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The *capability of prediction* of the *radar echoes* (radar cross sections) of *aerospace vehicles* is necessary for either *modifying (reducing)* such cross sections for the purpose of *reducing detectability*, and/or *categorizing* such cross sections for the purpose of *identification*. Such predictions can be accomplished by *measurements* (radar range, unechoic chamber, etc.). The purpose of the capability of computing radar cross sections (*vis-a-vis* measurements) is to reduce both *cost and time*. The *state-of-the-art* of such computational methods consists of computer solving the *scattering integral equations* by *matrix inversion* methods. The *matrix* nature of such formulations and solutions *restricts the size* of the targets for which radar cross sections can be calculated on even the largest and fastest existing computers to no more than the order of one wave length, and *renders the possible* solutions *computer-time consuming and costly*. The purpose of the *k-space method* is the capability of *rapid and cost-efficient* computer calculations of radar cross sections of aerospace vehicles, particularly those of *much larger size than one wave length*. The *technical* means by which this purpose if achieved is summarized next.

The *initial-value problem* is solved by means of a *k-space formulation of the field equations*, thereby replacing the conventional integral equation formulation by a *set of two simultaneous algebraic equations in two unknowns in two spaces* (the constitutive boundary condition being an algebraic equation in x-space). These equations are solved by an *iterative generalized-relaxation* method with the aid of the Fast Fourier Transform (FFT) algorithm connecting the two spaces, requiring trivial initial approximations. Since algebraic and FFT equations are used, the number of arithmetic multiply-add operations and storage allocations required for a numerical solution are reduced from the order of N^3 and N^2 respectively (for solving the matrix equations resulting from the conventional integral equations) to the order of $N\log_2 N$ and N respectively (where N is the number of data points required for the specification of the problem). The convergence rate of the iterative process is optimized by generalizing the conventional relaxation factors to a *relaxation function* and/or its *generalized inverse* determined by the Eigen values of the appropriate Green's function. These Eigen values are obtained numerically in $N\log_2 N$ operations (*vis-a-vis* the conventionally required N^3 operations) by means of the FFT of the Green's function cast into a circulant matrix form. The advantage gained in speed and storage is thus of the order of $N^2/\log_2 N$ and N respectively. This method is thus considerably more efficient, and permits exact numerical solutions for much larger problems, than is possible with the conventional integral equation - matrix inversion method. Arguments are presented towards the view that the field equations are more fundamental in k-space. The *physical and mathematical* meaning of both the continuous and discrete k-space representations are discussed. The details and some numerical results of the application of this method to three-dimensional electromagnetic scattering are presented. It is shown that the scattered far fields are yielded *directly* in k-space.

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K-SPACE FORMULATION OF THE ELECTROMAGNETIC SCATTERING PROBLEM

NORBERT N. BOJARSKI

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FOREWORD

This unclassified document reports the work performed by Norbert N. Bojarski - Physicist, Consultant, 16 Circle Drive, Moorestown, New Jersey 08057, (609 235 3001), under USAF Contract F33615-71-C-1576, *Fast Fourier Transform Applied to Radar Scattering.*

This report covers the effort performed during the period from 1 April 1971 to 31 May 1972. This report was submitted by the author on June 1972.

This report was prepared by Norbert N. Bojarski

This technical report has been reviewed and is approved for publication.

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Assistant Chief
Electronic Warfare Division

ABSTRACT

The *capability of prediction* of the *radar echoes* (radar cross sections) of *aerospace vehicles* is necessary for either *modifying (reducing)* such cross sections for the purpose of *reducing detectability*, and/or *categorizing* such cross sections for the purpose of *identification*. Such predictions can be accomplished by *measurements* (radar range, unechoic chamber, *etc.*). The purpose of the capability of *computing* radar cross sections (*vis-a-vis* measurements) is to reduce both *cost and time*. The *state-of-the-art* of such computational methods consists of *computer solving* the *scattering integral equations* by *matrix inversion* methods. The *matrix* nature of such formulations and solutions *restricts the size* of the targets for which radar cross sections can be calculated on even the largest and fastest existing computers to no more than the order of one wave length, and *renders the possible* solutions *computer-time consuming and costly*. The purpose of the *k-space method* is the capability of *rapid and cost-efficient* computer calculations of radar cross sections of aerospace vehicles, particularly those of *much larger size than one wave length*. The *technical* means by which this purpose is achieved is summarized next.

The *initial-value problem* is solved by means of a *k-space formulation of the field equations*, thereby replacing the conventional integral equation formulation by a *set of two simultaneous algebraic equations in two unknowns in two spaces* (the constitutive boundary condition being an algebraic equation in x-space). These equations are solved by an *iterative generalized-relaxation* method with the aid of the Fast Fourier Transform (FFT) algorithm connecting the two spaces, requiring trivial initial approximations. Since algebraic and FFT equations are used, the number of arithmetic multiply-add operations and storage allocations required for a numerical solution are reduced from the order of N^3 and N^2 respectively (for solving the matrix equations resulting from the conventional integral equations) to the order of $N\log_2 N$ and N respectively (where N is the number of data points required for the specification of the problem). The convergence rate of the iterative process is optimized by generalizing the conventional relaxation factors to a *relaxation function* and/or its *generalized inverse* determined by the Eigen values of the appropriate Green's function. These Eigen values are obtained numerically in $N\log_2 N$ operations (*vis-a-vis* the conventionally required N^3 operations) by means of the FFT of the Green's function cast into a circulant matrix form. The advantage gained in speed and storage is thus of the order of $N^2/\log_2 N$ and N respectively. This method is thus considerably more efficient, and permits exact numerical solutions for much larger problems, than is possible with the conventional integral equation - matrix inversion method. Arguments are presented towards the view that the field equations are more fundamental in k-space. The *physical and mathematical* meaning of both the continuous and discrete k-space representations are discussed. The details and some numerical results of the application of this method to three-dimensional electromagnetic scattering are presented. It is shown that the scattered far fields are yielded *directly* in k-space.

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INTRODUCTION

The main subjects of this report are a *Generalized Relaxation* method and a *Generalized Inverse* method for iteratively solving large integral equations. These methods are specifically applicable to the *k-space formulation* of such integral equations. This k-space formulation was developed earlier by this author* under a previous contract, and was specifically addressed to the problem of *electromagnetic scattering*. The purpose of the generalized relaxation method and the generalized inverse method is the acceleration of the convergence rate and the enlargement of the domain of applicability of the previously developed iterative method of solution of the k-space formulation. Since the basic k-space formulation and its method of solution is applicable to all the *initial value* problems of mathematical physics which conventionally lead to integral equations, this report consists of a brief development of the *k-space formulation of the initial value problem*, with brief treatment of the *electromagnetic scattering problem* as a special case; and with detailed emphasis on the development of the generalized relaxation function method and the generalized inverse method (the reader is referred to the above cited earlier report for a detailed and tutorial presentation of the k-space formulation of the electromagnetic scattering problem).

The organization of this report is as follows. In Section 1, the conventional matrix formulation of the initial value problem is briefly reviewed. In Section 2, the k-space formulation of the initial value problem and its conventional iterative relaxation method of solution are formally

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developed. In Section 3, the special case of the k-space formulation of the general wave scattering problem is formally developed. Section 4 consists of the detailed development of the matrix theory perspective of the k-space formulation, necessary for the subsequent development of the generalized relaxation method and the generalized inverse method. Section 5 consists of the detailed development of the generalized relaxation method, and Section 6 consists of the detailed development of the generalized inverse method. In Section 7, the special case of the k-space formulation of the electromagnetic scattering problem is formally developed, and a brief review of earlier obtained numerical results in summarized. In Section 8, a summary of the state-of-development of the method, its present limitations, and recommendations for future needed research, are presented. The applicability of the method to the non-monochromatic (wide band) electromagnetic scattering problem, electromagnetic problems other than scattering, and the general initial value problems of mathematical physics, are also discussed.

This author is indebted to Dr. Charles H. Krueger, Jr., of the Air Force Avionics Laboratory, Wright-Patterson Air Force Base, Ohio, for developing the details, programming and verifying, the two-dimensional k-space formulation of the electromagnetic scattering problem.

This author is also indebted to Messrs. Leslie E. Whitford and William Hopkins of the Computing and Information Systems Division, Computer Science Center, Wright-Patterson Air Force Base, Ohio, for directing the programming effort and performing the actual programming respectively, of the one- and three-dimensional k-space formulations of the electromagnetic scattering problem, as well as for several important suggestions.

1. THE CONVENTIONAL MATRIX FORMULATION OF THE INITIAL VALUE PROBLEM

Consider the n-dimensional scalar, vector, or tensor field $\phi(X)$ and the source density $w(X)$, governed by the linear m-th order differential *field equation*

$$L_\phi \, \phi(X) = - L_w \, w(X) \; , \tag{1}$$

where the n-dimensional linear differential scalar, vector, or tensor operators L_ϕ and L_w are of the form

$$L_\phi = \sum_{i=0}^{m} a_i \left(\frac{\partial}{\partial x_j}\right)^i \quad ; \; i=1,2,\cdots,n \tag{2}$$

$$L_w = \sum_{i=0}^{m} b_i \left(\frac{\partial}{\partial x_j}\right)^i \quad , \tag{3}$$

subject to the n-dimensional scalar, vector, or tensor *constitutive equation*

$$w(X) = \sigma(X) \, \phi(X) \; . \tag{4}$$

The conventional n-dimensional *integral representation* of the generalized *initial value* problem associated with the field equation (1) is

$$\phi(X) = \int_D g(X|X') \, w(X') \, d^n x' + \phi^i(X) \quad , \tag{5}$$

subject to the constitutive equation (4) where $\phi^{i}(X)$ is the externally imposed field (initial value field, the source distribution of which is external to the problem defined by the *constitutive equation* (4)), D is the domain of non-vanishing $\sigma(X)$, and $g(X)$ is the appropriate Green's function, vector, or tensor, satisfying the differential equation

$$L_{\phi}\, g(X) = -\, L_{W}\, \delta(X) \ . \tag{6}$$

The constitutive equation (4) can also be viewed as the *boundary conditions* imposed by a specific physical situation on the *differential field equation* (1), which is invariant to the specific physical situation.

The conventional numerical method of solution of this initial value problem is by means of numerical matrix inversion methods [1], applied to the Fredholm *integral equation* of the Second Kind, formed by combining (4) and (5), *i.e.*,

$$\phi(X) - \int_{D} K(X|X')\, \phi(X')\, d^{n}x' = \phi^{i}(X) \ , \tag{7}$$

where the integral transform kernel $K(X|X')$ is given by

$$K(X|X') \equiv g(X|X')\, \sigma(X') \ . \tag{8}$$

(It should be noted that in cartesian coordinates, this kernal $K(X|X')$ is always a *compound* kernel of the form $K(X|X') = g(X-X')\, \sigma(X')$; *i.e.*, composed of a *difference* kernel and a *separable product* kernel).

Such matrix inversion methods require of the order of N^2 computer storage allocations, and of the order of N^3 arithmetic multiply-add operations (for matrix inversion) for the execution of a numerical solution, where N is the number of data points required for the numerical specification of the constitutive equation (4) (the specification of the non-vanishing portion of $\sigma(x)$). The practical size limit with state-of-the-art computers is for N of the order of several hundred.

2. THE K-SPACE FORMULATION OF THE GENERAL INITIAL VALUE PROBLEM

The *k-space representation* and solution of the generalized n-dimensional initial value problem is presented next.

The n-dimensional Fourier Transform of the differential field equation (1) yields the *local algebraic* scalar, vector, or tensor *k-space field equation*

$$L_\phi(k)\ \Phi(k) = -\ L_w(k)\ W(k) \tag{9}$$

where

$$\Phi(k) = \int_{-\infty}^{\infty} e^{ik\cdot x}\ \phi(x)\ d^n x\ , \quad etc., \tag{10}$$

and where, by virtue of (2) and (3), the quantities $L_\phi(k)$ and $L_w(k)$ are polynomials of the form

$$L_\phi(k) = \sum_{j=o}^{m} a_j\ (ik)^j\ , \tag{11}$$

$$L_w(k) = \sum_{j=o}^{m} b_j\ (ik)^j \tag{12}$$

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The *k-space representation of the generalized n-dimensional initial value problem*, consistent with the x-space integral representation (5), thus is the *algebraic* scalar, vector, or tensor equation (*vis-a-vis* the conventional *integral* or *differential* equation representation)

$$\Phi(k) = G(k)\, W(k) + \Phi^{I}(k) \ , \tag{13}$$

subject to the algebraic x-space constitutive equation (4), *i.e.*,

$$w(x) = \sigma(x)\, \phi(x) \ , \tag{14}$$

where,

$$G(k) = -\frac{L_w(k)}{L_\phi(k)} \tag{15}$$

which clearly can be taken as the Green's function (or vector or tensor) in k-space, *i.e.*,

$$G(k) \leftrightarrow g(x) \tag{16}$$

The generalized n-dimensional initial value problem is thus reduced to a set of two local algebraic (scalar, vector, or tensor) equations in two unknowns in two spaces, i.e., (13) and (14).

The k-space representation (13) *et seq.* also clearly follows from (5) and the n-dimensional convolution theorem since

$$g(x|x') = g(x-x') \tag{17}$$

in any cartesian coordinate system.

The unique existence [2] of this k-space representation is restricted to media for which

$$\int |\sigma(x)| \, d^n x < \infty \quad . \tag{18}$$

If $\sigma(x)$ is in general non-vanishing only in a finite n-dimensional x-domain, then the pair of algebraic equations (13) and (14) can be solved numerically with the aid of the n-dimensional Fast Fourier Transform (FFT) algorithm [3] as the *connection between the two spaces*, by the following iterative relaxation method [4]; the recursion relationship for which is

$$W_n(k) = F(k|x) \, w_n(x) \tag{19.1}$$

$$\Phi_n(k) = G(k) \, W_n(k) + \Phi^i(k) \tag{19.2}$$

$$\phi_n(x) = F(x|k) \, \Phi_n(k) \tag{19.3}$$

$$w_{n+1}(x) = \alpha \, \sigma(x) \, \phi_n(x) + (1-\alpha) \, w_n(x) \quad , \tag{19.4}$$

where α is an appropriately chosen relaxation coefficient (best numerical results to date were obtained for $\alpha=\frac{1}{2}$), and where $F(k|x)$ and $F(x|k)$ designate the Fast Fourier Transform algorithm operator and its inverse respectively.

The initial approximation $w_0(x)$ can be taken as any known simply programmable approximation to the problem, including the trivial case $w_0(x) = 0$, the computer programming of which consumes virtually no signifi-

cant, much needed for data, core storage allocation (which is not the case with any other non-trivial initial approximation, which, at best, would reduce by two the total number of iterations required).

In order to avoid the numerical difficulties arising from *aliasing*, *i.e.*, the *fictitious periodic* nature of the FFT (which is a *discrete* and *finite* Fourier transform, *vis-a-vis* the continuous and infinite Fourier transform implied by (10)), and the possible singularities in the Green's function (or vector or tensor) in k-space, it becomes necessary to choose an n-dimensional hyper-rectangular box of *twice the size* (in each dimension) of the smallest hyper-rectangular box in which the non-vanishing $\sigma(\mathbf{x})$ is imbeddable as the x-domain for the FFT, and take the Green's function, vector, or tensor as (in conventional FFT notation [5])

$$G(\mu,\nu,\cdots,\sigma) = -\Delta^n x \sum_{\alpha=-\frac{N_1}{2}}^{\frac{N_1}{2}-1} \sum_{\beta=-\frac{N_2}{2}}^{\frac{N_2}{2}-1} \cdots \sum_{\gamma=-\frac{N_n}{2}}^{\frac{N_n}{2}-1} W_{N_1}^{-\mu\alpha} W_{N_2}^{-\nu\beta} \cdots W_{N_n}^{-\sigma\gamma}\, g(\alpha,\beta,\cdots,\gamma) \;, \quad (20)$$

where

$$W_{N_j} = e^{2\pi i/N_j} \;, \quad j=1,2,\cdots,n \quad (21)$$

$$\Delta x_j \, \Delta k_j = 2\pi/N_j \quad (22)$$

$$\Delta^n x = \Delta x_1 \, \Delta x_2 \cdots \Delta x_n \quad (23.1)$$

$$N = N_1 \, N_2 \cdots N_n \quad , \textit{etc.}, \quad (23.2)$$

and where appropriate use must be made of the n-fold *periodicity properties* of the FFT [6] for both (20), as well as the desired placement of $\sigma(\mathbf{x})$ and $\phi^i(\mathbf{x})$ in the hyper-rectangular FFT x-domain.

The numerical difficulty arising from the possible singularity of the Green's function at the origin of the x-space (*i.e.*, $g(0) = \infty$) can be alleviated by taking advantage of the appropriate *principal value* integral representation of the field equation (5), (*e.g.*, [7]), *i.e.*,

$$\phi(x) = \ell\, w(x) + p\int_D g(x|x')\, w(x')\, d^n x' + \phi^i(x) \quad . \tag{24}$$

It thus immediately follows that $g(0)$ for (20) can be taken as

$$g(0) = \left(\frac{\ell}{\Delta^n x}\right) \quad . \tag{25}$$

Since algebraic and FFT equations are used, the number of arithmetic multiply-add operations required [8] for a solution is reduced from the order of N^3 (required for the numerical matrix inversion needed for solving the matrix equations resulting from the conventional integral equation representation of the problem) to the order of $N \log_2 N$, and the storage requirement is reduced from the order of N^2 (required for storing the matrix associated with the Green's function needed for the matrix method of solution) to the order of N (required for storing the k- and x-space functions). The advantage gained in speed and storage is thus of the order of $N^2 \log_2 N$ and N respectively. This method is thus considerably more efficient, and permits exact numerical solutions for much larger problems. To date, acoustic scattering problems of the order of $N=10^4$ have been successfully solved, and problems of the order of 10^7 are feasible with state-of-the-art computers.

3. THE K-SPACE FORMULATION OF THE GENERAL WAVE SCATTERING PROBLEM

For the general n-dimensional wave scattering problem, the (range and phase normalized) scattered far fields in x-space are in general simply and algebraically related to the n-dimensional Fourier Transform of the induced (by the incident field $\phi^{I}(x)$) source distribution $w(x)$, *i.e.*, $W(k)$, which is clearly yielded directly by the iterative solution (19) without additional computations. Since this is not the case with the conventional matrix method of solution of the integral equation representation of the scattering problem, the k-space method of solution presented is particularly and additionally attractive when applied to scattering problems.

For the special case of the n-dimensional Helmholtz (time-reduced) wave equations for which

$$L_{\phi} = \sum_{j=o}^{n} \left(\frac{\partial}{\partial x_j}\right)^2 + k_o^2 \quad , \tag{26}$$

where $k_o \equiv \frac{\omega}{c}$, and c is the wave velocity in the *free space* (this deviation from conventional notation is for the purpose of distinction from k, the Fourier Transform variable of x), the n-dimensional k-space Green's function, in the notation of (9), (15), and (16), is clearly

$$G(k) = \Psi(k)\, L_w(k) \quad , \tag{27.1}$$

$$\Psi(k) = \frac{1}{k^2 - k_o^2} \quad . \tag{27.2}$$

The form of the Green's function $\Psi(k)$ in k-space is clearly *invariant to the dimensionality* n of the space, which is *not* the case for the Green's function $\psi(x)$ in x-space.

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Thus, by virtue of the previously stated relationship between the (range and phase normalized) scattered far fields in x-space and the source distribution W(k) in k-space, and conservation of energy considerations (mathematically equivalent to Parseval's formula) for passive media of finite spatial extent (see (18)), it follows that the *Radiation Condition* (the *boundary condition* or *constitutive equations* at infinity) for the Helmholtz equation Green's function in k-space can be stated as

$$\left| \frac{\Phi(k_s)}{\Psi(k_s)} \right| < \infty \quad , \; |k_s| = k_o \quad , \tag{28}$$

where k_s is the propagation wave number vector of the scattered far fields.

It can thus be argued that the *field equations* of mathematical physics are more fundamental in *k-space* because of the simple *local algebraic* nature of these equations in k-space (*vis-a-vis* the global nature of the integral or differential representation of these field equations in x-space), and the invariance of the form of the k-space Green's function for the Helmholtz equation with respect to the dimensionality of the space; particularly when bearing in mind that the Fourier Transform is the only transform known for which a *fast algorithm* exists. However, the constitutive equations (or boundary conditions) are more fundamental in x-space because of their local algebraic nature in x-space.

4. THE MATRIX THEORY PERSPECTIVE OF THE K-SPACE FORMULATION

The equivalent *matrix theory perspective* of the k-space formulation can best be developed from the integral equation (7) for the one-dimensional problem, with the generalization to n-dimensional integral equations then becoming obvious; *i.e.*, by (7) and (8)

$$\phi(x) - \int_0^a g(x|x')\,\sigma(x')\,\phi(x')\,dx' = \phi^I(x) \ . \tag{29}$$

If x is discreteized in M equal intervals Δx in the domain (o,a), then

$$x_\alpha = \alpha\,\Delta x \quad ; \ \alpha = 0,1,2,\cdots,M \quad ; \tag{30}$$

and (29) yields

$$\phi(x_\alpha) - \sum_{\alpha'=0}^{M-1} g(x_\alpha|x_{\alpha'})\,\sigma(x_{\alpha'})\,\phi(x_{\alpha'})\,\Delta x = \phi^I(x_\alpha) \ . \tag{31}$$

Introducing the M-dimensional *vectors* ϕ_α and ϕ^I_α as

$$\phi_\alpha \equiv \phi(x_\alpha) \tag{32}$$

$$\phi^I_\alpha \equiv \phi^I(x_\alpha) \quad , \tag{33}$$

the M-dimensional (square) *diagonal matrix* $\sigma_{\alpha\beta}$ as

$$\sigma_{\alpha\beta} \equiv \Delta x \; \delta_{\alpha\beta} \; \sigma(x_\alpha) \; , \tag{34}$$

$$\delta_{\alpha\beta} = \begin{cases} 1 \; ; \; \alpha = \beta \\ 0 \; ; \; \alpha \neq \beta \; , \end{cases} \tag{35}$$

and the M-dimensional *matrix* $g_{\alpha\beta}$ as

$$g_{\alpha\beta} \doteq g(x_\alpha | x_\beta) \; , \tag{36}$$

then in *cartesian* tensor notation (with summation convention implied), (31) becomes

$$\phi_\alpha - g_{\alpha\beta} \; \sigma_{\beta\gamma} \; \phi_\gamma = \phi_\alpha^{\dagger} \quad ; \tag{37}$$

which, in conventional matrix notation becomes

$$\phi - g\sigma\phi = \phi^{\dagger} \tag{38}$$

$$(I - g\sigma)\phi = \phi^{\dagger} \quad ; \tag{39}$$

where I is the *identity* matrix.

(The conventional *matrix inversion* solution of the problem is accomplished by forming a matrix $s \equiv g\sigma$ directly, without the separate introduction of the diagonal matrix σ, and numerically inverting the matrix (I-s), yielding the solution $\phi=(I-s)^{-1}\phi'$).

Introducing the M-dimensional *generatrix vector* γ_α (generatrix of the matrix g)

$$\gamma_\alpha \equiv g(x_\alpha) \ , \tag{40}$$

and recalling that the Green's function $g(x|x')$ is always a *difference function* (see text subsequent to 8) of the form

$$g(x|x') = g(|x-x'|) \ , \tag{41}$$

yields with the aid of (36)

$$g_{\alpha\beta} = \gamma_{\alpha-\beta} \tag{42.1}$$

$$= \gamma_{\beta-\alpha} \tag{42.2}$$

$$= \gamma_{|\alpha-\beta|} \ . \tag{42.3}$$

It thus follows that the (MxM) Green's matrix g is a *symmetric matrix* derivable from the *M-dimensional Green's generatrix vector* γ by (42).

Introducing a (2M+1) dimensional space, *i.e.*, an N-dimensional space of dimension (2M+1), *i.e.*

$$N = 2M + 1 \ , \tag{43}$$

for the vectors ϕ, $\phi^{\dagger}$, and γ in which the first M components are defined as per (30) through (42), and the remaining components are defined by the *periodic* relationships

$$\phi_\alpha = \phi_{N-\alpha} \quad ; \quad M \leqslant \alpha \leqslant N-1 \ , \tag{44}$$

$$\phi^{\dagger}_\alpha = \phi_{N-\alpha} \quad ; \quad M \leqslant \alpha \leqslant N-1 \ , \tag{45}$$

$$\gamma_\alpha = \gamma_{N-\alpha} \quad ; \quad M \leqslant \alpha \leqslant N-1 \ , \tag{46}$$

and in which the first MxM components of the diagonal of the matrix σ are defined as per (34) and (35), and the remaining components of the diagonal are defined as zero, *i.e.*,

$$\sigma_{\alpha\beta} \equiv \begin{cases} \Delta x \ \delta_{\alpha\beta} \ \sigma(x_\alpha) & ; \ 0 \leqslant \alpha,\beta < M \\ 0 & ; \ M \leqslant \alpha,\beta \leqslant N-1 \ , \end{cases} \tag{47}$$

clearly leaves the now N-dimensional matrix equation

$$\phi - g\sigma\phi = \phi^{\dagger} \tag{48}$$

consistent with (31), (38), and (39).

However, the NxN Green's matrix g resulting from the N-dimensional now *periodic generatrix* vector γ, still given by (42), is now not only *symmetric*, but also has the additional property of being a *circulant matrix* [9].

Some well known special properties of circulant matricies [10] will now be recast into a somewhat more convenient notation and general form; for this purpose, however, it becomes convenient to introduce the NxN *Fourier matrix* F, defined as

$$F_{\mu\alpha} \equiv \frac{1}{\sqrt{N}} e^{-\frac{2\pi I}{N}\mu\alpha} , \tag{49}$$

which clearly is a *symmetric unitary* matrix, *i.e.*

$$F_{\mu\alpha} = F_{\alpha\mu} \tag{50}$$

$$F\,F^{\dagger} = I\ . \tag{51}$$

It should be noted that this Fourier matrix has only N distinct elements (thus requiring only N storage allocations). The matrix multiplication of this matrix by a vector requires only $\frac{1}{2}N\log_2 N$ arithmetic multiply-add operations if the FFT algorithm is used.

The special properties of a circulant matrix can now be stated as follows:

1. The *Fourier matrix* is the *unitary transformation matrix* which *diagonalizes* a *circulant matrix*, *i.e.*

$$F\,g\,F^{\dagger} = g' \tag{52}$$

where g' is the *diagonal Eigen matrix* of g. The rationale for choosing the factor $1/\sqrt{N}$ in (49) is now evident; namely, the conventional definition without this factor would have lead to merely a

similarity transformation matrix, which is clearly far less general.

2. The *Eigen values* λ of a *circulant matrix* g are related to the *generatrix vector* γ of the incident matrix and the *Fourier matrix* by

$$\lambda_\mu = \sqrt{N}\, F_{\mu\alpha}\, \gamma_\alpha \quad , \tag{53}$$

I.e., The vector formed by the set of Eigen values of a circulant matrix is the Fourier transform of the generatrix vector of this circulant matrix. The practical significance of this equation (53) is that whereas it takes N^3 arithmetic multiply-add operations to compute the complete set of Eigen values of a general matrix, it takes only $\frac{1}{2}N\log_2 N$ arithmetic multiply-add operations to compute the complete set of Eigen values of a circulant matrix, since (53) can be computed with the aid of the FFT algorithm (matrix multiplication by the Fourier matrix can always be accomplished with the FFT algorithm). Furthermore, whereas N^2 storage allocations are needed for a general matrix, only N storage allocations are needed for the generatrix vector completely and uniquely defining a circulant matrix.

3. The *Eigen vectors* $e_\mu^{(\alpha)}$ of a *circulant matrix* are totally independent of the (N independent) elements of the circulant matrix (and the elements of its generatrix vector), and are proportional to the vectors formed by the *rows* or *columns* of the *Fourier Matrix; i.e.*

$$e_\mu^{(\alpha)} = \sqrt{N}\, F_{\mu\alpha} \quad . \tag{54}$$

The practical significance of this equation (54) is that the Eigen vectors of a circulant matrix can be generated directly by inspection of the Fourier matrix (49); *i.e.*, the N N-*th roots of unity*.

4. The *inverse* of a *circulant matrix* g is also a *circulant matrix*, and is given by

$$g^{-1} = F^{\dagger} \left\{\frac{1}{\lambda}\right\} F \quad , \tag{55}$$

where the quantity $\frac{1}{\lambda}$ is a *diagonal matrix*, the diagonal elements of which consist of the *reciprocals* of the Eigen values λ of the circulant matrix g.

An alternative formulation of (55), in terms of the generatrix γ of the circulant matrix g, as per (53), in a consistent notation, is

$$g^{-1} = F^{\dagger}\left\{\frac{1}{\sqrt{N}\ F\ \gamma}\right\} F \quad ; \tag{56}$$

i.e., the quantity {} in (56) is a diagonal matrix, the diagonal elements of which consist of the *reciprocals* of the elements of the vector ($\sqrt{N}\ F\gamma$).

The practical significance of (55) and (56) is similar to those stated subsequent to (53).

The proof of (55) and (56) follows directly from (49) *et seq.*

All the preceding properties of a circulant matrix can be extended to n-dimensional spaces by the introduction of the *n-dimensional N×N Fourier matrix* (see 49) as

$$F_{\mu_j \alpha_j} \equiv \prod_{j=1}^{n} \frac{1}{\sqrt{N_j}} e^{-\frac{2\pi i}{N_j} \mu_j \alpha_j} \quad ; \tag{57}$$

where the *matrix multiplication* of this n-dimensional NxN Fourier matrix by an n-dimensional vector (of N elements) is accomplished numerically with the aid of the n-dimensional FFT algorithm. (For the limiting case of $N \to \infty$; *i.e.*, a Hilbert space, the preceding properties simply revert back to the Fourier integral transform properties, where it is now clear that the *Eigen value spectrum* $\lambda(k)$ of a difference function $g(x-x')$ is the integral ourier transform of the generatrix function $g(x)$).

The iterative solution (19) can now be developed in matrix theory perspective form (48) *et seq.* By (48) and (52)

$$\phi - F^{+} g' F \sigma \phi = \phi^{i} \quad , \tag{58}$$

where now both σ and g' are diagonal, where the matrix F has only N distinct elements requiring storage allocations, and where all the implied matrix multiplications can be executed with the aid of the FFT algorithm.

Choosing any vector ϕ_n for ϕ clearly yields the *self consistent* vector $\tilde{\phi}_n$ by (48) as

$$\tilde{\phi}_n = g\sigma\phi_n + \phi^{i} \quad . \tag{59}$$

If ϕ_n is taken as the n-th approximation of ϕ, then clearly $\tilde{\phi}_n$ can be taken as the (n+1)th approximation of ϕ; *i.e.*

$$\phi_{n+1} = g\sigma\phi_n + \phi^{i} \quad . \tag{60}$$

This recursion relationship (60) for the iterative solution of (48) is equivalent to the *Neumann Series expansion* of (48), the *sufficient condition* for the convergence of which is that the *norm* of the matrix operator $g\sigma$ be less than unity; *i.e.*

$$|g\sigma| < 1 \quad . \tag{61}$$

By (53), and the diagonality of σ, this norm can clearly be computed in $N\log_2 N$ operations.

For those cases for which (61) is not satisfied, the iterative *relaxation method* can be developed as follows; let α be a *real scalar* relaxation factor such that

$$\phi_{n+1} = \alpha\tilde{\phi}_n + (1-\alpha)\phi_n \quad , \tag{62}$$

which, with the aid of (59) yields

$$\phi_{n+1} = [\ I - \alpha(I-g\sigma)\ ]\phi_n + \alpha\phi^{i} \quad . \tag{63}$$

An alternative more rigorous derivation of (63) is as follows; (48) can be written as

$$(I-g\sigma)\phi = \phi^{i} \quad . \tag{64}$$

thus

$$\alpha(I-g\sigma)\phi = \alpha\phi^{i} \tag{65}$$

$$\phi - \phi + \alpha(I-g\sigma)\phi = \alpha\phi^{i} \tag{66}$$

$$\phi - [\, I - \alpha(I-g\sigma)]\phi = \alpha\phi^{I} \quad . \tag{67}$$

The Neumann series expansion of (67) is

$$\phi = \sum_{j=0}^{\infty} [\, I - \alpha(I-g\sigma)]^{j} \, (\alpha\phi^{I}) \quad ; \tag{68}$$

And, consistent with (60), the recursion relationship for the iterative solution of (67) is

$$\phi_{n+1} = [\, I - \alpha(I-g\sigma)]\phi_{n} + \alpha\phi^{I} \quad , \tag{69}$$

which is consistent with the iterative k-space solution (19). The sufficient condition for the convergence of (69) is

$$|\, I - \alpha(I-g\sigma) \,| < 1 \quad . \tag{70}$$

The conventional relaxation method thus reduces to finding the relaxation factor α which minimizes $|I-\alpha(I-g\sigma)|$, and thus maximizes the convergence rate of (69).

The essence of the k-space formulation, from a matrix theory perspective, is in (52), (58), and (63), in which only the *diagonal matrices* g' and σ need be stored, and all matrix multiplications (executed by the FFT algorithm) are via the *Fourier matrix* F. This becomes most evident if (63) and (52) are combined into

$$\phi_{n+1} = [\, I - \alpha(I - F^{\dagger} g' F\sigma)]\phi_n + \alpha\phi^{I} \,, \tag{71}$$

which is consistent with the iterative k-space solution (19).

The mathematical meaning (*vis-a-vis* the physical meaning discussed earlier) of the k-space formulation is thus in having found the transformation matrix (the Fourier matrix) which diagonalizes the discrete field equations (5) in a new coordinate system (the k-space), and possessing the means of executing this transformation economically in speed and storage (*i.e.*, with the FFT algorithm).

5. THE GENERALIZED RELAXATION METHOD

The difficulty with the conventional relaxation method is the difficulty of finding a relaxation factor which optimizes convergence rate, as well as the fasic question of the existence of such a factor which assures convergence at all.

These difficulties are alleviated if the relaxation factor is *not* restricted to being a *real scalar*, but is generalized to being a *complex matrix*.

The *trivial and naive* such relaxation matrix α clearly is

$$\alpha = (I-g\sigma)^{-1} \quad , \tag{72}$$

for which the iterative recursion relationship (69) and the convergence condition (70) respectively become

$$\phi_{n+1} = \alpha\phi^{1} \tag{73}$$

$$\left| \; I - \alpha(I-g\sigma) \; \right| = 0 < 1 \quad ; \tag{74}$$

namely; the iterative process converges in one iteration to the exact solution. This trivial and naive choice of α as the solution to the problem was presented merely for the purpose of making the following argument:

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The best known approximation to the solution of the problem should be chosen as a relaxation matrix, and not as the initial approximation, in the iterative process (since convergence is determined solely by the relaxation matrix and is independent of the initial approximation chosen).

However, prior to searching for a practically suitable choice of the matrix α, certain practical limitations must be imposed on the properties of this matrix. A completely arbitrary matrix α will negate all the storage and speed advantages of the k-space method, since such an arbitrary matrix will require of the order of N^2 storage allocations and arithmetic multiply-add operations for the execution of the implied matrix multiplications in (69).

A natural such choice of special properties for the matrix α, which will preserve the speed and storage advantages of the k-space method, consistent with the matrix theory perspective of the k-space formulation, clearly are that the matrix α be *diagonal*, *circulant*, or *compound* (the product of a diagonal and a circulant matrix).

(The conventional *scalar* relaxation method can now be viewed as the special case of the *generalized relaxation matrix* being *fully degenerate*).

The general required properties of the relaxation matrix will be examined next.

The *norm* implied in the *convergence condition* (74) must clearly be taken as the *Euclidean norm* or the *spectral radius*, *i.e.*, the largest magnitude (absolute value) of the Eigen value of the matrix operator A in (74) where

$$A \equiv I - \alpha(I-g\sigma) \quad . \tag{75}$$

It now becomes convenient to introduce the matrix B, defined as

$$B \equiv g\sigma \quad ; \tag{76}$$

thus

$$A = I - \alpha(I-B) \quad . \tag{77}$$

The Eigen values of A are given by the determinental equation

$$det\ (\ A - \lambda_A\ I\ ) = 0 \quad . \tag{78}$$

The convergence condition (74) for optimal convergence rate thus is

$$max\ |\lambda_A| < 1 \tag{79}$$
$$\rightarrow 0 \quad .$$

If the set of all Eigen values $\{\lambda_A\}$ is chosen as exactly zero, then clearly (74) is satisfied in such a fashion that the resulting α yields an exact solution at the first iteration (as was the case with the previously presented naive example of $\alpha=(I-g\sigma)^{-1}$); *i.e.*, a *numerically closed form solution* of the form (see 71)

$$\phi = \alpha\phi^{\dagger} \quad . \tag{80}$$

For this choice, the determinental equation (78) thus becomes

$$det\ A = 0 \quad , \tag{81}$$

which, with the aid of (77) becomes

$$det\ [\ I - \alpha(I-B)\ ] = 0 \quad , \tag{82}$$

which can be rewritten as

$$det\ (I-\alpha+\alpha B) = 0$$

$$det\ [\alpha(\alpha^{-1}-I+B)] = 0$$

$$det\ \alpha \cdot det\ [\ B - (I-\alpha^{-1})\ ] = 0 \quad , \tag{83}$$

which is satisfied by

$$det\ [\ B - (I-\alpha^{-1})\ ] = 0 \quad . \tag{84}$$

The determinental equation which determines the Eigen values λ_B of the matrix B is

$$det\ (\ B - \lambda_B\ I\ ) = 0 \quad , \tag{85}$$

It thus follows from (84) and (85) that if the matrix $(I-\alpha^{-1})$ is chosen as the *diagonal* matrix $\{\lambda_B\}$, *i.e.*, consistent with the previously introduced notation $\{\}$,

$$I - \alpha^{-1} = \{\lambda_B\} \quad . \tag{86}$$

then (81) is satisfied identically. It thus follows from (86) that α is the diagonal matrix

$$\alpha = \left\{ \frac{1}{1-\lambda_B} \right\} \quad . \tag{87}$$

Examination of (76) thus reveals that knowledge (or a rapid means of computing) of the Eigen values of the matrix $g\sigma$ would yield a numerically closed form solution of the problem; *i.e.*, (80). Such knowledge or means are clearly *not* available. However, since σ is a diagonal matrix, an approximation for the Eigen values of the matrix $g\sigma$ *are* available; namely, the product of the Eigen values of the matricies g and σ (which, in general, for non-diagonal σ, *are not* equal to the Eigen values of the product of the matricies), *i.e.*,

$$\lambda_B \simeq \lambda_g \, \lambda_\sigma \quad , \tag{88}$$

where the Eigen values λ_σ are the diagonal elements of the diagonal matrix σ, and the Eigen values λ_g are given by (53), and are computable easily and rapidly. To the extent to which (88) is a good approximation, a good choice for the diagonal matrix α is thus

$$\alpha = \left\{ \frac{1}{1 - \lambda_g \, \lambda_\sigma} \right\} \tag{89}$$

It can be shown that for α chosen as per (89), the *norm* of A is always less than unity.

In the k-space notation of (19), with the aid of (53), (89) yields

$$\alpha(x) = \frac{1}{1 - \sigma(x)\ G(x)} \tag{90}$$

where G(x) is G(k) *colocated* in x-space; *i.e.*, G(k) evaluated at k=x.

For purposes of computer programming (storage allocations) the solution (19)-(90) can be simplified by introducing ϕ^s defined by

$$\phi \equiv \phi^i + \phi^s \quad , \tag{91}$$

and $\beta(x)$ defined by $\beta(x) \equiv \sigma(x)\ \alpha(x)$; *i.e.*, by (90),

$$\beta(x) \equiv \frac{\sigma(x)}{1 - \sigma(x)\ G(x)} \quad . \tag{92}$$

The recursion relationship for the generalized relaxation function iterative solution (19)-(90) thus becomes

$$W_n(k) = F(k|x)\ w_n(x) \tag{93.1}$$

$$\phi_n^s(k) = G(k)\ W_n(k) \tag{93.2}$$

$$\phi_n^s(x) = F(x|k)\ \phi_n^s(k) \tag{93.3}$$

$$w_{n+1}(x) = \beta(x)\ [\ \phi_n^s(x) + \phi^i(x) - G(x)\ w_n(x)\ ] \quad . \tag{93.4}$$

An artificial test case of N=8 was programmed, with the following numerical results. For cases for which the conventional relaxation method succeeded, convergence was accelerated from about 30 iterations by the conventional method to about 6 iterations by the generalized method. For cases for which the conventional relaxation method failed, convergence was reached in about 10 iterations by the generalized method for those cases for which the magnitude of the largest element of α was less than about 10^3, and the generalized method failed for those cases for which the magnitude of the largest element of α approached about 10^3.

It thus seems that the *generalized relaxation function* method cannot overcome the difficulties presented by a *norm* of A of about 10^3 and larger (*vis-a-vis* a *norm* A $\lesssim$ 1 for the conventional method). Although higher precision in the computations could undoubtedly overcome this difficulty, such higher computational precision is totally undesirable since it defeats the very purpose of the basic method; *i.e.*, economy of speed and storage. It is to the alleviation of this difficulty that the next section is addressed.

6. THE GENERALIZED INVERSE METHOD

The generalized relaxation function method can now be put in the following perspective. The *Neumann series solution* (or the *method of successive approximations*) of (48), *i.e.*,

$$\phi - g\sigma\phi = \phi^{1} \tag{94}$$

is applicable if

$$\textit{norm}\ (g\sigma) < 1 \quad . \tag{95}$$

The *conventional matrix method* solution of (48) is applicable if

$$\textit{norm}\ (g\sigma) \lesssim 1 \quad , \tag{96}$$

and possesses the additional property of *accelerating* (over the Neumann series solution) and *uniforming* convergence if (96) is satisfied.

The *generalized relaxation method* of solution of (48) is applicable if

$$\textit{norm}\ (g\sigma) \lesssim 10^{3} \quad , \tag{97}$$

and further accelerates uniform convergence if (97) is satisfied.

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This section is addressed to the problem of solving (84) subject to the condition

$$\textit{norm}\ (g\sigma) > 1 \quad . \tag{98}$$

For that purpose let α be chosen as the *compound* matrix

$$\alpha \equiv (-g\sigma)^{(-1)} \quad , \tag{99}$$

i.e., the negative of the *generalized inverse* [11] of the product of the *circulant* matrix g and the *diagonal* matrix σ.

The recursion relationship for the iterative solution (69) thus becomes

$$\phi_{n+1} = (g\sigma)^{(-1)} \ (\ \phi_n - \phi^{\dagger} \ ) \quad , \tag{100}$$

subject to the *norm* condition (70), which, due to the choice (99) of α becomes

$$| \ (-g\sigma)^{(-1)} \ | < 1 \tag{101}$$

$$| \ g\sigma \ | > 1 \quad , \tag{102}$$

which is the desired condition (98).

Equation (100) reduces to

$$\phi_{n+1} = \sigma^{(-1)} g^{(-1)} (\phi_n - \phi^i) \quad . \tag{103}$$

Since the determinant of g is in general non-zero, the inverse of g can be computed economically and efficiently (with N storage allocations and $\frac{1}{2}N\log_2 N$ operations) by (55); the *generalized* inverse of g can thus be taken as the conventional inverse g^{-1}. Since σ is a diagonal matrix with one-half of its diagonal elements consisting of zeros, the determinant of σ vanishes; it is thus necessary to take its generalized inverse. A suitable such generalized inverse is clearly (in the previously introduced notation) the diagonal matrix $\{\frac{1}{\sigma}\}$; *i.e.*, a diagonal matrix, the diagonal elements of which consist of the reciprocals of the elements of the diagonal matrix σ for those values for which the diagonal elements of σ do not vanish, and zero otherwise (thus yieldind a *diagonal generalized inverse* with again one-half of its diagonal elements consisting of zeros). Equation (103) can thus be written as

$$\phi_{n+1} = \{\frac{1}{\sigma}\} g^{-1} (\phi_n - \phi^i) \quad . \tag{104}$$

Close examination of the derivation of (48) from (38) reveals that (105) could also have been derived directly from (38) by a process similar to the one that lead to (48).

Convergence of (104) under conditions (102) can be further accelerated by the re-application of the method of Sect. 4; *i.e.*,

$$\tilde{\phi}_n = \{\frac{1}{\sigma}\} g^{-1} (\phi_n - \phi^i) \tag{105.1}$$

$$\phi_{n+1} = \alpha\tilde{\phi}_n + (1-\alpha)\phi_n \tag{105.2}$$

where, consistent with the derivation of (90) and its notation, α is chosen as

$$\alpha(x) = \frac{1}{1 - \frac{1}{\sigma(x)\ G(x)}} \tag{105.3}$$

For computer programming purposes, (105) can clearly be put into the simplified concise form of (93).

With the results of this and the previous sections, it is thus always possible to determine the *norm* of $(g\sigma)$ economically and efficiently, and choose the appropriate economical and efficient method of solution of (48), no matter what that norm is.

7. K-SPACE FORMULATION OF THE ELECTROMAGNETIC SCATTERING PROBLEM

Three-dimensional electromagnetic monochromatic scattering by passive inhomogeneous media, including perfect conductors, of finite spatial extent and arbitrary shape, is considered.

The time-reduced electric and magnetic field wave equations, valid for all linear inhomogeneous media, in terms of the total current density [12], are respectively

$$\nabla\times\nabla\times E(x) - k_o^2\, E(x) = i\omega\mu_o\, J(x) \tag{106}$$

$$\nabla\times\nabla\times H(x) - k_o^2\, H(x) = \nabla\times J(x) \quad , \tag{107}$$

which, with the aid of Maxwell's first and second equations, and the equation of continuity for the total charge and current density, can be written as

$$\nabla^2\, E - k_o^2\, E = -i\omega\mu_o(J + \frac{1}{k_o^2}\, \nabla\nabla\cdot J) \tag{108}$$

$$\nabla^2\, H - k_o^2\, H = -\nabla\times J \quad . \tag{109}$$

For non-magnetic media and perfectly conducting media, the appropriate constitutive equations for the total volume and surface current density $J(x)$ and $K(x)$ respectively are

$$J = (\sigma_f - i\omega\chi_e) E \tag{110}$$

$$K = n \times H \tag{111}$$

where σ_f and χ_e are the free charge conductivity and electric susceptibility of a (non-magnetic) medium respectively, and n is the outward surface unit vector of a perfectly conducting medium.

The latter (111) is usually regarded as a boundary condition for perfect conductors, but in the context of this paper, this equation must be taken as a constitutive equation in the truest sense, particularly if regarded as a geometrically constraining condition on the flow of all charges.

By the relationship between surface and volume current densities, consistent with the FFT notation, the volume current density for a perfectly conducting medium, can be written as

$$J = \frac{ds}{dv} K \tag{112}$$

$$= \frac{\Delta s}{\Delta v} n \times H \tag{113}$$

$$= \frac{\Delta \mathbf{s}}{\Delta^3 x} \times H \tag{114}$$

where Δs is the finite differential vector surface area in the FFT cell of volume $\Delta v = \Delta^3 x$.

The conventional magnetic [13] and electric [14] field *principal value* scattering integral equations

$$H - 2\, p\!\int \nabla g \times (n \times H)\, ds = 2H^{i} \tag{115}$$

$$E - \frac{1}{3i\omega\varepsilon_o} J - i\omega\mu_o\, p\!\int (I + \frac{1}{k_o^2} \nabla\nabla)g \cdot J\, dv = E^{i} \tag{116}$$

can thus be re-formulated into a form consistent with the k-space formulation (13, (14), *et seq.*, *i.e.*,

$$H(k) = 2\, G(k) \times J(k) + 2\, H^{i}(k) \tag{117.1}$$

$$J(x) = \sigma(x) \times H(x) \quad , \tag{117.2}$$

where

$$\sigma(x) \equiv \frac{\Delta S}{\Delta^3 x} \tag{117.3}$$

$$G(k) = F(k|x)\, \nabla g(x) \tag{117.4}$$

$$\nabla g(x) = \begin{cases} \nabla\left(\dfrac{e^{ik_o r}}{4\pi r}\right) & ;\ x \neq 0 \\ 0 & ;\ x = 0 \end{cases} \tag{117.5}$$

$$r \equiv |x| \tag{117.6}$$

and

$$E(k) = \Gamma(k)\cdot J(k) + E^{i}(k) \tag{118.1}$$

$$J(x) = [\ \sigma_f(x) - i\omega\chi_e(x)\ ]\ E(x) \quad , \tag{118.2}$$

where

$$\Gamma(k) = F(k|x)\ \gamma(x) \tag{118.3}$$

$$\gamma(x) = \begin{cases} i\omega\mu_0(I + \frac{1}{k_0^2}\nabla\nabla)g & ;\ x \neq 0 \\ \frac{1}{3i\omega\varepsilon_0} I & ;\ x = 0 \end{cases} \quad . \tag{118.4}$$

Equations (117) and (118) can now be numerically solved economically and efficiently by the methods of the preceding sections.

Defining the range and phase normalized scattered far-field $S(k_s)$ as

$$S(k_s) \equiv \underset{r \to \infty}{Limit} \sqrt{4\pi r^2}\ F^s(x)\ e^{-ik_s\cdot x} \tag{119}$$

where $F^s(x)$ is any scattered field satisfying the relationship $F = F^i + F^s$) which is consistent with the conventional definition [15] of the radar power cross section σ and the relationship

$$\sigma = S\cdot S^* \tag{120}$$

readily reveals that the range and phase normalized electric and magnetic scattered far-fields S_e and S_m are given directly by the k-space current density distributions

$$S_e(k_s) = \frac{ik_o z_o}{\sqrt{4\pi}} \left[J(k) - \hat{k}\hat{k}\cdot J(k) \right] \Bigg|_{k=k_s} \tag{121}$$

$$S_m(k_s) = \frac{i}{\sqrt{4\pi}} k\times J(k) \Bigg|_{k=k_s} , \tag{122}$$

where z_o is the impedence of free space. (As dictated by the transversality of the scattered far-fields in free space, S_e, S_m and K_s are indeed all orthogonal to each other).

It can thus be shown that the conventionally defined [16] electric polarization scattering matrix $\rho_{\eta\xi}$ is given *directly in k-space* by

$$\rho_{\eta\xi} = \frac{ik_o z_o}{\sqrt{4\pi}} \eta_s \cdot J(k) \Bigg|_{k=k_s} , \tag{123}$$

where, in the conventional notation for spherical coordinates, η_s are the Eigen-polarizations (spherical coordinate unit base vectors) ϕ_s and θ_s associated with the scattered far-field propagation vector k_s, and $J(k)$ is the k-space current density induced by an electric incident plane wave field of the form and polarization

$$E^i(x) = \xi\, e^{ik_i\cdot x} , \tag{124}$$

where ξ are the Eigen polarizations (spherical coordinate unit base vectors) ϕ_i and θ_i associated with the incident propagation vector k_i.

The solution to the k-space formulation of the three-dimensional scattering problem (for the electric field equations (118) for non-magnetic media, and the magnetic field equations (117) for perfect conductors) has been

numerically computer executed for a limited number of cases by the iterative method of solution (19), with final results within about one db. of exact known analytic closed form solutions after about 30 iterations. For example, see Fig. 1-5 for comparison of this technique with the exact solution of Mie [17] for the perfectly conducting sphere (of radius a). The failure of this k-space technique in the near-vicinity of $k_o a = 2.75$ (see Fig. 1) is due to the fact that $k_o a = 2.75$ is the occurrence of the first Eigen frequency (internal resonance of perfectly spherical shell). This difficulty can be readily and simply alleviated by the appropriate incorporation of the method of Mitzner [18] into the k-space method. However, since the objective of this project was to prove the feasibility and merits of the k-space method, and not the generation of an operational user-library of computer programs, such an incorporation was taken as beyond the scope of this project.

For the results of the application of the k-space method to two-dimensional electromagnetic scattering, the reader is referred to the work of Krueger [19].

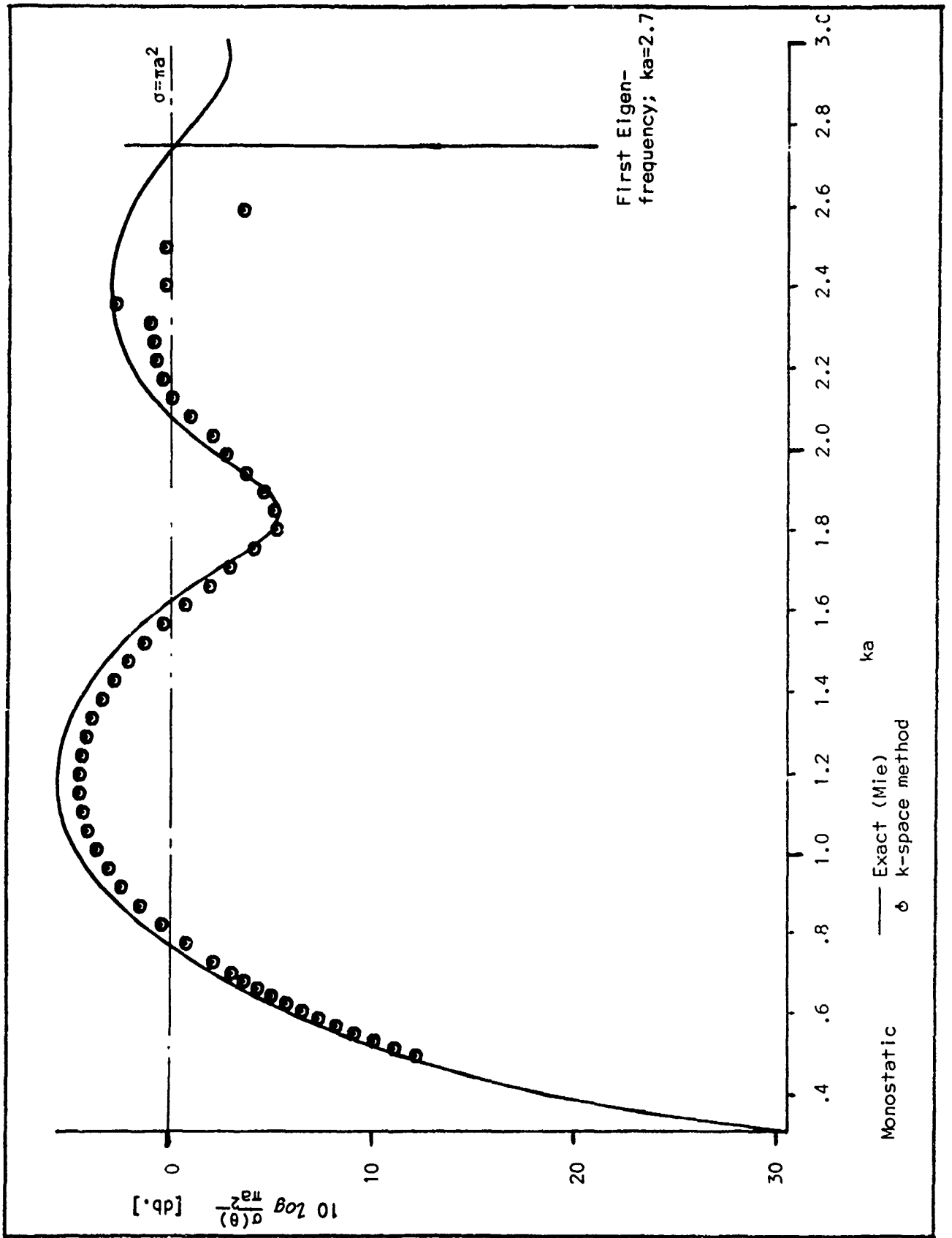

FIGURE 1 SPHERE; CROSS SECTION *vs.* FREQUENCY

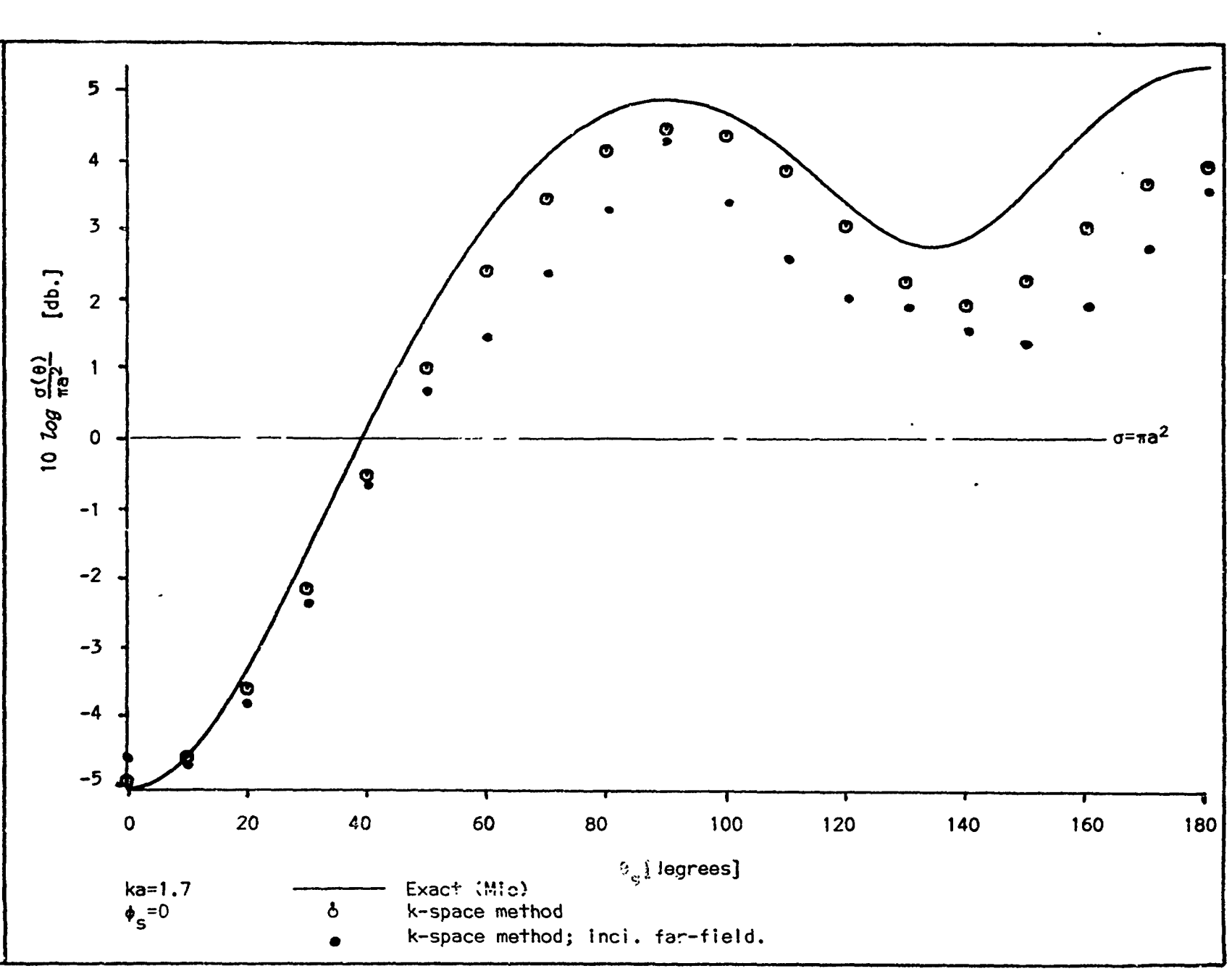

FIGURE 2 SPHERE; CROSS SECTION vs. ASPECT ANGLE

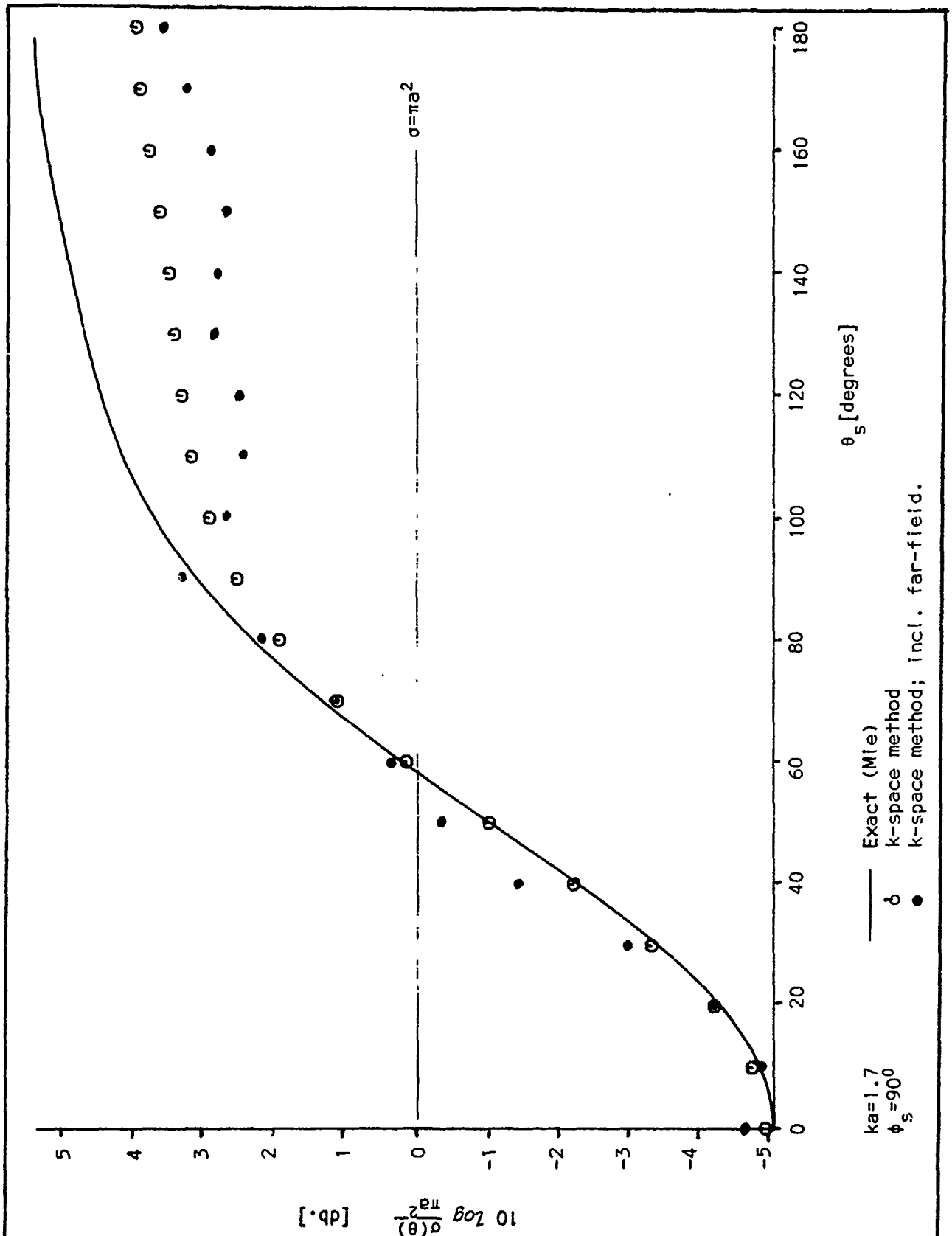

FIGURE 3 SPHERE; CROSS SECTION *vs.* ASPECT ANGLE

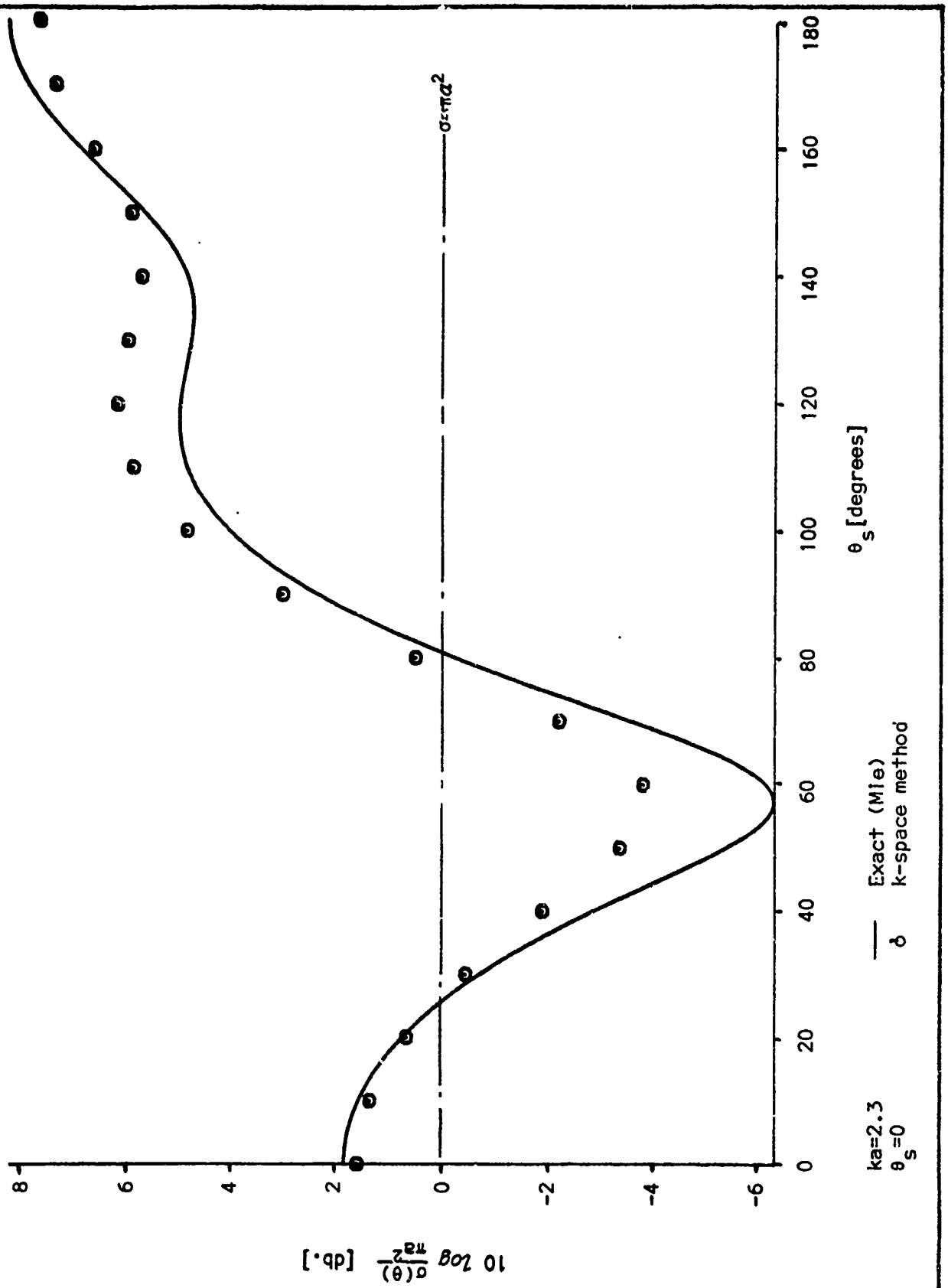

FIGURE 4 SPHERE; CROSS SECTION *vs.* ASPECT ANGLE

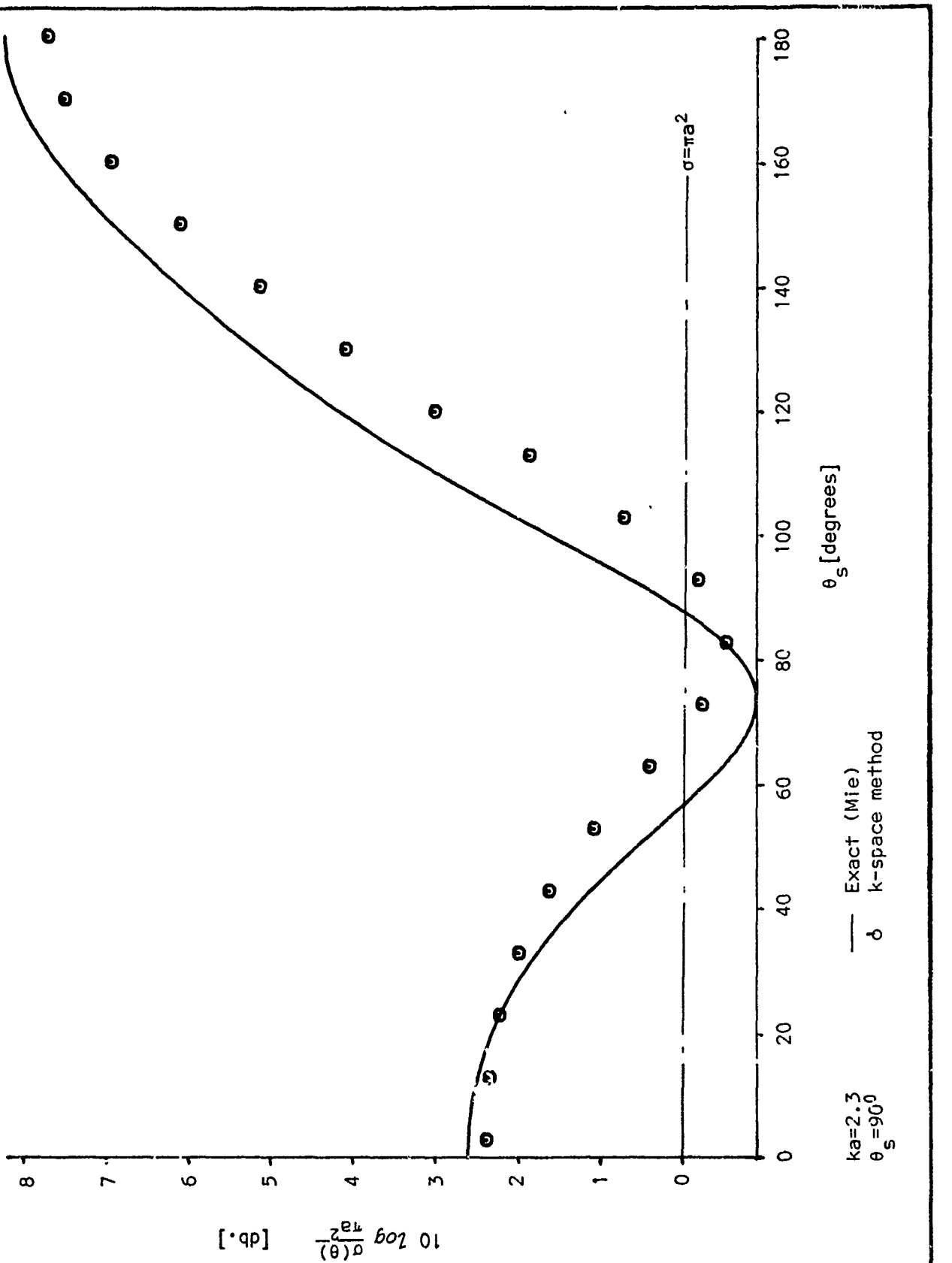

FIGURE 5 SPHERE; CROSS SECTION *vs.* ASPECT ANGLE

8. SUMMARY OF CONCLUSIONS AND RECOMMENDATIONS

The *basic feasibility and merits* of the application of the *k-space method* to *electromagnetic scattering* has essentially been demonstrated sufficiently to warrant the following conclusions and recommendations.

For *small scatterers* ($N<10^2$), the preference of the k-space method over the conventional integral equation - matrix inversion method *cannot yet be justified.* For *medium-sized scatterers* ($N\sim300$), the k-space method is indeed *more efficient* than the integral equation - matrix inversion method. For *large scatterers* ($N>10^3$), the k-space method is capable of yielding useful results in *realistic* computer time; whereas the integral equation - matrix inversion method *cannot* even be computer implemented for such sizes.

For *present-day, state-of-the-art* computer size and speed, the *size limit* for which the k-space method could be implemented is of the order of $N\sim10^7$ (*i.e.*, about 10,000 times as many data points than possible with the conventional integral equation - matrix inversion method).

For the generation of a *user-library type* computer program of the k-space method that would utilize a maximum of the basic inherent advantages and applicabilities of the method, the following effort would be needed.

1. Implement appropriately and efficiently the *generalized relaxation function* and the *generalized inverse* methods into the *k-space formulation of the electromagnetic scattering program.*
2. Develop a *completely general* computer program for reading *arbitrary shapes* and *arbitrary electromagnetic properties* into the k-space method.
3. Conduct a thorough error analysis such that *error bounds* will be available to the *user* of such a system.

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4. Develop (or implement an existing) Fast Fourier Transform (FFT) program operative on *disc-to-core-to-disc* data, which would be capable of handling of the order of 10^7 data points; *i.e.*, the state-of-the-art size limit of discs (*vis-a-vis* the presently utilized *in-core* FFT limited to about 10^4 data points).

5. Investigate the feasibility and desirability of replacing a disc-to-core-to-disc *soft-wired* (software compiled) FFT program (see item 4 above) with a *hard-wired* (hardware compiled) *Fast Fourier Analyzer* (FFA).

6. Implement a *mixed-mode* k-space formulation capable of solving *three simultaneous equations in three unknowns* (*i.e.*, the electric and magnetic fields simultaneously with the current densities), applicable simultaneously to volume, surface, and line current density representations of the electromagnetic scattering problem; thus realizing solutions for large *complex compound* scattering problems (*e.g.*, perfect conductors with absorbing materials and wire antennas).

The preceding conclusions and recommendations dealt with *monochromatic* electromagnetic scattering; the subject to which this report was addressed. It is now reasonable to conclude that the k-space method is also applicable with similar advantages to *wide band electromagnetic scattering*. This can be accomplished by a *four-dimensional* k-space formulation of the relativistic four-dimensional time-dependent Poisson equation; *vis-a-vis* the presently implemented k-space formulation of the Helmholtz equation (time-reduced wave equation). The *additional* advantage of such a formulation is that the relativistically correct Doppler shifted spectrum for arbitrary time-dependent motion (including time-dependent rotation, acceleration and deformation) would be yielded directly and efficiently.

Furthermore, the k-space method could similarly be applied with full advantage to electromagnetic problems *other* than scattering; *e.g.*, *radiation* (antennas) problems, *propagation* problems, *etc.*

Since the k-space method does *not* require the constitutive boundary equation conditions to be linear algebraic, this method becomes applicable with full advantage to (electromagnetic) *initial value problems* resulting from *interactive systems* that conventionally yield *several coupled simultaneous non-linear integro-differential equations* (*e.g.*, magneto-ionic

plasma with acoustic waves and thermodynamic coupling); whereas the conventional matrix inversion method is not applicable to such non-linear integral equations.

In conclusion, it is noteworthy that the k-space formulation is applicable with full advantage to *all the initial value problems of mathematical physics* that arise from linear field equations subject to linear or non-linear constitutive boundary condition equations; which conventionally lead to (linear or non-linear) integral equations, including multiple simultaneous such equations governing interactive systems.

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