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INTERNAL WAVE-SURFACE WAVE INTERACTIONS REVISITED

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ABSTRACT

The interaction of internal waves and surface waves in water is explored in the regions where the effects of the interaction are small. Explicit solutions for an arbitrary internal wave are given for various kinds of boundary conditions, with a detailed discussion of the conditions under which they are valid. The connection of the solutions with the conservation equations is made explicit, and the modifications produced by viscous damping are outlined.

CONTENTS

I. INTRODUCTION

This paper is to be viewed as a sequel to an earlier work on the internal wave-surface wave interaction in water, which appeared in IDA/JASON Study S-334.* We wish to enlarge on the previous analysis in the following ways:

1. Clarify the assumptions involved in the calculation.
2. Correct the result in (I) by including a term which was unjustifiably neglected.
3. Extend the solution to ranges of the parameters not treated in (I).
4. Establish the connection with the approach to the problem based on the conservation equations, and
5. Include the effect of viscous damping of the surface waves.

Let us begin with a brief recapitulation of the problem which we wish to discuss. We assume the existence originally of a surface wave, that is, of a displacement of the surface of an infinite ocean, given by $h_0(x,t) = A_0 \exp i(k_0 x - \omega_0 t)$ with $\omega_0 = \sqrt{gk_0}$.** An internal wave with velocity U--which is

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*The complete reference is IDA/JASON Study S-334, Generation and Airborne Detection of Internal Waves from an Object Moving Through a Stratified Ocean, JASON 1968 summer study, Vol. II, April 1969, p. 69. We shall refer to this as (I) in what follows.

**We shall, for simplicity, assume there is only one horizontal dimension (denoted x). The depth variable is called z, and is measured positive up. We are also going to confine ourselves to gravity waves--hence (since the water is assumed to be infinitely deep) we take $\omega_0 = \sqrt{gk_0}$. This limits us to values of $k < 3\ \text{cm}^{-1}$.

assumed to be a given, but arbitrary, function of space and time--is turned on, and this produces, for later times, a change $\delta h(x,t)$ in the surface displacement. The problem is to calculate δh.

We shall carry out the calculation under the following assumptions:

1. Small amplitude surface waves:

   $$kA << 1 \qquad (1.1)$$

2. Small internal wave velocities:

   $$U/c_g << 1 \qquad (1.2)$$

   where $c_g = \frac{1}{2}\sqrt{g/k}$ is the group velocity of the surface wave.

3. Long wavelength and long period internal waves:

   $$K << k \text{ and } \Omega << \omega \qquad (1.3)$$

   where

   $$K \sim \frac{1}{U}\frac{\partial U}{\partial X} \text{ and } \Omega \sim \frac{1}{U}\frac{\partial U}{\partial t}$$

   are the wave number and frequency of the internal wave.

4. The internal wave effects on the surface wave are small:

   $$\frac{\delta A}{A} << 1,\ \delta k/k << 1,\ \frac{\delta\omega}{\omega} << 1 \qquad (1.4)$$

   Thus a perturbation treatment of the effect is appropriate.

5. In (I), we also made the assumption that the time of interaction between the surface wave and internal wave was short:

   $$(K/k)\ (\omega t) << 1 \qquad (1.5)$$

   If viscosity damps the surface wave in a time $1/\sigma$, this condition is satisfied if $(K/k)(\omega/\sigma) << 1$.

Assumption (5) was made for convenience in (I), but was not essential to the treatment. We shall therefore, here, also obtain the solution without imposing this constraint. We shall, however, still find it useful in practice to keep a weaker limit on the time, namely $(K/k)^4(\omega t)^2 << 1$; to relax this requirement too is possible, but prevents us from exhibiting the solution in a very explicit form.

A typical set of numbers with which we may be concerned is as follows:

$k = 1\ cm^{-1}$

$A = 0.1\ cm$

$U = 1\ cm/sec$

$K = 10^{-4}\ cm^{-1}$

$C = 15\ cm/sec$

For $k = 1\ cm^{-1}$, we have $\sigma = 3 \times 10^{-2}/sec$, $c_g = 15\ cm/sec$ and $\omega = 30\ sec^{-1}$.

With these parameters all of our conditions would appear to be met: we have

$$kA = 0.1$$

$$U/c_g = 0.07$$

$$K/k = 10^{-4}$$

$$\Omega/\omega = 5 \times 10^{-5}$$

$$(K/k) \cdot \omega/\sigma = 0.1$$

The question of how well this idealized problem imitates the real ocean is of course a serious one. Probably the most important physical effect that has been ignored entirely is wind. We have pretended that the wind enters only indirectly, in that it is responsible for the generation of the surface waves, but that once generated, the surface wave is no longer seriously influenced by the wind and is merely eroded by viscosity in some finite distance--or time--depending on its wavelength. This is evidently a great idealization.

The next most serious difficulty would appear to be in the validity of the $kA \ll 1$ assumption. Nonlinear effects in the surface wave amplitude may well become important as $\delta A/A$ increases from zero, and invalidate the treatment given here even when $\delta A/A$ is still quite small.

II. BASIC EQUATIONS

To obtain our basic equations we first write the equations for a free surface:

$$\frac{\partial \bar{h}}{\partial t} + \frac{\partial \bar{\phi}}{\partial x}\frac{\partial \bar{h}}{\partial x} = \frac{\partial \bar{\phi}}{\partial z}$$

$$\frac{\partial \bar{\phi}}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \bar{\phi}}{\partial x}\right)^2 + \left(\frac{\partial \bar{\phi}}{\partial z}\right)^2\right] = -g\bar{h} \ ,$$

both evaluated at $z = \bar{h}$. The notation is obvious: $\bar{h}$ is the surface displacement and $\bar{\phi}$ is the velocity potential. We have, in addition

$$\nabla^2\bar{\phi} = 0$$

and the boundary condition that $\bar{\Phi} \to \Phi$, where $\vec{\nabla}\Phi = \vec{U}$, as $z \to -\infty$. Φ is assumed to be a given function, and represents the velocity potential of the internal wave.

The next step is to expand the surface equations to second order in $\bar{h}$. Then we write

$$\bar{h} = h + H$$

$$\bar{\phi} = \phi + \Phi$$

and keep all first-order terms in either h or ϕ. We choose H and Φ (which are of course to be identified with the internal wave) to satisfy the free surface equations by themselves (to first order in H--and we furthermore assume that the z-dependence in the internal wave is weak.) We then obtain equations for h and ϕ, viz:

$$\frac{\partial h}{\partial t} + U\frac{\partial h}{\partial x} + h\frac{\partial U}{\partial x} = \frac{\partial \phi}{\partial z} + \frac{1}{g}\frac{\partial \phi}{\partial x}\frac{\partial U}{\partial t} \tag{2.1}$$

$$\frac{\partial\phi}{\partial t} + U\frac{\partial\phi}{\partial x} = -gh \tag{2.2}$$

evaluated at z = 0, together with

$$\nabla^2\phi = 0$$

and the boundary condition $\phi \to 0$ as $z \to -\infty$. The symbol U stands for the x component of the internal wave velocity $\vec{U}$.

We have thus far made use of assumption 1.1, the small amplitude assumption.

Equations 2.1 and 2.2 are our starting point; comparing with (I) we note the presence in Eq. 2.1 of the term $h\ \partial U/\partial x$. In (I) this was neglected in accord with assumption 1.3 with the argument that it was small compared to $U\ \partial h/\partial x$. In magnitude, this is true; but the two terms are of different phase, and hence it is not legitimate to ignore $h\ \partial U/\partial x$. This fact will become more transparent later.

The next convenient thing to do is to eliminate ϕ between Eqs. 2.1 and 2.2 to obtain an equation for h alone, which is, after all, the quantity of primary interest. This is made feasible through $\nabla^2\phi = 0$ and the fact that $\phi \to 0$ as $z \to -\infty$, which permits us to write

$$\phi(x,z,t) = \int \frac{dk}{2\pi}\ \phi(k,t)\ e^{ikx}\ e^{|k|z}$$

Hence (we take k positive) we have

$$\partial\phi/\partial z = -i\ \partial\phi/\partial x \tag{2.3}$$

We can now get rid of ϕ. Equations 2.1 and 2.2, together with assumptions 1.2 and 1.3, yield

$$\left(\frac{\partial^2}{\partial t^2} - ig\frac{\partial}{\partial x}\right)h = -2U\frac{\partial^2 h}{\partial x\partial t} - 2\frac{\partial U}{\partial t}\frac{\partial h}{\partial x} - 2\frac{\partial U}{\partial x}\frac{\partial h}{\partial t} \tag{2.4}$$

We have neglected all quadratic terms in U, and all second and higher derivatives of U, in obtaining this result. The boundary condition which we associate with Eq. 2.4 is that for $t > 0$, $h = h_o(x,t)$, some specified surface wave present before the internal wave was turned on. Normally, we shall take h_o to be simply

$$h_o = A_o \, e^{i(k_o x - \omega_o t)} \tag{2.5}$$

where

$$\omega_o = \sqrt{g k_o}$$

III. PERTURBATION SOLUTION

We assume that the effect of the internal wave on the surface wave which was present initially will be small (Assumption 1.4). We may then write

$$h = h_o + \delta h, \; h_o >> \delta h \tag{3.1}$$

where, to first order in the perturbation,

$$\left(\frac{\partial^2}{\partial t^2} - ig\frac{\partial}{\partial x}\right)\delta h = -2U\frac{\partial^2 h_o}{\partial x \partial t} - 2\frac{\partial U}{\partial t}\frac{\partial h_o}{\partial x} - 2\frac{\partial U}{\partial x}\frac{\partial h_o}{\partial t} \; ;$$

for the simple choice (2.5), this simplifies to

$$\left(\frac{\partial^2}{\partial t^2} - ig\frac{\partial}{\partial x}\right)\delta h = \left\{-2k_o\omega_o U + 2i\omega_o\frac{\partial U}{\partial x} - 2ik_o\frac{\partial U}{\partial t}\right\} h_o$$

$$\equiv Z(x,t)\, h_o\,(x,t) \tag{3.2}$$

This will be recognized as the same equation as that used in (I) except that the coefficient of $\partial U/\partial x$ in (I) was $i\omega_o$ rather than $2i\omega_o$. The factor of 2 reflects the existence of the $h\,\partial U/\partial x$ term in Eq. 2.1.

It may be objected that the derivation of Eq. 3.2 is invalid, since it relied on the assumption that $h_o >> \delta h$, and for certain values of x and t (for which h_o vanishes) this inequality is false. However, these values constitute a very small range of x and t, and do not, in fact, affect the correctness of Eq. 3.2. This will be made explicit in Section IV, where we will obtain the same result without use of the assumption that $h_o >> \delta h$.

In any event, our problem is now a straightforward one, namely to solve Eq. 3.2 with the boundary condition that $\delta h = 0$ for $t < 0$. This can be done directly by the method used in (I). Let Δ be the retarded Green's function, so that

$$\left(\frac{\partial^2}{\partial t^2} - i\omega \frac{\partial}{\partial x}\right) \Delta = \delta(x)\ \delta(t) \qquad (3.3)$$

with $\Delta = 0$ for $t < 0$.
Then

$$\delta h(x,t) = \iint dx'dt'\ \Delta(x-x',t-t')\ Z(x',t')\ h_o(x',t')$$

and using the (obvious) solution to Eq. 3.3 together with Eq. 2.5, we find

$$\frac{\delta h(x,t)}{h_o(x,t)} = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} \frac{dk}{2\pi}\ Z(k,t')\ e^{ikx} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\ \frac{e^{-i\omega(t-t')}}{gk-2\omega\omega_o - \omega^2} \qquad (3.4)$$

where

$$Z(k,t) \equiv \int_{-\infty}^{\infty} dx\ e^{-ikx}\ Z(x,t)\ .$$

The integral over $d\omega$ in Eq. 3.4 is to be taken above the poles of the denominator in the complex ω plane because Δ is the retarded Green's function.

Up to this point we have invoked assumptions 1.1, 1.2, 1.3 and 1.4 but have made no use of 1.5. Equation 3.4 constitutes a complete solution to the problem; however for a given (arbitrary) U some numerical integrations will be required to obtain a solution. It is therefore of some interest to try to simplify 3.4.

To do this we note that from assumption 1.3, U, and hence Z, may be expected to have only small frequency components present. Hence we may expect ω to be small, so we may expand the denominator in Eq. 3.4:

$$\frac{1}{gk-2\omega\omega_o-\omega^2} = \frac{1}{gk-2\omega\omega_o} + \frac{\omega^2}{(gk-2\omega\omega_o)^2} + \ldots \tag{3.5}$$

The first term in this expansion, after evaluation of the contour integral over $d\omega$, yields directly

$$\frac{\delta h}{h_o} = \frac{i}{2\omega_o}\int_{-\infty}^{t} dt'\, Z\left(x - c_g(t-t'),\ t'\right) \tag{3.6}$$

Let us assess the error involved in stopping at this point. To do this it is important to note that, from Eq. 3.2, Re Z is bigger than Im Z by a factor $\sim k/K$. Since the next term in the expansion is out of phase by 90^o with this term, the first term alone will be an excellent approximation to Im $\delta h/h_o$ but a less good one for Re $\delta h/h_o$. In fact the errors are evidently of order $(\omega_o t)\ (K/k_o)^3$ and $(\omega_o t)\ (K/k_o)$ for the imaginary and real parts, respectively.

The first term is thus a good approximation for the real part of $\delta h/h_o$; that is, for $\delta A/A_o$, if the "short time assumption," 1.5, is valid. It is a good approximation for the imaginary part, that is for $\delta\omega$ and δk, in any event.

In the case of short wavelength internal waves, or long wavelength surface waves, where assumption 1.5 may not be satisfied, we may keep the next term in the expansion 3.5. This leads to the result

$$\frac{\delta h}{h_o} = \frac{i}{2\omega_o} \int_{-\infty}^{t} Z\left(x - c_g\,(t-t'),\ t'\right)\,dt'$$

$$+ \frac{1}{4\omega_o{}^2} \int_{-\infty}^{t} (t-t')\,\frac{\partial^2}{\partial t'^2}\,Z(x',t')\Big|_{x'=x-c_g(t-t')}\,dt' \qquad (3.7)$$

We may again estimate the error involved in stopping here, and because of the alteration in phase by 90^o of successive terms in the expansion, it is clear that this is now an excellent approximation for both real and imaginary parts.

The successive terms are of the following magnitudes:

Imaginary part: $\left[1 + \left(\frac{t\Omega^2}{\omega_o}\right)\left(\frac{K}{k_o}\right) + \left(\frac{t\Omega^2}{\omega_o}\right)^2 + \ldots\right]$

Real part: $\left[\left(\frac{K}{k_o}\right) + \left(\frac{t\Omega^2}{\omega_o}\right) + \left(\frac{t\Omega^2}{\omega_o}\right)^2\left(\frac{K}{k_o}\right) + \ldots\right]$

Thus the neglected terms are of order $(\omega_o t)^2\ (\Omega/\omega_o)^4$ in both cases.

Finally, we may extract from Eq. .7 expressions for $\delta A/A_o$ and $\delta k/k_o$, the quantities of real interest. We find, using the definition of Z,

$$\frac{\delta k}{k_o} = - \int_{-\infty}^{t} \frac{\partial}{\partial x}\,U\left(x - c_g\,(t-t'),\ t'\right)\,dt' \qquad (3.8)$$

and

$$\frac{\delta A}{A_o} = \frac{U}{4c_g} - \int_{-\infty}^{t} \frac{\partial}{\partial x}\,U\left(x - c_g\,(t-t'),\ t'\right)\,dt'$$

$$- \frac{c_g}{4} \int_{-\infty}^{t} (t-t')\,\frac{\partial^2}{\partial x^2}\,U\left(x - c_g\,(t-t'),\ t'\right)\,dt' \qquad (3.9)$$

The last term in 3.9 may be dropped if the short time assumption is valid; we have neglected the corresponding term in $\delta k/k_0$ in any event.

Comparing 3.8 and 3.9 with the results given in (I), we note that the expression for δk is the same, but that for δA differs by factors of 2 (apart from the $\partial^2U/\partial x^2$ term); these changes are caused by the h $\partial U/\partial x$ term in the original equation with which we began.

IV. CONSERVATION EQUATIONS

We now return to our starting point, Eq. 2.4. Instead of making a perturbation expansion of this immediately, as we did in Section III, let us use it to derive equations for the amplitude, wave number and frequency directly. This is easily done with the following definitions:

$$h = Ae^{iX} \tag{4.1}$$

$$k = \partial X/\partial x \tag{4.2}$$

$$\omega = -\partial X/\partial t \tag{4.3}$$

The definitions 4.2 and 4.3 lead immediately to the so-called "conservation of waves":

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0 \tag{4.4}$$

We substitute 4.1 to 4.3 into 2.4, and separate 2.4 into real and imaginary parts, and find after a small amount of algebraic juggling that

$$\omega^2 - gk - \frac{1}{A}\frac{\partial^2 A}{\partial t^2} = 2k\omega\, U + 2\frac{\partial U}{\partial t}\frac{1}{A}\frac{\partial A}{\partial x} + 2\frac{\partial U}{\partial x}\frac{1}{A}\frac{\partial A}{\partial t} + 2U\left(\frac{1}{A}\frac{\partial^2 A}{\partial x \partial t}\right) \tag{4.5}$$

and

$$\frac{1}{A}\left(\frac{\partial A}{\partial t} + \frac{g}{2\omega}\frac{\partial A}{\partial x}\right) + \frac{1}{2\omega}\frac{\partial \omega}{\partial t}$$

$$= \frac{kU}{\omega}\frac{1}{A}\frac{\partial A}{\partial t} - U\frac{1}{A}\frac{\partial A}{\partial x} - \frac{U}{\omega}\frac{\partial \omega}{\partial x}$$

$$+ \frac{k}{\omega}\frac{\partial U}{\partial t} - \frac{\partial U}{\partial x} \qquad (4.6)$$

The first of these is the real part of 2.4, the second the imaginary part.

These two equations are recognizable as the dispersion relation and the conservation of energy, respectively. (One should multiply the second through by A^2 to make this obvious.)

Let us now introduce the perturbation expansion in U. Thus we write $A = A_o + \delta A$, $k = k_o + \delta k$, $\omega = \omega_o + \delta\omega$ and assume $\delta A << A_o$, etc. The zero-order equations obviously tell us that A_o = const, $\omega_o = \sqrt{gk_o}$ and k_o = const. The first-order equations are

$$\delta\omega = k_o U + c_g \delta k \qquad (4.7)$$

$$\left(\frac{\partial}{\partial t} + c_g\frac{\partial}{\partial x}\right)\delta k = -k_o\frac{\partial U}{\partial x} \qquad (4.8)$$

$$\left(\frac{\partial}{\partial t} + c_g\frac{\partial}{\partial x}\right)\frac{\delta A}{A_o} - \frac{c_g}{4k_o}\frac{\partial}{\partial x}\delta k = -\frac{3}{4}\frac{\partial U}{\partial x} + \frac{1}{4c_g}\frac{\partial U}{\partial t} \qquad (4.9)$$

where we define $c_g = g/2\omega_o = (1/2)\sqrt{g/k_o}$.

We have used 4.4 to eliminate $\delta\omega$ in deriving these, and have also neglected higher derivatives of δA.

First of all, these equations are immediately seen to coincide with Eq. 3.2, thereby confirming the use of $h_o >> \delta h$, in spite of the small range of x and t over which this may not be true. Second, these coincide with the usual results obtained from the conservation equations.*

Finally, the solutions may be written down by inspection, (incorporating the boundary conditions that δA and δk vanish when $t < 0$) and are seen to coincide exactly with Eqs. 3.8 and 3.9. The fact that the solutions to 4.8 and 4.9 agree exactly with 3.8 and 3.9, which are only the first two terms in the expansion of the solution Eq. 3.4 to 3.2, stems from the neglect of higher derivatives of δA in deriving 4.8 and 4.9. This observation should help to clarify what is being dropped when higher terms in the expansion 3.5 are neglected.

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*O. M. Phillips, Dynamics of the Upper Ocean, Cambridge, 1966, p. 50. Note that we have an additional term in $\partial U/\partial t$ not given by Phillips. Presumably he omits this because of the time averaging he uses to obtain the conservation equations.

V. PROPERTIES OF THE SOLUTIONS

Let us discuss some special cases. First, suppose the internal wave has a constant phase speed C, and is turned on at some time t_o, so that $U(x,t) = U(x - Ct)\ \theta(t - t_o)$. Then

$$\frac{\delta k}{k_o} = -\frac{1}{c_g - C} \int_{x-c_g t+(c_g-C)t_o}^{x-Ct} \frac{\partial U(y)}{\partial y}\, dy$$

$$= \frac{U(x - Ct) - U\big(x - c_g t + (c_g - C)\ t_o\big)}{C - c_g}$$

Furthermore,

$$\delta\omega = k_o\ U + c_g\ \delta k$$

$$= \frac{Ck_o}{C-c_g}\ U - \frac{c_g k_o}{C-c_g}\ U_o$$

Hence

$$\delta\omega - C\ \delta k = -\frac{c_g k_o}{C-c_g}\ U\big(x - c_g t + (c_g - C)\ t_o\big)$$

Thus if $t_o \to -\infty$, $\delta\omega - C\ \delta k \to 0$. Hence infinitely long after the turn-on time, the quantity $\omega - Ck$ is a constant. This, of course, is just the frequency in a system moving with the internal wave phase speed, and the statement that this is constant is part of the "steady-state solution" to the internal wave-surface wave interaction problem.

The fact that at finite times after turn on, the steady-state solution is not reached is because the presence of $\theta(t - t_0)$ in the form chosen for U violates the assumption of the steady-state theory, that U is <u>only</u> a function of x - Ct. It would thus not seem to be possible to reconcile the steady-state case with an initial value problem.

Next suppose we go to resonance, that is, we take $C = c_g$. Then (choosing $t_0 = 0$ for convenience)

$$\frac{\delta k}{k_0} = -\int_0^t \frac{\partial}{\partial x} U\big(x - C(t-t') - Ct'\big)\, dt'$$

$$= -t\, \frac{\partial U}{\partial x} \qquad (5.1)$$

and similarly,

$$\frac{\delta A}{A_0} = \frac{U}{4c_g} - t\,\frac{\partial U}{\partial x} - \frac{t^2 c_g}{8}\,\frac{\partial^2 U}{\partial x^2} \qquad (5.2)$$

where we've kept the last term in $\delta A/A_0$ in case the "short time" approximation is invalid.

These results constitute the expression of resonant enhancement--they depend only on the form $U = U(x - Ct)\,\theta(t)$ and the choice $C = c_g$. The linear (or quadratic) growth with t obviously means that the expressions break down eventually, in that the requirements $\delta k/k_0$ and $\delta A/A_0 \ll 1$ fail.

VI. OTHER BOUNDARY CONDITIONS

Experiments on internal wave-surface wave interactions tend to replace the initial value problems we have dealt with up to now with a boundary condition specifying the surface wave at a point in space, the point where the surface wave is generated. If it is generated with constant frequency, these conditions would say that at $x = 0$, $\delta\omega = \delta A = 0$. Under these conditions, solutions can easily be obtained by exactly the same methods used before, and they are:

$$\frac{\delta k}{k_o} = -\frac{U}{c_g} + \frac{1}{c_g^{\,2}} \int_o^x dx' \frac{\partial}{\partial t} U(x', t - \frac{x-x'}{c_g}) \tag{6.1}$$

and

$$\frac{\delta A}{A_o} = -\frac{1}{c_g} [U(x,t) - U(0, t - x/c_g)] + \frac{3}{2c_g^{\,2}} \int_o^x dx' \frac{\partial}{\partial t} U(x', t - \frac{x-x'}{c_g}) - \frac{1}{4c_g^{\,3}} \int_o^x (x-x')\, dx' \frac{\partial^2}{\partial t^2} U(x', t - \frac{x-x'}{c_g}) \tag{6.2}$$

These are very similar to the solutions obtained before, for the initial value problem.

For the special case of $U = U(x - Ct)$, and at resonance, where $C = c_g$, the solutions reduce to

$$\frac{\delta k}{k_o} = -\frac{U}{c_g} + \frac{k_o}{c_g^{\,2}} \times \frac{\partial U}{\partial t} \tag{6.3}$$

and

$$\frac{\delta A}{A_o} = \frac{3}{2c_g^{\,2}} \times \frac{\partial U}{\partial t} - \frac{1}{8c_g^{\,3}} x^2 \frac{\partial^2 U}{\partial t^2} \tag{6.4}$$

A special case of interest is evidently that of a sinusoidal internal wave

$$U = U_o \sin K(x-Ct)\theta(t) \tag{6.5}$$

In this situation, (6.4), for example, simplifies to

$$\frac{\delta A}{A_o} = -\frac{3Kx}{2C} U_o \cos K(x-Ct)$$

$$-\frac{(Kx)^2}{8C} U_o \sin K(x-Ct) \tag{6.6}$$

The amplitude enhancement thus oscillates with the internal wave, with an envelope given by

$$\left(\frac{\delta A}{A_o}\right)_{Max} = \frac{3}{2}\left(\frac{U_o}{C}\right) Kx\sqrt{1 + \frac{(Kx)^2}{144}} \tag{6.7}$$

where x is the distance from the point at which the surface wave is generated. The growth is thus linear in distance for small x (i.e. where $Kx \ll 12$) and becomes quadratic in x for larger x (when $Kx \gg 12$). The region of linear behavior corresponds to the "short-time" situation in the initial value problem; for larger Kx the situation is the same as that in which the correction term derived in Eq. (3.7) is required.

The result (6.7) is compared, in Fig. 1, with an experimental measurement.* The agreement is evidently excellent, and remarkable in that our perturbation theory result holds even up to very large enhancements.

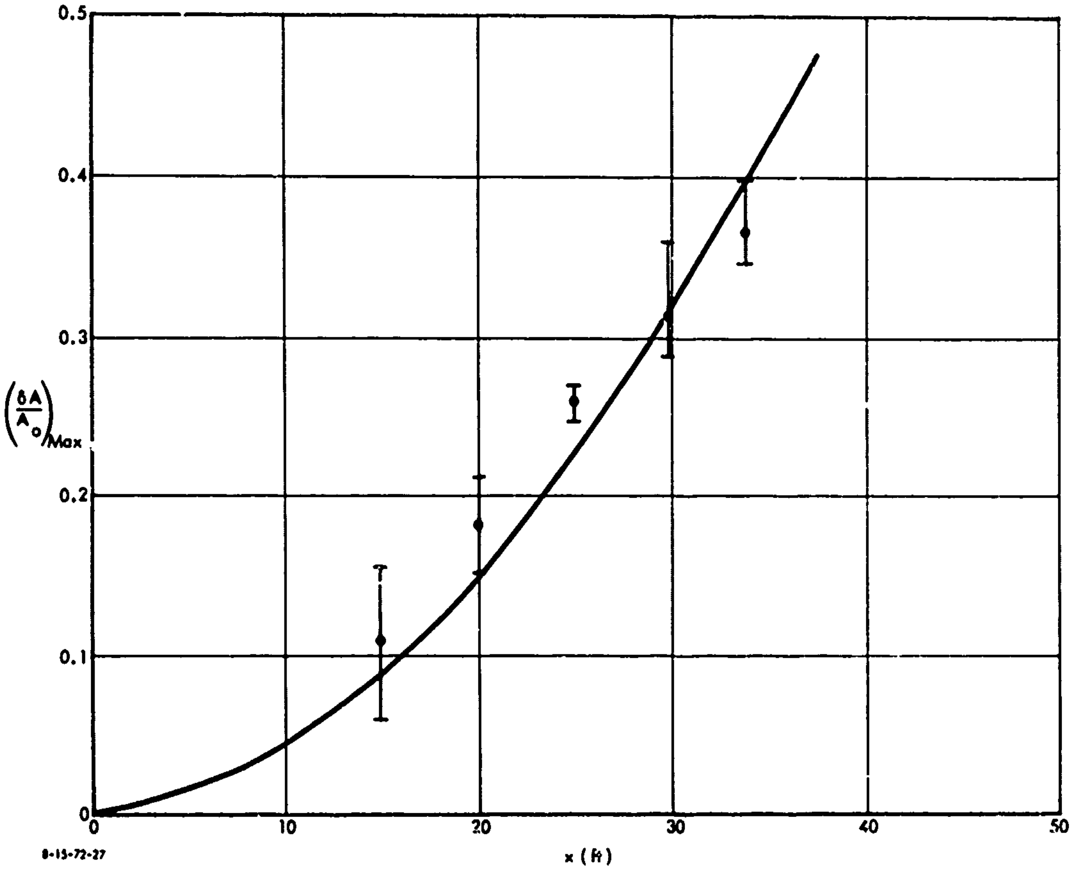

FIGURE 1. Comparison of Equation 6.7 with Measurements

*John E. Lewis, "Experimental Investigation of the Interaction of Internal Waves and Surface Gravity Waves," TRW Systems Group Report No. 18202-6001-RO-00, October 31, 1971.

VII. VISCOUS DAMPING

Viscosity is most easily introduced into the energy conservation equation, and it will appear as an additional constant term σ on the right-hand side of Eq. 4.6, where 1/σ is the viscous damping time.* Defining a new amplitude A' by

$$A = A' e^{-\sigma t}$$

removes the term σ, so that A' satisfies Eq. 4.6 as written, without the σ. Hence the solution for δA' is given in 3.9, and the solution for A, including viscosity, is simply

$$\frac{\delta A}{A_o} = \left[\frac{U}{4c_g} - \int_{-\infty}^{t} \left(\frac{\partial U}{\partial x}\right)_{\text{retarded}} - \frac{c_g}{4} \int_{-\infty}^{t} (t-t') \left(\frac{\partial^2 U}{\partial x^2}\right)_{\text{retarded}}\right] e^{-\sigma t} \quad (7.1)$$

In particular, for U = U(x - Ct) and at resonance, we have

$$\frac{\delta A}{A_o} = \left(\frac{U}{4c_g} - t\,\frac{\partial U}{\partial x} - \frac{t^2 c_g}{8}\,\frac{\partial^2 U}{\partial x^2}\right) e^{-\sigma t} \quad (7.2)$$

in conformity with Eq. (5.2).

*Phillips, p. 39, gives numerical values for σ.