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DESIGN AND MAXIMUM ERROR ESTIMATION FOR SMALL ERROR LOW PASS FILTERS

W. D. Hibler III

September 1972

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By using standard spectral windows small error low pass nonrecursive filters may be designed with transition bandwidths inversely proportional to the number of filter weights. The maximum ripple error outside the transition band for any low pass filter using discrete smoothing by the three most standard spectral windows is estimated. Consequently, the straightforward design equations may be used to calculate low pass digital filter weights with a guaranteed maximum error of less than 0.9% or 0.05% depending on how wide the transition band is made. Filters designed in this way have errors comparable to or smaller than those of filters designed by existing techniques and have the advantage that the maximum error is known beforehand.

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PREFACE

This report was prepared by Dr. W.D. Hibler III, Research Physicist, of the Snow and Ice Branch, Research Division, U.S. Army Cold Regions Research and Engineering Laboratory (USA CRREL).

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DESIGN AND MAXIMUM ERROR ESTIMATION FOR SMALL ERROR LOW PASS FILTERS

by

W.D. Hibler III

Introduction

There exist many situations in which it is desirable to filter out certain frequency components from geophysical time series with a sharp frequency cutoff. When using a finite nonrecursive digital filter for this purpose, some smoothing of the frequency cutoff must be employed to reduce the magnitude of spurious frequency ripples in both the stop band and the pass band. One type of smoothing for this purpose was developed by Martin (1957). In later work numerical search techniques for finding the optimal type of smoothing in terms of minimizing the ripple error for a given transition bandwidth were discussed by Gold and Jordan (1969). However, in both these techniques the design procedure depends upon estimating errors numerically for individual filters. For convenience it would be quite useful to have a design procedure allowing the accurate estimation of ripple error for any low pass filter designed by using the procedure. This is especially important for filters consisting of large numbers of weights because the error calculation is time consuming and cumbersome in such cases.

Such a formulation and an error estimation are carried out here. The smoothing is carried out by using standard spectral windows, developed for purposes of spectral analysis (Blackman and Tukey 1958). Specific examples illustrated later show that such a smoothing procedure yields low pass filters with smaller errors (by a factor of five in one case) than those of comparable filters designed by the Martin (1957) technique. As a result of the error estimation, the straightforward working equations may be used to quickly calculate convolution weights for symmetric, unity gain low pass filters with a guaranteed ripple amplitude outside the transition band of less than 0.9% or 0.05% depending on how wide the transition band is made. Moreover, in the working equations presented later the width of the transition band scales inversely with N (the number of convolution weights), so that the frequency cutoff for a given error may be made arbitrarily sharp by increasing N. Once the convolution weights are calculated, the convolution may be performed by conventional techniques or by the aperiodic fast Fourier transform (FFT) procedure (Stockham 1966), a much more rapid technique. (The aperiodic FFT procedure should not be confused with simply doing a FFT, removing certain frequencies and then transforming back to real space, because such an operation effects a periodic convolution with spurious end point effects). For completeness we have included the basic equations for the FFT aperiodic convolution procedure in Appendix A.

In the next section we shall formulate the design procedure in a manner that allows the numerical estimate of maximum error. Maximum errors will then be calculated for the three most standard spectral windows. A specific example and working equations will also be presented for those interested in using low pass filters without going through the complete detail of the design procedure.

Filter design and error estimation

A convenient way of viewing the frequency response of a given finite filter is to consider the frequency spectrum of the filter an "estimate" of the true spectrum of the infinite length filter. Viewed in this manner, spurious frequency ripples result from the convolution of the Fourier transform of the data window with the true spectrum. A way to improve the situation is to use a non-rectangular data window which produces smaller amplitude ripples, but which also smooths out the frequency cutoff.

Let us consider a symmetric digital filter with weights $C(n)$:

$$C(n) = \begin{cases} H(n) & n = 0, \pm 1, \ldots, \pm N - 1 \\ \dfrac{H(n)}{2} & n = \pm N. \end{cases}$$

The frequency response of this filter is (Holloway 1958):

$$\tilde{H}(f) = 2 \sum_{n=1}^{N-1} H(n) \cos(2\pi f n \Delta t) + H(0) + \cos(2\pi f N \Delta t) H(N) \tag{1}$$

where Δt is the data interval. We shall denote functions in frequency space by a tilda variable with f ranging from 0 to 0.5 cycles/data interval. If $\tilde{H}(f)$ is known at the discrete frequencies $n/(2T)$ (where $T = N\Delta t$), then eq 1 may be inverted

$$H(n) = \frac{1}{N} \sum_{k=1}^{N-1} \cos \frac{\pi k n}{N} \tilde{H}(k/2T) + \frac{\tilde{H}(0)}{2N} + \cos(\pi n) \frac{\tilde{H}(N/2T)}{2N} . \tag{2}$$

Introducing the infinite Dirac comb

$$\nabla(t; \Delta t) = \sum_{q=-\infty}^{\infty} \delta(t - q\Delta t) \tag{3}$$

(where $\delta(t)$ is the Dirac delta function), eq 1 may be rewritten in the form

$$\tilde{H}(f) = \int_{-\infty}^{\infty} dt' \cos(2\pi f t') W(t') \nabla(t'; \Delta t) \int_{-\infty}^{\infty} H_1(t' - t_1) \nabla(t_1; 2T) dt_1 \tag{4}$$

where

$$W(t) = \begin{cases} 0 & |t'| > T \\ 1/2 & |t'| = T \\ 1 & |t'| < T \end{cases}$$

$$H_1(t) = \begin{cases} H(n) & n = 0, \ldots, \pm(N-1) \\ H(n)/2 & n = \pm N \\ 0 & |t| > T. \end{cases}$$

(The values of $H_1(t)$ at other values of t are not important because of the multiplication by $\nabla(t'; \Delta t)$ in eq 4. To verify that eq 4 is in fact equal to eq 1, it is convenient to note that the convolution of

$H_1(t)$ with the infinite Dirac comb is equal to $\sum_{q=-\infty}^{\infty} H_1(t' - 2qT)$ so that for $|t'| < N\Delta t$ this sum over q reduces to $H_1(t')$.)

Since all terms in the integral in eq 4 are even functions of t', we may replace $\cos(2\pi f t')$ by $e^{i2\pi f t'}$. Using the identity

$$\int_{-\infty}^{\infty} e^{i2\pi f t'} \nabla(t'; \Delta t) dt' = \frac{1}{\Delta t} \nabla(f; 1/\Delta t) \tag{5}$$

and the convolution theorem, eq 4 becomes

$$\tilde{H}(f) = \sum_{n=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \tilde{W}(f - n/2T - q/\Delta t)\tilde{H}(n/2T) \tag{6}$$

where

$$\tilde{W}(f) = \frac{\sin 2T\pi f}{2T\pi f} \equiv Q_0(f). \tag{7}$$

Equation 6 illustrates how the aliased Fourier transform of the spectral window is convoluted with the discrete frequency response to give the complete frequency response. We are certainly free to change the form of $\tilde{W}(f)$ by using some nonrectangular spectral window, call it $W_1(t)$, zero outside of the interval $[-T, T]$. If we substitute such a data window in eq 4, then eq 6 is the same with $\tilde{W}(f)$ being replaced by $\tilde{W}_1(f)$ and $\tilde{H}(f)$ being a different frequency response, call it $\tilde{H}_1(f)$.

We would like to use a spectral window $W_1(t)$, having as small side lobes as possible, while causing minimum smoothing of the spectral content. A general class of windows developed for power spectral calculations with these considerations in mind is of the form

$$W_i(t) = \begin{cases} a_{i0} + 2\sum_{j=1}^{\infty} a_{ij} \cos \frac{j\pi t}{T} & |t| < T \\ 0.5[a_{i0} + 2\sum_{j=1}^{\infty} (-1)^j a_{ij}] & |t| = T \\ 0 & |t| > T. \end{cases} \tag{8}$$

The Fourier transform of any window of this type is given by (Blackman and Tukey 1958, p. 99)

$$Q_i(f) = a_{i0}Q_{i0}(f) + \sum_{j=1}^{\infty} a_{ij}[Q_0(f + j/2T) + Q_0(f - j/2T)] \tag{9}$$

where $Q_0(f)$ is as defined in eq 7. In particular, we will consider only cases where $a_{ij} = 0$ for $j > 2$. In this case, since $Q_0(m/2T) = 0$ for an integer m with magnitude > 0, we have

$$\sum_{q=-\infty}^{\infty} Q_i(n/2T - Nq/T) = Q_i(n/2T) \tag{10}$$

for $|n| \leq N - 3$. Consequently, eq 6 [with $\tilde{W}(f)$ replaced by $\tilde{W}_1(f)$] takes on the simple form at discrete frequency values $n/2T$:

$$\tilde{H}_1(n/2T) = \sum_{m=-2}^{2} a_{1m}\tilde{H}\left(\frac{n-m}{2T}\right) \tag{11}$$

if $\tilde{H}(n/2T) = 0$ for $|n| \geq N - 3$. Equation 11 can thus be used to determine smoothed discrete frequency response values $H_1(n/2T)$ given ideal frequency response values $H(n/2T)$ and some spectral window coefficients a_{im}. Filter weights may then be calculated from eq 2.

We will consider here the three most standard spectral windows (Blackman and Tukey 1958) which have the listed nonzero coefficients a_{im}:

1. Hanning $a_{1,0} = 0.5$, $a_{1,-1} = a_{1,+1} = 0.25$
2. Hamming $a_{2,0} = 0.54$, $a_{2,-1} = a_{2,+1} = 0.23$
3. Blackman $a_{3,0} = 0.42$, $a_{3,-1} = a_{3,+1} = 0.25$, $a_{3,-2} = a_{3,2} = 0.04$.

Using eq 11 the design procedure is to choose initially $\tilde{H}(n/2T)$ equal to 1 up to $|n| = p$, and zero for $|n| > p$. We will call p the design cutoff. Smoothed frequency weights $H_1(n/2T)$ are determined by eq 11 and the smoothed frequency weights are converted to digital filter weights by eq 2. Clearly the transition band will extend from $(p - 1)/2T$ to $(p + 2)/2T$ for the Hamming and Hanning spectral windows and from $(p - 2)/2T$ to $(p + 3)/2T$ for the Blackman window. Errors are any deviation from 1 in the pass band and deviations from 0 in the stop band.

The basic equation for estimating the maximum error is eq 6. An equivalent form of this equation obtained by rearranging the double sum is (frequencies are measured in units of $1/2T$):

$$H_1(f) = \sum_{k=-\infty}^{\infty} W_1(k)\tilde{H}_A(k - f) \tag{12}$$

where $H_A(f) = \sum_{q=-\infty}^{\infty} \tilde{H}(f - 2qN)$ is now periodic. Keeping the periodicity of $H_A(f)$ in mind, we see that if we search numerically for the maximum error we need consider only low pass filters with design cutoff values $p \leq N/2$. This follows because a filter with a higher design cutoff creates the same sawtooth function (with a shift in the origin) for $H_A(f)$ as for $1 - \tilde{H}_A'(f)$, where $H_A'(f)$ is an appropriate set of design weights with design cutoff p less than $N/2$. But the maximum error for $1 - H_A'(f)$ is the same as for $H_A'(f)$ because convolving $W(k)$ with 1 yields 1 identically. (As before, by maximum error we mean the greatest deviation from 1 in the pass band or from zero in the stop band, whichever is larger. Thus, changing the origin of a given sawtooth function does not change the maximum error defined in this way.)

Now for design cutoffs less than or equal to $N/2$ we may write eq 12 in the form

$$H(f) = \sum_{k=-N_0}^{N_0} W_1(k)\tilde{H}_A(k - f) + E \tag{13}$$

where

$$E = \sum_{k=N_0+1}^{\infty} W(k)\tilde{H}_A(k - f) + \sum_{k=-N_0-1}^{-\infty} W(k)\tilde{H}_A(k - f) \tag{14}$$

where $N_0 < N/2$. The first sum is quite tractable for numerical calculation, so we need to estimate the maximum error introduced by neglecting E. To do this we first note that $W(k)$ is of the form $g(k)$ sin πk where $g(k)$ is a monotonically decreasing function past a certain value of k. For example, for the Hamming spectral window $g(k)$ is given by $1/\pi[0.08k^2 - 0.54/k(k^2 - 1)]$. Determining the extremum values of $g(k)$ by differentiation, we see that $g(k)$ is monotonically decreasing for $k \geq 10$ for all three spectral windows considered in this paper. Consequently, we see that if $N_0 \geq 10$ the expression for E represents the sum of two alternating decreasing series. This follows from the form of $W(p)$ and the fact that consecutive ones in the periodic function $H_A(k)$ occur in pairs. But Leibnitz's inequality (Gradshteyn and Ryzhik 1965) states that the magnitude of such a series is always less than the magnitude of the first term so that $E \leq 2|W(N_0)|$.

Therefore, we see that we may estimate the error of any filter with a number of weights greater than $4N_0 + 1$, within an additive constant of $2|W(N_0)|$, by evaluating

$$\sum_{k=-N_0}^{N_0} W(k)H(K - f) = W(f - p) + W(f - p + 1) + \ldots + W(f + p) \tag{15}$$

(where p is the design cutoff value) for values of $p = 1, 2, \ldots, N_0$ and values of f up to N_0. Such a calculation was performed numerically by evaluating eq 15 at the discrete frequency points $f = 0$, $1/20, 2/20, \ldots, N_0$ for values of $p = 1, 2, \ldots, N_0$. The maximum error was determined by taking the absolute value of $[\tilde{H}(f) - 1]$ for f in the pass band and the absolute value of $\tilde{H}(f)$ for f in the stop band. Since $\tilde{H}(f)$ typically goes through one oscillation for $\Delta f = 1$ [because $W(f)$ is of the form $g(f)$ sin (πf)], the sampling interval used in the numerical calculations is small compared with the rate of change in $\tilde{H}(f)$. Consequently, errors due to slightly missing the maximum side lobe were small.

To evaluate the error for filters with numbers of weights $2N + 1$ less than (or equal to) $4N_0 + 1$, we used eq 1 directly, again evaluating the frequency response at values of the frequency $f = 0, 1/20, \ldots, N$ [units in $1/2T$(cycles/data interval)] and for each value of f sweeping through values of the design cutoff p from 1 to $N/2 + 1$ for the Hamming and Hanning windows and from 2 to $N/2 + 1$ for the Blackman window. This process was begun at $N = 5$. The results are illustrated in Figure 1, which can be used to estimate the maximum error of any filter with $N \geq 5$, and with the 100% and 0% cutoff values both contained in the frequency interval 0 to 0.5 cycles/data interval.

In Figure 1 the filter error for N greater than a given cutoff value is illustrated by a straight dashed line and was evaluated from eq 10 by the method described above. The maximum error probably represents a somewhat higher error than would actually be obtained in practice. We have also included in Figure 1 maximum error estimates for a trapezoidal type of smoothing for comparison. The trapezoidal smoothing consists of using smoothing weights of $a_0 = 2/3$, $a_{-1} = a_{+1} = 1/3$. These weights give the smoothed low pass filter frequency response a trapezoidal shape. Errors using a sharp cutoff are of course even larger (about 15% in some cases). To summarize the results, low pass filters designed using the Hanning, Hamming and Blackman spectral windows have respective errors always less than 1.14%, 0.89% and 0.048%.

Working equations

To illustrate more specifically the filtering procedure we will design a specific low pass filter and calculate its frequency response. Taking a filter with $2N + 1$ weights, we would like to pass all frequencies below $(N_1/2N)$ cycles/data interval. To do this we take frequency weights of (at integer values of i)

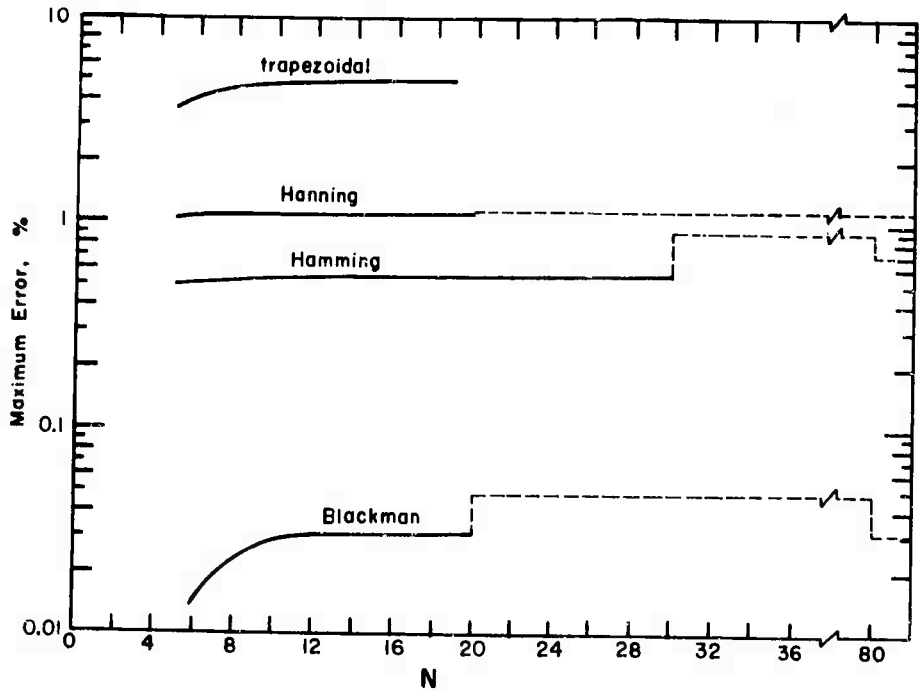

Figure 1. Maximum error versus number of filter weights for low pass digital filters designed using standard spectral windows. The dotted lines denote maximum error calculated according to eq 15 and represent an upper limit that is never exceeded. As can be seen, the upper limit becomes more refined for large N. For a given N, there are 2N + 1 digital filter weights.

$$\tilde{H}(i) = \begin{cases} 1, & i \leq N_1 \\ 0.77, & i = N_1 + 1 \\ 0.23, & i = N_1 + 2 \\ 0, & i > N_1 + 2 \end{cases} \tag{16}$$

for the Hamming window, and frequency weights:

$$\tilde{H}(i) = \begin{cases} 1, & i \leq N_1 \\ 0.96, & i = N_1 + 1 \\ 0.71, & i = N_1 + 2 \\ 0.29, & i = N_1 + 3 \\ 0.04, & i = N_1 + 4 \\ 0, & i > N_1 + 4 \end{cases} \tag{17}$$

for the Blackman window. The transition band frequencies extend (units of cycles/data interval) from $N_1/2N$ to $(N_1 + 3)/2N$ for the Hamming filter and from $N_1/2N$ to $(N_1 + 5)/2N$ for the Blackman filter. The filter weights $C(n)$ are obtained from eq 18

$$C'(n) = \frac{1}{N} \sum_{i=1}^{N-1} \cos\left(\frac{\pi n i}{N}\right) \tilde{H}(i) + \frac{\tilde{H}(0)}{2N} + \frac{\tilde{H}(N)}{2N} \cos \pi n \tag{18}$$

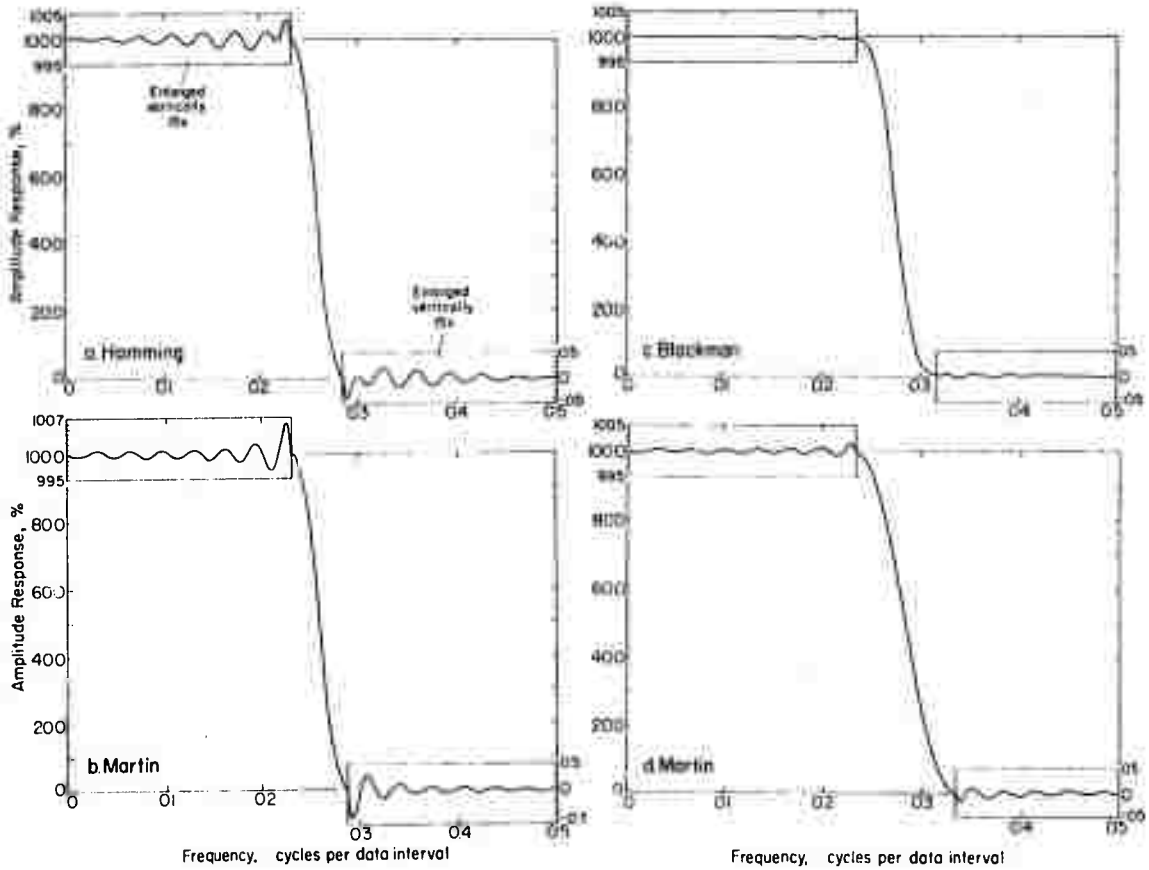

Figure 2. Amplitude response of selected nonrecursive digital filters consisting of 61 symmetric weights. The error and transition bandwidth (Δf) in cycles/data interval for the various filters are: 1. (Hamming) 0.42%, $\Delta f = 0.050$; 2. (Martin) 0.62%, $\Delta f = 0.054$; 3. (Blackman) 0.03%, $\Delta f = 0.083$; and 4. (Martin) 0.17%, $\Delta f = 0.10$. For the Martin filters b and d, the r_c parameter is 0.234 and the H parameters are respectively 0.27, 0.051.

where

$$C(n) = C'(n) \text{ for } n = 0, \pm 1, \pm 2, \ldots, \pm(N - 1)$$

and

$$C(n) = C'(n)/2 \text{ for } n = \pm N.$$

Note that for numerical computation the sum over i in eq 18 need only extend up to the last nonzero value of $H(i)$. The resulting weights $C(n)$ can then be used to filter some time series $\eta(i)$ by the convolution

$$\eta_f(i) = \sum_{j=-N}^{N} C(j)\, \eta(i - j) \tag{19}$$

with $\eta_f(i)$ being the filtered result. The filtering procedure necessitates the loss of $2N$ points. As long as the transition band is contained between zero and the Nyquist frequency (1/2 cycles/data interval), the maximum error is less than 0.9% for the Hamming filter and less than 0.05% for the Blackman filter (Fig. 1).

In particular, taking $N = 30$ and $N_1 = 14$, Figure 2 illustrates the frequency response of the filters designed using eq 16-18. We have also illustrated typical filters generated according to the

procedure described by Martin (1957). The working equations for the Martin filters are also given by Davis (1971), a somewhat more accessible reference. For the Martin filters there are two parameters r_c and H, where r_c denotes the 100% cutoff frequency and H determines the amount of smearing of the frequency cutoff. (Values of r_c and H are given with the figure caption.) Both the Hamming and Blackman windows generate filters with less side lobe error than the Martin filters, with comparable transition bands. The Blackman window has decidedly less error than the comparable Martin filter (smaller by a factor of 5). Moreover, the transition band is considerably smaller in the Blackman case, with a width equal to 5/61 cycles/data interval as opposed to about 6/61 cycles/data interval for the Martin filter. A Martin filter with a smaller transition bandwidth would be expected to have an even greater error.

Application example

To illustrate the application of a low pass filter to digitized data we have applied a low pass filter to laser profilometer data taken from an aircraft flying over the arctic pack ice. The results are illustrated in Figure 3. The illustrated data consist of 1800 digitized points, at 1-yd intervals, representing a straight-line profile of the upper surface of the ice pack. The original record length was 2720 points as the filtering process necessitated the loss of 460 end points at each end of the record. The low frequency trends in the data are due to the aircraft's change in altitude (about 20 ft) over the 1800-yard record. The smooth curve represents the low-pass filtered result.

The low pass filter was designed using the Hamming window and consisted of 921 weights with the transition band extending from 3/460 to 6/460 cycles/yd. By reference to Figure 1 the error is less than 0.6%. The filter weights were convolved with the record using the aperiodic fast Fourier transform procedure with a Fourier series length of 2048 points. In this particular case, the FFT aperiodic convolution took about 45 sec (requiring 5 applications of the FFT algorithm for 2048 points), which was about five times faster than the convolution time using a conventional program.

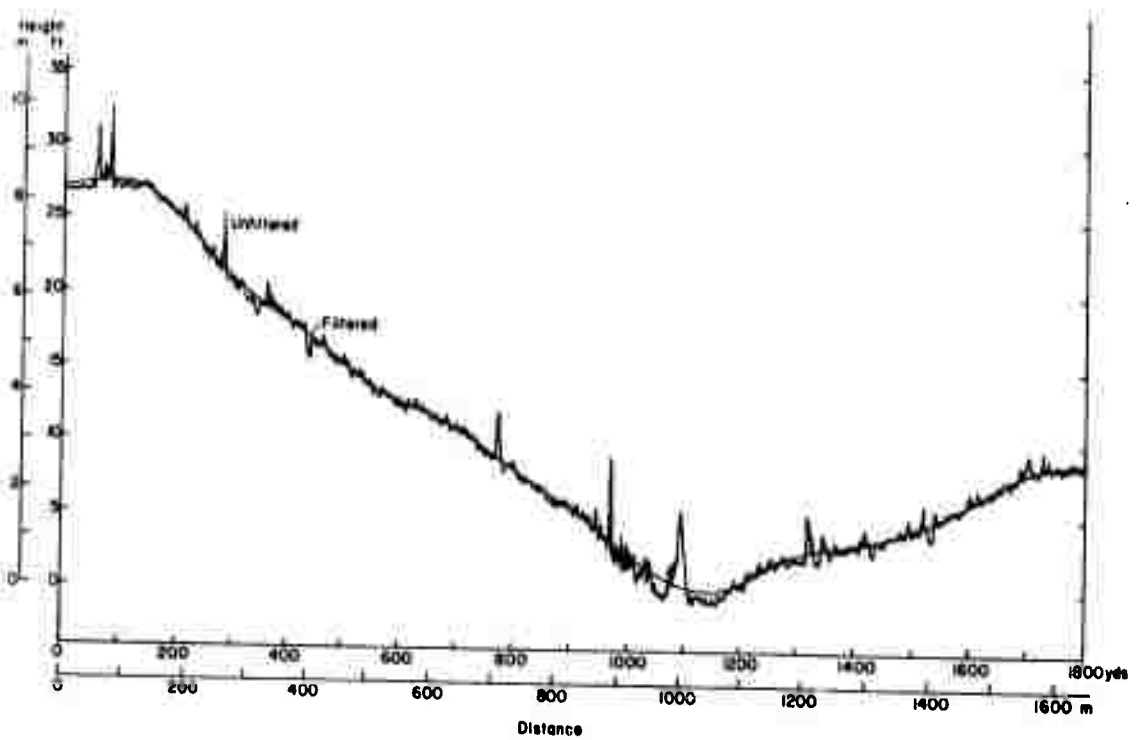

Figure 3. Example of low pass filtering. The illustrated data consist of 1800 points at 1-yd intervals. The smooth curve is the low pass filtered result using a Hamming low pass filter consisting of 921 weights with a transition band extending from 3/920 to 6/920 cycles/yd.

We remark that the low pass filter gives a reasonable estimate of the aircraft motion (especially in this example), but not in general ideal because there is an overlap between the surface roughness spectrum and the aircraft motion spectrum. [A more detailed filtering process which bypasses the spectral overlap problem is described by Hibler (in prep).] Figure 3 is a good example, however, of where small filter errors in the pass band are important because of the much larger amplitude of the low frequency components compared with high frequency components. When, for example, the low pass filtered result is subtracted from the initial profile, large errors in the pass band leave residual low frequencies with amplitudes commensurate with the surface roughness. We note in passing that it is clear that rather than low pass filter the data and then subtract the filtered points the same end result could be obtained by applying the high pass filter consisting of weights $C_1(n)$, where $C_1(n) = \delta_{n,0} - C(n)$ with $C(n)$ being the above low pass filter weights.

Conclusion

In conclusion, the working equations described in eq 16-18 give a rapid method of determining low pass filters with sharp frequency cutoffs and small spurious side lobe errors. Moreover, since the maximum filter error is known, individual filter errors do not have to be checked – a laborious procedure for large N. In practice, the procedure generates filters with small errors, especially for the Blackman window, which are smaller than those of comparable filters generated by existing techniques. It is probably true that for a particular cutoff and convolution length a more accurate nonrecursive filter could be designed, for example, by using a slightly different spectral window. However, in most applications eq 16-18 give adequate filters; and these equations are convenient to use because a maximum filter error estimate is known.

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APPENDIX A. FFT APERIODIC CONVOLUTION TECHNIQUE

Given a set of filter weights $C(n)$, we would like to filter the space series $\eta(i)$ by performing the convolution

$$\eta_f(i) = \sum_{j=-N}^{N} C(j)\eta(i - j) \tag{A1}$$

with $\eta_f(i)$ being the filtered result. The filtering procedure necessitates the loss of $2N$ end points. For a filter consisting of a large number of weights, a conventional convolution program takes prohibitive amounts of time. Consequently, it is necessary to use the aperiodic fast Fourier transform high-speed convolution technique (Stockham 1966). The aperiodic FFT convolution procedure should not be confused with simply carrying out a fast Fourier transform, removing certain frequencies, and then transforming back to real space as this results in a periodic convolution with spurious end point effects. The aperiodic fast Fourier transform is performed as follows. The discrete Fourier transform of $\eta(i)$ [denoted by $\tilde{\eta}(k)$]:

$$\tilde{\eta}(k) = \sum_{j=0}^{N_1-1} e^{2\pi ijk/N_1}\,\eta(j) \tag{A2}$$

may be performed rapidly using the FFT algorithm for N_1, a power of 2 (Cooley and Tukey 1965). For a filter with $2N + 1$ weights we take $N_1 > N$ and define

$$C_1(n) = \begin{cases} C(n + N) & n = 0, \ldots, 2N + 1 \\ 0 & n > (2N + 1) \end{cases} . \tag{A3}$$

We then take the discrete Fourier transform of $C_1(n)$ [denoted by $\tilde{C}_1(k)$] and $\tilde{\eta}(k)$ and form $\tilde{H}(k) = \tilde{C}_1(k)\,\tilde{\eta}(k)$. The discrete Fourier transform of $\tilde{H}(k)$ is given by

$$H(n) = \frac{1}{N_1} \sum_{k=0}^{N_1-1} e^{-2\pi ink/N_1}\tilde{H}(k) \tag{A4}$$

and is equal to the aperiodic convolution

$$H(n) = \sum_{m=0}^{N_1-1} \eta_p(n - m)C(m) \tag{A5}$$

where $\eta_p(i)$ is periodic; i.e., $\eta_p(i \pm qN_1) = \eta(i)$, q an integer. Because of the zeros in $C(j)$, however, the aperiodic convolution values $\eta_f(i)$ defined in eq 2 are given by $\eta_f(i) = H(n + N)$ for $i = N$ to $N_1 - N - 1$. Thus, $N_1 - 2N$ filtered points are obtained. Note that if the convolution length, $2N + 1$, is nearly equal to N_1 then only a few points will be accurately filtered.

Clearly the above procedure may be used for filtering a long record by operating with the FFT algorithm on overlapping sections of record with the overlap depending on the convolution length. In our application the FFT aperiodic technique (using $N_1 = 2048$) is about five times faster than conventional methods.