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The Expansion of Physical Quantities in Terms of the Irreducible Representations of the Scale-Euclidean Group and Applications to the Construction of Scale-Invariant Correlation Functions

Part II: Three-Dimensional Problems; Generalizations of the Helmholtz Vector Decomposition Theorem

H.E. MOSES
A.F. QUESADA

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As in Part I of this work (Concepts: One-Dimensional Problems), the correlations that are introduced are invariant under changes of frames of reference. Correlations are set up between tensors of different ranks and dimensions. A correlation that measures a degree of isotropy is defined.

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L. G. HANSCOM FIELD, BEDFORD, MASSACHUSETTS

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Abstract

The irreducible representations of the scale-Euclidean group in three dimensions are introduced, and the general tensor is expanded in terms of these representations. The cases of zero-rank tensor (scalar), rank-1 tensor (vector), and rank-2 tensor, are studied in detail. The expansion is shown to be a generalization of the Helmholtz expansion of a vector into rotational and irrotational parts.

As in Part I of this work (Concepts: One-Dimensional Problems), the correlations that are introduced are invariant under changes of frames of reference. Correlations are set up between tensors of different ranks and dimensions. A correlation that measures a degree of isotropy is defined.

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The Expansion of Physical Quantities in Terms of the Irreducible Representations of the Scale-Euclidean Group and Applications to the Construction of Scale-Invariant Correlation Functions

Part II: Three-Dimensional Problems; Generalizations of the Helmholtz Vector Decomposition Theorem

1. INTRODUCTION. THE THREE-DIMENSIONAL SCALE-EUCLIDEAN GROUP

In Part I (Moses and Quesada), we introduced the scale-Euclidean group in one dimension and expanded physical quantities in terms of the irreducible representations of the group. The expansion was a kind of separation of variables which makes it easier to solve linear differential equations, which are invariant under the group, and to define correlations between physical quantities in such a way that they are independent of the frame of reference and units of measurement. The motivation for introducing the scale-Euclidean group is also valid for the three-dimensional scale-Euclidean group.

This paper, Part II, a direct extension of Part I, deals with the three-dimensional group. The three-dimensional group is considerably richer than the one-dimensional group. Because of its greater mathematical complexity, we give our results in the body of the paper, and present the proofs in the appendixes.

We shall be led very naturally into generalizations of Fourier transformations of physical quantities. In particular, we shall expand scalars, vectors, and tensors of rank 2, in terms of the irreducible representations of the group. We shall also give the relations between the representations that appear in the expansion of scalars, vectors, and rank-2 tensors that are obtained from one another by taking

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gradients or divergences. It will be shown that the expansion in terms of the scale-Euclidean group is a generalization of the Helmholtz decomposition of vectors into rotational and irrotational parts.

The rotation group is a subgroup of the scale-Euclidean group. Since it plays such an important role, we shall discuss the rotation group and its irreducible representations briefly before defining the scale-Euclidean group.

1.1 The Rotation Group and Its Irreducible Representations; Generalized Surface Harmonics

In a three-dimensional space, rotations can be parametrized by a vector $\underset{\sim}{\theta}$, where $\theta = |\underset{\sim}{\theta}|$ gives the angle of rotation and the unit vector $\underset{\sim}{\theta}/\theta$ gives the direction of the axis of rotation. If the frame of reference undergoes a rotation described by $\underset{\sim}{\theta}$, then the components of a vector $\underset{\sim}{x}$ given by x_i $(i = 1, 2, 3)$ in the original frame are given by x_i' in the new frame, with

$$x_i' = \sum_j R_{ij}(\underset{\sim}{\theta})x_j \,, \tag{1}$$

where $R_{ij}(\underset{\sim}{\theta})$ are the elements of the rotation matrix $R_M(\underset{\sim}{\theta})$

$$R_{ij}(\underset{\sim}{\theta}) = \delta_{ij}\cos\theta + \frac{1-\cos\theta}{\theta^2}\,\theta_i\theta_j + \sum_k \epsilon_{ijk}\theta_k\frac{\sin\theta}{\theta}. \tag{2}$$

In Eq. (2), ϵ_{ijk} is the usual antisymmetric tensor, that is,

$$\epsilon_{jik} = \epsilon_{ikj} = -\epsilon_{ijk}, \quad \epsilon_{123} = 1. \tag{2a}$$

It will often be convenient to regard the components x_i' of the vector $\underset{\sim}{x}$ in the rotated frame as the components of another vector $\underset{\sim}{x}'$ in the original frame and write Eq. (1) as a transformation in vector space:

$$\underset{\sim}{x}' = R_M(\underset{\sim}{\theta})\underset{\sim}{x}. \tag{1a}$$

The product of two rotation matrices $R_M(\underset{\sim}{\theta})$ and $R_M(\underset{\sim}{\theta}')$ is another rotation matrix $R_M(\underset{\sim}{\theta}'')$:

$$R_M(\underset{\sim}{\theta})R_M(\underset{\sim}{\theta}') = R_M(\underset{\sim}{\theta}''). \tag{3}$$

Thus, the matrices form a matrix group. The rotation group is the group with the same multiplication properties as the matrix group. We shall denote the abstract group element by $R(\underset{\sim}{\theta})$.

The irreducible representations of the rotation group have been studied exhaustively. (See, for example, Wigner; Edmonds). The matrix $R_M(\underset{\sim}{\theta})$ can be written:

$$R_M(\underset{\sim}{\theta}) = e^{i\underset{\sim}{\theta}\cdot\underset{\sim}{K}}, \tag{4}$$

where

$$\underset{\sim}{\theta}\cdot\underset{\sim}{K} = \sum_{i=1}^{3} \theta_i K_i, \tag{4a}$$

and the Hermitian matrices K_i (called the infinitesimal generators of the rotation matrix group) are

$$K_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5}$$

(see, for example, Moses, 1965a).
These matrices satisfy the commutation rules:

$$[K_1, K_2] = iK_3 \text{ (cyc)}. \tag{6}$$

The irreducible sets of Hermitian matrices S_i that satisfy the commutation relations (6) can be used to construct the irreducible unitary representations of the rotation group.

As is well known, each of the irreducible representations of the infinitesimal generators is labeled j, which can take on any of the values 0, 1/2, 1, 3/2, 2, ... Let $S_i^{(j)}$ be the matrices that give the irreducible representations. Then

$$S_i^{(j)} = \left\{S_i^{(j)}(m,m')\right\}, \tag{7}$$

where the quantities in braces are the matrix elements. Here the labels m, m' take on the values j, j-1, j-2, ..., -j+1, -j. On defining $S_\pm^{(j)}$ by

$$S_\pm^{(j)} = S_1^{(j)} \pm iS_2^{(j)}, \tag{8}$$

with a corresponding notation for the matrix elements, we have

$$S_3^{(j)}(m,m') = m\delta_{m,m'} ,$$

$$S_{\pm}^{(j)}(m,m') = \left[(j\mp m')(j\pm m'+1)\right]^{1/2}\delta_{m,m'\pm 1}. \tag{9}$$

The irreducible representations are unique within a unitary transformation.

The matrices that give the irreducible unitary representations of the rotation group element $R(\underset{\sim}{\theta})$ are denoted by $R^{(j)}(\underset{\sim}{\theta})$, with

$$R^{(j)}(\underset{\sim}{\theta}) = \left\{R^{(j)}(\underset{\sim}{\theta})\ (m,m')\right\}, \tag{10}$$

and

$$R^{(j)}(\underset{\sim}{\theta}) = \exp\left[i\underset{\sim}{\theta}\cdot\underset{\sim}{S}^{(j)}\right]. \tag{11}$$

The matrix elements $R^{(j)}(\underset{\sim}{\theta})(m,m')$ have been given explicitly by Moses(1965a; 1966). For the sake of completeness, they are also given here in Appendix A (Sec. A2).

The multiplication law for the matrices $R^{(j)}(\underset{\sim}{\theta})$ corresponding to the multiplication law for the rotation matrices (Moses, 1965a) is

$$R^{(j)}(\underset{\sim}{\theta})R^{(j)}(\underset{\sim}{\theta}') = R^{(j)}(\underset{\sim}{\theta}'') \tag{3a}$$

for j an integer (the case in which we are interested); but for $j = 1/2, 3/2, 5/2, \ldots$, the mutiplication law is

$$R^{(j)}(\underset{\sim}{\theta})R^{(j)}(\underset{\sim}{\theta}') = \pm R^{(j)}(\underset{\sim}{\theta}''), \tag{3b}$$

where the sign used in (3b) depends on $\underset{\sim}{\theta}$ and $\underset{\sim}{\theta}'$. In this latter case the representation is said to be a <u>ray</u> representation.

We find it convenient to introduce generalized surface harmonics, which are closely related to the matrices that represent the rotation group in its irreducible representations. Let θ, ϕ be angular variables $0<\theta<\pi$, $0<\phi<2\pi$. The generalized surface harmonics of these variables $Y_j^{m,m'}(\theta,\phi)$ were introduced by Moses in 1967; in this paper their properties are summarized in Appendix A. The numbers j,m,m' take on the same values as in the irreducible representations of the rotation group. They are called generalized surface harmonics because $Y_j^{m,0}(\theta,\phi) = Y_{jm}(\theta,\phi)$ for integer j, where $Y_{jm}(\theta,\phi)$ are the usual surface harmonics in, say, Edmonds' (1957) notation.

1.2 The Three-Dimensional Scale-Euclidean Group

The transformations of the three-dimensional scale-Euclidean group are rotations, translations, and dilations. The rotations have already been discussed. The group element is denoted, as before, by $R(\underset{\sim}{\theta})$. The translation $T_j(a)$ along the jth

axis is defined by

$$x_i' = x_i - a\delta_{ij}, \tag{12}$$

The dilation transformation $S(\lambda)$ is defined by

$$x_i' = e^{\lambda} x_i. \tag{13}$$

The multiplication laws for the group are:

$$T_i(0) = S(0) = R(0) = I \tag{14}$$

(where I is the identity transformation),

$$T_i(a)T_i(b) = T_i(a+b); \quad S(\lambda)S(\mu) = S(\lambda+\mu), \tag{15}$$

$$R(\underset{\sim}{\theta})R(\underset{\sim}{\theta}') = R(\underset{\sim}{\theta}''). \tag{16}$$

In particular,

$$R(\underset{\sim}{\theta})R(-\underset{\sim}{\theta}) = I. \tag{16a}$$

Also,

$$T_i(a)T_j(b) = T_j(b)T_i(a), \tag{17}$$

$$S(\lambda)R(\underset{\sim}{\theta}) = R(\underset{\sim}{\theta})S(\lambda). \tag{17a}$$

Finally,

$$S(\lambda)T_i(a) = T_i(e^{\lambda}a)S(\lambda), \tag{18}$$

$$R(\underset{\sim}{\theta})T(\underset{\sim}{a}) = T(\underset{\sim}{a}')R(\underset{\sim}{\theta}), \tag{19}$$

where

$$\begin{aligned} T(\underset{\sim}{a}) &= T_1(a_1)T_2(a_2)T_3(a_3), \\ \underset{\sim}{a} &= (a_1, a_2, a_3), \\ a_i' &= \sum_j R_{ij}(\underset{\sim}{\theta})a_j. \end{aligned} \tag{19a}$$

In a representation, the group elements $T_i(a)$, $R(\underset{\sim}{\theta})$, $S(\lambda)$ are represented by operators acting in a Hilbert space. These operators we also denote by $T_i(a)$, $R(\underset{\sim}{\theta})$, and $S(\lambda)$, respectively. They should not be confused with the abstract group elements. When the representation is unitary and acts on a separable Hilbert space, we may introduce the Hermitian infinitesimal generators of the representation P_i, J_i, D by writing

$$T_j(a) = \exp\left[iaP_j\right]; \quad R(\underset{\sim}{\theta}) = \exp\left[i\underset{\sim}{\theta}\cdot\underset{\sim}{J}\right]; \quad S(\lambda) = \exp\left[i\lambda D\right]. \tag{20}$$

In Eq. (20), $\underset{\sim}{\theta}\cdot\underset{\sim}{J} = \sum_i \theta_i J_i$.

The multiplication rules for the group lead to the following commutation rules for the infinitesimal generators:

$$\left[P_i, P_j\right] = 0; \quad \left[D, J_i\right] = 0; \quad \left[P_j, D\right] = iP_j; \quad \left[P_i, J_i\right] = 0;$$

$$\left[J_1, J_2\right] = iJ_3 \text{ (cyc)}; \quad \left[P_1, J_2\right] = \left[J_1, P_2\right] = iP_3 \text{ (cyc)}. \tag{21}$$

2. TRANSFORMATIONS OF TENSOR FIELD QUANTITIES

The physical quantities that we wish to expand are r-rank tensor functions of the space variables $\underset{\sim}{x}$ and, generally, functions of the time t. The $\underset{\sim}{x}$-dependence interests us primarily and will be indicated explicitly, whereas the time-dependence will be suppressed in the notation.

Let a tensor $G(\underset{\sim}{x})$ have the components

$$G(\underset{\sim}{x}) = \left\{G_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x})\right\}. \tag{22}$$

Since the tensor is a physical quantity it will have dimensions of length to some dimension, say, N:

$$G_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x}) \sim L^N. \tag{23}$$

In Sec. 5 we go into great detail for cases where the tensor is of rank zero (the scalar case), of rank 1 (the vector case), and of rank 2. Examples of scalars and vectors are fluid densities and fluid velocities, respectively. An example of a physical quantity that is of a rank 2 tensor is a stress tensor.

We now show how tensors transform under the three-dimensional scale-Euclidean group. Under the various transformations of the group the components

$G_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x})$ in the original frame go into the components $G'_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x})$ in the new frame.

Under the transformations $T(\underset{\sim}{a})$ and $S(\lambda)$, we have

$$G'_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x}) = G_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x} + \underset{\sim}{a}) \tag{24}$$

and

$$G'_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x}) = e^{N\lambda} G_{i_1 i_2 i_3 \ldots i_r}(e^{-\lambda}\underset{\sim}{x}), \tag{25}$$

respectively. Under the rotation $R(\underset{\sim}{\theta})$,

$$G'_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x}) = \sum_{j_1 \ldots j_r} R_{i_1 j_1}(\underset{\sim}{\theta}) R_{i_2 j_2}(\underset{\sim}{\theta}) \times$$

$$\times\ R_{i_3 j_3}(\underset{\sim}{\theta}) \ldots R_{i_r j_r}(\underset{\sim}{\theta}) G_{j_1 j_2 j_3 \ldots j_r}\left[R_M(-\underset{\sim}{\theta})\underset{\sim}{x}\right]. \tag{26}$$

For general discussions it is convenient to construct a column vector $G_c(\underset{\sim}{x})$ from the tensor components $G_{i_1 i_2 i_3 \ldots i_r}(\underset{\sim}{x})$, in which the rows are labeled by taking the set of indices $i_1, i_2, i_3 \ldots i_r$ through their entire range in an obvious way. We can also introduce the matrix $\hat{R}(\underset{\sim}{\theta})$ [acting on the column vector $G_c(\underset{\sim}{x})$] which is the $\underline{r}$th direct product of the matrices $R_M(\underset{\sim}{\theta})$ appearing in Eq. (26) compatible with construction of the column vector from the tensor components. Then under the transformations $T(\underset{\sim}{a})$, $S(\lambda)$, and $R(\underset{\sim}{\theta})$ the column vector tranforms thus:

$$G'_c(\underset{\sim}{x}) = G_c(\underset{\sim}{x} + \underset{\sim}{a}), \tag{24a}$$

$$G'_c(\underset{\sim}{x}) = e^{N\lambda} G_c(e^{-\lambda}\underset{\sim}{x}), \tag{25a}$$

$$G'_c(\underset{\sim}{x}) = \hat{R}(\underset{\sim}{\theta}) G_c\left[R_M(-\underset{\sim}{\theta})\underset{\sim}{x}\right], \tag{26a}$$

respectively.

Just as in the one-dimensional case, it is clear that the space of tensors of rank r is the carrier space of a linear representation of the group.

3. IRREDUCIBLE REPRESENTATIONS OF THE THREE-DIMENSIONAL SCALE-EUCLIDEAN GROUP

We now give the irreducible unitary representations of the three-dimensional scale-Euclidean group. The first two representations are representations in the usual separable Hilbert space. The third representation uses a nonseparable Hilbert space and is derived from the second through replacement of the inner product. Using the three representations enables us to make very general expansions.

That the operators acting on the carrier spaces do indeed constitute irreducible unitary representations of the group is easily verified by direct computation. That the representations in the separable Hilbert spaces are the <u>only</u> representations in the separable spaces will be proved in Appendix B when we construct the representations from the multiplication rules of the group.

3.1 The Irreducible Rotation Group Representations

Each of these representations is characterized by an irreducible representation of the rotation group corresponding to the representation labeled j and any other real number d. Each element or vector of the carrier space consists of complex numbers $f(m)$, where m takes on the values $-j, -j+1, \ldots, j-1, j$. The inner product of the two vectors $f^{(1)}$ and f is given by

$$(f^{(1)}, f) = \sum_{m=-j}^{j} f^{(1)*}(m) f(m). \tag{27}$$

We denote the operators representing the elements of the group by $T(\underset{\sim}{a})$, $R(\underset{\sim}{\theta})$, $S(\lambda)$. The representations are then:

$$T(\underset{\sim}{a}) f(m) = f(m),$$

$$R(\underset{\sim}{\theta}) f(m) = \sum_{m'=-j}^{j} R^{(j)}(\underset{\sim}{\theta})(m, m') f(m'),$$

$$S(\lambda) f(m) = e^{i\lambda d} f(m). \tag{28}$$

The rotation $R(\underset{\sim}{\theta})$ is represented by the matrix $R^{(j)}(\theta)$, which explains the name given the representation. The translation $T(\underset{\sim}{a})$ is represented by the identity.

3.2 The Continuous Helicity Representations

These representations are characterized by a single real number α, which can take on any positive or negative integer value or zero. The Hilbert space on which the operators of the representation act is a set of complex functions $\{f(\underset{\sim}{p})\}$ defined over the entire three-dimensional $\underset{\sim}{p}$-space such that the inner product is given by

$$(f^{(1)}, f) = \int f^{(1)*}(\underset{\sim}{p}) f(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^3}, \quad (p = |\underset{\sim}{p}|). \tag{29}$$

The representation is then given by

$$\begin{aligned} T(\underset{\sim}{a}) f(\underset{\sim}{p}) &= e^{i\underset{\sim}{a}\cdot\underset{\sim}{p}} f(\underset{\sim}{p}), \\ R(\underset{\sim}{\theta}) f(\underset{\sim}{p}) &= \exp\left[2i\alpha\Phi(\underset{\sim}{\theta}, \underset{\sim}{\eta})\right] f\left(R_M(-\underset{\sim}{\theta})\underset{\sim}{p}\right), \\ S(\lambda) f(\underset{\sim}{p}) &= f(e^{\lambda}\underset{\sim}{p}), \end{aligned} \tag{30}$$

where $\Phi(\underset{\sim}{\theta}, \underset{\sim}{\eta})$ is the principal branch of

$$\tan\Phi(\underset{\sim}{\theta}, \underset{\sim}{\eta}) = \frac{(\underset{\sim}{\theta}\cdot\underset{\sim}{\eta} + \theta_3)\tan(\theta/2)}{\theta(1+\eta_3) + (\underset{\sim}{\theta}\times\underset{\sim}{\eta})_3 \tan(\theta/2)},$$

$$\underset{\sim}{\eta} = \underset{\sim}{p}/p. \tag{30a}$$

The quantity α is called the <u>helicity</u> of the representation. The representation of the rotation $R(\underset{\sim}{\theta})$ is called the helicity representation of the rotation group. Such representations are discussed in detail in Moses (1967, 1970).

It is to be noted that this Hilbert space is separable.

3.3 The Discrete Helicity Representations

The Hilbert space upon which the operators act is the set of complex functions $\{f(\underset{\sim}{p})\}$ defined over the entire three-dimensional $\underset{\sim}{p}$-space such that $f(0) = 0$ and the functions $f(\underset{\sim}{p})$ vanish at all but a denumerable set of $\underset{\sim}{p}$. The inner product of two vectors in the space is defined by

$$(f^{(1)}, f) = \sum_{\underset{\sim}{p}} f^{(1)*}(\underset{\sim}{p}) f(\underset{\sim}{p}), \tag{31}$$

where the summation is taken over all values of $\underset{\sim}{p}$ for which the summand does not vanish. This Hilbert space is thus a nonseparable space.

The operators representing the group elements act on the vectors of the Hilbert space as in Eq. (30).

4. REDUCTION OF PHYSICAL QUANTITIES IN TERMS OF THE IRREDUCIBLE UNITARY REPRESENTATIONS OF THE SCALE-EUCLIDEAN GROUP

4.1 The General Case

We now discuss the general case in which the field quantity is the column vector $G_c(\underset{\sim}{x})$ as in Eqs. (24a) to (26a). It is convenient to introduce the index γ to label the rows for $G_c(\underset{\sim}{x})$ and the rows and columns of the matrices $\hat{R}(\underset{\sim}{\theta})$:

$$G_c(\underset{\sim}{x}) = \left\{G(\underset{\sim}{x},\gamma)\right\},$$

$$\hat{R}(\underset{\sim}{\theta}) = \left\{\hat{R}(\underset{\sim}{\theta})(\gamma,\gamma')\right\}. \tag{32}$$

In terms of components, Eq. (26a) then becomes:

$$G'(\underset{\sim}{x},\gamma) = \sum_{\gamma'} \hat{R}(\underset{\sim}{\theta})(\gamma,\gamma')\, G(R(-\underset{\sim}{\theta})\underset{\sim}{x},\gamma'). \tag{26b}$$

The matrices $\hat{R}(\underset{\sim}{\theta})$ [the $\underline{r}$th direct product of the matrices $R_M(\underset{\sim}{\theta})$] constitute a reducible unitary representation of the rotation group. Accordingly, there exists a unitary matrix W that reduces $\hat{R}(\underset{\sim}{\theta})$ to the irreducible representations $R^{(j)}(\underset{\sim}{\theta})$. Let W be written in terms of its matrix elements:

$$W = \left\{W(\gamma \mid j,m,n)\right\}, \tag{33}$$

where j labels the irreducible representations, $m = -j, -j+1, \ldots, j-1, j$, and n is used as an additional label if the irreducible representation j occurs more than once. Then

$$\sum_{\gamma'} \hat{R}(\underset{\sim}{\theta})(\gamma,\gamma')\, W(\gamma' \mid j,m,n) = \sum_{j,m',n} W(\gamma \mid j,m',n)\, R^{(j)}(\underset{\sim}{\theta})(m',m),$$

$$\sum_{\gamma} W^*(\gamma \mid j,m,n)\, W(\gamma \mid j',m',n') = \delta_{j,j'}\,\delta_{m,m'}\,\delta_{n,n'},$$

$$\sum_{j,m,n} W(\gamma \mid j,m,n)\, W^*(\gamma' \mid j,m,n) = \delta_{\gamma,\gamma'} \tag{34}$$

It is useful to define the column vector $W(j,m,n)$ in terms of its components:

$$W(j,m,n) = \left\{ W(\gamma | j,m,n) \right\} . \tag{34a}$$

Let us also define the column vectors $Q(\underset{\sim}{p},j,n,\alpha)$ with components:

$$Q(\underset{\sim}{p},j,n,\alpha) = \left\{ Q(\gamma | \underset{\sim}{p},j,n,\alpha) \right\} , \tag{35}$$

$$Q(\underset{\sim}{p},j,n,\alpha) = \sum_m \left[4\pi/(2j+1) \right]^{1/2} W(j,m,n) Y_j^{m,\alpha *}(\theta,\phi) . \tag{36}$$

In Eq. (36), θ and ϕ are the polar angles of $\underset{\sim}{p}$. It is to be noted that m and α take on the values of $-j, -j+1, \ldots, j-1, j$. In consequence of the orthogonality relations between the last two lines of Eq. (34) and Eq. A14 (Appendix A), the following orthogonality and completeness relations hold:

$$\sum_\gamma Q^*(\gamma | \underset{\sim}{p},j,n,\alpha) Q(\gamma | \underset{\sim}{p},j',n',\alpha') = \delta_{j,j'} \delta_{n,n'} \delta_{\alpha,\alpha'} ,$$

$$\sum_{j,n,\alpha} Q(\gamma | \underset{\sim}{p},j,n,\alpha) Q^*(\gamma' | \underset{\sim}{p},j,n,\alpha) = \delta_{\gamma,\gamma'} . \tag{37}$$

We first write a general Fourier expansion of $G_c(\underset{\sim}{x})$ and show that by additionally expanding the Fourier amplitudes in terms of the column vectors $Q(\underset{\sim}{p},j,n,\alpha)$ we obtain an expansion in terms of the irreducible representations of the scale-Euclidean group.

A very general Fourier expansion of the column vector $G_c(\underset{\sim}{x})$, which is bounded for $|\underset{\sim}{x}| \to \infty$, is

$$G_c(\underset{\sim}{x}) = C + (2\pi)^{-3/2} \int g_c(\underset{\sim}{p}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} d\underset{\sim}{p} + \sum_n A_n \exp\left[i\underset{\sim}{k}_n \cdot \underset{\sim}{x} \right] , \tag{38}$$

where C is a constant column vector, $g_c(\underset{\sim}{p})$ is a column vector that is a function of $\underset{\sim}{p}$, A_n is a column vector for each n, and $\underset{\sim}{k}_n$ are discrete propagation vectors; Eq. (38) thus decomposes the column vector into Fourier transformed modes. The constant column vector C is the mean, the integral is the portion of the column vector that decays as $|\underset{\sim}{x}| \to \infty$, and the sum is the sum of oscillatory terms in $\underset{\sim}{x}$.

It is convenient to define the column vector $g_d(\underset{\sim}{p})$ by

$$g_d(\underset{\sim}{k}_n) = A_n ,$$

$$g_d(\underset{\sim}{p}) = 0, \text{ if } \underset{\sim}{p} \neq \text{any } \underset{\sim}{k}_n . \tag{39}$$

Then Eq. (38) becomes

$$G_c(\underset{\sim}{x}) = C + (2\pi)^{-3/2} \int g_c(\underset{\sim}{p}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} dp + \sum_{\underset{\sim}{p}} g_d(\underset{\sim}{p}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} . \tag{38a}$$

The column vectors C, g_c, g_d, can be found from $G_c(\underset{\sim}{x})$ in the following way. We note that

$$\lim_{X\to\infty} (2X)^{-3} \int_{-X}^{+X} \int_{-X}^{+X} \int_{-X}^{+X} e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} d\underset{\sim}{x} = \delta_{\underset{\sim}{p}\underset{\sim}{k}} . \tag{40}$$

Thus,

$$\lim_{X\to\infty} (2X)^{-3} \int_{-X}^{+X} \int_{-X}^{+X} \int_{-X}^{+X} e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} G_c(\underset{\sim}{x}) d\underset{\sim}{x} = g_d(\underset{\sim}{p}) . \tag{41}$$

Also,

$$\lim_{|\underset{\sim}{x}|\to\infty} \left[G_c(\underset{\sim}{x}) - \sum_{p} g_d(\underset{\sim}{p}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \right] = C . \tag{42}$$

Finally,

$$(2\pi)^{-3/2} \int \left[G_c(\underset{\sim}{x}) - \sum_{p} g_d(\underset{\sim}{p}) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} - C \right] e^{i\underset{\sim}{k}\cdot\underset{\sim}{x}} d\underset{\sim}{x} = g_c(\underset{\sim}{k}) . \tag{43}$$

We now introduce the functions $f_c^{j,n}(\underset{\sim}{p},\alpha)$ and $f_d^{j,n}(\underset{\sim}{p},\alpha)$ and the numbers $c^{j,n}(m)$

$$g_c(\underset{\sim}{p}) = (p)^{-(N+3)} \sum_{j,n,\alpha} Q(\underset{\sim}{p},j,n,\alpha) f_c^{j,n}(\underset{\sim}{p},\alpha),$$

$$g_d(\underset{\sim}{p}) = (p)^{-N} \sum_{j,n,\alpha} Q(\underset{\sim}{p},j,n,\alpha) f_d^{j,n}(\underset{\sim}{p},\alpha),$$

$$C = \sum_{j,m,n} W(j,m,n) c^{j,n}(m). \tag{44}$$

Denoting the components of $g_c(\underset{\sim}{p})$, $g_d(\underset{\sim}{p})$, and C, by $g_c(\underset{\sim}{p},\gamma)$, $g_d(\underset{\sim}{p},\gamma)$, and $C(\gamma)$, respectively, we then have, from Eqs. (34) and (37),

$$f_c^{j,n}(\underset{\sim}{p},\alpha) = p^{(N+3)} \sum_{\gamma} Q^*(\gamma \mid \underset{\sim}{p}, j,n,\alpha) g_c(\underset{\sim}{p},\gamma),$$

$$f_d^{j,n}(\underset{\sim}{p},\alpha) = p^N \sum_\gamma Q^*(\gamma \mid \underset{\sim}{p}, j, n, \alpha) g_d(\underset{\sim}{p},\gamma),$$

$$c^{j,n}(m) = \sum_\gamma W^*(\gamma \mid j, m, n) C(\gamma). \tag{44a}$$

In Eqs. (44) and (44a) it is often convenient to regard $f_c^{j,n}(\underset{\sim}{p},\alpha)$ and $f_d^{j,n}(\underset{\sim}{p},\alpha)$ as defined for α any positive or negative integer or zero but to require that $f_c^{j,n}(\underset{\sim}{p},\alpha) \equiv f_d^{j,n}(\underset{\sim}{p},\alpha) \equiv 0$ for $|\alpha| > j$. Similarly, $c^{j,n}(m)$ is considered to be defined for all integer or zero m with the requirement that $c^{j,n}(m) = 0$ when $|m| > j$. The expansion Eq. (38) now becomes

$$G_c(\underset{\sim}{x}) = \sum_{j,m,n} W(j,m,n) c^{j,n}(m) +$$

$$+ (2\pi)^{-3/2} \sum_{j,n,\alpha} \int Q(\underset{\sim}{p}, j, n, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_c^{j,n}(\underset{\sim}{p},\alpha) \frac{d\underset{\sim}{p}}{p^{N+3}} +$$

$$+ \sum_{j,n,\alpha} \sum_{\underset{\sim}{p}} Q(\underset{\sim}{p}, j, n, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d^{j,n}(\underset{\sim}{p},\alpha) p^{-N}. \tag{45}$$

When G_c undergoes transformations (24a), (25a), (26a) of the scale-Euclidean group, the functions $f_c^{j,n}(\underset{\sim}{p},\alpha)$ and $f_d^{j,n}(\underset{\sim}{p},\alpha)$ transform under the irreducible representations corresponding to the continuous and discrete helicity representations, where α is the helicity. The constants $c^{j,n}(m)$ transform under the irreducible rotation group representation corresponding to j. The quantity d of Eq. (28) is given by $d = -iN$, however. Hence, the rotation group representations, although irreducible, are not unitary because the scale transformation is not unitary. Expression (45) is thus the expansion of a physical quantity in terms of the irreducible representations of the scale-Euclidean group.

The irreducible representations are unitary if the rotation group representations are absent. For application in differential equations that are invariant under the scale-Euclidean group, it is useful to retain the rotation group representations as a constant contribution. For use in constructing invariant correlations, however, we need only unitary representations.

Since the constant (that is, $\underset{\sim}{x}$-independent) term corresponds to a mean value, we can eliminate it from the expansion by measuring G_c as a deviation from the mean value. Thus, correlations of velocities, for example, will be made for velocities from which the mean value has been subtracted so that the correlations will be invariant under the transformations of the scale-Euclidean group. Similarly, correlations can be constructed for density fluctuations, and so on.

That the expansion (45) is indeed an expansion in terms of the irreducible representations, and that it is the only such expansion (within unitarity), is the subject of Appendix C.

We now consider the expansion of scalars, vectors, and tensors of rank 2.

4.2 The Scalar Case

For scalars, the index γ takes on only one value. Hence, it need not be indicated. The rotation matrix $\hat{R}(\underset{\sim}{\theta})$ is just unity since the representation of the rotation group for the scalar is, by definition, the scalar representation. Thus,

$$\hat{R}(\underset{\sim}{\theta}) = 1 ,$$

$$W = 1 ,$$

$$Q(\underset{\sim}{p}, j, n, \alpha) = 1 ,$$

$$j = m = \alpha = 0 . \tag{46}$$

The expansion corresponding to Eq. (45) is

$$G(\underset{\sim}{x}) = c + (2\pi)^{-3/2} \int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_c(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^{N+3}} +$$

$$+ \sum_{p} e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d(\underset{\sim}{p})(p)^{-N} . \tag{47}$$

The quantity c and the amplitudes $f_c(\underset{\sim}{p})$ and $f_d(\underset{\sim}{p})$ are found as follows:

$$f_d(\underset{\sim}{p}) = p^N \lim_{X\to\infty} (2X)^{-3} \int_{-X}^{+X} \int_{-X}^{+X} \int_{-X}^{+X} e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} G(\underset{\sim}{x})\, d\underset{\sim}{x} , \tag{47a}$$

$$c = \lim_{X\to\infty} \left\{ G(\underset{\sim}{x}) - \sum_{\underset{\sim}{p}} e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d(\underset{\sim}{p}) p^{-N} \right\} , \tag{47b}$$

$$f_c(\underset{\sim}{p}) = (2\pi)^{-3/2} p^{N+3} \int e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} \left\{ G(\underset{\sim}{x}) - c - \right.$$

$$\left. - \sum_{\underset{\sim}{p}'} e^{i\underset{\sim}{p}'\cdot\underset{\sim}{x}} f_d(\underset{\sim}{p}')(p')^{-N} \right\} d\underset{\sim}{x} . \tag{47c}$$

In Eqs. (47), all superfluous labels have been omitted. It is noted that the expansion is just a generalized Fourier transformation, weighted so that the amplitudes transform properly under the scale transformations.

We can give requirements on the amplitudes that are necessary and sufficient to make $G(\underset{\sim}{x})$ real. It is readily seen that

$$c = c^*, \quad f_c(-\underset{\sim}{p}) = f_c^*(\underset{\sim}{p}), \quad f_d(-\underset{\sim}{p}) = f_d^*(\underset{\sim}{p}) . \tag{48}$$

When (48) is used in (47),

$$G(\underset{\sim}{x}) = 2\,\mathrm{Re}\left[c + (2\pi)^{-3/2} \int_+ e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_c(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^{N+3}} + \right.$$

$$\left. + \sum_{\underset{\sim}{p}}{}_+ \, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d(\underset{\sim}{p})(p)^{-N}\right] , \tag{49}$$

where the plus subscript on the integral and on the sum means that only the half of the $\underset{\sim}{p}$-space for which $p_z > 0$ is to be used. Any other half of the $\underset{\sim}{p}$-space may be used in the integration, however.

4.3 The Vector Case

In the vector case, the index γ takes on three values 1, 2, 3, which are identified with the x, y, z components of the vector. The matrix $\hat{R}(\underset{\sim}{\theta})$ is just the matrix whose components are given by Eq. (2), and is unitarily equivalent to the irreducible representation of the rotation group of Eq. (10) for which j = 1.

At this point it is convenient to introduce the unitary matrix V, which transforms the representation of Eq. (2) into that of Eq. (10). Let the components of V be denoted by V_{im}, as

$$V = \left\{ V_{im} \right\} , \tag{50}$$

where the row index i goes through the values 1, 2, 3, and the column vector index goes through the values 1, 0, -1. Explicitly,

$$V = (2)^{-1/2} \begin{pmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & -(2)^{1/2} & 0 \end{pmatrix} . \tag{51}$$

Then

$$V^{\dagger} = V^{-1}$$

and

$$K_i = V S_i^{(1)} V^{\dagger} , \tag{52}$$

where the dagger means Hermitian adjoint. Clearly,

$$W(\gamma \mid j,m,n) = V_{\gamma m} , \tag{53}$$

where j = 1, with n taking on only one value and hence disregarded.

Let us now introduce the vectors $\underset{\sim}{V}_m$, which are the column vectors of the matrix V written in vector notation:

$$\underset{\sim}{V}_m = (V_{1m}, V_{2m}, V_{3m}) . \tag{54}$$

We also introduce the vectors $\underset{\sim}{Q}(\underset{\sim}{p},\alpha)$ for $\alpha = 0, \pm 1$, which in vector form are the column vectors $\underset{\sim}{Q}(\underset{\sim}{p},j,n,\alpha)$. From Eq. (36),

$$\underset{\sim}{Q}(\underset{\sim}{p},\alpha) = (4\pi/3)^{1/2} \sum_m \underset{\sim}{V}_m Y_1^{m,\alpha *}(\theta,\phi) . \tag{55}$$

Let us denote the components of $\underset{\sim}{Q}(\underset{\sim}{p},\alpha)$ by $Q_i(\underset{\sim}{p},\alpha)$ where i = 1, 2, 3:

$$\underset{\sim}{Q}(\underset{\sim}{p},\alpha) = \left\{ Q_i(\underset{\sim}{p},\alpha) \right\} . \tag{55a}$$

Then the vectors $\underset{\sim}{Q}(\underset{\sim}{p},\alpha)$ satisfy the following completeness and orthogonality relations:

$$\underset{\sim}{Q}^*(\underset{\sim}{p},\alpha) \cdot \underset{\sim}{Q}(\underset{\sim}{p},\alpha') = \delta_{\alpha,\alpha'} ,$$

$$\sum_{\alpha} Q_i^*(\underset{\sim}{p},\alpha) Q_{i'}(p,\alpha) = \delta_{i,i'} . \tag{55b}$$

Another useful property of the vectors $\underset{\sim}{Q}(\underset{\sim}{p},\alpha)$ is

$$\underset{\sim}{p}\times\underset{\sim}{Q}(\underset{\sim}{p},\alpha) = -ip\alpha\,\underset{\sim}{Q}(\underset{\sim}{p},\alpha)\,,$$

$$\underset{\sim}{p}\cdot\underset{\sim}{Q}(\underset{\sim}{p},\alpha) = (|\alpha|-1)p\ . \tag{55c}$$

The generalized surface harmonics $Y_1^{m,\alpha}(\theta,\phi)$ are given in Appendix D. It is seen that $\underset{\sim}{Q}(\underset{\sim}{p},\alpha)$ can be given explicitly in terms of $\underset{\sim}{p}$ in the following way.

$$\underset{\sim}{Q}(\underset{\sim}{p},0) = -\frac{\underset{\sim}{p}}{p}$$

$$\underset{\sim}{Q}(\underset{\sim}{p},\alpha) = -\alpha(2)^{-1/2}\left[\frac{p_1(p_1+i\alpha p_2)}{p(p+p_3)}-1,\ \frac{p_2(p_1+i\alpha p_2)}{p(p+p_3)}-i\alpha,\ \frac{p_1+ip_2}{p}\right],$$

$$\text{for } \alpha = \pm 1\ . \tag{56}$$

When $G_c(\underset{\sim}{x})$ is written as a vector $\underset{\sim}{G}(\underset{\sim}{x})$, the expansion (45) for the vector field becomes

$$\underset{\sim}{G}(\underset{\sim}{x}) = \sum_m \underset{\sim}{V}_m c(m) + (2\pi)^{-3/2}\sum_\alpha \int \underset{\sim}{Q}(\underset{\sim}{p},\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}f_c(\underset{\sim}{p},\alpha)\frac{d\underset{\sim}{p}}{p^{N+3}} +$$

$$+\sum_\alpha\sum_{\underset{\sim}{p}} \underset{\sim}{Q}(\underset{\sim}{p},\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}f_d(\underset{\sim}{p},\alpha)(p)^{-N}\ . \tag{57}$$

The amplitudes $c(m)$, $f_c(\underset{\sim}{p},\alpha)$, and $f_d(\underset{\sim}{p},\alpha)$ can be obtained from $G(\underset{\sim}{x})$ as follows:

$$f_d(\underset{\sim}{p},\alpha) = p^N \lim_{X\to\infty}(2X)^{-3}\int_{-X}^{+X}\int_{-X}^{+X}\int_{-X}^{+X} e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\underset{\sim}{Q}^*(\underset{\sim}{p},\alpha)\cdot\underset{\sim}{G}(\underset{\sim}{x})d\underset{\sim}{x}, \tag{57a}$$

$$c(m) = \underset{\sim}{V}_m^*\cdot\lim_{|\underset{\sim}{x}|\to\infty}\left\{\underset{\sim}{G}(\underset{\sim}{x}) - \sum_\alpha\sum_{\underset{\sim}{p}}\underset{\sim}{Q}(\underset{\sim}{p},\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}f_d(\underset{\sim}{p},\alpha)p^{-N}\right\}, \tag{57b}$$

$$f_c(\underset{\sim}{p},\alpha) = (2\pi)^{-3/2}p^{N+3}\int e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\underset{\sim}{Q}^*(\underset{\sim}{p},\alpha)\cdot$$

$$\cdot\left\{\underset{\sim}{G}(\underset{\sim}{x}) - \sum_m \underset{\sim}{V}_m c(m) - \sum_{\alpha'}\sum_{\underset{\sim}{p}'}\underset{\sim}{Q}(\underset{\sim}{p}',\alpha')e^{i\underset{\sim}{p}'\cdot\underset{\sim}{x}}f_d(\underset{\sim}{p}',\alpha')(p')^{-N}\right\}d\underset{\sim}{x}\ . \tag{57c}$$

We now discuss the necessary and sufficient conditions on the irreducible representations for $\underset{\sim}{G}(\underset{\sim}{x})$ to be real. We note :

$$\underset{\sim}{V}_m^* = (-1)^m \underset{\sim}{V}_{-m} ,$$

$$\underset{\sim}{Q}^*(-\underset{\sim}{p}, \alpha) = - \frac{p_1 - i\alpha p_2}{p_1 + i\alpha p_2} \underset{\sim}{Q}(\underset{\sim}{p}, \alpha) . \tag{58}$$

Equations (58) lead to the following necessary and sufficient conditions for the reality of $\underset{\sim}{G}(\underset{\sim}{x})$:

$$c(-m) = (-1)^m c^*(m) ,$$

$$f_c(-\underset{\sim}{p}, \alpha) = - \frac{p_1 - i\alpha p_2}{p_1 + i\alpha p_2} f_c^*(\underset{\sim}{p}, \alpha) ,$$

$$f_d(-\underset{\sim}{p}, \alpha) = - \frac{p_1 - i\alpha p_2}{p_1 + i\alpha p_2} f_d^*(\underset{\sim}{p}, \alpha) . \tag{59}$$

Then

$$\underset{\sim}{G}(\underset{\sim}{x}) = \underset{\sim}{V}_0 c(0) + 2\,\mathrm{Re}\,\Big[\underset{\sim}{V}_1 c(1) + (2\pi)^{-3/2} \times$$

$$\times \sum_{\alpha} \int_{+} \underset{\sim}{Q}(\underset{\sim}{p}, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_c(\underset{\sim}{p}, \alpha) \frac{d\underset{\sim}{p}}{p^{N+3}} +$$

$$+ \sum_{\alpha} \sum_{p}{}_{+} \underset{\sim}{Q}(\underset{\sim}{p}, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d(\underset{\sim}{p}, \alpha)(p)^{-N}\Big] . \tag{60}$$

We now return to the general expansion (57). We want to show that the decomposition of the $\underset{\sim}{x}$-dependent part into a sum over the helicity variable α is a generalization of the well-known Helmholtz decomposition of a vector into a rotational and irrotational part.

After defining $\underset{\sim}{G}(\underset{\sim}{x}, \alpha)$ by

$$\underset{\sim}{G}(\underset{\sim}{x}, \alpha) = (2\pi)^{-3/2} \int \underset{\sim}{Q}(\underset{\sim}{p}, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_c(\underset{\sim}{p}, \alpha) \frac{d\underset{\sim}{p}}{p^{N+3}} +$$

$$+ \sum_{\underset{\sim}{p}} \underset{\sim}{Q}(\underset{\sim}{p}, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d(\underset{\sim}{p}, \alpha)(p)^{-N} \tag{61}$$

so that

$$\underset{\sim}{G}(\underset{\sim}{x}) = \sum_m \underset{\sim}{V}_m c(m) + \sum_\alpha \underset{\sim}{G}(\underset{\sim}{x}, \alpha) , \tag{61a}$$

we use (55c) and have, by explicit calculation,

$$\underset{\sim}{\nabla} \times \underset{\sim}{G}(\underset{\sim}{x}, 0) = 0 ,$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{G}(\underset{\sim}{x}, \alpha) = 0 , \text{ for } \alpha = \pm 1 . \tag{62}$$

Except for a constant term, Eq. (61a) is an expansion of a vector field into rotational and irrotational fields as in the Helmholtz theorem. The theorem is sharpened, however, in that there are two rotational fields corresponding to $\alpha = \pm 1$. Thus the expansion of the vector field into the irreducible representations of the scale-Euclidean group leads to a generalization of the Helmholtz Theorem. Moses (1971) approached the generalized Helmholtz theorem primarily from the rotation point of view, and applied it to a variety of physical problems. In that paper (1971) he introduced spherical coordinates but did not use the dilation operator.

The general expansion of a physical quantity [Eq. (45)] may be regarded as a still further generalization of the Helmholtz decomposition theorem for vectors.

4.4 The Case of Tensors of Rank 2

The quantities $W(\gamma | j, m, n)$ of Eq. (34) will now be written $W_{ik}(j, m)$. The variable n takes on only one value for each j and need not be indicated. From the discussion of the general case, it is seen that

$$W_{ik}(j, m) = \sum_{q, q'} V_{iq} V_{kq'} (1, q, 1, q' | 1, 1, j, m) , \tag{63}$$

where V_{iq} are the matrix elements of the matrix V [Eq. (50) and (51)] and $(1, q, 1, q' | 1, 1, j, m)$ is a Clebsch-Gordan coefficient in a standard notation for the rotation group.

We find it convenient to regard the quantities $W_{ik}(j, m)$ as matrix elements of a matrix $W_M(j, m)$:

$$W_M(j, m) = \left\{ W_{ik}(j, m) \right\} , \tag{64}$$

and to regard the components $G_{ik}(\underset{\sim}{x})$ of the tensor of rank 2 as components of matrix $G_M(\underset{\sim}{x})$:

$$G_M(\underset{\sim}{x}) = \left\{G_{ik}(\underset{\sim}{x})\right\} . \tag{65}$$

From Eq. (63),

$$W_M(0,0) = -(3)^{-1/2} I_M , \tag{66}$$

where I_M is the 3×3 identity matrix ;

$$W_M(1,0) = -(2)^{-1/2} K_3 ,$$

$$W_M(1,m) = \frac{m}{2}(K_1 + i\,m\,K_2), \quad \text{for } m = \pm 1 , \tag{67}$$

where K_i are the matrices of Eq. (5).

Finally,

$$W_M(2,0) = (6)^{-1/2} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix} ,$$

$$W_M(2,m) = -\frac{m}{2} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & im \\ 1 & im & 0 \end{pmatrix} , \quad \text{for } m = \pm 1 ,$$

$$W_M(2,2) = W_M^*(2,-2) = \frac{1}{2} \begin{pmatrix} 1 & i & 0 \\ i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , \tag{68}$$

We note that $W_M(0,0)$ is proportional to the identity matrix, $W_M(1,m)$ are antisymmetric matrices, and $W_M(2,m)$ are symmetric matrices. Thus,

$$W_M^t(1,m) = -W_M(1,m) ,$$

$$W_M^t(2,m) = W_M(2,m) , \tag{69}$$

where the superscript $\underline{t}$ means <u>transpose.</u> Furthermore, the traces of the matrices are :

$$\mathrm{tr}\, W_M(0,0) = -(3)^{1/2} ,$$

$$\mathrm{tr}\, W_M(j,m) = 0 , \quad \text{for } j = 1,2 . \tag{70}$$

The matrices $W_M(j,m)$ satisfy the orthogonality and completeness relations :

$$\mathrm{tr}\, W_M^{\dagger}(j,m) W_M(j',m') = \delta_{j,j'} \delta_{m,m'} ,$$

$$\sum_{j,m} W_{ik}^{*}(j,m) W_{i'k'}(j,m) = \delta_{i,i'} \delta_{k,k'} . \tag{71}$$

In the first of Eqs. (71) matrix multiplication is meant and the dagger means Hermitian adjoint.

The quantity $Q(\gamma \mid \underset{\sim}{p}, j, n, \alpha)$ of the general case is now denoted by $Q_{ik}(\underset{\sim}{p}, j, \alpha)$. The matrix $Q_M(\underset{\sim}{p}, j, \alpha)$ is defined as having the components $Q_{ik}(\underset{\sim}{p}, j, \alpha)$:

$$Q_M(\underset{\sim}{p}, j, \alpha) = \left\{ Q_{ik}(\underset{\sim}{p}, j, \alpha) \right\}$$

$$= \sum_m \left[4\pi/(2j+1) \right]^{1/2} W_M(j,m) Y_j^{m,\alpha}(\theta, \phi) . \tag{72}$$

The matrices $Q_M(\underset{\sim}{p}, j, \alpha)$ satisfy the orthogonality and completeness relations :

$$\mathrm{tr}\, Q_M^{\dagger}(\underset{\sim}{p}, j, \alpha) Q_M(\underset{\sim}{p}, j', \alpha') = \delta_{j,j'} \delta_{\alpha,\alpha'} ,$$

$$\sum_{j,\alpha} Q_{ik}^{*}(\underset{\sim}{p}, j, \alpha) Q_{i'k'}(\underset{\sim}{p}, j, \alpha) = \delta_{i,i'} \delta_{k,k'} . \tag{73}$$

Also,

$$Q_M(\underset{\sim}{p}, 0, 0) = -(3)^{-1/2} I_M ,$$

$$\mathrm{tr}\, Q_M(\underset{\sim}{p}, 0, 0) = -(3)^{1/2} ,$$

$$\mathrm{tr}\, Q_M(\underset{\sim}{p}, 1, \alpha) = \mathrm{tr}\, Q_M(\underset{\sim}{p}, 2, \alpha) = 0 ,$$

$$Q_M^{t}(\underset{\sim}{p}, 1, \alpha) = -Q_M(\underset{\sim}{p}, 1, \alpha) ,$$

$$Q_M^{t}(\underset{\sim}{p}, 2, \alpha) = Q_M(\underset{\sim}{p}, 2, \alpha) . \tag{74}$$

The expansion of the matrix $G_M(\underset{\sim}{x})$ is

$$G_M(\underset{\sim}{x}) = \sum_{j,m} W_M(j,m)c^j(m) + (2\pi)^{-3/2} \sum_{j,\alpha} \int Q_M(\underset{\sim}{p},j,\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \times$$

$$\times\ f_c^j(\underset{\sim}{p},\alpha)\frac{d\underset{\sim}{p}}{p^{N+3}} + \sum_{j,\alpha}\sum_{\underset{\sim}{p}} Q_M(\underset{\sim}{p},j,\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d^j(\underset{\sim}{p},\alpha)p^{-N}. \tag{75}$$

The amplitudes in the expansion are obtained as follows :

$$f_d^j(\underset{\sim}{p},\alpha) = p^N \lim_{X\to\infty}(2X)^{-3}\int_{-X}^{+X}\int_{-X}^{+X}\int_{-X}^{+X} e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}} \times$$

$$\times\ \mathrm{tr}\, Q_M^\dagger(\underset{\sim}{p},j,\alpha)G_M(\underset{\sim}{x})d\underset{\sim}{x}, \tag{75a}$$

$$c^j(m) = \mathrm{tr}\, W_M^\dagger(j,m) \lim_{|\underset{\sim}{x}|\to\infty}\Big\{G_M(\underset{\sim}{x}) -$$

$$- \sum_{j,\alpha}\sum_{\underset{\sim}{p}} Q_M(\underset{\sim}{p},j,\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d^j(\underset{\sim}{p},\alpha)p^{-N}\Big\}, \tag{75b}$$

$$f_c^j(\underset{\sim}{p},\alpha) = (2\pi)^{-3/2}p^{N+3}\int e^{-i\underset{\sim}{p}\cdot\underset{\sim}{x}}\,\mathrm{tr}\, Q_M^\dagger(\underset{\sim}{p},j,\alpha) \times$$

$$\times\Big\{G_M(\underset{\sim}{x}) - \sum_{j',m} W_M(j',m)c^{j'}(m) - \sum_{j',\alpha'}\sum_{p'} Q_M(\underset{\sim}{p}',j',\alpha') \times$$

$$\times\ e^{i\underset{\sim}{p}'\cdot\underset{\sim}{x}} f_d^{j'}(\underset{\sim}{p}',\alpha')(p')^{-N}\Big\}\, d\underset{\sim}{x}. \tag{75c}$$

The expansion (75) may be written

$$G_M(\underset{\sim}{x}) = \sum_j G_M^j(\underset{\sim}{x}), \tag{76}$$

where

$$G_M^j(\underset{\sim}{x}) = \sum_m W_M(j,m)c^j(m) + (2\pi)^{-3/2}\sum_\alpha \int Q_M(\underset{\sim}{p},j,\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \times$$

$$\times\ f_c^j(\underset{\sim}{p},\alpha)\frac{d\underset{\sim}{p}}{p^{N+3}} + \sum_\alpha\sum_p Q_M(\underset{\sim}{p},j,\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d^j(\underset{\sim}{p},\alpha)p^{-N}. \tag{76a}$$

It is well known that a tensor of arbitrary rank 2 can be written as the sum of a tensor that is proportional to the unit tensor, an antisymmetric tensor, and a symmetric tensor of zero trace. In terms of the matrix notation for the rank-2 tensor we have, in fact,

$$G_M(\underset{\sim}{x}) = G_{tr}(\underset{\sim}{x}) + G_A(\underset{\sim}{x}) + G_S(\underset{\sim}{x}) , \tag{76b}$$

where the matrices G_{tr}, G_A, and G_S are given by

$$G_{tr}(\underset{\sim}{x}) = \left[\mathrm{tr}\, G_M(\underset{\sim}{x})/3\right] I_M ,$$

$$G_A(\underset{\sim}{x}) = \frac{1}{2}\left[G_M(\underset{\sim}{x}) - G_M^t(\underset{\sim}{x})\right] ,$$

$$G_S(\underset{\sim}{x}) = \frac{1}{2}\left(G_M(\underset{\sim}{x}) + G_M^t(\underset{\sim}{x}) - \frac{2}{3}\left[\mathrm{tr}\, G_M(\underset{\sim}{x})\right] I_M\right) . \tag{76c}$$

It is seen that Eqs. (76) and (76a) represent generalizations of the decompositions (76b) and (76c). Furthermore,

$$G_M^0(\underset{\sim}{x}) = G_{tr}(\underset{\sim}{x}), \quad G_M^1(\underset{\sim}{x}) = G_A(\underset{\sim}{x}), \quad G_M^2(\underset{\sim}{x}) = G_S(\underset{\sim}{x}) . \tag{76d}$$

From Eqs. (76) and (76a) it follows that

$$G_M^0(\underset{\sim}{x}) = -(3)^{-1/2} I_M G(\underset{\sim}{x}) , \tag{77}$$

where

$$G(\underset{\sim}{x}) = c + (2\pi)^{-3/2} \int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_c(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^{N+3}} + \sum_{\underset{\sim}{p}} e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d(\underset{\sim}{p}) p^{-N} . \tag{77a}$$

In Eq. (77a), $c \equiv c^0(0)$, $\quad f_c(\underset{\sim}{p}) \equiv f_c^0(\underset{\sim}{p}, 0)$, $\quad f_d(\underset{\sim}{p}) \equiv f_d^0(\underset{\sim}{p}, 0)$.

From Eq. (77) it is readily seen that when $G_M^0(\underset{\sim}{x})$ transforms under the rank-2 representation of the scale-Euclidean group, $G(\underset{\sim}{x})$ transforms under the scalar representation of the group. The converse is also true. Moreover, Eq. (77a) is just the expansion (47) of a scalar in the irreducible representations of the group. Hence, the discussion of $G_M^j(\underset{\sim}{x})$ for $j = 0$ is equivalent to the discussion of the

scalar case.

It should be noted that

$$G(\underset{\sim}{x}) = -(3)^{-1/2}\,\mathrm{tr}\,G_M(\underset{\sim}{x})\,. \tag{77a}$$

We now show that the decomposition of the antisymmetric tensor $G_M^1(\underset{\sim}{x})$ into the irreducible representations of the group is entirely equivalent to the decomposition of the vector representation. By explicit computation,

$$W_{23}(1,m) = -i(2)^{-1/2}V_{1m}\,,$$

$$W_{31}(1,m) = -i(2)^{-1/2}V_{2m}\,,$$

$$W_{12}(1,m) = -i(2)^{-1/2}V_{3m}\,. \tag{78}$$

In Eq. (78), V_{im} is the matrix element of Eq. (51). Since the matrices $W_M(1,m)$ are antisymmetric matrices, the components of the matrices that appear in Eq. (78) are the only nonzero ones.

From Eqs. (72) and (55) it follows that

$$Q_{23}(\underset{\sim}{p},1,\alpha) = -i(2)^{-1/2}Q_1(\underset{\sim}{p},\alpha)\,,$$

$$Q_{31}(\underset{\sim}{p},1,\alpha) = -i(2)^{-1/2}Q_2(\underset{\sim}{p},\alpha)\,,$$

$$Q_{12}(\underset{\sim}{p},1,\alpha) = -i(2)^{-1/2}Q_3(\underset{\sim}{p},\alpha)\,. \tag{79}$$

Again, because of the antisymmetry of the matrices $Q_M(\underset{\sim}{p},1,\alpha)$, the only nonzero independent components are those that appear in Eq. (79).

Now, a necessary and sufficient condition that $G_M^1(\underset{\sim}{x})$ be antisymmetric is that there exist functions $G_i(\underset{\sim}{x})$ such that

$$G_M^1(\underset{\sim}{x}) = -i(2)^{-1/2}\begin{pmatrix} 0 & G_3(\underset{\sim}{x}) & -G_2(\underset{\sim}{x}) \\ -G_3(\underset{\sim}{x}) & 0 & G_1(\underset{\sim}{x}) \\ G_2(\underset{\sim}{x}) & -G_1(\underset{\sim}{x}) & 0 \end{pmatrix}\,, \tag{80}$$

Let us define $\underset{\sim}{G}(\underset{\sim}{x})$ by

$$\underset{\sim}{G}(\underset{\sim}{x}) = \left\{ G_1(\underset{\sim}{x}), G_2(\underset{\sim}{x}), G_3(\underset{\sim}{x}) \right\} . \tag{81}$$

From Eqs. (76a), (78), and (79),

$$\underset{\sim}{G}(\underset{\sim}{x}) = \sum_m \underset{\sim}{V}_m c^1(m) + (2\pi)^{-3/2} \sum_\alpha \int \underset{\sim}{Q}(\underset{\sim}{p}, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_c^1(\underset{\sim}{p}, \alpha) \frac{d\underset{\sim}{p}}{p^{N+3}} +$$

$$+ \sum_\alpha \sum_p \underset{\sim}{Q}(\underset{\sim}{p}, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f_d^1(\underset{\sim}{p}, \alpha) p^{-N} . \tag{82}$$

A necessary and sufficient condition that the antisymmetric tensor $G_M^1(\underset{\sim}{x})$ transform under the rank-2 representation of the scale-Euclidean group is that $\underset{\sim}{G}(\underset{\sim}{x})$ transform under the vector representation. The expansion (82) is just Eq. (57), that is, the expansion of a vector in the irreducible representations of the group. Thus, the expansion of the general rank-2 tensor includes as special cases the expansion of scalar and vector functions.

The remaining tensor $G_M^2(\underset{\sim}{x})$ of the expansion (76) is symmetric and has zero trace. The matrices $Q_M(\underset{\sim}{p}, 2, \alpha)$ can be obtained from Eq. (72) using the explicit forms of $Y_2^{m,\alpha}(\theta, \phi)$ as given in Appendix D.

For convenience we define

$$\sigma = \operatorname{sgn} \alpha ,$$

$$\underset{\sim}{\eta} = \underset{\sim}{p}/p ,$$

$$\eta_\sigma = \eta_1 + i\sigma\eta_2 . \tag{83}$$

$$Q_M(\underset{\sim}{p}, 2, 0) = -(6)^{-1/2} \begin{pmatrix} 1-3\eta_1^2 & -3\eta_1\eta_2 & -3\eta_1\eta_3 \\ -3\eta_1\eta_2 & 1-3\eta_2^2 & -3\eta_2\eta_3 \\ -3\eta_1\eta_3 & -3\eta_2\eta_3 & 1-3\eta_3^2 \end{pmatrix} . \tag{84}$$

$$Q_M(\underset{\sim}{p},2,\alpha) = -\frac{\sigma\,\eta_\sigma}{1-\eta_3} \times$$

$$\times \begin{pmatrix} \eta_1\left(\eta_1 - \frac{\eta_\sigma}{1+\eta_3}\right) & \eta_1\eta_2 + i\sigma\frac{\eta_\sigma^{\,2}}{2(1+\eta_3)} & -\frac{1}{2}\left[\eta_1(1-2\eta_3) + \frac{\eta_3\eta_\sigma}{1+\eta_3}\right] \\ \eta_1\eta_2 + i\sigma\frac{\eta_\sigma^{\,2}}{2(1+\eta_3)} & \eta_2\left(\eta_2 + i\sigma\frac{\eta_\sigma}{1+\eta_3}\right) & -\frac{1}{2}\left[\eta_2(1-2\eta_3) - i\sigma\frac{\eta_3\eta_\sigma}{1+\eta_3}\right] \\ -\frac{1}{2}\left[\eta_1(1-2\eta_3) + \frac{\eta_3\eta_\sigma}{1+\eta_3}\right] & -\frac{1}{2}\left[\eta_2(1-2\eta_3) - i\sigma\frac{\eta_3\eta_\sigma}{1+\eta_3}\right] & -(1-\eta_3)\eta_3 \end{pmatrix}$$

for $\alpha = \pm 1$. (85)

The independent elements of $Q_M(\underset{\sim}{p},2,\alpha)$ for $\alpha = \pm 2$ are the following :

$$Q_{11}(\underset{\sim}{p},2,\alpha) = \frac{1}{2}\frac{\eta_\sigma^{\,2}}{(1-\eta_3)^2}\left(\eta_1 - \frac{\eta_\sigma}{1+\eta_3}\right)^2 ,$$

$$Q_{22}(\underset{\sim}{p},2,\alpha) = \frac{1}{2}\frac{\eta_\sigma^{\,2}}{(1-\eta_3)^2}\left(\eta_2 + i\sigma\frac{\eta_\sigma}{1+\eta_3}\right)^2 ,$$

$$Q_{33}(\underset{\sim}{p},2,\alpha) = \frac{1}{2}\frac{\eta_\sigma^{\,2}}{(1-\eta_3)^2}(1-\eta_3)^2 ,$$

$$Q_{12}(\underset{\sim}{p},2,\alpha) = \frac{1}{2}\frac{\eta_\sigma^{\,2}}{(1-\eta_3)^2}\left(\eta_1 - \frac{\eta_\sigma}{1+\eta_3}\right)\left(\eta_2 + i\sigma\frac{\eta_\sigma}{1+\eta_3}\right),$$

$$Q_{13}(\underset{\sim}{p},2,\alpha) = -\frac{1}{2}\frac{\eta_\sigma^{\,2}}{(1-\eta_3)^2}\left(\eta_1 - \frac{\eta_\sigma}{1+\eta_3}\right)(1-\eta_3) ,$$

$$Q_{23}(\underset{\sim}{p},2,\alpha) = -\frac{1}{2}\frac{\eta_\sigma^{\,2}}{(1-\eta_3)^2}\left(\eta_2 + i\sigma\frac{\eta_\sigma}{1+\eta_3}\right)(1-\eta_3) . \qquad (86)$$

We now discuss the case where the tensor is real. Since $G_M^0(\underset{\sim}{x})$ and $G_M^1(\underset{\sim}{x})$ are equivalent to the scalar and vector representations, which already have been discussed, we need concern ourselves only with $G_M^2(\underset{\sim}{x})$. From Eq. (68),

$$W_M^*(2,m) = (-1)^m W_M(2,-m) . \tag{87}$$

Furthermore, from Eqs. (87), (72), and (A16),

$$Q_M^*(-\underset{\sim}{p},2,\alpha) = \frac{p_1 - i\alpha p_2}{p_1 + i\alpha p_2} Q_M(\underset{\sim}{p},2,\alpha) . \tag{88}$$

For $G_M^2(x)$ to be real, we have

$$c^2(-m) = (-1)^m \left[c^2(m)\right]^* ,$$

$$f_c^2(-\underset{\sim}{p},\alpha) = \frac{p_1 - i\alpha p_2}{p_1 + i\alpha p_2} \left[f_c^2(\underset{\sim}{p},\alpha)\right]^* ,$$

$$f_d^2(-\underset{\sim}{p},\alpha) = \frac{p_1 - i\alpha p_2}{p_1 + i\alpha p_2} \left[f_d^2(\underset{\sim}{p},\alpha)\right]^* . \tag{89}$$

For real $G_M^2(\underset{\sim}{x})$ we can then write

$$G_M^2(\underset{\sim}{x}) = W_M(2,0)c^2(0) + 2\,\mathrm{Re}\Big[W_M(2,1)c^2(1) + W_M(2,2)c^2(2) + (2\pi)^{-3/2}\sum_\alpha \int_+ Q_M(\underset{\sim}{p},2,\alpha)e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \times f_c^2(\underset{\sim}{p},\alpha)\frac{d\underset{\sim}{p}}{p^{N+3}} + \sum_\alpha \sum_{\underset{\sim}{p}_+} Q_M(\underset{\sim}{p},2,\alpha) f_d^2(\underset{\sim}{p},\alpha)(p)^{-N}\Big] . \tag{90}$$

5. RELATIONSHIP OF THE IRREDUCIBLE REPRESENTATIONS IN THE EXPANSIONS OF SCALARS, VECTORS, AND RANK-2 TENSORS, OBTAINED FROM ONE ANOTHER THROUGH DIFFERENTIATION: SOLUTION OF THE FUNDAMENTAL EQUATIONS

5.1 Introduction

The gradient of a scalar yields a vector, and it is useful to know the relationship between the irreducible representations of the scale-Euclidean group that appear in the expansion of the scalar and those that appear in the expansion of the vector. The divergence of a rank-2 tensor with respect to one of its indices also leads to a vector, and it is also useful to relate the irreducible representations appearing in the expansion of the tensor and those of the vector.

More generally, divergences and gradients of physical quantities lead to quantities that have different transformation properties, and it is useful to know the relationship between the expansions of the original quantities and these obtained through the use of the derivative operators. The usefulness is particularly apparent when it becomes necessary to solve invariant linear partial differential equations such as occur in electrodynamics or linearized fluid mechanics, where gradient, divergence, and curl operators appear. When the expansions for the physical quantities are introduced, the differential operators modify the amplitudes in what is seen to be a simple way. The differential equation then becomes a relatively simple equation for the amplitudes.

To obtain the vector whose divergence is a given scalar, we obtain a relationship between the amplitudes of the vector and the scalar. We may in fact solve for the amplitudes occurring in the expansion of the vector in terms of those for the scalar. Effectively, this is to solve a differential equation for the vector in terms of its divergence. Similarly, differential relationships between tensors and vectors can be considered as differential equations which are solved by means of the expansions in terms of the irreducible representations.

For simplicity, we assume that the $\underset{\sim}{x}$-independent term (that is, the mean) is zero and also that there are no periodic terms. In terms of the general expansion (45) we take

$$c^{j,n}(m) = 0, \quad f_d^{j,n}(\underset{\sim}{p}, a) = 0 \tag{91}$$

We then have the condition $\lim_{|\underset{\sim}{x}| \to \infty} G_c(\underset{\sim}{x}) = 0$. This case is of particular interest when it is necessary to solve differential equations in which the solutions have finite 'energy.' The more general case is also treatable and uses a straightforward extension of the techniques that follow.

5.2 Vector-Scalar Relations; Scalar Potentials

Let $\underset{\sim}{u}(\underset{\sim}{x})$ be a vector of dimensions L^N. Under the conditions discussed in Sec. 5.1, it will have the expansion

$$\underset{\sim}{u}(\underset{\sim}{x}) = (2\pi)^{-3/2} \sum_{\alpha} \int \underset{\sim}{Q}(\underset{\sim}{p},\alpha)\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f(\underset{\sim}{p},\alpha) \frac{d\underset{\sim}{p}}{p^{N+3}} . \tag{92}$$

Let us define the scalar $\rho(\underset{\sim}{x})$ by

$$\rho(\underset{\sim}{x}) = \underset{\sim}{\nabla}\cdot\underset{\sim}{u}(\underset{\sim}{x}) . \tag{93}$$

The scalar $\rho(\underset{\sim}{x})$ will have the dimension L^{N-1}, and its expansion is thus

$$\rho(\underset{\sim}{x}) = (2\pi)^{-3/2} \int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} g(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^{N+2}} . \tag{94}$$

Our objective is to find the relationship between the amplitudes $g(\underset{\sim}{p})$ and $f(\underset{\sim}{p},\alpha)$. From Eqs. (92) and (93) and the second of Eqs. (55$\underset{\sim}{c}$),

$$\underset{\sim}{\nabla}\cdot\underset{\sim}{u}(\underset{\sim}{x}) = (2\pi)^{-3/2} i \sum_{\alpha} \int \left[\underset{\sim}{p}\cdot\underset{\sim}{Q}(\underset{\sim}{p},\alpha)\right] e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f(\underset{\sim}{p},\alpha) \frac{d\underset{\sim}{p}}{p^{N+3}}$$

$$= (2\pi)^{-3/2} (-i) \int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f(\underset{\sim}{p},0) \frac{d\underset{\sim}{p}}{p^{N+2}} . \tag{95}$$

From Eqs. (93), (94), and (95) we obtain

$$g(\underset{\sim}{p}) = -\, i f(\underset{\sim}{p},0) . \tag{96}$$

Equation (96) is the result we were seeking. It should be noted that the two amplitudes are proportional, the constant of proportionality being independent of $\underset{\sim}{p}$ and hence a pure number. The simplicity of the result is due to our being careful to use the proper expansion for the vector and the scalar. It is also to be observed that Eq. (96) can be used to solve the differential Eq. (93) for $\underset{\sim}{u}(\underset{\sim}{x})$ when $\rho(\underset{\sim}{x})$ is given; however, $\underset{\sim}{u}(\underset{\sim}{x})$ is not unique since $f(\underset{\sim}{p}, \pm 1)$ are arbitrary.

Let us now consider the case where the vector $\underset{\sim}{u}(\underset{\sim}{x})$ is obtained as the gradient of a scalar $\phi(\underset{\sim}{x})$, that is,

$$\underset{\sim}{u}(\underset{\sim}{x}) = \underset{\sim}{\nabla}\phi(\underset{\sim}{x}) . \tag{97}$$

As before, we take $\underset{\sim}{u}(\underset{\sim}{x})$ to have the dimensions L^N and the expansion (92). But now $\phi(\underset{\sim}{x})$ has the dimensions L^{N+1}, and its expansion is

$$\phi(\underset{\sim}{x}) = (2\pi)^{-3/2} \int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} h(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^{N+4}} . \tag{98}$$

From the first of Eqs. (56),

$$\underset{\sim}{\nabla}\phi(\underset{\sim}{x}) = (2\pi)^{-3/2} i \int \underset{\sim}{p} e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} h(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^{N+4}}$$

$$= - (2\pi)^{-3/2} i \int \underset{\sim}{Q}(\underset{\sim}{p}, 0) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} h(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^{N+3}} . \tag{99}$$

Hence, the desired relation between the amplitudes is

$$f(\underset{\sim}{p}, \alpha) \equiv 0, \quad \text{for } \alpha = \pm 1$$

$$f(\underset{\sim}{p}, 0) = -ih(\underset{\sim}{p}) . \tag{100}$$

From the first of Eqs. (55c) it is clear that

$$\underset{\sim}{\nabla}\times\underset{\sim}{u}(\underset{\sim}{x}) = 0 , \tag{101}$$

because generally

$$\underset{\sim}{\nabla}\times\underset{\sim}{u}(\underset{\sim}{x}) = (2\pi)^{-3/2} \sum_{\alpha} \int p\alpha\, \underset{\sim}{Q}(\underset{\sim}{p}, \alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f(\underset{\sim}{p}, \alpha) \frac{d\underset{\sim}{p}}{p^{N+3}} . \tag{102}$$

The converse is also clearly true : If $\underset{\sim}{u}$ is irrotational, that is, satisfies (101), then the first of Eqs. (100) holds. So a necessary and sufficient condition for a vector to be irrotational is that the first of Eqs. (100) holds.

If the vector $\underset{\sim}{u}$ is irrotational, the second of Eqs. (100) can be used to find a scalar potential $\phi(\underset{\sim}{x})$ such that Eq. (97) holds. Thus, for a scalar potential $\phi(\underset{\sim}{x})$ to be found such that Eq. (97) holds, a necessary and sufficient condition is that $\underset{\sim}{u}(\underset{\sim}{x})$ be an irrotational vector. This statement is of course part of the Helmholtz theorem.

5.3 Vector-Vector Relations; Vector Potentials

Let $\underset{\sim}{u}(\underset{\sim}{x})$ be the general vector expanded as in (92). We wish to relate the representations of $\underset{\sim}{u}(\underset{\sim}{x})$ to those of $\underset{\sim}{v}(\underset{\sim}{x})$, where

$$\underset{\sim}{v}(\underset{\sim}{x}) = \underset{\sim}{\nabla} \times \underset{\sim}{u}(\underset{\sim}{x}) . \tag{103}$$

Since $\underset{\sim}{v}(\underset{\sim}{x})$ is of dimension L^{N-1}, its expansion has the form

$$\underset{\sim}{v}(\underset{\sim}{x}) = (2\pi)^{-3/2} \sum_{\alpha} \int \underset{\sim}{Q}(\underset{\sim}{p},\alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} k(\underset{\sim}{p},\alpha) \frac{d\underset{\sim}{p}}{p^{N+2}} . \tag{104}$$

Using Eqs. (102) and (103) we obtain the relationship between the representations, which is :

$$k(\underset{\sim}{p},\alpha) = \alpha\, f(\underset{\sim}{p},\alpha) , \tag{105}$$

and in particular

$$k(\underset{\sim}{p},0) \equiv 0 . \tag{105a}$$

From Eqs. (104), (105a), and (95), with N replaced by N - 1, it follows that

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{v}(\underset{\sim}{x}) = 0 , \tag{106}$$

that is, $\underset{\sim}{v}(\underset{\sim}{x})$ is solenoidal. Conversely, if we wish to solve Eq. (103) for $\underset{\sim}{u}(\underset{\sim}{x})$ when $\underset{\sim}{v}(\underset{\sim}{x})$ is given, we must have (105a), or equivalently, (106). We may consider $\underset{\sim}{u}(\underset{\sim}{x})$ as a vector potential from which $\underset{\sim}{v}(\underset{\sim}{x})$ is obtained.

We have thus proved the following portion of the Helmholtz theorem: A necessary and sufficient condition that $\underset{\sim}{u}(\underset{\sim}{x})$ be a vector potential for $\underset{\sim}{v}(\underset{\sim}{x})$ is that Eq. (106) hold.

We note, however, that in constructing $\underset{\sim}{u}(\underset{\sim}{x})$ from $\underset{\sim}{v}(\underset{\sim}{x})$, $\underset{\sim}{u}(\underset{\sim}{p},0)$ is arbitrary if Eq. (106) or, equivalently, (105a) is satisfied. The contribution to $\underset{\sim}{u}(\underset{\sim}{x}) = \sum_{\alpha} \underset{\sim}{u}(\underset{\sim}{x},\alpha)$ [see Eq. (57a) for notation],

$$\underset{\sim}{u}(\underset{\sim}{x},0) = \int \underset{\sim}{Q}(\underset{\sim}{p},0) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} f(\underset{\sim}{p},0) \frac{d\underset{\sim}{p}}{p^{N+3}} , \tag{107}$$

is thus also arbitrary.

The term $\underset{\sim}{u}(\underset{\sim}{x}, 0)$ is the gauge of the vector potential, which is usually chosen so that $\underset{\sim}{u}(\underset{\sim}{x})$ satisfies desired conditions in addition to Eq. (103). Our procedure enables us to explicitly separate the gauge portion of the vector potential from the part that is essential for obtaining $\underset{\sim}{v}(\underset{\sim}{x})$.

If, as in electromagnetic theory, we identify $\underset{\sim}{v}(\underset{\sim}{x})$ with the magnetic field [for which (106) always holds] , $\underset{\sim}{u}(\underset{\sim}{x})$ is called the electromagnetic vector potential and the choice of the gauge is of some importance in simplifying problems.

5.4 Vector-Tensor Relations

Let us consider the tensor $G_M(\underset{\sim}{x}) = \left\{G_{ij}(\underset{\sim}{x})\right\}$. The quantities $u_i(\underset{\sim}{x})$ and $v_i(\underset{\sim}{x})$, defined by

$$u_i(\underset{\sim}{x}) = \sum_j \frac{\partial G_{ji}(\underset{\sim}{x})}{\partial x_j}, \quad v_i(\underset{\sim}{x}) = \sum_j \frac{\partial G_{ij}(\underset{\sim}{x})}{\partial x_j}, \tag{108}$$

are components of vectors $\underset{\sim}{u}(\underset{\sim}{x}) = \left\{u_i(\underset{\sim}{x})\right\}$, $\underset{\sim}{v}(\underset{\sim}{x}) = \left\{v_i(\underset{\sim}{x})\right\}$. The relationships between the irreducible representations contained in the expansion of $G_M(\underset{\sim}{x})$ and those contained in the expansions of $\underset{\sim}{u}(\underset{\sim}{x})$ and $\underset{\sim}{v}(\underset{\sim}{x})$ will now be given. The results can also be used to solve Eq. (108) for $G_M(\underset{\sim}{x})$ when either $\underset{\sim}{u}(\underset{\sim}{x})$ or $\underset{\sim}{v}(\underset{\sim}{x})$ are given, although $G_M(\underset{\sim}{x})$ will not be unique.

Actually, we obtain a better result. We decompose the tensor into its unit, antisymmetric and symmetric parts, as in Eq. (76b) or, equivalently, Eq. (76), and find the vector expansion corresponding to the use of (108) for each part of the tensor. Let us take $G_M(\underset{\sim}{x})$ to have the dimensions L^N, and write

$$G_M(\underset{\sim}{x}) = \sum_j G_M^j(\underset{\sim}{x}),$$

$$G_M^j(\underset{\sim}{x}) = (2\pi)^{-3/2} \sum_\alpha \int Q_M(\underset{\sim}{p}, j, \alpha)\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}}\, g^j(\underset{\sim}{p}, \alpha) \frac{d\underset{\sim}{p}}{p^{N+3}}. \tag{109}$$

We define the vectors $\underset{\sim}{u}^j(\underset{\sim}{x})$ and $\underset{\sim}{v}^j(\underset{\sim}{x})$ by

$$\underset{\sim}{u}^j(\underset{\sim}{x}) = \left\{u_i^j(\underset{\sim}{x})\right\}, \quad u_i^j(\underset{\sim}{x}) = \sum_k \frac{\partial G_{ki}^j(\underset{\sim}{x})}{\partial x_k}.$$

$$\underset{\sim}{v}^j(\underset{\sim}{x}) = \left\{v_i^j(\underset{\sim}{x})\right\}, \quad v_i^j(\underset{\sim}{x}) = \sum_k \frac{\partial G_{ik}^j(\underset{\sim}{x})}{\partial x_k}: \tag{110}$$

The vectors $\underset{\sim}{u}^j(\underset{\sim}{x})$ and $\underset{\sim}{v}^j(\underset{\sim}{x})$ will have expansions :

$$\underset{\sim}{u}^j(\underset{\sim}{x}) = (2\pi)^{-3/2} \sum_\alpha \int Q(\underset{\sim}{p},\alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} r^j(\underset{\sim}{p},\alpha) \frac{d\underset{\sim}{p}}{p^{N+2}},$$

$$\underset{\sim}{v}^j(\underset{\sim}{x}) = (2\pi)^{-3/2} \sum_\alpha \int \underset{\sim}{Q}(\underset{\sim}{p},\alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} s^j(\underset{\sim}{p},\alpha) \frac{d\underset{\sim}{p}}{p^{N+2}}. \tag{111}$$

Then it can be shown that

$$r^0(\underset{\sim}{p},\alpha) = s^0(\underset{\sim}{p},\alpha) = (3)^{-1/2} i\delta_{\alpha,0} g^0(\underset{\sim}{p}), \tag{112}$$

$$r^1(\underset{\sim}{p},\alpha) = -s^1(\underset{\sim}{p},\alpha) = (2)^{-1/2} i\alpha\, g^1(\underset{\sim}{p},\alpha), \tag{113}$$

$$r^2(\underset{\sim}{p},0) = s^2(\underset{\sim}{p},0) = -(2/3)^{1/2} i g^2(\underset{\sim}{p},0), \tag{114}$$

$$r^2(\underset{\sim}{p},\alpha) = {}^2(\underset{\sim}{p},\alpha) = -(2)^{-1/2} i g^2(\underset{\sim}{p},\alpha), \quad \text{for } \alpha = \pm 1 . \tag{115}$$

In Eq. (112), $g^0(\underset{\sim}{p}) \equiv g^0(\underset{\sim}{p},0)$. Equation (112) is proved trivially.

To prove Eq. (113) we write $G_M^1(\underset{\sim}{x})$ as in Eq. (80), with

$$\underset{\sim}{G}(\underset{\sim}{x}) = (2\pi)^{-3/2} \sum_\alpha \int Q(\underset{\sim}{p},\alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} g^1(p,\alpha) \frac{d\underset{\sim}{p}}{p^{N+3}} \tag{116}$$

[see Eq. (82)]. Then for j = 1, The first of Eqs. (110) becomes

$$\underset{\sim}{u}^1(\underset{\sim}{x}) = -\underset{\sim}{v}^1(\underset{\sim}{x}) = (2)^{-1/2} i\, \underset{\sim}{\nabla}\times\underset{\sim}{G}(\underset{\sim}{x}) . \tag{117}$$

Equation (113) follows from Eq. (117) in the same way that Eq. (105) follows from Eq. (103).

Equations (114) and (115) are proved as follows. From the first of Eqs. (110), (111), and (109), together with Eq. (72), we have, after taking Fourier transforms,

$$\sum_\alpha \underset{\sim}{Q}(\underset{\sim}{p},\alpha) r^2(\underset{\sim}{p},\alpha) = \sum_\alpha \sum_m i(4\pi/5)^{1/2} \underset{\sim}{\sigma}(\underset{\sim}{p},m) Y_2^{m,\alpha *}(\theta,\phi) g^2(\underset{\sim}{p},\alpha), \tag{118}$$

where $\underset{\sim}{\sigma}(\underset{\sim}{p},m) = \left\{\sigma_i(\underset{\sim}{p},m)\right\}$ is a vector whose components are defined by

$$\sigma_i(\underset{\sim}{p},m) = \sum_k (p_k/p) W_{ki}(2,m) . \tag{118a}$$

By explicit computation, we can show that

$$\underset{\sim}{Q}^*(\underset{\sim}{p},0) \cdot \underset{\sim}{\sigma}(\underset{\sim}{p},m) = -(2/3)^{1/2}(4\pi/5)^{1/2} Y_2^{m,0}(\theta,\phi) ,$$

$$\underset{\sim}{Q}^*(\underset{\sim}{p},\alpha) \cdot \underset{\sim}{\sigma}(\underset{\sim}{p},m) = -(1/2)^{1/2}(4\pi/5)^{1/2} Y_2^{m,\alpha}(\theta,\phi) , \quad (\alpha = \pm 1) . \tag{119}$$

In Eqs. (119), θ, ϕ are the polar angles of $\underset{\sim}{p}$, as usual.

The relationship between $r^2(\underset{\sim}{p},\alpha)$ and $g^2(\underset{\sim}{p},\alpha)$ follows from the first of Eqs. (55b) and (119) and the orthogonality relation (A11). The other relationships of (114) and (115) are similarly obtained.

Let us now consider the case where a tensor is formed by differentiating a vector. Let the components of the tensor $G_M(\underset{\sim}{x}) = \left\{G_{ik}(\underset{\sim}{x})\right\}$ be obtained from the components of the vector $\underset{\sim}{w}(\underset{\sim}{x})$ $\left\{w_i(\underset{\sim}{x})\right\}$ as follows :

$$G_{ik}(\underset{\sim}{x}) = \frac{\partial w_i(\underset{\sim}{x})}{\partial x_k} . \tag{120}$$

Let us assume that $G_M(\underset{\sim}{x})$ has the dimensions L^N and therefore that the expansion (109) is valid. The vector $\underset{\sim}{w}(\underset{\sim}{x})$ will have the dimensions L^{N+1} and hence the expansion

$$\underset{\sim}{w}(\underset{\sim}{x}) = (2\pi)^{-3/2} \sum_\alpha \int \underset{\sim}{Q}(\underset{\sim}{p},\alpha) e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} \underset{\sim}{k}(\underset{\sim}{p},\alpha) \frac{d\underset{\sim}{p}}{p^{N+4}} . \tag{121}$$

We shall show that the expansions of $G_M(\underset{\sim}{x})$ and $\underset{\sim}{w}(\underset{\sim}{x})$ are related by

$$g^0(\underset{\sim}{p}) = (3)^{-1/2} i\underset{\sim}{k}(\underset{\sim}{p},0) , \tag{122}$$

$$g^1(\underset{\sim}{p},\alpha) = -(2)^{-1/2} i\alpha \underset{\sim}{k}(\underset{\sim}{p},\alpha) , \quad \text{for } \alpha = \pm 1, \tag{123}$$

$$g^1(\underset{\sim}{p},0) = 0 , \tag{124}$$

$$g^2(\underset{\sim}{p},0) = -(2/3)^{-1/2} i\underset{\sim}{k}(\underset{\sim}{p},0) , \tag{125}$$

$$g^2(\underset{\sim}{p},\alpha) = -(2)^{-1/2}\, i k(\underset{\sim}{p},\alpha) \text{ for } \alpha = \pm 1, \tag{126}$$

$$g^2(\underset{\sim}{p},\alpha) = 0 \text{ , for } \alpha = \pm 2 \text{ .} \tag{127}$$

To prove the above results we first note from Eqs. (76c), (76d), and (120) that

$$G_M^0(\underset{\sim}{x}) = (1/3)\, I_M \underset{\sim}{\nabla} \cdot \underset{\sim}{w}(\underset{\sim}{x}) \text{ ,} \tag{128}$$

$$G_{ik}^1(\underset{\sim}{x}) = \frac{1}{2}\left[\frac{\partial w_i(\underset{\sim}{x})}{\partial x_k} - \frac{\partial w_k(\underset{\sim}{x})}{\partial x_i}\right] , \tag{129}$$

$$G_{ik}^2(\underset{\sim}{x}) = \frac{\partial w_i(\underset{\sim}{x})}{\partial x_k} + \frac{\partial w_k(\underset{\sim}{x})}{\partial x_i} - \frac{2}{3}\delta_{ik}\underset{\sim}{\nabla} \cdot \underset{\sim}{w}(\underset{\sim}{x}) \text{ .} \tag{130}$$

To prove Eq. (122) we use Eq. (77), with

$$G(\underset{\sim}{x}) = (2\pi)^{-3/2}\int e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} g^0(\underset{\sim}{p})\frac{d\underset{\sim}{p}}{p^{N+3}} \text{ .} \tag{131}$$

Then Eq. (128) leads to

$$G(\underset{\sim}{x}) = -(3)^{-1/2}\underset{\sim}{\nabla} \cdot \underset{\sim}{w}(\underset{\sim}{x}) \text{ .} \tag{132}$$

Equation (122) then follows from Eqs. (132) and (121) just as Eq. (96) follows from Eq. (93).

To prove Eqs. (123) and (124) we use Eq. (80), with

$$\underset{\sim}{G}(\underset{\sim}{x}) = (2\pi)^{-3/2}\sum_{\alpha}\int \underset{\sim}{Q}(\underset{\sim}{p},\alpha)\, e^{i\underset{\sim}{p}\cdot\underset{\sim}{x}} g^1(\underset{\sim}{p},\alpha)\frac{d\underset{\sim}{p}}{p^{N+3}} \text{ .} \tag{133}$$

It is easy to see that Eq. (129) is the same as

$$\underset{\sim}{G}(\underset{\sim}{x}) = -(2)^{-1/2}\, i\, \underset{\sim}{\nabla} \times \underset{\sim}{w}(\underset{\sim}{x}) \text{ .} \tag{134}$$

Then Eqs. (123) and (124) follow from (134) just as Eq. (105) follows from Eq. (103).

To prove Eqs. (125), (126), and (127) we introduce the matrix $C_M(\underset{\sim}{p},\alpha) = \left\{C_{ik}(\underset{\sim}{p},\alpha)\right\}$:

$$C_{ik}(\underset{\sim}{p},\alpha) = \frac{1}{2}\left[\eta_k Q_i(\underset{\sim}{p},\alpha) + \eta_i Q_k(\underset{\sim}{p},\alpha) + \frac{2}{3}\delta_{ik}\,\delta_{\alpha,0}\right], \quad \underset{\sim}{\eta} = \underset{\sim}{p}/p. \tag{135}$$

Equation (130) is equivalent (after taking an obvious Fourier transform) to

$$(4\pi/5)^{1/2}\sum_{\alpha}\sum_{m} W_M(2,m)\,Y_2^{m,\alpha *}(\theta,\phi)\,g^2(p,\alpha)$$

$$= i\sum_{\alpha} C_M(\underset{\sim}{p},\alpha)\,k(\underset{\sim}{p},\alpha)\,. \tag{136}$$

But by explicit calculation we can show that

$$\mathrm{tr}\,W_M^{\dagger}(2,m)\,C_M(\underset{\sim}{p},\alpha) = -(2)^{-1/2}(4\pi/5)^{1/2}\,Y_2^{m,\alpha *}(\theta,\phi)\,, \quad \alpha = \pm 1\,,$$

$$\mathrm{tr}\,W_M^{\dagger}(2,m)\,C_M(\underset{\sim}{p},0) = -(2/3)^{1/2}(4\pi/5)^{1/2}\,Y_2^{m,0 *}(\theta,\phi)\,. \tag{137}$$

Equations (125) to (127) then follow directly from Eq. (71).

6. INNER PRODUCTS, CORRELATIONS, AUTOCORRELATIONS, MAGNITUDES OF PHYSICAL QUANTITIES

6.1 Correlations Between Tensors of the Same Rank

Let us consider two tensors of the same rank, which we write as column vectors $G_c(\underset{\sim}{x})$ and $H_c(\underset{\sim}{x})$. Let $G_c(\underset{\sim}{x})$ be of dimension L^N and $H_c(\underset{\sim}{x})$ be of dimension L^M. The quantity $G_c(\underset{\sim}{x})$ will be taken to have the expansion (45); $H_c(\underset{\sim}{x})$ will be taken to have a similar expansion but with $c^{j,n}(m)$, $f_c^{j,n}(\underset{\sim}{p},\alpha)$, $f_d^{j,n}(\underset{\sim}{p},\alpha)$, and N replaced by $d^{j,n}(m)$, $k_c^{j,n}(\underset{\sim}{p},\alpha)$, $k_d^{j,n}(\underset{\sim}{p},\alpha)$, and M respectively.

We wish to introduce an invariant inner product between $G_c(\underset{\sim}{x})$ and $H_c(\underset{\sim}{x})$. It was seen that $c^{j,n}(m)$, and thus $d^{j,n}(m)$, transform as nonunitary representations of the scale-Euclidean group. Hence we measure $G_c(\underset{\sim}{x})$ and $H_c(\underset{\sim}{x})$ from the mean and set $c^{j,n}(m) \equiv d^{j,n}(m) \equiv 0$. An obvious inner product that is invariant under all transformations of the group—including the scale transformation— and satisfies

the usual requirements of an inner product in Hilbert space is

$$(H,G) = \sum_{j,n} \sum_{\alpha} \left[\int k_c^{*j,n}(\underset{\sim}{p},\alpha) f_c^{j,n}(\underset{\sim}{p},\alpha) \frac{d\underset{\sim}{p}}{p^3} + \right.$$

$$\left. + \sum_{\underset{\sim}{p}} k_d^{*j,n}(\underset{\sim}{p},\alpha) f_d^{j,n}(\underset{\sim}{p},\alpha) \right] . \tag{138}$$

In particular, $(G,G)^{1/2} = M(G)$ will be called the magnitude of $G_c(\underset{\sim}{x})$ and used as a measure of this quantity, which is independent of the frame of reference and the units of length used.

We take over the arguments of Part I for using the inner product (H,G) as a correlation. Accordingly, we define the correlation $K(H,G)$ between $H_c(\underset{\sim}{x})$ and $G_c(\underset{\sim}{x})$ as

$$K(H,G) = (H,G) . \tag{139}$$

Further, we define the correlation coefficient $C(H,G)$ as

$$C(H,G) = K(H,G) \,/\, \left[M(H) M(G) \right] . \tag{140}$$

For each transformation of the group we introduce an autocorrelation and an autocorrelation coefficient. The translation autocorrelation (TAC), written $A(G,\underset{\sim}{a})$, and the translation autocorrelation coefficient (TACC), written $A_c(G,\underset{\sim}{a})$, are defined like their one-dimensional analogs:

$$A(G,\underset{\sim}{a}) = K\left(G, T(a)G\right) ,$$

$$A_c(G,\underset{\sim}{a}) = C\left(G, T(a)G\right) . \tag{141}$$

These quantities compare a tensor with itself when it is measured in a coordinate system whose origin with respect to the original coordinate system is shifted by the vector distance $\underset{\sim}{a}$.

Again, as in Part I, the scale autocorrelation (SAC), written $B(G,\lambda)$, and the scale autocorrelation coefficient (SACC), written $B_c(G,\lambda)$, are defined by :

$$B(G,\lambda) = K\left(G, S(\lambda)G\right) ,$$

$$B_c(G,\lambda) = C\left(G, S(\lambda)G\right) . \tag{142}$$

These quantities compare a tensor with itself when it is 'stretched.'

The third autocorrelation, which we call the isotropy autocorrelation (IAC), is denoted by $I(G,\underset{\sim}{\theta})$, and the corresponding isotropy autocorrelation coefficient (IACC) by $I_c(G,\underset{\sim}{\theta})$. These are defined by:

$$I(G,\underset{\sim}{\theta}) = K\left(G, R(\underset{\sim}{\theta})G\right),$$

$$I_c(G,\underset{\sim}{\theta}) = C\left(G, R(\underset{\sim}{\theta})G\right). \tag{143}$$

These quantities compare a tensor with itself when it is measured in a rotated frame of reference. If the tensor is isotropic, then IACC is unity for all $\underset{\sim}{\theta}$. (Although it can be shown that such tensors have trivial properties, we do not go into this in detail in the present paper.) More generally, the range of θ for which the IACC is near unity gives a measure of the isotropy of the tensor—the larger the range, the greater the isotropy.

We believe that the IAC and IACC will be very useful in defining a degree of isotropy of tensors and we hope to make use of them in later papers. The one-dimensional analog of the IAC and IACC is the comparatively simple autocorrelation and autocorrelation coefficient denoted by D and D_c respectively.

6.2 Detailed Correlations; Correlations Between Tensors of Different Ranks

In Sec. 6.1 we introduced correlations between tensors of the same rank. We used an invariant inner product (138) that resembles Parseval's theorem for Fourier series or integrals. There is, however, a more fundamental inner product that is invariant, namely, the inner product (29) or (31) for the irreducible representations. This inner product is, in an obvious sense, the <u>minimal</u> invariant inner product.

We now introduce the notion of sets of detailed correlations and detailed correlation coefficients for two tensors. Let $G_c(\underset{\sim}{x})$ and $H_c(\underset{\sim}{x})$ be column vectors corresponding to two tensors, with the same expansion as before. Instead of requiring the two tensors to be of the same rank, however, we now allow them to be of different ranks, and therefore have different values, for the variable n. We form pairs of irreducible representations, one member of each pair being taken from the expansion of $G_c(\underset{\sim}{x})$ and the other from $H_c(\underset{\sim}{x})$. To each pair we assign a correlation, as follows. The correlation is zero if the value of the helicity variable α of one member is different from that of the second member, or if one member is a discrete representation and the other is a continuous representation. Otherwise, the correlation between the pair $k_c^{j,n}(\underset{\sim}{p},\alpha)$ and $f_c^{j',n'}(\underset{\sim}{p},\alpha)$, denoted by $K_c(k,j,n;\ f,j',n';\alpha)$, is:

$$K_c(k,j,n;f,j',n':\alpha) = \int k_c^{*j,n}(\underset{\sim}{p},\alpha)\, f_c^{j',n'}(\underset{\sim}{p},\alpha)\frac{d\underset{\sim}{p}}{p^{N+3}} \,. \tag{144}$$

Likewise, the correlation between the pair $k_d^{j,n}(\underset{\sim}{p},\alpha)$ and $f_d^{j',n'}(\underset{\sim}{p},\alpha)$ is given by

$$K_d(k,j,n;f,j',n':\alpha) = \sum_p k_d^{*j,n}(\underset{\sim}{p},\alpha)\, f_d^{j',n'}(\underset{\sim}{p},\alpha) \,. \tag{144a}$$

The set of detailed correlations for the tensors $G_c(\underset{\sim}{x})$ and $H_c(\underset{\sim}{x})$ is the set of correlations thus constructed. We can now ask for the set of detailed correlations between a scalar and a tensor of rank 2, for example, or for the set of correlations between a vector and a tensor.

A special case is that for which $H_c(\underset{\sim}{x}) \equiv G_c(\underset{\sim}{x})$. Among the correlations for a tensor of rank 2, for example, will be $j = 2$, $\alpha = 1$, and $j = 1$, $\alpha = 1$. Correlations of this type are clearly significant.

We can also introduce magnitudes of the irreducible representations in the following obvious way :

$$M_c(f,j,n:\alpha) = \left[K_c(f,j,n;f,j,n:\alpha)\right]^{1/2} ,$$

$$M_d(f,j,n:\alpha) = \left[K_d(f,j,n;f,j,n:\alpha)\right]^{1/2} . \tag{145}$$

We now also introduce sets of detailed correlation coefficients between $H_c(\underset{\sim}{x})$ and $G_c(\underset{\sim}{x})$. The correlation coefficients are zero if the correlations are zero. Otherwise,

$$C_c(k,j,n;f,j',n':\alpha) = \frac{K_c(k,j,n;f,j',n':\alpha)}{M_c(k,j,n:\alpha)M_c(f,j',n':\alpha)} ,$$

$$C_d(k,j,n;f,j',n':\alpha) = \frac{K_d(k,j,n;f,j',n':\alpha)}{M_d(k,j,n:\alpha)M_d(f,j',n':\alpha)} . \tag{146}$$

Sets of detailed autocorrelations and autocorrelation coefficients associated with the transformations of the group can be defined in an obvious way. For the sake of brevity we omit the details.

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Appendix A

Irreducible Representations of the Rotation Group; Generalized Surface Harmonics

A1. JACOBI POLYNOMIALS

The Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ are defined and their properties given elsewhere (see, for example, Szego, 1959). For convenience, we define the closely related polynomials $S(j,m,m',x)$ as:

$$S(j,m,m',x) = P_{j-m}^{(m-m',m+m')}(x) , \tag{A1}$$

where j is any nonnegative integer or half-odd integer, and m and m' each take on the values of $-j, -j+1, \ldots, j-1, j$. From Rodrigues' formula for the Jacobi polynomials (Szego, 1959):

$$S(j,m,m',x) = (-1)^{j-m} \frac{2^{m-j}}{(j-m)!} (1-x)^{-(m-m')} (1+x)^{-(m+m')} \times$$

$$\times \frac{d^{j-m}}{dx^{j-m}} \left[(1-x)^{j-m'} (1+x)^{j+m'}\right] . \tag{A2}$$

An alternative expression, given in Moses (1965b), is:

$$S(j,m,m',x) = (-1)^{j+m'} \frac{2^{m-j}}{(j+m)!} \frac{d^{j+m}}{dx^{j+m}} \left[(1-x)^{j+m'}(1+x)^{j-m'}\right] . \qquad \text{(A3)}$$

A2. THE IRREDUCIBLE REPRESENTATIONS OF THE ROTATION GROUP

The expression for the matrix elements $R^{(j)}(\underset{\sim}{\theta})(m,m')$ is

$$R^{(j)}(\underset{\sim}{\theta})(m,m') = \left[\frac{(j-m)!(j+m)!}{(j-m')!(j+m')!}\right]^{1/2} \left(\sin\frac{\theta}{2}\right)^{m-m'} \times$$

$$\times \left[\frac{\theta_2 + i\,\theta_1}{\theta}\right]^{m-m'} \left(\cos\frac{\theta}{2} + i\,\frac{\theta_3}{\theta}\sin\frac{\theta}{2}\right)^{m+m'} S(j,m,m',z) , \qquad \text{(A4)}$$

where

$$\theta = |\underset{\sim}{\theta}| , \qquad z = \left[1 - (\theta_3/\theta)^2\right]\cos\theta + (\theta_3/\theta)^2 . \qquad \text{(A4a)}$$

A3. GENERALIZED SURFACE HARMONICS AND THEIR PROPERTIES

Generalized surface harmonics are defined by

$$Y_j^{m,m'}(\theta,\phi) = (-1)^{m-m'}(1/2)^{m+1}\left[(2j+1)/\pi\right]^{1/2} \times$$

$$\times \left[\frac{(j-m)!(j+m)!}{(j-m')!(j+m')!}\right]^{1/2} e^{i(m-m')\phi} \left[\sin\theta\right]^{m-m'} \times$$

$$\times \left[1+\cos\theta\right]^{m'} S(j,m,m',\cos\theta) , \qquad \text{(A5)}$$

where $0 < \theta < \pi$ and $0 < \phi < 2\pi$.

Let $\underset{\sim}{\theta}$ be a vector:

$$\underset{\sim}{\theta} = \theta(\cos\phi, \sin\phi, 0) . \qquad \text{(A6)}$$

Then

$$R^{(j)}(\underset{\sim}{\theta})(m,m') = (-i)^{m-m'}\left[4\pi/(2j+1)\right]^{1/2} Y_j^{m,m'*}(\theta,\phi) . \qquad (A6\underline{a})$$

That the surface harmonics reduce the infinitesimal generators of the rotation group in the helicity representation is perhaps their most interesting property. Let θ and ϕ be the polar angles of a vector $\underset{\sim}{x} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. The operators J_i are defined by

$$J_3 = \left[-i(\underset{\sim}{x}\times\underset{\sim}{\nabla})_3 + m'\right] ,$$

$$J_k = \left[-i(\underset{\sim}{x}\times\underset{\sim}{\nabla})_k + \frac{x_k}{r+x_3} m'\right] , \quad \text{for } k = 1,2 . \qquad (A7)$$

The operators J_i satisfy the commutation rules Eq. (6) for the infinitesimal generators of the rotation group.

In terms of polar coordinates,

$$J_1 = \left[i(\sin\phi\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}) + m'\cos\phi\tan\frac{\theta}{2}\right] ,$$

$$J_2 = \left[-i(\cos\phi\frac{\partial}{\partial\theta} - \cot\theta\sin\phi\frac{\partial}{\partial\phi}) + m'\sin\phi\tan\frac{\theta}{2}\right] ,$$

$$J_3 = \left[-i\frac{\partial}{\partial\phi} + m'\right] . \qquad (A7\underline{a})$$

Then

$$J_i Y_j^{m,m'}(\theta,\phi) = \sum_{n=-j}^{j} Y_j^{n,m'}(\theta,\phi) S_i^{(j)}(n,m) . \qquad (A8)$$

The integrated form of (A8) can also be given. Let $\underset{\sim}{\eta}$ be the unit vector, which in polar coordinates is

$$\underset{\sim}{\eta} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) . \qquad (A9)$$

Let $\underset{\sim}{\eta}'$ be the unit vector obtained from $\underset{\sim}{\eta}$ through the rotation

$$\underset{\sim}{\eta}' = R_M(-\underset{\sim}{\Omega})\underset{\sim}{\eta} , \qquad (A9\underline{a})$$

and let θ' and ϕ' be the polar angles obtained from $\underset{\sim}{\eta}'$ through Eq. (A9). Then the integrated form of (A8) is

$$Y_j^{m,m'}(\theta',\phi') = \exp\left[-2im'\Phi(\underset{\sim}{\Omega},\underset{\sim}{\eta})\right] \sum_{n=-j}^{j} Y_j^{n,m'}(\theta,\phi) R^{(j)}(\underset{\sim}{\Omega})(n,m), \quad \text{(A9b)}$$

where $\Phi(\underset{\sim}{\Omega},\underset{\sim}{\eta})$ is given by Eq. (31a) .

Further properties of the generalized surface harmonics are :

$$Y_j^{m,0}(\theta,\phi) = Y_{jm}(\theta,\phi), \quad j \text{ an integer}, \quad \text{(A10)}$$

where $Y_{jm}(\theta,\phi)$ are the usual surface harmonics in the notation of, for example, Edmonds (1957),

$$\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta\, Y_j^{m,n}(\theta,\phi) Y_{j'}^{m',n*}(\theta,\phi) = \delta_{j,j'}\delta_{m,m'} . \quad \text{(A11)}$$

$$\sum_{j=|n|}^{\infty} \sum_{m=-j}^{j} Y_j^{m,n}(\theta,\phi) Y_j^{m,n*}(\theta',\phi') \sin\theta' = \delta(\theta-\theta')\delta(\phi-\phi') . \quad \text{(A12)}$$

$$\sum_{m=-j}^{j} Y_j^{m,n}(\theta,\phi) Y_j^{m,n'*}(\theta,\phi) = \left[(2j+1)/4\pi\right]\delta_{n,n'} . \quad \text{(A13)}$$

$$\sum_{n=-j}^{j} Y_j^{m,n}(\theta,\phi) Y_j^{m',n*}(\theta,\phi) = \left[(2j+1)/4\pi\right]\delta_{m,m'} . \quad \text{(A14)}$$

$$Y_j^{m,n}(\theta,\phi) = (-1)^{n-m} Y_j^{n,m*}(\theta,\phi) . \quad \text{(A15)}$$

$$Y_j^{m,n}(\pi-\theta,\pi+\phi) = (-1)^{j+m+2n} e^{-2in\phi} Y_j^{-m,n*}(\theta,\phi). \quad \text{(A16)}$$

Appendix B

Derivation of the Irreducible Representations of the Scale-Euclidean Group

We now obtain all irreducible unitary representations of the scale-Euclidean group in the usual separable Hilbert space. These are the irreducible rotation group representations of Sec. 3.1 and the continuous helicity representations of Sec. 3.2. The discrete helicity representations (Sec. 3.3) are obtained from the continuous helicity representations by going to a nonseparable Hilbert space. The irreducible unitary representations in the separable Hilbert space are the only ones in such spaces. There may, however, be other representations in nonseparable Hilbert spaces. In the nonseparable Hilbert spaces, the particular representations that we have chosen allow us to include functions that have periodicities among the physical quantities we have been studying.

B.1 PRELIMINARIES

We shall be using the infinitesimal generators of the representations of Eq. (20). From Eqs. (17a) and (18) it follows that

$$\left[S(\lambda), J_i\right] = 0, \quad S(\lambda) P_i = e^{\lambda} P_i S(\lambda) . \tag{B1}$$

Furthermore, from the multiplication rule for the rotation group [Eq. (16)],

$$R(\underset{\sim}{\Omega})R(\underset{\sim}{\theta}) = R(\underset{\sim}{\theta})R(\underset{\sim}{\Omega}') , \tag{B2}$$

where

$$\underset{\sim}{\Omega}' = R_M(-\underset{\sim}{\theta})\underset{\sim}{\Omega} . \tag{B2a}$$

Thus,

$$R(-\underset{\sim}{\theta})(\underset{\sim}{\Omega}\cdot\underset{\sim}{J})R(\underset{\sim}{\theta}) = \underset{\sim}{\Omega}'\cdot\underset{\sim}{J} , \tag{B2b}$$

which finally leads to :

$$R(-\underset{\sim}{\theta})\underset{\sim}{J}R(\underset{\sim}{\theta}) = R_M(\underset{\sim}{\theta})\underset{\sim}{J} . \tag{B2c}$$

From Eq. (19), we have

$$R(-\underset{\sim}{\theta})\underset{\sim}{P}R(\underset{\sim}{\theta}) = R_M(\underset{\sim}{\theta})\underset{\sim}{P} . \tag{B3}$$

B2. THE IRREDUCIBLE ROTATION GROUP REPRESENTATIONS

Since the Hermitian infinitesimal generators P_i commute, they can be diagonalized simultaneously. We consider two mutually exclusive cases. In the first case, the simultaneous eigenvalues of P_i, which we denote by $\underset{\sim}{p} = (p_1, p_2, p_3)$, have the value $\underset{\sim}{p} = 0$ in the spectrum. In the second case, they have the value $\underset{\sim}{p} \neq 0$ in the spectrum. The first case leads to the irreducible rotation group representations, and the second leads to the continuous helicity representations.

We consider the first case and designate by $|\gamma\rangle$ all linearly independent simultaneous eigenkets of P_i having the simultaneous eigenvalue $\underset{\sim}{p} = 0$. This variable γ should not be confused with the variable γ that was used to label the rows of G_c, as in Eq. (32). Then

$$P_i|\gamma\rangle = 0 \tag{B4}$$

or, equivalently,

$$\underset{\sim}{P}|\gamma\rangle = 0 . \tag{B4a}$$

Lemma

The ket $R(\underset{\sim}{\theta})|\gamma\rangle$ is also a simultaneous eigenket of P_i that has the eigenvalue $\underset{\sim}{p} = 0$. Likewise, $S(\lambda)|\gamma\rangle$ is such an eigenket.

Proof

From (B4a) and (B3),

$$\underset{\sim}{P} R(\underset{\sim}{\theta})|\gamma\rangle = R(\underset{\sim}{\theta}) \left[R_M(\underset{\sim}{\theta})\underset{\sim}{P}\right]|\gamma\rangle = 0 . \tag{B5}$$

Likewise, from (B1),

$$\underset{\sim}{P} S(\lambda)|\gamma\rangle = e^{-\lambda} S(\lambda, \underset{\sim}{P}|\gamma\rangle = 0 , \tag{B6}$$

which completes the proof.

Thus, the Hilbert space is an invariant space corresponding to the eigenvalue 0 of $\underset{\sim}{P}$. This is the carrier space for the rotation group and for the scale operator $S(\lambda)$. Since the one-dimensional scale operator commutes with the elements of the rotation group, we can diagonalize it and simultaneously reduce the rotation group. Clearly, if there is more than one representation of the scale group and one of the rotation group, the representation of the direct product of the representations of the rotation and scale groups—and hence the representation of the corresponding scale-Euclidean group—is reducible. Thus, the only irreducible representations are those for which the rotation group has an irreducible representation characterized by the number j and for which the scale group has an irreducible representation such that the infinitesimal generator D has the single real value d in its spectrum.

B3. THE CONTINUOUS HELICITY REPRESENTATIONS

In this case the operators P_i have a simultaneous eigenvalue $\underset{\sim}{q} \neq 0$. We denote the simultaneous eigenket by $|\underset{\sim}{q}\rangle$. Then

$$P_i |\underset{\sim}{q}\rangle = q_i |\underset{\sim}{q}\rangle . \tag{B7}$$

Lemma

The point $\underset{\sim}{q}_o = (0, 0, 1)$ is in the simultaneous spectrum of P_i.

Proof

Let the ket $|W\rangle$ be defined by

$$|W\rangle = R(-\underset{\sim}{\Lambda})S(\tau)|\underset{\sim}{q}\rangle, \tag{B8}$$

where $\underset{\sim}{\Lambda}$ is given by

$$\Lambda_3 = 0, \quad \Lambda = |\underset{\sim}{\Lambda}| \quad , \quad q_1 = -q\frac{\Lambda_2}{\Lambda}\sin\Lambda, \quad q_2 = q\frac{\Lambda_1}{\Lambda}\sin\Lambda,$$

$$q_3 = q\cos\Lambda, \quad q = |\underset{\sim}{q}|, \tag{B9}$$

and

$$\tau = \log q . \tag{B10}$$

Then from (B1) and (B3),

$$P_i|W\rangle = e^{-\tau}\sum_j R_{ij}(\underset{\sim}{\Lambda})R(-\underset{\sim}{\Lambda})S(\tau)P_j|\underset{\sim}{q}\rangle$$

$$= e^{-\tau}\sum_j R_{ij}(-\underset{\sim}{\Lambda})q_j|W\rangle . \tag{B11}$$

Then from Eq. (2), (B9), and (B10) it follows that

$$P_1|W\rangle = P_2|W\rangle = 0, \quad P_3|W\rangle = |W\rangle . \tag{B12}$$

Hence, W is an eigenstate of the operators P_1, with the simultaneous eigenvalues given by $\underset{\sim}{q}_o$, and the lemma is proved.

We now let $|\gamma\rangle$ be all the linearly independent simultaneous eigenkets of P_i having the simultaneous eigenvalue $\underset{\sim}{q}_o$. The variable γ is used to label the linearly independent kets, as before. We then define the kets $|\underset{\sim}{p},\gamma\rangle$:

$$|\underset{\sim}{p},\gamma\rangle = R(\underset{\sim}{\omega})S(-\mu)|\gamma\rangle , \tag{B13}$$

where $\underset{\sim}{\omega}$ and μ are given in terms of $\underset{\sim}{p}$ as a one-to-one correspondence by

$$\omega_3 = 0, \quad \omega = |\underset{\sim}{\omega}| \quad , \quad p_1 = -p\frac{\omega_2}{\omega}\sin\omega \, , \quad p_2 = p\frac{\omega_1}{\omega}\sin\omega \, ,$$

$$p_3 = p\cos\omega, \quad \mu = \log p \, . \tag{B14}$$

Theorem

The kets $|\underset{\sim}{p},\lambda\rangle$ are simultaneous eigenkets of P_i, having the eigenvalues $\underset{\sim}{p}$, that is,

$$P_i\,|\underset{\sim}{p},\gamma\rangle = p_i\,|\underset{\sim}{p},\gamma\rangle \, . \tag{B15}$$

Proof

The proof is very similar to the proof of the preceding theorem.

The following theorem is obvious for separable Hilbert spaces.

Theorem

The spectrum of the operators P_i is continuous. The simultaneous eigenvalue $\underset{\sim}{p}$ ranges over the entire three-dimensional $\underset{\sim}{p}$-space. Any eigenvector $|\underset{\sim}{p})$ that satisfies $P_i|\underset{\sim}{p}) = p_i|\underset{\sim}{p})$ is a linear combination in γ of the kets $|\underset{\sim}{p},\gamma\rangle$.

Strictly speaking, $\underset{\sim}{p} = 0$ is not in the spectrum of the operators P_i; however, one point in the continuous spectrum is of zero measure and of no importance.

From Eqs. (B15), (19$\underset{\sim}{a}$), and (20),

$$T(\underset{\sim}{a})\,|\underset{\sim}{p},\gamma\rangle = e^{i\underset{\sim}{a}\cdot\underset{\sim}{p}}\,|\underset{\sim}{p},\gamma\rangle . \tag{B16}$$

Equation (B16) shows us how the group element $T(\underset{\sim}{a})$ is represented in this basis. We now find the representations of the remaining elements of the group in the same basis. We first find the representation of $S(\lambda)$. From the commutation rules (17$\underset{\sim}{a}$) and Eq. (B13),

$$S(\lambda)\,|\underset{\sim}{p},\gamma\rangle = R(\underset{\sim}{\omega})S(\lambda-\mu)|\gamma\rangle = |\underset{\sim}{p}',\gamma\rangle \, ,$$

where, from Eq. (B14),

$$\underset{\sim}{p}' = e^{-\lambda}\,\underset{\sim}{p}$$

or finally,

$$S(\lambda)|\underset{\sim}{p},\gamma\rangle = |e^{-\lambda}\underset{\sim}{p},\gamma\rangle . \tag{B17}$$

To find $R(\underset{\sim}{\theta})$ in this representation for arbitrary $\underset{\sim}{\theta}$ is much more difficult. We first find the infinitesimal generators J_i and then integrate.

Lemma

The ket $J_3|\gamma\rangle$ is a simultaneous eigenvector of the operators P_i, with the eigenvalues given by $\underset{\sim}{q}_o = (0, 0, 1)$.

Proof

We have $P_i J_3|\gamma\rangle = [P_i, J_3]|\gamma\rangle + J_3 P_i|\gamma\rangle$. The lemma then follows from the commutation rules of Eq. (21).

Since the kets $|\gamma\rangle$ are all linearly independent kets having the eigenvalue $\underset{\sim}{q}_o$, it follows from the lemma that $J_3|\gamma\rangle$ is a linear combination of the kets $|\gamma\rangle$. Hence, we write

$$J_3|\gamma\rangle = \sum_{\gamma'} M(\gamma',\gamma)|\gamma'\rangle . \tag{B18}$$

In Eq. (B18) we have assumed that γ is a discrete variable. Actually, we can use any measure function for γ that is compatible with the separable Hilbert space.

Definition

Let us define the operator M acting on the kets $|\underset{\sim}{p},\gamma\rangle$:

$$M|\underset{\sim}{p},\gamma\rangle = \sum_{\gamma'} M(\gamma',\gamma)|\underset{\sim}{p},\gamma'\rangle . \tag{B19}$$

It is to be noted that M commutes with P_i and $S(\lambda)$. To find J_3, Eq. (B2) is used to obtain

$$\exp[i\beta J_3]\,|\underset{\sim}{p},\gamma\rangle = \exp[i\beta J_3]\,R(\underset{\sim}{\omega})\exp[-i\beta J_3]\,S(-\mu)\exp[i\beta J_3]\,|\gamma\rangle$$

$$= R(\underset{\sim}{\omega}')\,S(-\lambda)\exp[i\beta M]\,|\gamma\rangle ,$$

or,

$$\exp\left[i\beta J_3\right] | \underset{\sim}{p}, \gamma \rangle = \exp\left[i\beta M\right] | \underset{\sim}{p}', \gamma \rangle , \tag{B20}$$

where

$$\omega_1' = \omega_1 \cos\beta + \omega_2 \sin\beta ,$$

$$\omega_2' = \omega_2 \cos\beta - \omega_1 \sin\beta ,$$

$$\omega_3' = \omega_3 , \tag{B20a}$$

and $\underset{\sim}{p}'$ is related to $\underset{\sim}{\omega}'$ as $\underset{\sim}{p}$ is to $\underset{\sim}{\omega}$ [see Eq. (B14)] . Hence, by Eq. (B20a),

$$p_1' = p_1 \cos\beta + p_2 \sin\beta ,$$

$$p_2' = p_2 \cos\beta - p_1 \sin\beta ,$$

$$p_3' = p_3 . \tag{B20b}$$

Now, from Eq. (B20),

$$J_3 | \underset{\sim}{p}, \gamma \rangle = \left\{ -i \frac{\partial}{\partial\beta} \exp\left[i\beta J_3\right] | \underset{\sim}{p}, \gamma \rangle \right\}_{\beta = 0}$$

$$= \left\{ -i \frac{\partial}{\partial\beta} | \underset{\sim}{p}', \gamma \rangle \right\}_{\beta = 0} + M | \underset{\sim}{p}, \gamma \rangle .$$

Finally,

$$J_3 | \underset{\sim}{p}, \gamma \rangle = \left\{ i (\underset{\sim}{p} \times \underset{\sim}{V})_3 + M \right\} | \underset{\sim}{p}, \gamma \rangle , \tag{B21}$$

since

$$\left\{ \frac{\partial}{\partial\beta} | \underset{\sim}{p}', \gamma \rangle \right\}_{\beta = 0} = \sum_i \frac{\partial}{\partial p_i} | \underset{\sim}{p}, \gamma \rangle \cdot \left\{ \frac{\partial p_i'}{\partial\beta} \right\}_{\beta = 0} . \tag{B22}$$

We now find J_1 and J_2. From Eqs. (B13) and (20),

$$\underset{\sim}{\omega}\cdot\underset{\sim}{J}|\underset{\sim}{p}\gamma\rangle = \left\{-i\frac{\partial}{\partial\beta}R(\beta\underset{\sim}{\omega})S(-\mu)|\gamma\rangle\right\}_{\beta=1}$$

$$= \left\{-i\frac{\partial}{\partial\beta}|\underset{\sim}{p}',\gamma\rangle\right\}_{\beta=1} = \sum_k \frac{\partial}{\partial p_k}|\underset{\sim}{p},\gamma\rangle\cdot\left\{-i\frac{\partial p_k'}{\partial\beta}\right\}_{\beta=1}, \tag{B23}$$

where

$$p_1' = -p\frac{\omega_2}{\omega}\sin\beta\omega, \quad p_2' = p\frac{\omega_1}{\omega}\sin\beta\omega, \quad p_3' = p\cos\beta\omega. \tag{B23a}$$

Finally, from Eq. (B14),

$$(p_1J_2 - p_2J_1)|\underset{\sim}{p},\gamma\rangle = i\left[-p^2\frac{\partial}{\partial p_3}|\underset{\sim}{p},\gamma\rangle + p_3(\underset{\sim}{p}\cdot\underset{\sim}{\nabla})|\underset{\sim}{p},\gamma\rangle\right]. \tag{B24}$$

Having obtained $(p_1J_2 - p_2J_1)|p,\gamma\rangle$, we now find $(p_1J_1 + p_2J_2)|\underset{\sim}{p},\gamma\rangle$. We shall then have two equations in $J_1|\underset{\sim}{p},\gamma\rangle$ and $J_2|\underset{\sim}{p},\gamma\rangle$ which we can then solve simultaneously.

From Eq. (B2c),

$$\exp\left[i\underset{\sim}{\omega}\cdot\underset{\sim}{J}\right](\underset{\sim}{\omega}\times\underset{\sim}{J})_3\exp\left[-\underset{\sim}{\omega}\cdot\underset{\sim}{J}\right] = (\underset{\sim}{\omega}\times\underset{\sim}{J})_3\cos\omega - \omega J_3\sin\omega,$$

$$\exp\left[-i\underset{\sim}{\omega}\cdot\underset{\sim}{J}\right](\underset{\sim}{\omega}\cdot\underset{\sim}{J})_3\exp\left[i\underset{\sim}{\omega}\cdot\underset{\sim}{J}\right] = (\underset{\sim}{\omega}\times\underset{\sim}{J})_3\cos\omega + \omega J_3\sin\omega. \tag{B25}$$

From Eqs. (B13) and (B25),

$$(\underset{\sim}{\omega}\times\underset{\sim}{J})_3|\underset{\sim}{p},\gamma\rangle = \exp\left[i\underset{\sim}{\omega}\cdot\underset{\sim}{J}\right]\left\{\exp\left[-i\underset{\sim}{\omega}\cdot\underset{\sim}{J}\right](\underset{\sim}{\omega}\times\underset{\sim}{J})_3\exp\left[i\underset{\sim}{\omega}\cdot\underset{\sim}{J}\right]S(-\mu)\right\}|\gamma\rangle$$

$$= \cos\omega\left\{\exp\left[i\underset{\sim}{\omega}\cdot\underset{\sim}{J}\right](\underset{\sim}{\omega}\times\underset{\sim}{J})_3S(-\mu)|\gamma\rangle\right\} + \omega\sin\omega\, M|p,\gamma\rangle$$

$$= \cos^2\omega(\underset{\sim}{\omega}\times\underset{\sim}{J})_3|\underset{\sim}{p},\gamma\rangle - \omega J_3\sin\omega\cos\omega|\underset{\sim}{p},\gamma\rangle + \omega\sin\omega\, M|\underset{\sim}{p},\gamma\rangle. \tag{B26}$$

We can now solve Eq. (B26) for $(\underset{\sim}{\omega}\times\underset{\sim}{J})_3|\underset{\sim}{p},\gamma\rangle$. Using Eq. (B21) for $J_3|\underset{\sim}{p},\gamma\rangle$, and Eq. (B14) to replace $\underset{\sim}{\omega}$ by $\underset{\sim}{p}$, we have

$$(p_1 J_1 + p_2 J_2)|\underset{\sim}{p},\gamma\rangle = \left[-i p_3 (\underset{\sim}{p}\times\underset{\sim}{\nabla})_3 + (p - p_3) M\right] |\underset{\sim}{p},\gamma\rangle . \tag{B27}$$

Solving Eqs. (B24) and (B27) for $J_1|\underset{\sim}{p},\gamma\rangle$ and $J_2|\underset{\sim}{p},\gamma\rangle$, we finally get

$$J_1|\underset{\sim}{p},\gamma\rangle = \left[i(\underset{\sim}{p}\times\underset{\sim}{\nabla})_1 - \frac{p_1}{p+p_3} M\right] |\underset{\sim}{p},\gamma\rangle ,$$

$$J_2|\underset{\sim}{p},\gamma\rangle = \left[i(\underset{\sim}{p}\times\underset{\sim}{\nabla})_2 + \frac{p_2}{p+p_3} M\right] |\underset{\sim}{p},\gamma\rangle . \tag{B27}$$

From Eqs. (B21) and (B27),

$$(\underset{\sim}{P}\cdot\underset{\sim}{J})|\underset{\sim}{p},\gamma\rangle = p M |\underset{\sim}{p},\gamma\rangle , \tag{B28}$$

or,

$$M = (\underset{\sim}{P}\cdot\underset{\sim}{J})(P)^{-1} , \tag{B28a}$$

where the operator P is defined by $P|\underset{\sim}{p},\gamma\rangle = p|\underset{\sim}{p},\gamma\rangle$. From (B28a) it is clear that M is a Hermitian operator. Simce M is independent of P_i, it is possible to find a transformation in the γ-space such that M is diagonal, that is, spectrally represented. With this choice,

$$M|\underset{\sim}{p},\gamma\rangle = \alpha(\gamma)|\underset{\sim}{p},\gamma\rangle , \tag{B29}$$

where $\alpha(\gamma)$ is a real function of γ. We are led to the following lemmas.

Lemma

For an irreducible representation the variable γ can take on only one value. Hence, $\alpha(\gamma) \equiv \alpha$ is simply a real number.

Proof

If γ took on more than one value the operators $T(\underset{\sim}{a})$, $S(\lambda)$, and J_i would map into themselves subspaces corresponding to two different values of γ. By definition, then, the representation would be reducible.

Lemma

The constant α can take on only integer values (positive, negative, or zero).

Proof

For irreducible representations, Eq. (B20) now reads

$$\exp\left[i\beta J_3\right] |\underset{\sim}{p}\rangle = e^{i\beta a} |\underset{\sim}{p}'\rangle , \tag{B30}$$

where p' is given by Eq. (B20b). From the group multiplication laws, however, $\exp\left[i\,2\pi J_3\right] = I$, where I is the identity operator. On setting $\beta = 2\pi$, the lemma follows immediately.

We finally give $R(\theta)$ in this representation:

$$R(\underset{\sim}{\theta})|\underset{\sim}{p}\rangle = \exp\left[-2\,i\alpha\Phi(-\underset{\sim}{\theta},\underset{\sim}{\eta})\right] | R_M(\underset{\sim}{\theta})\underset{\sim}{p}\rangle . \tag{B31}$$

In Eq. (B31), $\Phi(\underset{\sim}{\theta},\underset{\sim}{\eta})$ is given by Eq. (30a) .

To verify Eq. (B31) we write

$$|\underset{\sim}{p},\theta,\underset{\sim}{\sigma}\rangle = R(\underset{\sim}{\theta})|\underset{\sim}{p}\rangle , \tag{B32}$$

where $\theta = |\underset{\sim}{\theta}|$, as usual, and $\underset{\sim}{\sigma} = \underset{\sim}{\theta}/\theta$.

A first-order differential equation for $|\underset{\sim}{p},\theta,\underset{\sim}{\sigma}\rangle$ is, on using $R(\underset{\sim}{\theta}) = \exp\left[i\underset{\sim}{\theta}\cdot\underset{\sim}{J}\right]$,

$$\frac{\partial}{\partial\theta}|\underset{\sim}{p},\theta,\underset{\sim}{\sigma}\rangle = i(\underset{\sim}{\sigma}\cdot\underset{\sim}{J})\,|\underset{\sim}{p},\theta,\underset{\sim}{\sigma}\rangle . \tag{B33}$$

It is readily shown that the only solution of Eq. (B33), subject to the condition that $|\underset{\sim}{p},0,\underset{\sim}{\sigma}\rangle = |\underset{\sim}{p}\rangle$, is the righthand side of Eq. (B31)

Since the kets $|\underset{\sim}{p}\rangle$ span the Hilbert space, the identity is given by

$$I = \int |\underset{\sim}{p}\rangle\langle\underset{\sim}{p}|\,\frac{d\underset{\sim}{p}}{a(\underset{\sim}{p})} , \tag{B34}$$

where $a(\underset{\sim}{p})$ is a positive function of its arguments, determined by the requirement that the representation be unitary.

Lemma

The function $a(\underset{\sim}{p})$ is given by

$$a(\underset{\sim}{p}) = p^3 . \tag{B35}$$

Proof

From Eqs. (B31) and (B34)

$$R(\underset{\sim}{\theta}) = R(\underset{\sim}{\theta}) \cdot \underset{\sim}{I} = \int R(\underset{\sim}{\theta}) |\underset{\sim}{p}\rangle\langle\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{a(\underset{\sim}{p})}$$

$$= \int \exp\left[-2i\alpha\Phi(-\underset{\sim}{\theta},\underset{\sim}{\eta})\right] | R_M(\underset{\sim}{\theta})\underset{\sim}{p}\rangle\langle\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{a(\underset{\sim}{p})} . \tag{B36}$$

Hence,

$$R^{\dagger}(\underset{\sim}{\theta}) = \int \exp\left[2i\alpha\,\Phi(-\underset{\sim}{\theta},\underset{\sim}{\eta})\right] | \underset{\sim}{p}\rangle\langle R_M(\underset{\sim}{\theta})\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{a(\underset{\sim}{p})} . \tag{B37}$$

But

$$R(\underset{\sim}{\theta})R^{\dagger}(\underset{\sim}{\theta}) = \int \exp\left[2i\alpha\Phi(-\underset{\sim}{\theta},\underset{\sim}{\eta})\right] \; R(\underset{\sim}{\theta}) | \underset{\sim}{p}\rangle\langle R_M(\underset{\sim}{\theta})\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{a(\underset{\sim}{p})}$$

or

$$R(\underset{\sim}{\theta})R^{\dagger}(\underset{\sim}{\theta}) = \int | R_M(\underset{\sim}{\theta})\underset{\sim}{p}\rangle\langle R_M(\underset{\sim}{\theta})\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{a(\underset{\sim}{p})} = I . \tag{B38}$$

In Eq. (B38) we define the new variable integration $\underset{\sim}{k}$ as

$$\underset{\sim}{k} = R_M(\underset{\sim}{\theta})\underset{\sim}{p}, \quad d\underset{\sim}{p} = d\underset{\sim}{k} . \tag{B39}$$

After substituting into Eq. (B38), we replace the dummy variable $\underset{\sim}{k}$ by $\underset{\sim}{p}$. Then Eq. (B38) becomes

$$\int | \underset{\sim}{p}\rangle\langle\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{a(R_M(-\underset{\sim}{\theta})\underset{\sim}{p})} = I . \tag{B40}$$

Comparing Eqs. (B40) and (B34) shows

$$a(\underset{\sim}{p}) = a(R_M(-\underset{\sim}{\theta})\underset{\sim}{p}) \tag{B41}$$

for all $\underset{\sim}{\theta}$. Let us pick $\underset{\sim}{\theta} = \underset{\sim}{\omega}$ of Eq. (B14). Then

$$a(\underset{\sim}{p}) = a(0,0,p) \equiv c(p) , \tag{B42}$$

that is, $a(\underset{\sim}{p})$ depends on $p = |\underset{\sim}{p}|$ only. Thus, we now write

$$I = \int |p\rangle\langle p| \frac{d\underset{\sim}{p}}{c(p)} . \tag{B43}$$

Now, from Eq. (B17),

$$S(\lambda) = S(\lambda)\underset{\sim}{\cdot} I = \int S(\lambda)|\underset{\sim}{p}\rangle\langle \underset{\sim}{p}| \frac{d\underset{\sim}{p}}{c(p)}$$

$$= \int |e^{-\lambda}\underset{\sim}{p}\rangle\langle \underset{\sim}{p}| \frac{d\underset{\sim}{p}}{c(p)} . \tag{B44}$$

Thus,

$$S^{\dagger}(\lambda) = \int |\underset{\sim}{p}\rangle\langle e^{-\lambda}\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{c(p)} \tag{B45}$$

and

$$S(\lambda)S^{\dagger}(\lambda) = \int S(\lambda)|\underset{\sim}{p}\rangle\langle e^{-\lambda}\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{c(p)} = I ,$$

or,

$$I = \int |e^{-\lambda}\underset{\sim}{p}\rangle\langle e^{-\lambda}\underset{\sim}{p}| \frac{d\underset{\sim}{p}}{c(p)} . \tag{B46}$$

In Eq. (B46) let us define the new variables of integration $\underset{\sim}{k}$ as

$$\underset{\sim}{k} = e^{-\lambda}\underset{\sim}{p}, \quad d\underset{\sim}{k} = e^{-3\lambda} d\underset{\sim}{p} . \tag{B47}$$

After using the new variable of integration in Eq. (B46) and after replacing the dummy variable $\underset{\sim}{k}$ by $\underset{\sim}{p}$, we have

$$I = \int |\underset{\sim}{p}\rangle\langle \underset{\sim}{p}| \frac{d\underset{\sim}{p}}{c(e^{\lambda}p)} \cdot e^{3\lambda} . \tag{B48}$$

Then, from Eq. (B43),

$$c(p) = e^{-3\lambda} c(e^{\lambda} p) . \tag{B49}$$

Equation (B49) is true for all λ. Let us pick $\lambda = -\mu$, where μ is given by Eq. (B14). Then

$$c(p) = p^3 c(1) = p^3 D , \tag{B50}$$

where $D \equiv c(1)$ is a positive constant. On replacing the kets $|\underset{\sim}{p}\rangle$ by $D^{1/2}|\underset{\sim}{p}\rangle$ the lemma is proved.

The adjoint relations of Eqs. (B16), (B17), and (B31) are

$$\langle \underset{\sim}{p} | T(\underset{\sim}{a}) = e^{i\underset{\sim}{a}\cdot\underset{\sim}{p}} \langle \underset{\sim}{p} | ,$$

$$\langle \underset{\sim}{p} | S(\lambda) = \langle e^{\lambda} \underset{\sim}{p} | ,$$

$$\langle \underset{\sim}{p} | R(\underset{\sim}{\theta}) = \exp\left[2 i \alpha \, \Phi(\underset{\sim}{\theta}, \underset{\sim}{n})\right] \langle R_M(-\underset{\sim}{\theta}) \underset{\sim}{p} | . \tag{B51}$$

Let us denote an abstract vector in the Hilbert space by $|\chi\rangle$ and its representative in the basis by $f(\underset{\sim}{p})$, as

$$f(\underset{\sim}{p}) = \langle \underset{\sim}{p} | \chi \rangle . \tag{B52}$$

Then, since

$$T(\underset{\sim}{a}) f(\underset{\sim}{p}) = \langle \underset{\sim}{p} | T(\underset{\sim}{a}) | \chi \rangle , \quad S(\lambda) f(\underset{\sim}{p}) = \langle \underset{\sim}{p} | S(\lambda) | \chi \rangle ,$$

$$R(\underset{\sim}{\theta}) f(\underset{\sim}{p}) = \langle \underset{\sim}{p} | R(\underset{\sim}{\theta}) | \chi \rangle ,$$

Eq. (3) follows immediately from Eq. (B51).
Likewise, the invariant inner product of two vectors $|\chi\rangle$ and $|\chi'\rangle$ is, from Eqs. (B34) and (B35),

$$\langle \chi' | \chi \rangle = \langle \chi' | I | \chi \rangle$$

$$= \int f'^*(\underset{\sim}{p}) f(\underset{\sim}{p}) \frac{d\underset{\sim}{p}}{p^3} ,$$

where $f'(\underset{\sim}{p}) = \langle \underset{\sim}{p} | \chi' \rangle$ is just the inner product Eq. (29).

B4. THE DISCRETE HELICITY REPRESENTATIONS

The discrete helicity representations that we shall use are given in a nonseparable Hilbert space. They are obtained from the continuous helicity representations by replacing the continuous variable $\underset{\sim}{p}$ in the function $f(\underset{\sim}{p})$ by a discrete variable and by requiring $f(\underset{\sim}{p})$ to vanish except for a denumerable set of values of $\underset{\sim}{p}$. The group elements in these representations are required to have the same form as in the continuous helicity representations but the inner product is changed to correspond to Eq. (31) in order that the representations may be unitary.

Without going into detail, we remark that the infinitesimal generators P_i exist in these representations. They are Hermitian and have a discrete spectrum consisting of the entire real axis.

Appendix C

Derivation of the Expansion of Physical Quantities in the Irreducible Representations of the Scale-Euclidean Group

We shall now derive Eq. (45), which is the expansion of a physical quantity in terms of the irreducible representations of the scale-Euclidean group. The expansion is unique if we restrict ourselves to expansions in a separable Hilbert space, that is, to those for which $f_d^{j,n}(\underset{\sim}{p}, \alpha) \equiv 0$.

Let us first expand $G_c(\underset{\sim}{x})$ of Eq. (32), as follows:

$$G(\underset{\sim}{x}, \gamma) = \sum_{j,m,n} W(\gamma \mid j,m,n) H^{j,n}(\underset{\sim}{x}, m) . \tag{C1}$$

We now expand $H^{j,n}(\underset{\sim}{x}, m)$ in the irreducible representations of the scale-Euclidean group and then substitute into (C1) to obtain the expansion of the physical quantity $G_c(\underset{\sim}{x})$. For simplicity, let us drop the superscripts j, n on $H^{j,n}(\underset{\sim}{x}, m)$.

We regard $H(\underset{\sim}{x}, m)$ as the abstract vector $|\chi\rangle$ given in a representation in a space spanned by the kets $|\underset{\sim}{x}, m\rangle$. Thus,

$$H(\underset{\sim}{x}, m) \equiv \langle \underset{\sim}{x}, m \mid \chi \rangle . \tag{C2}$$

From Eqs. (24a), (25a), (26a), and (34), the group acts on the bras $\langle \underset{\sim}{x}, m|$ in the following way:

$$\langle \underset{\sim}{x}, m | T(\underset{\sim}{a}) = \langle \underset{\sim}{x}+\underset{\sim}{a}, m | ,$$

$$\langle \underset{\sim}{x}, m | S(\lambda) = e^{N\lambda} \langle e^{-\lambda} \underset{\sim}{x}, m | ,$$

$$\langle \underset{\sim}{x}, m | R(\underset{\sim}{\theta}) = \sum_{m'} R^{(j)}(\underset{\sim}{\theta})(m,m') \langle R_M(-\underset{\sim}{\theta}) \underset{\sim}{x}, m' | . \tag{C3}$$

The expansion of $\Pi(\underset{\sim}{x}, m)$ is

$$\langle \underset{\sim}{x}, n | \chi \rangle = \sum_{k,m'} \langle \underset{\sim}{x}, m | k, m' \rangle \langle k, m' | \chi \rangle + \sum_{\alpha} \int \langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha \rangle \langle \underset{\sim}{p}, \alpha | \chi \rangle \frac{d\underset{\sim}{p}}{p^3} +$$

$$+ \sum_{\alpha} \sum_{\underset{\sim}{p}} \langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha) (\underset{\sim}{p}, \alpha | \chi \rangle . \tag{C4}$$

In Eq. (C4) the kets $|k, m\rangle$ are the kets spanning the space for the irreducible rotation group representations of the scale-Euclidean group, where k denotes the representation of the rotation group that is involved and m is the variable m = -k, -k+1, ..., k-1, k. The representative $\langle k, m | \chi \rangle$ is $c^k(m)$ in the expansion (45). We have suppressed the index n here and shall often suppress it in what follows.

The kets $|\underset{\sim}{p}, \alpha\rangle$ are the kets of the irreducible continuous helicity representations for helicity α. The kets $|\underset{\sim}{p}, \alpha)$ are the corresponding kets for the discrete helicity representations. The representatives $\langle \underset{\sim}{p}, \alpha | \chi \rangle$ and $(\underset{\sim}{p}, \alpha | \chi \rangle$ are the functions $f_c^{j,n}(\underset{\sim}{p}, \alpha)$ and $f_d^{j,n}(\underset{\sim}{p}, \alpha)$, respectively, in the expansion (45).

Our objective is to find the transformation kernels $\langle \underset{\sim}{x}, m | k, m' \rangle$; $\langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha \rangle$; $\langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha)$.

To find $\langle \underset{\sim}{x}, m | k, m' \rangle$, we note from the discussion in Appendix B2 that $|k, m'\rangle$ can be so chosen that

$$T(\underset{\sim}{a}) | k, m \rangle = | k, m \rangle, \quad S(\lambda) | k, m \rangle = e^{i d\lambda} | k, m \rangle,$$

$$R(\underset{\sim}{\theta}) | k, m \rangle = \sum_{m'} R^{(k)}(\underset{\sim}{\theta})(m', m) | k, m' \rangle . \tag{C5}$$

From Eqs. (C3) and (C5),

$$\langle \underset{\sim}{x}, m | T(\underset{\sim}{a}) | k, m' \rangle = \langle \underset{\sim}{x} + \underset{\sim}{a}, m | k, m' \rangle = \langle \underset{\sim}{x}, m | k, m' \rangle \tag{C6}$$

for all $\underset{\sim}{a}$. In the second of Eq. (C6) we take $\underset{\sim}{a} = -\underset{\sim}{x}$. Then

$$\langle \underset{\sim}{x}, m | k, m' \rangle = G^{(k)}(m, m') , \tag{C7}$$

and is independent of $\underset{\sim}{x}$.

Again, from Eqs. (C3) and (C5)

$$\langle \underset{\sim}{x}, m | R(\underset{\sim}{\theta}) | k, m' \rangle = \sum_{m''} R^{(j)}(\underset{\sim}{\theta})(m, m'') \langle R_M(-\underset{\sim}{\theta}) \underset{\sim}{x}, m'' | k, m' \rangle$$

$$= \sum_{m''} R^{(k)}(\underset{\sim}{\theta})(m'', m') \langle \underset{\sim}{x}, m | k, m'' \rangle . \tag{C8}$$

On using Eq. (C7), we get

$$\sum_{m''} R^{(j)}(\underset{\sim}{\theta})(m, m'') G^{(k)}(m'', m') = \sum_{m''} R^{(k)}(\underset{\sim}{\theta})(m'', m') G^{(k)}(m, m''). \tag{C9}$$

Because the matrices that represent the rotation group in different representations are linearly independent, as are the rows and columns of such matrices, it follows that

$$G^{(k)}(m, m') = C \delta_{m, m'} \delta_{j, k} , \tag{C10}$$

where C is a constant that may depend on j and n. This constant can always be set equal to unity by absorbing it into $\langle k, m | \chi \rangle$ in the expansion (C4). Thus, our final expression is

$$\langle \underset{\sim}{x}, m | k, m' \rangle = \delta_{m, m'} \delta_{j, k} . \tag{C11}$$

We now want to find the value of d in Eq. (C5), which will complete the description of the representation of the scale-Euclidean group. From Eqs. (C3) and (C5),

$$\langle \underset{\sim}{x}, m | S(\lambda) | k, m' \rangle = e^{N\lambda} \langle e^{-\lambda} \underset{\sim}{x}, m | k, m' \rangle = e^{i d \lambda} \langle \underset{\sim}{x}, m | k, m' \rangle . \tag{C12}$$

On using Eq. (C11) we have, finally,

$$d = -iN . \tag{C13}$$

We shall now find $\langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha \rangle$. Let us use $|\alpha\rangle$ to denote the ket for which $\underset{\sim}{p} = (0, 0, 1) = \underset{\sim}{q}_o$. Then, from Eqs. (C16) and (C3),

$$\langle \underset{\sim}{x}, m | T(\underset{\sim}{a}) | \alpha \rangle = \langle \underset{\sim}{x} + \underset{\sim}{a}, m | \alpha \rangle = \exp\left[i a_3\right] \langle \underset{\sim}{x}, m | \alpha \rangle \tag{C14}$$

for all $\underset{\sim}{a}$. Now let $\underset{\sim}{a} = -\underset{\sim}{x}$. Then

$$\langle \underset{\sim}{x}, m | \alpha \rangle = e^{iz} G(m, \alpha) , \tag{C15}$$

where $G(m, \alpha) = \langle 0, m | \alpha \rangle$ is independent of $\underset{\sim}{x}$.

From Eqs. (C30) and (C3),

$$\langle \underset{\sim}{x}, m | \exp\left[i \theta J_3\right] | \alpha \rangle = e^{i \theta m} \langle \underset{\sim}{x}', m, \alpha \rangle = e^{i \theta \alpha} \langle \underset{\sim}{x}, m | \alpha \rangle , \tag{C16}$$

where

$$x_3' = x_3 - z, \quad x_1' = x_1 \cos\theta - x_2 \sin\theta , \quad x_2' = x_2 \cos\theta + x_1 \sin\theta . \tag{C16a}$$

Using (C15) and (C16), we now have

$$\exp\left[i(m - \alpha)\theta\right] G(m, \alpha) = G(m, \alpha) \tag{C17}$$

for all θ. It follows that

$$G(m, \alpha) = C(\alpha)\, \delta_{m, \alpha}$$

and

$$\langle \underset{\sim}{x}, m | \alpha \rangle = e^{iz} C(\alpha)\, \delta_{m, \alpha} . \tag{C18}$$

Now, from Eqs. (C13) and (C3),

$$\langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha \rangle = \langle \underset{\sim}{x}, m | R(\underset{\sim}{\omega}) S(-\mu) | \alpha \rangle$$

$$= e^{-N\mu} \sum_{m'} R^{(j)}(\underset{\sim}{\omega})(m, m') \langle x', m' | \alpha \rangle \ , \tag{C19}$$

where

$$\underset{\sim}{x}' = e^{\mu} R_M(-\underset{\sim}{\omega}) \underset{\sim}{x} \ . \tag{C19a}$$

On using Eq. (C18) in Eq. (C19), and also using Eqs. (C14), (A5), and (A6a), we get

$$\langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha \rangle = p^{-N} C(\alpha) e^{i \underset{\sim}{p} \cdot \underset{\sim}{x}} R^{(j)}(\underset{\sim}{\omega})(m, \alpha) = p^{-N} C(\alpha) e^{i \underset{\sim}{p} \cdot \underset{\sim}{x}} Y_j^{m,\alpha *}(\theta, \phi) \left(\frac{4\pi}{2j+1} \right)^{1/2} , \tag{C20}$$

where θ, ϕ are the polar angles of $\underset{\sim}{p}$, as before. By redefining $\langle \underset{\sim}{p}, \alpha | \chi \rangle$, the quantity $C(\alpha)$ can be set equal to $(2\pi)^{-3/2}$.

The transformation function $\langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha \rangle$ is found in the same way as $\langle \underset{\sim}{x}, m | \underset{\sim}{p}, \alpha \rangle$ and has the same functional form.

Finally, on substituting the expansion (C4) into Eq. (C1) with the explicit forms for the transformation functions, the expansion Eq. (45) is obtained.

Appendix D

A Table of Generalized Surface Harmonics for j =0, 1, 2

$$Y_0^{0,0}(\theta,\phi) = (4\pi)^{-1/2}$$

$$Y_1^{1,1}(\theta,\phi) = \frac{1}{4}(3/\pi)^{1/2}(1+\cos\theta)$$

$$Y_1^{1,0}(\theta,\phi) = -(8)^{-1/2}(3/\pi)^{1/2}e^{i\phi}\sin\theta$$

$$Y_1^{1,-1}(\theta,\phi) = \frac{1}{4}(3/\pi)^{1/2}e^{2i\phi}(1-\cos\theta)$$

$$Y_1^{0,1}(\theta,\phi) = (8)^{-1/2}(3/\pi)^{1/2}e^{-i\phi}\sin\theta$$

$$Y_1^{0,0}(\theta,\phi) = \frac{1}{2}(3/\pi)^{1/2}\cos\theta$$

$$Y_1^{0,-1}(\theta,\phi) = -(8)^{-1/2}(3/\pi)^{1/2}e^{i\phi}\sin\theta$$

$$Y_1^{-1,1}(\theta,\phi) = \frac{1}{4}(3/\pi)^{1/2}e^{-2i\phi}(1-\cos\theta)$$

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$$Y_1^{-1,0}(\theta,\phi) = (8)^{-1/2}(3/\pi)^{1/2} e^{-i\phi} \sin\theta$$

$$Y_1^{-1,-1}(\theta,\phi) = \frac{1}{4}(3/\pi)^{1/2}(1+\cos\theta)$$

$$Y_2^{2,2}(\theta,\phi) = \frac{1}{8}(5/\pi)^{1/2}(1+\cos\theta)^2$$

$$Y_2^{2,1}(\theta,\phi) = -\frac{1}{4}(5/\pi)^{1/2} e^{i\phi} \sin\theta\,(1+\cos\theta)$$

$$Y_2^{2,0}(\theta,\phi) = (3/32)^{1/2}(5/\pi)^{1/2} e^{2i\phi} \sin^2\theta$$

$$Y_2^{2,-1}(\theta,\phi) = -\frac{1}{4}(5/\pi)^{1/2} e^{3i\phi} \sin\theta\,(1-\cos\theta)$$

$$Y_2^{2,-2}(\theta,\phi) = \frac{1}{8}(5/\pi)^{1/2} e^{4i\phi}(1-\cos\theta)^2$$

$$Y_2^{1,2}(\theta,\phi) = \frac{1}{4}(5/\pi)^{1/2} e^{-i\phi} \sin\theta\,(1+\cos\theta)$$

$$Y_2^{1,1}(\theta,\phi) = -\frac{1}{4}(5/\pi)^{1/2}(1+\cos\theta)(1-2\cos\theta)$$

$$Y_2^{1,0}(\theta,\phi) = -(3/8)^{1/2}(5/\pi)^{1/2} e^{i\phi} \sin\theta\cos\theta$$

$$Y_2^{1,-1}(\theta,\phi) = \frac{1}{4}(5/\pi)^{1/2} e^{2i\phi}(1-\cos\theta)(1+2\cos\theta)$$

$$Y_2^{1,-2}(\theta,\phi) = -\frac{1}{4}(5/\pi)^{1/2} e^{3i\phi} \sin\theta\,(1-\cos\theta)$$

$$Y_2^{0,2}(\theta,\phi) = (3/32)^{1/2}(5/\pi)^{1/2} e^{-2i\phi} \sin^2\theta$$

$$Y_2^{0,1}(\theta,\phi) = (3/8)^{1/2}(5/\pi)^{1/2} e^{-i\phi} \sin\theta\cos\theta$$

$$Y_2^{0,0}(\theta,\phi) = -\frac{1}{4}(5/\pi)^{1/2}(1-3\cos^2\theta)$$

$$Y_2^{0,-1}(\theta,\phi) = -(3/8)^{1/2}(5/\pi)^{1/2} e^{i\phi} \sin\theta \cos\theta$$

$$Y_2^{0,-2}(\theta,\phi) = (3/32)^{1/2}(5/\pi)^{1/2} e^{2i\phi} \sin^2\theta$$

$$Y_2^{-1,2}(\theta,\phi) = \frac{1}{4}(5/\pi)^{1/2} e^{-3i\phi} \sin\theta(1-\cos\theta)$$

$$Y_2^{-1,1}(\theta,\phi) = \frac{1}{4}(5/\pi)^{1/2} e^{-2i\phi}(1-\cos\theta)(1+2\cos\theta)$$

$$Y_2^{-1,0}(\theta,\phi) = (3/8)^{1/2}(5/\pi)^{1/2} e^{-i\phi} \sin\theta \cos\theta$$

$$Y_2^{-1,-1}(\theta,\phi) = -\frac{1}{4}(5/\pi)^{1/2}(1+\cos\theta)(1-2\cos\theta)$$

$$Y_2^{-1,-2}(\theta,\phi) = -\frac{1}{4}(5/\pi)^{1/2} e^{i\phi} \sin\theta(1+\cos\theta)$$

$$Y_2^{-2,2}(\theta,\phi) = \frac{1}{8}(5/\pi)^{1/2} e^{-4i\phi}(1-\cos\theta)^2$$

$$Y_2^{-2,1}(\theta,\phi) = \frac{1}{4}(5/\pi)^{1/2} e^{-3i\phi} \sin\theta(1-\cos\theta)$$

$$Y_2^{-2,0}(\theta,\phi) = (3/32)^{1/2}(5/\pi)^{1/2} e^{-2i\phi} \sin^2\theta$$

$$Y_2^{-2,-1}(\theta,\phi) = \frac{1}{4}(5/\pi)^{1/2} e^{-i\phi} \sin\theta(1+\cos\theta)$$

$$Y_2^{-2,-2}(\theta,\phi) = \frac{1}{8}(5/\pi)^{1/2}(1+\cos\theta)^2$$

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