FOURIER-MOTZKIN ELIMINATION AND ITS DUAL

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BY

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Research on linear inequalities systems prior to 1947 consisted of isolated efforts by a few investigators. A case in point is the elimination technique for reducing the number of variables in the system. A description of the method can be found in Motzkin's 1936 Ph.D. thesis. It differs from its analog for systems of equations in that (unfortunately) each step in the elimination can greatly increase the number of inequalities in the remaining variables. For years the method was referred to as the Motzkin Elimination Method. However, because of the odd grave-digging custom of looking for artifacts in long forgotten papers, it is now known as the Fourier-Motzkin Elimination Method. In this paper we review the elimination scheme and show that a dual form of the method is a technique for reducing the number of equations in a system of equations in non-negative variables. Some comments regarding its applicability to integer programs also made.
LINEAR INEQUALITY SYSTEMS
ELIMINATION OF VARIABLES
ELIMINATION OF EQUATIONS
INTEGER PROGRAMS

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Research on linear inequalities systems prior to 1947 consisted of isolated efforts by a few investigators. A case in point is the elimination technique for reducing the number of variables in the system. A description of the method can be found in Motzkin's 1936 Ph.D. thesis. It differs from its analog for systems of equations in that (unfortunately) each step in the elimination can greatly increase the number of inequalities in the remaining variables. For years the method was referred to as the Motzkin Elimination Method. However, because of the odd grave-digging custom of looking for artifacts in long forgotten papers, it is now known as the Fourier-Motzkin Elimination Method.

Given a system of linear inequalities: Find \( x = (x_1, \ldots, x_n) \) such that

\[
\sum_{j=1}^{n} a_{ij} x_j \geq b_i, \quad i = (1, \ldots, m).
\]

One may partition it into three sets of inequalities according to whether the coefficients of \( x_1 \) are positive, negative or zero. This permits rewriting (1) in the form:


\[
\begin{align*}
\begin{cases}
  x_1 &\geq D_1(\vec{x}) \\
  \vdots &\vdots \\
  x_1 &\geq D_p(\vec{x})
\end{cases}
\quad
\begin{cases}
  x_1 &\leq E_1(\vec{x}) \\
  \vdots &\vdots \\
  x_1 &\leq E_q(\vec{x})
\end{cases}
\quad
\begin{cases}
  0 &\leq F_1(\vec{x}) \\
  \vdots &\vdots \\
  0 &\leq F_r(\vec{x})
\end{cases}
\end{align*}
\]

where \( D_i(\vec{x}), E_j(\vec{x}), F_k(\vec{x}) \) are linear functions of \( \vec{x} = (x_2, \ldots, x_n) \).

It may be solved by first solving the reduced system: Find \( \vec{x} \) satisfying

\[
D_i(\vec{x}) \leq E_j(\vec{x}) \quad i = (1, \ldots, p); \quad j = (1, \ldots, q), \quad k = (1, \ldots, r)
\]

(3) \[
0 \leq F_k(\vec{x})
\]

and then finding an \( x_1 \), satisfying

\[
\max_{i} D_i(\vec{x}) \leq x_1 \leq \min_{j} E_j(\vec{x})
\]

(4) \[
\]

where \( x_1 \) always exists providing there exists an \( \vec{x} \) satisfying (3).

Proof: Given any \( (x_1, \vec{x}) \) satisfying (2), it is clear that (3) and (4) must hold. Conversely, given any \( \vec{x} \) satisfying (3), then \( \max_{i} D_i(\vec{x}) \leq \min_{j} E_j(\vec{x}) \) and we can always find an \( x_1 \) satisfying (4); hence \( (x_1, \vec{x}) \) satisfies (1).

System (3) is said to be the result of "eliminating" \( x_1 \) from system (2). If \( p+q \leq 4 \), the reduced system contains one less variable and no more inequalities. If \( p > 2, q > 2, r > 0 \), however, the process of elimination will greatly increase the number of inequalities. This is the chief reason given why it is not used as a practical solution.
method. It is worth noting, however, that (3) has special structure and that this might be used to advantage to develop it into a practical computational procedure.

Since (3) is a linear inequality system also, one could next proceed to eliminate $x_2$ etc. until one has eliminated all but a single variable, say $x_n$. The original system is solvable if and only if the final system $x_n < a_i, x_n \geq \beta_j, 0 \leq \gamma_k$ for $i = 1, \ldots, p'$, $j = 1, \ldots, q', k = 1, \ldots, r'$ is consistent, i.e., iff $a_i - \beta_j > 0$ and $\gamma_k > 0$ for all $i, j, k$. Another way to state this is

**Feasibility Theorem**: A necessary and sufficient condition that system (1) is solvable, is there exist no set of weights $(y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0)$ such that

\[(5) \quad \sum_{i=1}^{m} y_i b_i > 0 \quad \text{and} \quad \sum_{i=1}^{m} y_i a_{ij} = 0 \quad \text{for} \quad j = (1, \ldots, n).\]

**Proof (Abadie)**: Assume a solution $x$ to (1) exists and there exists weights $y_i \geq 0$ satisfying (5), then (1) implies

\[(6) \quad \sum_{j=1}^{n} \left( \sum_{i=1}^{m} y_i a_{ij} \right) x_j > \sum_{i=1}^{m} y_i b_i , \quad y_i > 0 ,\]

or $Ox > \sum y_i b_i > 0$, a contradiction. Thus the condition is necessary.

Assume no solution $x$ to (1) exists, then note each system generated by the elimination process, for example (3) from (2), is formed
by non-negative linear combinations of the inequalities of the previous system which in turn were formed by non-negative linear combinations of the system one before that, etc., back to the original system (1). Thus the condition for non-solvability, \( a_i - \beta_j < 0 \) or \( \gamma_k < 0 \) for some \( i, j \) or \( k \) (referred to earlier) could be derived directly by some non-negative linear combination of the inequalities of the original system.

This remarkably simple proof of the feasibility theorem based on Fourier-Motzkin elimination is due to Jean Abadie. From it one can derive easily (by trivial algebraic manipulations) the fundamental Duality Theorem of linear programming, Farkas Lemma, the various theorems of the alternatives, and the well known

Motzkin Transposition Theorem: Given the dual homogeneous linear program in partitioned form

\[
A_I x_I + A_{II} x_{II} = 0, \quad (x_I, x_{II}) \geq 0
\]

\[
\text{Dual:} \quad yA_I < 0, \quad yA_{II} < 0,
\]

then either there exists a solution to the dual such that \( yA_I < 0 \) (i.e., holds strictly in all components) or there exists a solution to the primal such that \( x_I \neq 0 \).
Proof: A solution to the dual such that \( y_A^I < 0 \) implies there exists a \( y \) such that

\[
yA^I_1 \leq -e, \quad e = (1, 1, \ldots, 1)
\]

\[
yA^I_II < 0
\]

If no such \( y \) exists satisfying (8), then by the feasibility theorem, there exists weights \( x^I > 0, x^II > 0 \) such that \( A^I x^I + A^II x^II = 0 \) and \( -e x^I < 0 \), i.e., \( x^I \neq 0 \).

The Dual of Fourier-Motzkin Elimination. Suppose we are given the homogeneous linear program

\[
\begin{align*}
x_i - D_i \bar{x} &\geq 0 & i = (1, \ldots, p) \\
-x_i + E_j \bar{x} &\geq 0 & j = (1, \ldots, q) \\
F_k \bar{x} &\geq 0 & k = (1, \ldots, r)
\end{align*}
\]

(9)

where \( \bar{x} = (x_2, \ldots, x_n) \) and \( D_i, E_j, F_k \) are \( 1 \times n \). The elimination of \( x_1 \) from (9) yields

\[
(E_j - D_i) \bar{x} \geq 0 \quad \text{for all } i,j
\]

\[
F_k \bar{x} \geq 0 \quad \text{for all } k.
\]

On the other hand the homogeneous dual of (9) is: To find \( u^I_1 > 0, v_j > 0, w_k > 0 \) such that
(a) \[ \sum_{i=1}^{p} u_i - \sum_{j=1}^{q} v_j = 0 \]  

(11)

(b) \[ - \sum_{i=1}^{p} u_i D_i + \sum_{j=1}^{q} v_j E_j + \sum_{k=1}^{r} w_k F_k = 0 \]

and the homogeneous dual of (10) is: To find \( \lambda_{ij} \geq 0, \ w_k \geq 0 \) such that:

(12) \[ \sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_{ij} (E_j - D_i) + \sum_{k=1}^{r} w_k F_k = 0 \]

Since (9) and its eliminated form (10) are in a sense equivalent systems, it seems natural to expect that their duals (11) and (12), are also equivalent in the same sense; i.e., from any solution to (11) we can derive a solution to (12) and conversely. Note that (11) has \( n \) equations corresponding to the \( n \) components of \( x \), whereas (12) has \( n-1 \) equations but would have (in general) far more variables. This suggests we have at hand a technique for reducing the number of equations in a linear program. Let us give a direct proof of this for the non-homogeneous system:

Find \( u_i \geq 0, v_j \geq 0, w_k \geq 0 \) satisfying:

(a) \[ \sum_{i=1}^{p} u_i - \sum_{j=1}^{q} v_j = 0 \]  

(13)

(b) \[ - \sum_{i=1}^{p} u_i D_i + \sum_{j=1}^{q} v_j E_j + \sum_{k=1}^{r} w_k F_k = g \]  

- 6 -
Let us introduce pq new variables $\lambda_{ij} \geq 0$ by setting

$$u_i = \sum_{j=1}^{q} \lambda_{ij}, \quad i = (1, \ldots, p)$$  

(14)

$$v_j = \sum_{i=1}^{p} \lambda_{ij}, \quad j = (1, \ldots, q)$$

Note that if $u_i$ and $v_j$ satisfy (13)(a), it is always easy to find $u_{ij} \geq 0$ satisfying (14). Even if $u_i > 0$ and $v_j > 0$ are constrained to be integers, it is easy to find integer $\lambda_{ij} > 0$ satisfying (14).

Substituting (14) into (13) we note that (13)(a) is automatically satisfied and we obtain the reduced system:

Find $\lambda_{ij} > 0, w_k > 0$ such that

(15)  

$$\sum_{i=1}^{p} \sum_{j=1}^{q} \lambda_{ij} (E_j - D_i) + \sum_{k=1}^{r} w_k F_k = g .$$

Conversely note that if we have a solution to (15), we can by regrouping the terms and substituting $u_i$ and $v_j$ for the resulting expression $\lambda_{ij}$, obtain a solution to (13). The solution will be in integers if $\lambda_{ij}$ is integral.

To apply the technique to a system of equations in non-negative variables, it is necessary to have one equation with a zero constant term to play the role of (13)(a) or to create an equation with a zero constant term by replacing one of the equations by some appropriate linear combination of the equations of the system. This will yield an equation of the form
and we could obtain a system of form (13) by a change of units. This may conveniently be done by replacing (14) by

\[
\begin{align*}
\alpha_i u_i &= \sum_{j=1}^{q} \lambda_{ij} \ , \\
\beta_j v_j &= \sum_{i=1}^{p} \lambda_{ij} 
\end{align*}
\]

where \( \alpha_i > 0, \beta_j > 0, \lambda_{ij} \geq 0, u_i \geq 0, v_j \geq 0 \).

**Application of the Dual of the Motzkin Elimination to Integer Programs**

As long as \( \alpha_i = 1, \beta_j = 1 \) for all \( i,j \) we have, as pointed out earlier, a reduced system of equations (16) in integer variables if \( u_i \) and \( v_j \) are integers. In general, however, for the case where \( \alpha_i > 0 \) and \( \beta_j > 0 \) are integers different from unity, we have to resort to more complicated substitutions. This will be illustrated below for a simple example. Suppose we have

\[
(u_1 + 2u_2) - (v_1 + v_2 + v_3) = 0
\]

Let us rewrite this

\[
(u_1 + u_2 + u_3) - (v_1 + v_2 + v_3) = 0
\]
where \( u_2 = u_3 \) and set as above

\[
u_1 = \sum_{j=1}^{3} \lambda_{ij}, \quad j = (1, 2, 3)\]

(20)

\[
v_j = \sum_{i=1}^{r} \lambda_{ij}, \quad i = (1, 2, 3)\]

The resulting integer reduced system is in \( \lambda_{ij} \geq 0 \) (as before) except we have the additional condition \( u_2 = u_3 \) which in terms of \( \lambda_{ij} \) becomes

\[
(\lambda_{21} + \lambda_{22} + \lambda_{23}) - (\lambda_{31} + \lambda_{32} + \lambda_{33}) = 0
\]

(21)

But (21) is in exactly the form we need for the integer reduction. We accordingly can introduce additional integer variables \( \mu_{ij} \geq 0 \), where

\[
\lambda_{21} = \sum_{j=1}^{3} \mu_{ij}, \quad i = 1, 2, 3
\]

(22)

\[
\lambda_{3j} = \sum_{i=1}^{3} \mu_{ij}, \quad j = 1, 2, 3
\]

Back substituting into (20), we have the desired integer substitution in terms of 12 auxiliary variables.
\[ u_1 = \sum_{j=1}^{3} \lambda_{1j}, \quad u_2 = \sum_{i=1}^{3} \sum_{j=1}^{3} \mu_{ij} \]

\[ v_1 = \lambda_{11} + \sum_{j=1}^{3} u_{1j} + \sum_{i=1}^{3} \mu_{i1} \]

(23)

\[ v_2 = \lambda_{12} + \sum_{j=1}^{3} u_{2j} + \sum_{i=1}^{3} \mu_{i2} \]

\[ v_3 = \lambda_{13} + \sum_{j=1}^{3} u_{3j} + \sum_{i=1}^{3} \mu_{i3} \]

By setting \[ \mu_{12} + \mu_{21} = \bar{\mu}_{12}, \quad \mu_{13} + \mu_{31} = \bar{\mu}_{13}, \quad \mu_{23} + \mu_{32} = \bar{\mu}_{23} \] we could simplify the above substitution to one involving nine non-negative integer variables \( \lambda_{1i}, \mu_{ii}, \bar{\mu}_{ij} \) where \( i, j = 1, 2, 3 \) and \( i \neq j \).

The problem in general of finding substitutions to replace (17) so as to reduce a linear system in non-negative integer variables to fewer equations is under study and will be the subject of a subsequent paper.