epsilon$ it is not a simple matter to solve (5) or (8) for either space-time varying dielectrics or plasmas (in fact, for plasmas we cannot generally solve (5) or (8) even when $\nabla \omega_p$ is along the direction of propagation). In many cases it is acceptable to use the approximation of (35), but for others this may not be possible. In such cases it appears most appropriate to approximate the temporal behavior of the medium in a stepwise fashion. For example, in a dielectric during the interval $0 \leq t \leq t_N$ the permittivity can be approximated by $\epsilon = \epsilon_1(\mathbf{x})$ for $0 \leq t < t_1$, $\epsilon = \epsilon_2(\mathbf{x})$ for $t_1 < t < t_2, \cdots, \epsilon = \epsilon_N(\mathbf{x})$ for $t_{N-1} < t \leq t_N$, where $|(\epsilon_j - \epsilon_{j-1})/\epsilon_j| \ll 1$. In a plasma we would approximate the electron plasma frequency $\omega_p(\mathbf{x}, t)$ by: $\omega_p = \omega_{p1}(\mathbf{x})$ for $0 \leq t < t_1$, $\omega_p = \omega_{p2}(\mathbf{x})$ for $t_1 < t < t_2, \cdots, \omega_p = \omega_{pN}(\mathbf{x})$ for $t_{N-1} < t < t_N$. It is assumed that $|(\omega_{pj} - \omega_{pj-1})/\omega_{pj}| \ll 1$ for $j = 1, 2, \cdots N$.

To illustrate the method we will consider a space-time varying plasma. We suppose that the plasma occupies the half-space $z \geq 0$, and is spatially stratified in the z direction only. We then assume that a signal with frequency spectrum sharply peaked about $\omega = \omega_1$ enters the medium at an angle θ_o relative to the z axis. We will follow the progress of two distinct points P and P' on the envelope of this signal (in a dielectric, P and P' could represent two separate values of phase, the progress of which we follow). If P is located at $z = 0$ when $t = t_o$, then by Snell's law we have that the ray path followed by P in the time interval (t_o, t_1) is determined from

$$[1 - \omega_{p1}^2(0)/\omega_1^2]^{1/2} \sin \theta_o = [1 - \omega_{p1}^2(z)/\omega_1^2]^{1/2} \sin \theta(z) \qquad (68)$$

The point P moves along this path with the group velocity $|V_1(z)| = c[1 - \omega_{p1}^2(z)/\omega_1^2]^{1/2}$ so that during the time interval $t_1 - t_o$ the distance S_1 travelled along the ray by this point is the solution of

$$t_1 - t_o = \int_0^{S_1} ds/V_1(z) \qquad (69)$$

where the integration is along the ray path (in isotropic media the ray and group paths are identical). Consider, also, another point P' on the signal envelope. If P' is at $z = 0$ at $t = t_o'$, then P' traverses the path given by (68), except that the distance S_1' travelled along the ray by this point during the interval $t_1 - t_o'$ is the solution of

$$t_1 - t_o' = \int_0^{S_1'} ds/V_1(z) \qquad (70)$$

Now suppose that at $t = t_1 - \delta(\delta \to 0)$ the point P is located at $\mathbf{x} = \mathbf{x}_1$ (i.e., $\mathbf{x} = \mathbf{x}_1$ is the coordinate of S_1) and the ray path at this point makes an angle θ_1 with the z axis. We also assume that at $t = t_1 - \delta$ the point P' is located at $\mathbf{x} = \mathbf{x}_1'$ at which point the ray makes an angle θ_1' with the z axis. At $t = t_1$ the plasma frequency is suddenly changed from $\omega_{p1}(z)$ to $\omega_{p2}(z)$. By virtue of (24), θ cannot change instantaneously (since $\mathbf{k}$ cannot change in either magnitude or direction at finite temporal discontinuities in the properties of the medium) so that $\theta(t - \delta) = \theta(t + \delta)$. Therefore, in the time interval $t_1 < t < t_2$ the ray path followed by P is determined from

$$\left\{1 - \left[\frac{\omega_{p2}(z_1)}{\omega_2}\right]^2\right\}^{1/2} \sin \theta_1 = \left\{1 - \left[\frac{\omega_{p2}(z)}{\omega_2}\right]^2\right\}^{1/2} \sin \theta(z) \qquad (71)$$

while the path followed by P' is given by

$$\left\{1 - \left[\frac{\omega_{p2}(z_1')}{\omega_2'}\right]^2\right\}^{1/2} \sin \theta_1' = \left\{1 - \left[\frac{\omega_{p2}(z)}{\omega_2'}\right]^2\right\}^{1/2} \sin \theta(z) \qquad (72)$$

where the new frequencies ω_2 and ω_2' associated with P and P' (recalling that since P and P' travel with the local group velocity, the frequency associated with these points remains constant during the interval $t_1 < t < t_2$) are

$$\omega_2 = [\omega_1^2 - \omega_{p1}^2(z_1) + \omega_{p2}^2(z_1)]^{1/2} \qquad (73)$$

$$\omega_2' = [\omega_1^2 - \omega_{p1}^2(z_1') + \omega_{p2}^2(z_1')]^{1/2} \qquad (74)$$

Therefore, in a space-time varying medium, not only will different portions of the signal acquire different instantaneous frequencies (even though both P and P' entered the medium with the same frequency), but different portions of the signal will also traverse different ray paths, as is evident from (71) and (72).

It is clear, then, that in a space-time varying plasma we must ray-trace independently for each point on the signal envelope, since the ray trajectories for different portions of the signal are different (except, of course, in the limit when the direction of propagation is along $\nabla\omega_p$ or $\nabla\epsilon$). For each point P followed, the above procedure can be repeated continually during other time intervals (i.e., $t_2 < t < t_3, t_3 < t < t_4, \ldots$) until the location and frequency of P at $t = t_N$ has been obtained. Of course, if in any regime of space (or time) we reach the situation where ω is close to ω_p, the WKB method is no longer valid, and we must perform a more careful analysis (see, e.g., *Kelso* [1964] and *Ginzburg* [1964]).

5. COMMENTS ON ENERGY FLOW

In a space-time varying medium we have found that values of ω and $\mathbf{k}$ are not propagated with $\mathbf{V} = \nabla_k\omega$; therefore, we should not be surprised to find that energy flux does not flow with this velocity either. To consider this problem let us first discuss lossless, isotropic dielectrics. We can then demonstrate, upon using (1) in Maxwell's equations and employing the slowly varying assumption, that $\mathbf{e}_o$ satisfies

$$\mathbf{k}(\nabla\psi\cdot\mathbf{e}_o) + \mathbf{e}_o(\nabla\cdot\mathbf{k}) + 2(\mathbf{k}\cdot\nabla)\mathbf{e}_o = -\mu_o\epsilon\{\mathbf{e}_o[(\partial\omega/\partial t) + 2\omega(\partial\psi/\partial t)] + 2\omega(\partial\mathbf{e}_o/\partial t)\} \qquad (75)$$

where $\psi = \ln \epsilon$. We note that (75) reduces to the results of section 3.1.3 in *Born and Wolf* [1959] in the limit when ϵ does not depend on time. We now consider the limit when $\mathbf{k} \times \nabla\epsilon = 0$. We then have from (75) that

$$(d/dt)(k^{1/2}e_o) = (k^{1/2}e_o)(\partial/\partial t)(\ln v^{3/2}) \quad (76)$$

where $v = [\mu_o\epsilon(z, t)]^{-1/2}$ and $d/dt = \partial/\partial t + v\,\partial/\partial z$. Equation (76) may be formally solved for e_o, by integrating along the characteristic to give

$$e_o(z, t) = \left[\frac{k(0, t_1)}{k(z, t)}\right]^{1/2} e_o(0, t_1) \cdot \exp\left[\int_{t_1}^{t} (\partial/\partial t')(\ln v^{3/2})\,dt'\right] \quad (77)$$

where t_1 is the time at which the signal present at (z, t) was located at $z = 0$, and the integral in (77) is along the characteristic. Using (77), we obtain for the flux $I = (\epsilon/\mu_o)^{1/2}|e_o|^2$

$$I = [k(0, t_1)]/(\omega\mu_o)\,|e_o(0, t_1)|^2 \cdot \exp\left[3\int_{t_1}^{t} (\partial/\partial t')(\ln v)\,dt'\right] \quad (78)$$

We next return to (8), specialized to a dielectric. The formal solution for ω is

$$\omega = \omega(0, t_1)\exp\left[\int_{t_1}^{t} (\partial/\partial t')(\ln v)\,dt'\right] \quad (79)$$

where the integral in (79) is again along the characteristic. Using (79) in (78) we find that

$$I/\omega^3 = [\epsilon(0, t_1)/\mu_o]^{1/2}\{|e_o(0, t_1)|^2/[\omega^2(0, t_1)]\} = \text{constant} \quad (80)$$

since t_1 is an invariant (i.e., $(\partial/\partial t + v\,\partial/\partial z)t_1 = 0$). Therefore I/ω^3 is an invariant of motion in a space-time varying dielectric, so that, once $\omega(z, t)$ has been determined, the energy flux $I(z, t)$ follows immediately from (80). For the case in which ϵ depends on z only, we have, from previous considerations, ω = constant, so that for time-invariant media we retrieve from (80) the well-known result that the flux I is invariant. In the limit when the dielectric depends on time, but not on position, then $k = \omega(\mu_o\epsilon)^{1/2}$ = constant. Using this in (80) gives the result that, in spatially homogeneous dielectrics, $\epsilon(t)I$ = constant.

Let us now consider the case of dispersive media and attempt to discover whether a result similar to (80) can be found. We consider a lossless, isotropic plasma in which the wavevector $\mathbf{k}$ lies along $\nabla\omega_p$. If we write the vector potential $\mathbf{A} = \mathbf{A}_o\exp(i\phi)$, where $\nabla\cdot\mathbf{A} = 0$ (radiation gauge) and $\mathbf{E} = -\partial\mathbf{A}/\partial t$ we obtain

$$(d/dt)(k^{1/2}A_o) = (\partial/\partial t + V\,\partial/\partial z)(k^{1/2}A_o) = (k^{1/2}A_o)(\partial/\partial t)(\ln V^{1/2}) \quad (81)$$

where $V = (c^2k/\omega)$ = group velocity, and $c^2k^2 = \omega^2 - \omega_p^2$. In obtaining (81) we have assumed that the plasma is electrically neutral so that $\nabla\cdot\mathbf{E} = 0$. If (81) now is formally integrated along the characteristic we obtain

$$A_o(z, t) = \left[\frac{k(0, t_1)}{k(z, t)}\right]^{1/2} A_o(0, t_1) \cdot \exp\left[\int_{t}^{t} (\partial/\partial t')(\ln V^{1/2})\,dt'\right] \quad (82)$$

where t_1 is the time when the group was located at $z = 0$. The energy flux (Poynting vector) $I = (k/\omega\mu_o)|e_o|^2 = (\omega k/\mu_o)|A_o|^2$ is given by

$$I/\omega^2 = \frac{k(0, t_1)}{\omega\mu_o}|A_o(0, t_1)|^2 \cdot \exp\left[\int_{t_1}^{t} (\partial/\partial t')(\ln V)\,dt'\right] \quad (83)$$

We note that in a plasma I/ω^2 is not an invariant since ω is no longer given by

$$\exp\left[\int_{t_1}^{t} (\partial/\partial t')(\ln V)\,dt'\right]$$

However, it is possible to obtain an approximate invariant involving the energy. Following *Stepanov* [1968] we consider a pulse with time duration $\hat{t}$, at position z, so short that ω can be considered constant from t to $t + \hat{t}$. We then have upon integrating (83) from t to $t + \hat{t}$

$$\omega^{-1}\int_{t}^{t+\hat{t}} I\,dt \simeq \mu_o^{-1}\int_{t}^{t+\hat{t}} dt'\,k(0, t_1)\,|A_o(0, t_1)|^2 \cdot \exp\left[\int_{t_1}^{t'} (\partial/\partial t'')(\ln V)\,dt''\right] \quad (84)$$

To perform the integral in (84) it is appropriate to change the variable of integration from t' to t_1. This requires calculating $(\partial t'/\partial t_1)$. The time, $t_1(z, t)$, at which the signal present at (z, t) was located at $z = 0$ is an invariant of motion, and therefore satisfies $(\partial/\partial t + V\,\partial/\partial z)t_1 = 0$. If we differentiate this equation with respect to time we find that

$$(\partial/\partial t + V\,\partial/\partial z)(\partial t_1/\partial t) = [(\partial/\partial t)\ln V](\partial t_1/\partial t) \quad (85)$$

Upon solving (85) for $(\partial t_1/\partial t)$ we have

$$(\partial t_1/\partial t) = \exp\left[\int_{t_1}^{t} (\partial/\partial t')(\ln V)\,dt'\right] \quad (86)$$

Using (86) in (84) we may write

$$W/\hbar\omega \simeq (\mu_o h)^{-1} \int_{t_1}^{t_1+\hat{t}_1} dt_1\, k(0, t_1)\, |A_o(0, t)|^2$$
$$= \text{constant} \qquad (87)$$

where $\hat{t}_1$ is the duration of the pulse at $z = 0$, h is Planck's constant, and $W \equiv \int_{t_1}^{t_1+\hat{t}_1} I\, dt'$ = energy in the pulse. Equation 87 states that, for a short pulse, the number of quanta in a wave packet is approximately invariant (since t_1 and $\hat{t}_1$ are invariants).

6. SUMMARY

We have studied the properties of the WKB solutions in lossless, isotropic space-time varying media. It was found that, in principle, one can always obtain constants of the motion which lead to a complete determination of the frequency and wavenumber, once appropriate boundary conditions have been specified. Once ω and k are known one can readily study the transmission through space-time varying media, as was illustrated in section 3.3. However, in general, the constants of motion are not always easily obtained. In such problems we have shown in section 4 that one can obtain solutions by modelling the medium by a series of temporal steps. That is, the index of refraction $n(\mathbf{x}, t)$ is approximated by $n_1(\mathbf{x})$ in $0 \le t \le t_1$, $n_2(\mathbf{x})$ in $t_1 < t < t_2$, etc., and ray-tracing techniques are applied during each time interval.

To keep our discussion relatively simple, we have avoided considering the effect of absorption. When dissipation is present equation 4 must still be valid, since the function $\phi(x, t)$ is required to be unique even if losses are present. However, when absorption is present both ω and k are complex numbers, if the medium varies with both position and time. In addition, the real and imaginary parts of the k vector may be in different directions. When absorption is present the group velocity no longer has any physical meaning. In fact $\mathbf{V} = \nabla_k \omega$ may be complex and may even have a magnitude greater than the speed of light, even in the limit of spatially-homogeneous, time-invariant media.

When the absorption is small it can be included by perturbation methods. For example in time-invariant media the ray paths are determined neglecting absorption. The absorption is then included by multiplying the field by $\exp(-\int \mathbf{k}'' \cdot d\mathbf{x})$ where the integral is along the ray path, and $\mathbf{k}''$ is the imaginary part of $\mathbf{k}$.

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