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Thermal Instabilities in a Viscoelastic Rod under Cyclic Loading

by

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THERMAL INSTABILITIES IN A
VISCOELASTIC ROD UNDER CYCLIC LOADING

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ABSTRACT

Thermal instabilities in a viscoelastic rod under cyclic loading are discussed by determining the stresses and temperature in a viscoelastic rod insulated on its lateral surface and driven by a sinusoidal stress at one end. Temperature dependence of the complex Young's modulus of the rod and the effect of thermomechanical coupling are included in the analysis. A method of finite differences is used to directly determine the steady state stresses and temperature without obtaining the complete time history of the process. The iterative algorithm used is very useful and converges rapidly for a wide range of driving stress amplitudes and frequencies. It is found that rapid rise of temperature to dangerous levels occurs for relatively low values of driving stress amplitudes, especially if the driving frequency is close to one of the critical frequencies of the rod. Drastic softening of the rod leads to large strains. Thus, failure of the rod could occur at low values of the driving stress.

INTRODUCTION

Thermal effects in viscoelastic materials under cyclic loading are of great practical importance. These materials are dissipative in nature. Continuous dissipation of mechanical energy into heat coupled with the fact that they are poor conductors of heat can lead to very high temperatures inside the materials and consequent failure. In addition, the mechanical properties of the materials are strongly temperature dependent, the complex elastic modulii, in general, being inversely proportional to some power of the temperature. This rapid softening of the material often leads to large strains for relatively small values of stress.

Several authors have studied thermal effects in viscoelastic media. Tormey and Britton [1]* carried out vibration tests of solid propellant rocket motors. Heating due to vibration near a resonant frequency for several hours caused the material to soften to such an extent that some of it flowed out of the motor. Hunter [2] derived a set of thermomechanically coupled equations for the propagation of stress, strain and temperature fields in viscoelastic solids. He assumed thermorheologically simple behavior and used a double time integral expression for the dissipation function. Petrof and Gratch [3] considered longitudinal wave propagation in a finite rod. They assumed that the material behaves like ideal rubber so that deformation occurs without any change in potential energy. They used an integral form of the constitutive equations. The transient response of the same problem was studied by

*Numbers in square brackets designate references at the end of this report.

Wolosewick and Gratch [4]. They found that due to the high damping in viscoelastic materials, the mechanical transients decay very quickly, and a steady state of oscillation with slowly increasing temperature can be reached within one or two cycles.

In a series of papers [5,6,7], Schapery derived field equations for viscoelastic media with thermomechanical coupling and solved some problems. In [6] he studied the problem of steady periodic shear oscillations using the complex modulus form of the constitutive equations. He, however, neglected inertia and his application was limited to lower frequencies than considered here. In [7] he developed and used variational principles to solve problems involving bodies that were either massless or with concentrated mass, and in his last example of a 'solid cylinder with distributed mass' he only gives a first approximation to the solution. This paper [7] has been discussed in more detail in [25].

Huang and Lee [8] studied longitudinal waves in a viscoelastic rod caused by a sinusoidal stress applied at one end. They included time as an independent variable and obtained time histories of the stress and temperature using the method of finite differences. Chang [9] has studied the transient temperature profile in an infinite viscoelastic medium exterior to a cylindrical cavity subjected to an oscillatory, axially symmetric boundary shear. More recently, Knauss [10], Hegemier and Morland [11], and Edelstein [12] have studied thermal effects in viscoelastic solids. Knauss considered the dynamic response of a long viscoelastic bar due to a step displacement at the end; Hegemier and Morland the response of a viscoelastic half space to the sudden application of a heat source distribution; and Edelstein an abalating cylinder. All three of them, however, neglected thermomechanical coupling, i.e., the heat source due to dissipation.

Studies of thermal effects in nonlinear viscoelastic media have been reported in [13]-[20]. In [14] Coleman and Gurtin discussed the velocity and the growth and decay of one-dimensional acceleration waves in conductors and in media which do not conduct heat. In [15] some of the findings of [14] were extended to three-dimensional acceleration waves. In [16] one-dimensional shock waves in non-conducting viscoelastic media were discussed. In [17] Achenbach, Vogel and Herrmann discussed the propagation of shock waves in a viscoelastic material with temperature dependent properties and thermomechanical coupling. The waves were caused by sudden application of surface tractions and the analysis was concerned with small strains and changes in temperature. Oden and Ramirez [18] manipulated their relations for finite difference application. McGuirt and Lianis [19] compared experimental results of finite uniaxial and equal biaxial tests for non-isothermal conductors with theory and reported good agreement; while Lianis [20] suggested integral constitutive equations for nonlinear thermoviscoelastic materials. Random temperature effects have been considered by Parkus and Bargmann [21] in a recent paper.

This report is concerned with efficiently obtaining steady state solutions for stress and temperature in longitudinal wave propagation in a viscoelastic rod with thermal coupling subjected to a sinusoidal stress at one end. Huang and Lee in [8] obtained time histories for the stress and temperature for this problem. They set up the governing nonlinear partial differential equations based on Schapery's model and solved them by a method of finite differences. In practice, the temperature in the rod increases with time owing to a continuous dissipation of mechanical energy into heat and a steady state might be reached if the rate of heat generation by the dissipative source equals the rate of heat flowing out of the rod. Such a steady state yields most severe temperature conditions which are of primary interest in design. In such cases it is much more efficient to obtain the steady state directly instead of following the complete time history of the process. In this report this is done by solving a set of nonlinear ordinary differential equations by a method of finite differences. A very useful algorithm is developed by 'partial linearization' of the equations and numerical solutions for a Lockheed solid propellant are obtained over a wide range of driving stresses and frequencies. For the cases studied, steady state solutions for stress and temperature are obtained. The very high temperatures reached, however, would cause failure of the rod. The word 'instability' in the title may thus be interpreted from an engineering (practical) point of view, although the existence of a mathematical instability cannot be ruled out since the complex modulus approaches zero as the temperature increases. In fact, if the driven end is insulated, instead of being kept at constant temperature as in our problem, unbounded increase of the compliance, corresponding to vanishing modulus, would lead to unbounded dissipation of power at the driven end. This would then cause infinite temperatures and a steady state temperature solution would no longer exist. These aspects of the problem are discussed in more detail later on in the report.

FORMULATION OF THE PROBLEM

The formulation of the problem closely follows that in [8] with the essential difference that the steady state problem is considered here. Variational principles were used by the author to obtain a numerical solution to this problem for one value of driving stress and frequency in [25].

Let us define stress, strain and displacement as the real parts of

$$\tilde{\sigma} = \sigma\, e^{i\omega t} \;,\quad \tilde{\epsilon} = \epsilon\, e^{i\omega t} \;,\quad \tilde{u} = u\, e^{i\omega t} \tag{1}$$

where $i = \sqrt{-1}$, ω is the frequency and t is time. The complex quantities σ , ϵ and u , functions of the space variable x , will be referred to simply as the stress, strain and temperature respectively.

Let us consider a viscoelastic rod of length ℓ insulated on its lateral surface, as in Fig. 1. The left end is free while the right end has a prescribed stress $\sigma_o \cos \omega t$ (σ_o real). The temperature of the vibrator is assumed constant at T_o while a radiation boundary condition is assumed at $x = 0$.

The boundary conditions, therefore, are

$$\begin{aligned} x = 0 \qquad & \sigma = 0 \\ & \frac{\partial T}{\partial x} = c(T - T_o) \\ x = \ell \qquad & \sigma = \sigma_o \\ & T = T_o \end{aligned} \tag{2}$$

Here T is the temperature and $c = h/K$ is the ratio of the surface conductance h to the thermal conductivity K of the viscoelastic material.

The equation of motion in one dimension can be written as

$$\frac{d^2\sigma}{dx^2} + \rho\, \omega^2 D^* \sigma = 0 \tag{3}$$

where ρ is the mass per unit volume and D^* is the complex tensile compliance which gives the strain in terms of the stress

$$\epsilon = D^* \sigma \tag{4}$$

D^* is a function of the frequency ω and the temperature T. The material is assumed to be thermorheologically simple (see [8]) so that

$$D^* = D^*(\omega') = D^*(a_T(T)\omega)$$

where ω' is the reduced frequency which is related to the actual frequency ω by a temperature-dependent shift factor a_T. The variable a_T represents the effect of temperature on viscosity.

Writing

$$\sigma = \sigma_1 + i\sigma_2$$

$$D^* = D_1 - iD_2$$

we can separate Eq. (3) into its real and imaginary parts

$$\frac{d^2\sigma_1}{dx^2} + \rho\, \omega^2 (D_1\sigma_1 + D_2\sigma_2) = 0 \tag{5}$$

$$\frac{d^2\sigma_2}{dx^2} + \rho\, \omega^2 (D_1\sigma_2 - D_2\sigma_1) = 0 \tag{6}$$

The steady state energy equation in one dimension for the cycle averaged temperature is given by (see [7,8,25])

$$K \frac{d^2T}{dx^2} + 2D = 0 \tag{7}$$

Here 2D is the cycle averaged value of the mechanical dissipation function

$$2D = \frac{\omega}{2\pi} \int_t^{t+\frac{2\pi}{\omega}} \mathrm{Re}(\sigma)\ \mathrm{Re}\left(\frac{d\varepsilon}{dt'}\right) dt'$$

$$= \frac{\omega}{2} D_2(\sigma_1^2 + \sigma_2^2)$$

where Re denotes the real part of the complex argument.

We note here that part of the mechanical dissipation is transformed into heat while the remaining causes increase of potential energy. Heat is also produced due to dilatational compression. These coupling terms due to dilatation and potential energy, however, are periodic and drop out if the energy equation is integrated over a cycle. Thus the temperature T here is the cycle averaged value of the true temperature. T is assumed to attain a steady state after a sufficiently long time and become independent of time. Strictly speaking, the temperature has small cyclic variations about a mean value as a result of the cyclic variations of the potential energy, dilatation and dissipation, but these are quite negligible.

We note also that while the mechanical transients vanish within one or two cycles as reported in [4], thermal transients, i.e., the slow increase of temperature with time, can continue over several hours before it settles down to a steady state, if, in fact, a steady state exists. In the numerical examples given in this report, such a steady state is found to exist and is obtained directly without integrating forward in time.

Equations (5), (6), and (7) constitute a set of three nonlinear ordinary differential equations for the unknowns σ_1 , σ_2 and T . The associated two point boundary conditions are given by Eq. (2). Thermomechanical coupling is caused by the nonlinear dissipative source D . The real and imaginary parts of the complex compliance D^* are, in general, nonlinear functions of temperature.

For a Lockheed solid propellant over a wide reduced frequency range we can express D_1 and D_2 by the following empirical formulae (see [8])

$$D_1 = c_1\omega^\beta(T - T_1)^\gamma$$
$$D_2 = c_2\omega^\beta(T - T_1)^\gamma \tag{8}$$

where c_1 , c_2 , β , γ and T_1 are constants.

Let us define the nondimensional quantities

$$q = \frac{x}{\ell} \ , \quad \tau = \frac{T - T_1}{T_o - T_1} \ , \quad \varkappa = c\ell \ , \quad \lambda = [2K\omega\rho(T_o - T_1)]^{-1/2} \ ,$$

$$s = \lambda\sigma \ , \quad s_o = \lambda\sigma_o \ , \quad s_1 = \lambda\sigma_1 \ , \quad s_2 = \lambda\sigma_2 \tag{9}$$

$$d_1 = \ell^2\omega^2\rho D_1 \ , \quad d_2 = \ell^2\omega^2\rho D_2$$

The governing differential equations (5), (6), and (7) reduce to the simplified form

$$s_1'' + d_1 s_1 + d_2 s_2 = 0 \tag{10}$$

$$s_2'' + d_1 s_2 - d_2 s_1 = 0 \tag{11}$$

$$\tau'' + d_2(s_1^2 + s_2^2) = 0 \tag{12}$$

where

$$d_1 = c_1 a \tau^\gamma \quad , \quad d_2 = c_2 a \tau^\gamma$$

$$a = \ell^2 \rho\, \omega^{2+\beta}(T_o - T_1)^\gamma, \text{ a constant}$$

and

$$(\ )' \equiv \frac{d}{dq}$$

The boundary conditions, Eq. (2), become

$$\begin{aligned} q = 0 : s_1 &= 0 \quad , \quad s_2 = 0 \quad , \quad \tau' = \kappa(\tau - 1) \\ q = 1 : s_1 &= s_o \quad , \quad s_2 = 0 \quad , \quad \tau = 1 \end{aligned} \tag{13}$$

We thus have a two point boundary value problem which is solved by a method of finite differences (see [22,23,24]).

METHOD OF SOLUTION

1. Partial Linearization

Equations (10) and (11) are linear in the stresses given the temperature distribution. However, Eq. (12) is not linear in temperature, given the stresses, since d_2 is a nonlinear function of temperature.

If Eq. (10) is multiplied by s_2 , Eq. (11) by s_1 , and the resulting second equation substracted from the first, we get

$$s_2 s_1'' - s_1 s_2'' + d_2(s_1^2 + s_2^2) = 0 \tag{14}$$

and using Eqs. (14) and (12) we have

$$\tau'' = s_2 s_1'' - s_1 s_2'' \tag{15}$$

Integrating once with respect to the independent variable q

$$\tau' - \tau'(0) = s_2 s_1' - s_1 s_2' \tag{16}$$

Now Eq. (16) is linear in temperature given the stress distribution and can be used to replace Eq. (12) in the original set of Eqs. (10), (11) and (12). Thus, the new set of governing differential equations ((10), (11), and (16)) are 'partially linear' and this set, together with the associated boundary conditions (Eq. (13)) is used to obtain solutions for the stress, strain and temperature distributions.

2. Finite Difference Algorithm

The iterative procedure works as follows:

Guess $\tau^{(0)}$

$\tau^{(j)}$ ==> Equations of Motion (10) and (11) + boundary conditions ⇒ $s_1^{(j)}$, $s_2^{(j)}$

==> Modified Energy Equation (16) + boundary conditions ⇒ $\tau^{(j+1)}$

where the superscript j denotes the jth approximation to the solution. Iteration is continued till convergence is achieved.

Let

$$\delta = \frac{1}{N+1} \quad , \quad q_k = k\delta \ (k = 0,1,\ldots,N+1)$$

Here δ is the step size and $N+1$ the number of mesh points. Hence

$$q_o = 0 \quad , \quad q_{N+1} = 1$$

Let us define, for $k = 0,1,\ldots,N+1$, the vectors

$$s_1(q_k) = s_k^R \quad , \quad s_2(q_k) = s_k^I \quad ,$$

$$d_1(q_k) = d_{1k} = c_1 a \tau_k{}^{\gamma} \ , \quad d_2(q_k) = d_{2k} = c_2 a \tau_k{}^{\gamma}$$

$$\tau(q_k) = \tau_k$$

The difference equations for the equations of motion (Eqs. (10) and (11)) become

$$\eta_{k-1} + B_k \eta_k + \eta_{k+1} = 0 \quad (k = 1,2,\ldots,N) \tag{17}$$

where η_k is a sequence of two component vectors

$$\eta_k = \begin{vmatrix} S_k^R \\ S_k^I \end{vmatrix} \quad (k = 0,1,\ldots N+1)$$

and B_k is a sequence of 2×2 matrices

$$B_k = \begin{bmatrix} \delta^2 d_{1k} - 2 & \delta^2 d_{2k} \\ -\delta^2 d_{2k} & \delta^2 d_{1k} - 2 \end{bmatrix} \quad (k = 1,2,\ldots,N)$$

The boundary conditions give

$$\eta_o = \begin{vmatrix} 0 \\ \\ 0 \end{vmatrix} \quad , \quad \eta_{N+1} = \begin{vmatrix} S_o \\ \\ 0 \end{vmatrix} \tag{18}$$

Let V_k be a sequence of 2×2 transfer matrices defined as

$$\eta_k = V_k \eta_{k+1} \qquad (k = 0,1,\ldots,N) \tag{19}$$

Now

$$V_o = \begin{bmatrix} 0 & 0 \\ & \\ 0 & 0 \end{bmatrix} \tag{20}$$

and substituting Eq. (19) into Eq. (17) we get the recurrence relation

$$V_k = - \left[B_k + V_{k-1} \right]^{-1} \qquad (k = 1,2,\ldots,N) \tag{21}$$

Now with V_o known from Eq. (20), Eq. (21) gives V_1 , V_2 , ..., V_N and then Eq. (19) gives η_N , η_{N-1} , ..., η_1 .

Thus, the equations of motion are completely solved for a certain guess of the temperature distribution. The use of transfer matrices is a big advantage since only 2×2 matrices need to be inverted.

The modified energy equation (16) and its associated boundary conditions become

$$\frac{(S^R_{k+1} - S^R_{k-1})\, S^I_k}{2\delta} - \frac{(S^I_{k+1} - S^I_{k-1})\, S^R_k}{2\delta} = \frac{(\tau_{k+1} - \tau_{k-1})}{2\delta}$$

$$- \varkappa(\tau_o - 1) \qquad (k = 1,2,\ldots,N)$$

$$-\tau_2 + 4\tau_1 - 3\tau_o = 2\delta\,\varkappa(\tau_o - 1)$$

$$\tau_{N+1} = 1 \tag{22}$$

Let

$$\tau_k = \beta_k\tau_o + \gamma_k \quad (k = 0,1,\ldots,N+1) \tag{23}$$

$$\omega_k = S^R_{k+1} - S^R_{k-1} \quad (k = 1,2,\ldots,N)$$

$$z_k = S^I_{k+1} - S^I_{k-1} \quad (k = 1,2,\ldots,N)$$

Substitution of Eq. (23) into Eq. (22) leads to

$$\beta_o = 1 \quad , \quad \beta_1 = 1 + \delta\varkappa$$

$$\gamma_o = 0 \quad , \quad \gamma_1 = \frac{1}{4}\ (S^I_1\omega_1 - S^R_1 z_1) - \delta\varkappa$$

and the recurrence relations

$$\begin{aligned} \beta_{k+1} &= \beta_{k-1} + 2\delta\varkappa \quad (k = 1,2,\ldots,N) \\ \gamma_{k+1} &= \gamma_{k-1} - 2\delta\varkappa + \omega_k S^I_k - z_k S^R_k \quad (k = 1,2,\ldots,N) \end{aligned} \tag{24}$$

and once β_k , γ_k $(k = 0,1,\ldots,N+1)$ are known, we have

$$\tau_o = \frac{(1 - \gamma_{N+1})}{\beta_{N+1}} \quad , \quad \tau_{N+1} = 1$$

and τ_k $(k = 1,2,\ldots,N)$ are obtained from Eq. (23).

Thus, the modified energy equation (16) is solved for the temperature, given the stresses.

The iterative procedure outlined in the beginning of this section is allowed to proceed till successive approximations for τ_k are within certain prescribed convergence limits. The accuracy of the solution is verified by back substitution into the original finite difference equations. Different mesh sizes are chosen in such a way that halving the step size causes a change of .5% or less in the results.

Thus, the steady state values of the real and imaginary parts of the nondimensional stress, s_1 and s_2 and the nondimensional temperature τ are obtained. The nondimensional stress at any time is

$$\mathrm{Re}(s\, e^{i\omega t}) = s_1 \cos \omega t - s_2 \sin \omega t$$

RESULTS AND CONCLUSIONS

Numerical calculations have been carried out for the following data for a Lockheed solid propellant [8]*

$$c_1 = 4.61 \times 10^{-11} \ (\mathrm{psi})^{-1}(\mathrm{sec})^{\beta}(^\circ\mathrm{F})^{-\gamma}$$

$$c_2 = 1.62 \times 10^{-11} \ (\mathrm{psi})^{-1}(\mathrm{sec})^{\beta}(^\circ\mathrm{F})^{-\gamma}$$

$$\beta = -\ 0.214 \quad , \quad \gamma = 3.21$$

$$\kappa = 1.0 \quad , \quad T_o = 65^\circ\mathrm{F}$$

$$T_1 = -125^\circ\mathrm{F} \quad , \quad \ell = 3 \text{ in.}$$

$$\ell^2\rho = 1.023 \times 10^{-4} \ \mathrm{psi\text{-}sec}^2$$

$$2K\rho(T_o - T_1) = 8.08 \times 10^{-4} \ \mathrm{psi}^2\text{-sec}$$

*In [8], κ should read 1.0 instead of 0.1.

Various values for the frequency ω and the driving stress s_o were used.

Figure 2 shows the value of τ as a function of q for various values of the driving stress s_o . The frequency ω is kept constant at 10^4 radians/second. The steady state temperature profiles are found to exist. However, the temperature response is markedly non-linear, increasing much more rapidly than stress and reaching dangerously high levels for low values of driving stress. The constitutive law used here assumed that the complex Young's modulus $E^* = 1/D^*$ approaches zero as the temperature $T \rightarrow \infty$ (see Eq. (8)). Schapery [6] has an experimental graph for $a_T(T)$ for our Lockheed propellant which shows that for large temperatures $E^* \rightarrow 0$ for a finite value of T . $E^* \rightarrow 0$ means zero wave velocity in the material and consequently failure. Thus, the temperature levels attained in our calculations (as shown in Fig. 2) would lead to melting and failure of the rod, even though a mathematical steady state solution exists and T is finite everywhere.

Other interesting observations can be made regarding Fig. 2. The rapid softening of the material of the rod leads to large values of strain (around 20% near the driven end for $s_o = 3.0$) so that further calculations for larger values of s_o would not be valid within the realm of linear viscoelasticity. With increasing stress the location of the temperature maximum asymptotically approaches the driven end of the rod since the larger damping at higher temperatures leads to rapid attenuation of the stress away from the driven end. Consequently, most of the mechanical energy is dissipated as heat near the driven end of the rod. Another consequence of softening of the rod is lower stress wave velocities and therefore smaller wave lengths since the driving frequency is kept the same. These effects on the s_1 and s_2 profiles are clearly evident in Figs. 3, 4 and 5 which show s_1 and s_2 as functions of q for $s_o = 0.5$, 1.5, and 2.5 with $\omega = 10^4$ radians/second. In Fig. 5 the stress is practically zero beyond about a third of the length of the rod from the driven end.

The effect of driving frequency on the maximum temperature is shown in Fig. 6 for two values of the driving stress. The maximum temperature is seen to have several peaks at certain critical frequencies. Increase of driving stress gives higher and more frequent peaks at different critical frequencies. This is because more input of mechanical energy leads to more dissipative heating and therefore larger temperatures. This, in turn, lowers E^* and the change of properties of the rod changes the critical frequencies. Also, softening of the material lowers the stress wave velocity in the rod and this leads to a lower fundamental frequency for higher driving stress.

The effect of driving stress is, of course, largest near these critical frequencies. As an example, the temperature and stress plots for $\omega = 2500$ radians/second (the fundamental critical frequency for $\sigma_o = 2.84$ psi) are shown for $\sigma_o = 1.42$ psi and $\sigma_o = 2.84$ psi in Figs. 7, 8, and 9. A very large temperature rise is obtained for a doubling of the driving stress and the s_1 and s_2 plots are considerably different for $\sigma_o = 2.84$ psi compared to those at 1.42 psi. Thus, these critical frequencies are of utmost importance in design.

An attempt was made to compare the critical frequencies of the coupled problem (as in Fig. 6) with the natural frequencies of the uncoupled problem where the temperature distribution in the rod is already specified. If a certain temperature distribution is prescribed in the rod, we have an inhomogenious viscoelastic rod where the complex Young's modulus is known as a function of the distance along it. We thus have a linear problem and the complex natural frequencies of the rod can be determined by solving the resultant eigenvalue problem. This was done using a method of finite differences. The details of the method are given in the appendix.

Let the temperature distribution from the coupled problem for $\sigma_o = 2.84$ psi , $\omega = 2500$ radians/second be prescribed in the rod. The table gives a comparison of the first three uncoupled natural frequencies with the critical frequencies of the coupled problem from Fig. 6.

	Uncoupled Frequency Radians/second	Coupled Frequency Radians/second
ω_1	1739.04 + 333.03i	2500
ω_2	4195.78 + 803.50i	5000
ω_3	6708.83 + 1284.75i	7500

Also, the fundamental uncoupled frequency in case the temperature distribution is prescribed for $\sigma_o = 1.42$ psi , $\omega = 4000$ radians/second is 3484.36 + 667.26i radians/second compared to the coupled fundamental critica' ·equency of 4000 radians/second.

The response of the uncoupled rod to forced oscillation at various frequencies is shown in Fig. 10. Here the prescribed temperature distribution is that obtained from the general coupled problem for σ_o = 2.84 psi , ω = 2500 radians/second. We see from Fig. 10 that owing to the presence of damping the peak is shallow and occurs around 1800 radians/second (compare with ω_1 in table). Thus the eigenvalues of the uncoupled problem give us an idea of those of the coupled problem and shed some light on the correlation between mechanical and thermal resonance.

This study shows that mechanical failure of viscoelastic polymers under cyclic loading can occur at very low values of driving stress, especially if the driving freqeuncy is close to one of the critical frequencies of the coupled problem. For purposes of design, the direct determination of the steady state solutions for temperature and stress appear superior to the method used by Huang and Lee in [8] where the complete time histories of the above mentioned quantities were determined. A lot of computational effort is saved since the method used here requires solution of ordinary rather than partial differential equations in a computer. 'Partial linearization' of the problem leads to a very useful algorithm which converges rapidly for a wide range of driving stresses and frequencies.

APPENDIX

Response of a Linear Inhomogenious Viscoelastic Rod with the Temperature Distribution $\tau(q)$ Prescribed

1. Determination of Natural Frequencies

The equations of motion of the rod can be written as

$$s'' + ds = 0 \tag{A-1}$$

where

$$s = s_1 + is_2 \quad , \quad d = d_1 - id_2$$

$$d(q) = (c_1 - ic_2)(T_o - T_1)^{\gamma} \ell^2 \rho(\tau(q))^{\gamma} \omega^{2+\beta}$$

Let

$$d(q) = r(q)\,\mu$$

where

$$\mu = \omega^{2+\beta}$$

$$p(q) = 1/r(q) = \frac{(c_1 + ic_2)}{(c_1^2 + c_2^2)\,(T_o - T_1)^{\gamma}\,\ell^2 \rho(\tau(q))^{\gamma}}$$

Now Eq. (A-1) becomes

$$p(q)\,s'' + \mu s = 0 \tag{A-2}$$

$$s(0) = s(1) = 0$$

which is an eigenvalue problem with p , μ and s complex.

Let

$$\delta = \frac{1}{N+1} \quad , \quad q_k = k\delta \qquad (k = 0,1,\ldots,N+1)$$

$$s(q_k) = s_k$$

$$p(q_k) = p_k \qquad (k = 0,1,\ldots,N+1)$$

and we have, from Eq. (A-2)

$$-p_k + 2p_k s_k - p_k s_{k+1} = \mu\delta^2 \, s_k \qquad (k = 1,2,\ldots,N)$$

$$s_o = s_{N+1} = 0$$

In matrix form, this can be written as

$$\underset{\sim}{P}\underset{\sim}{s} = \mu\delta^2 \, \underset{\sim}{s}$$

where

$$P \equiv \begin{bmatrix} 2p_1 & -p_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ -p_2 & 2p_2 & -p_2 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -p_3 & 2p_3 & -p_3 & \cdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & -p_{N-1} & 2p_{N-1} & -p_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & 0 & -p_N & 2p_N \end{bmatrix}$$

$$\underset{\sim}{s} = \begin{bmatrix} s_1 \\ s_2 \\ \cdot \\ \cdot \\ \cdot \\ s_N \end{bmatrix}$$

Thus, ν_i $(i = 1,2,\ldots,N)$ are the eigenvalues of P, the natural frequencies

$$\omega_i = \left(\frac{\nu_i}{\delta^2}\right)^{\frac{1}{2+\beta}} \qquad (i = 1,2,\ldots,N)$$

We know, from [22] that these numbers approximate the N smallest natural frequencies. Also, if arranged in an increasing order of magnitude, the approximation is very good for the first natural frequency ω_1 but the accuracy of the approximation decreases rather rapidly with increasing N.

In order to determine the complex eigenvalues of the complex matrix P, it is enough to solve for the eigenvalues of the real matrix P_1, since

$$P = P_1 + iP_2 = P_1 \left(1 + \frac{ic_2}{c_1}\right)$$

and, of course, the eigenvalues of P,

$$\nu_i = \left(1 + \frac{ic_2}{c_2}\right) \alpha_i$$

where α_i are the eigenvalues of P_1.

Since $\mathrm{Re}(r(q)) > 0$ for all q, all α_i are real and positive (see [22]).

The normal modes can be determined, if desired, by finding the eigenvalues of P.

2. Response to Forced Oscillations

This problem is a special case of the general problem with thermomechanical coupling and can be easily solved using a slight modification of the algorithm for the coupled problem.

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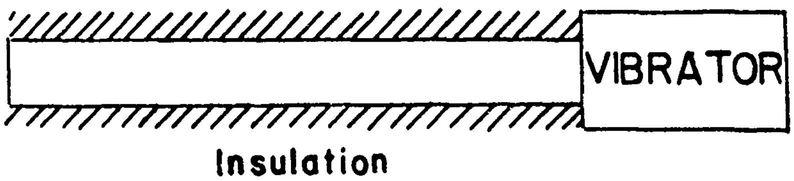

$x = 0$	$x = \ell$
$\sigma = 0$	$Re(\tilde{\sigma}) = \sigma_0 \cos\omega t$
$\frac{dT}{dx} = c(T - T_0)$	$T = T_0$

Figure 1. Boundary conditions for the problem.

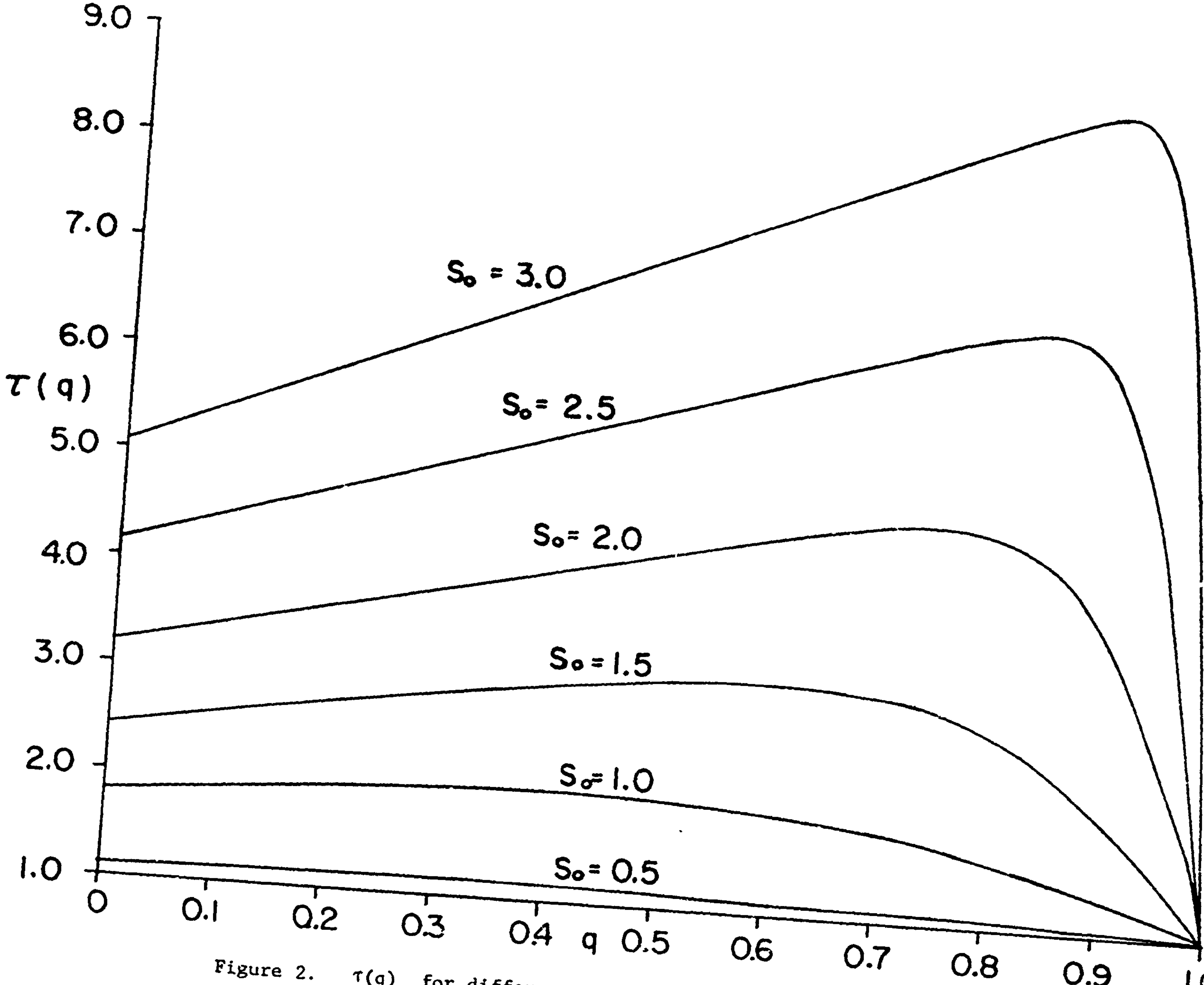

Figure 2. $\tau(q)$ for different values of s_o ($c_o = 2.84 \times s_o$ psi) for $\omega = 10^4$ rad./sec.

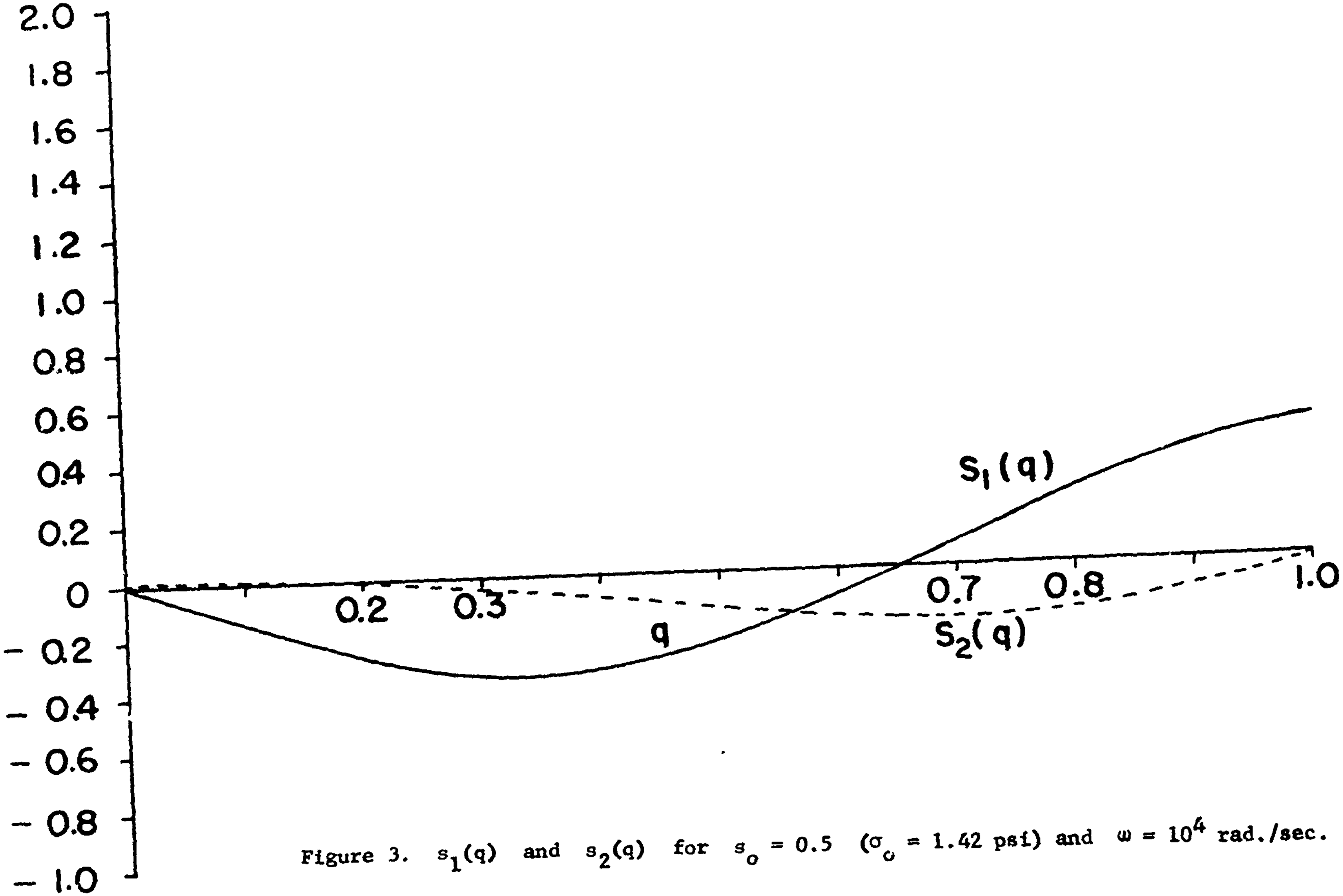

Figure 3. $s_1(q)$ and $s_2(q)$ for $s_o = 0.5$ ($\sigma_o = 1.42$ psi) and $\omega = 10^4$ rad./sec.

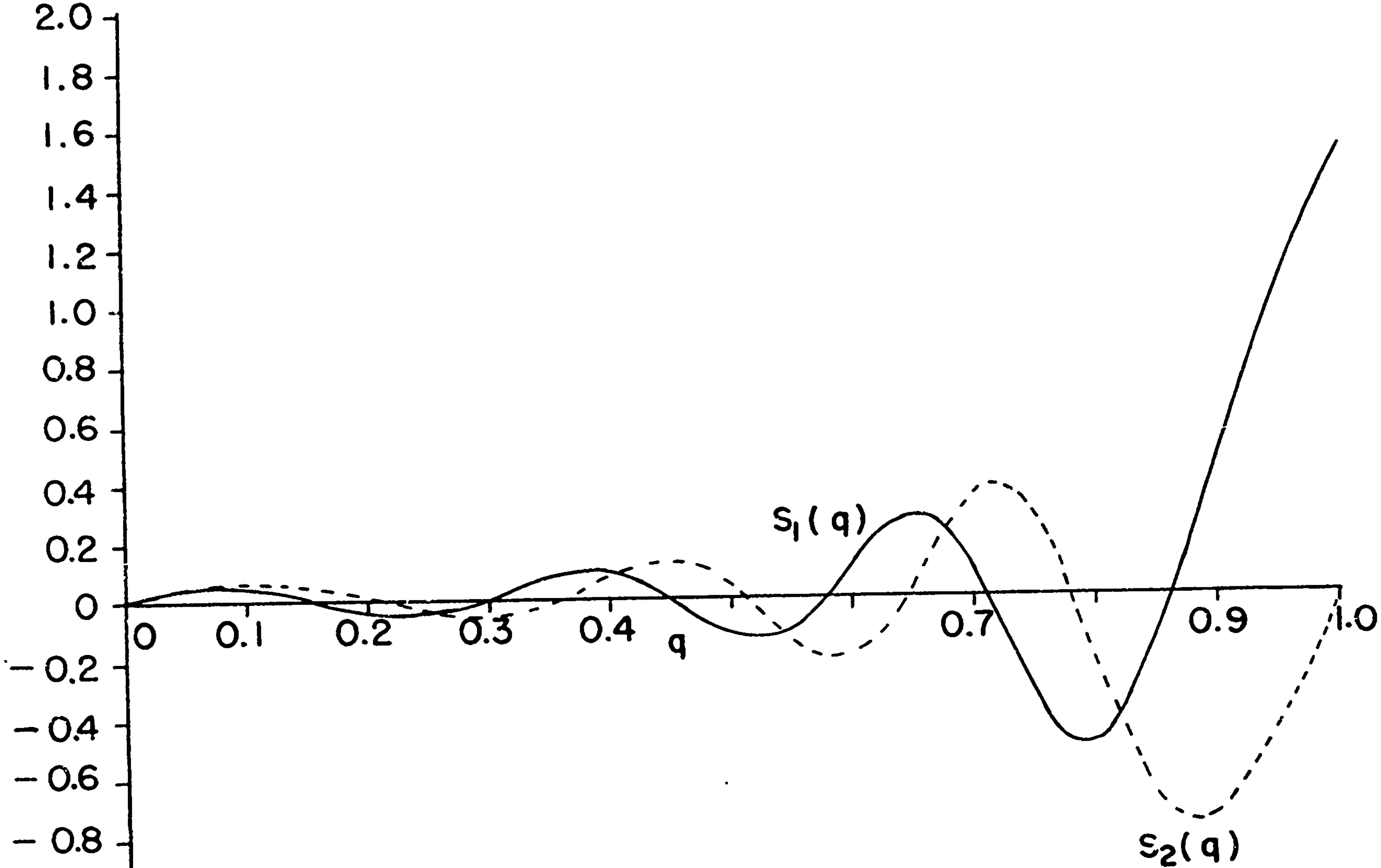

Figure 4. $s_1(q)$ and $s_2(q)$ for $s_o = 1.5$ ($\sigma_o = 4.26$ psi) and $\omega = 10^4$ rad./sec.

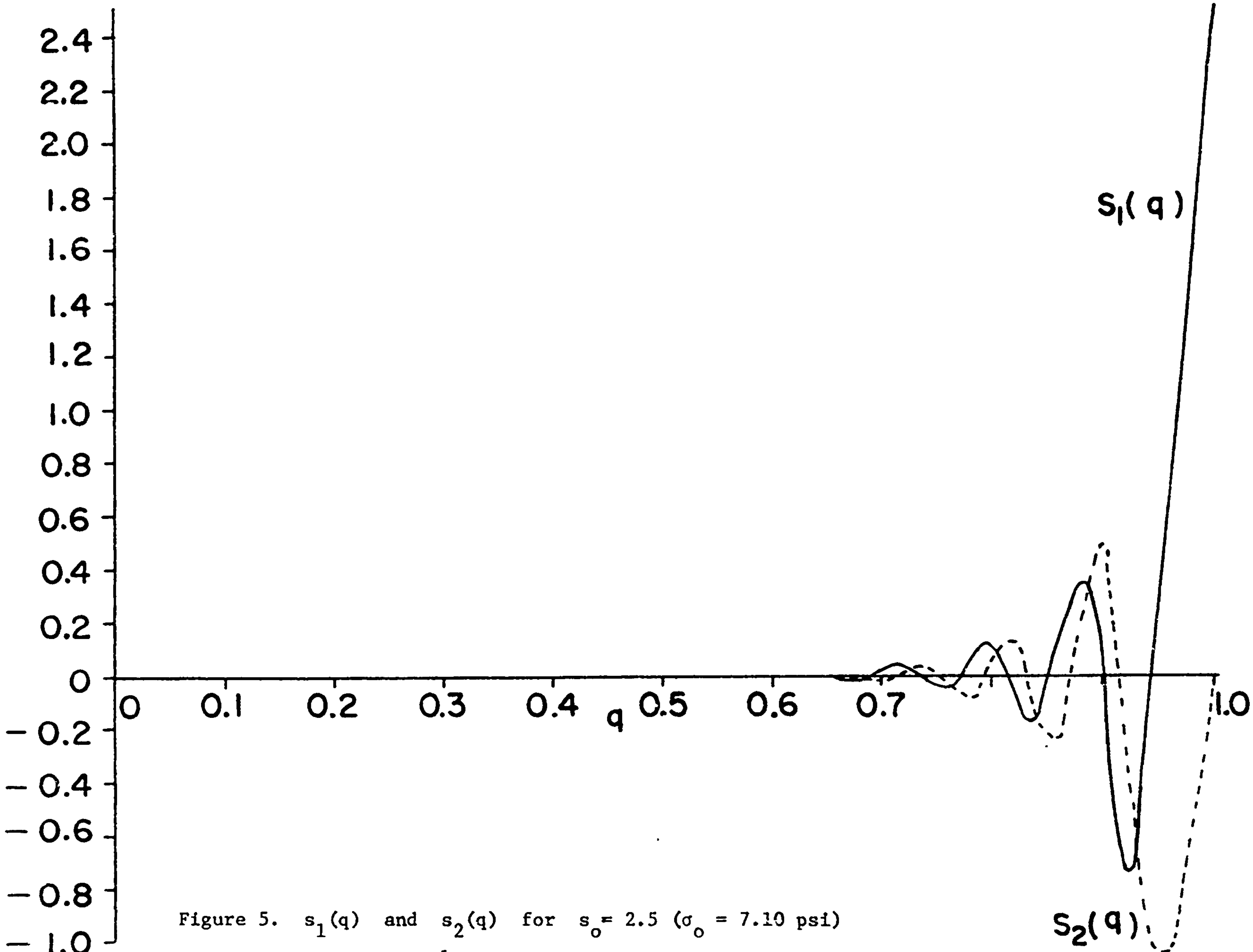

Figure 5. $s_1(q)$ and $s_2(q)$ for $s_o = 2.5$ ($\sigma_o = 7.10$ psi) and $\omega = 10^4$ rad./sec.

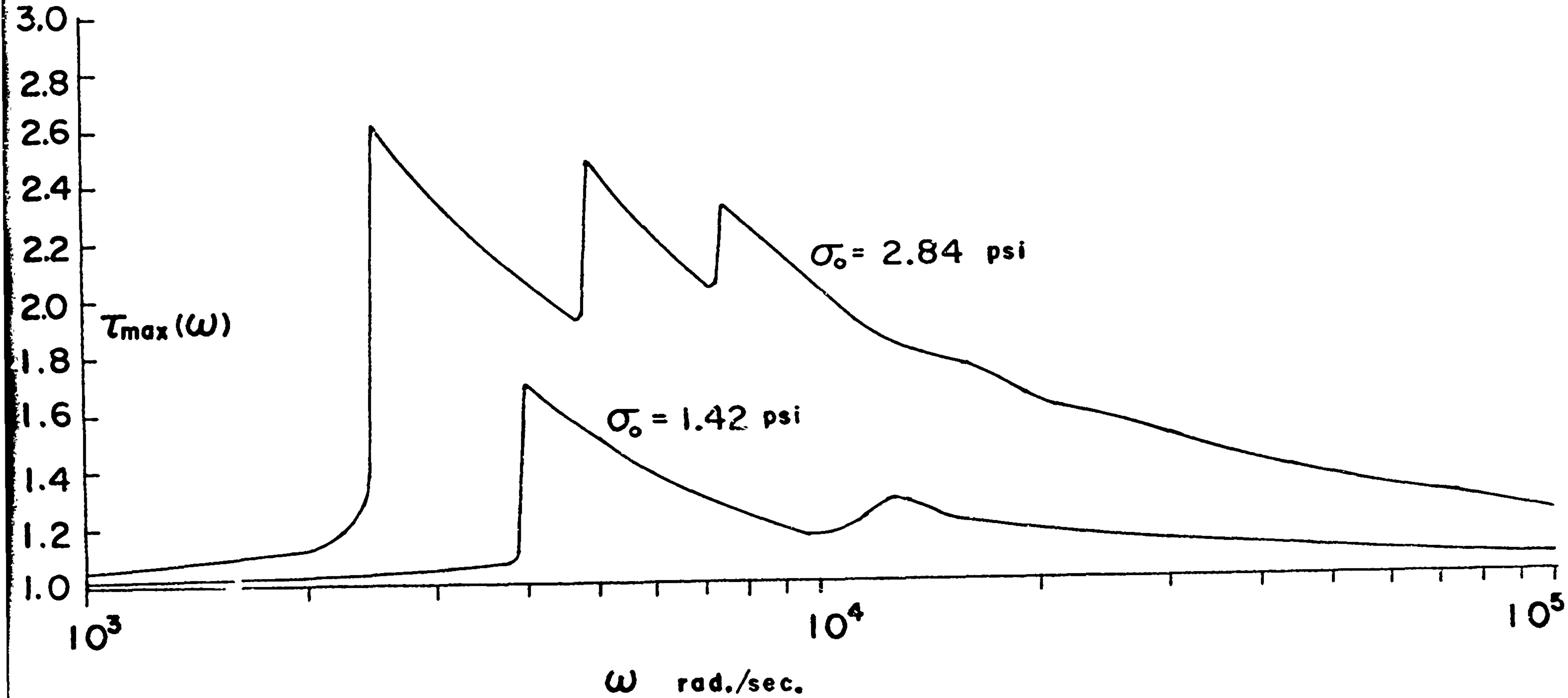

Figure 6. Peak temperature τ_{max} versus frequency ω for different values of σ_o .

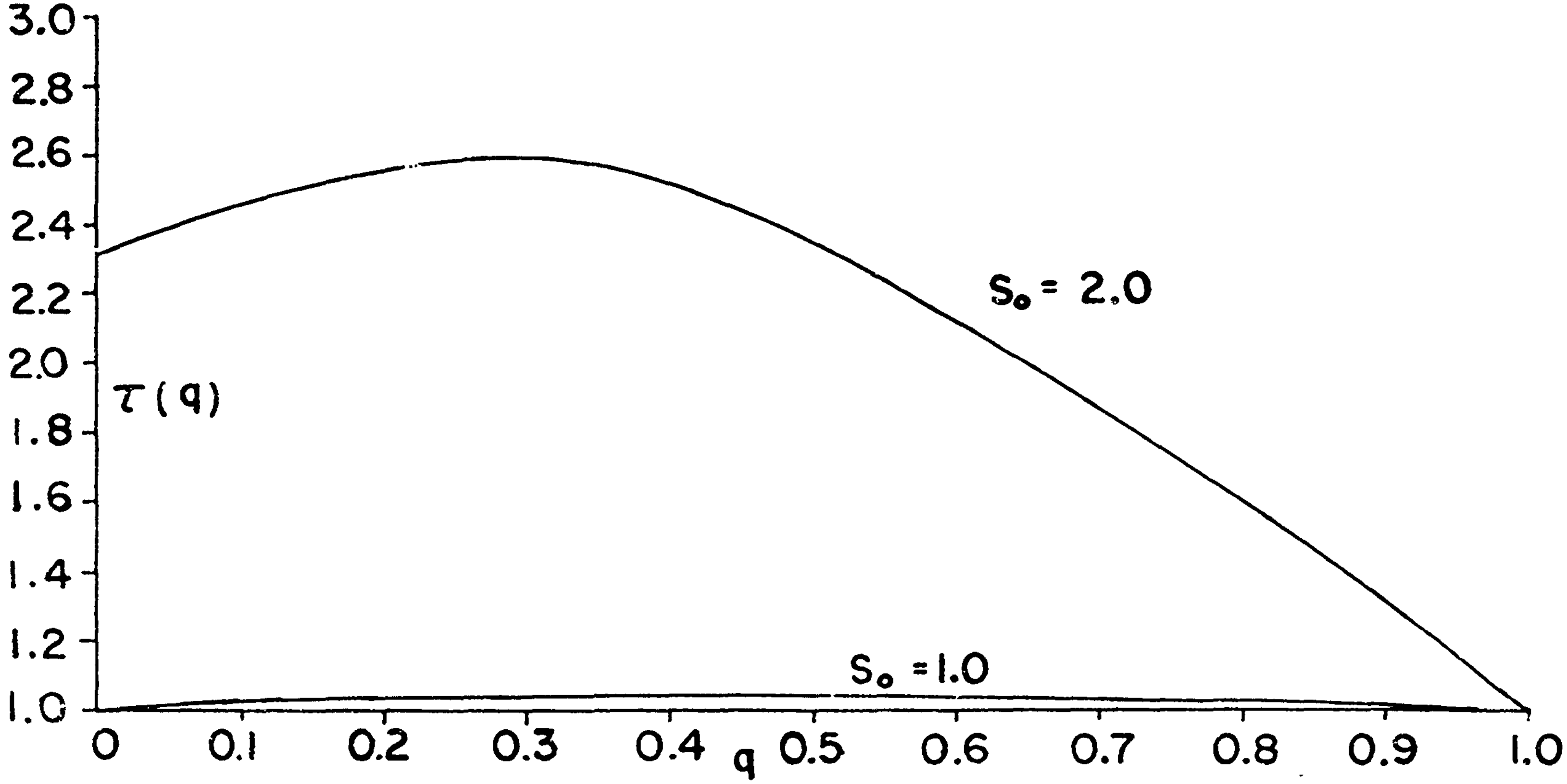

Figure 7. $\tau(q)$ for different values of s_o ($\sigma_o = 1.42 \times s_o$ psi) for $\omega = 2500$ rad./sec.

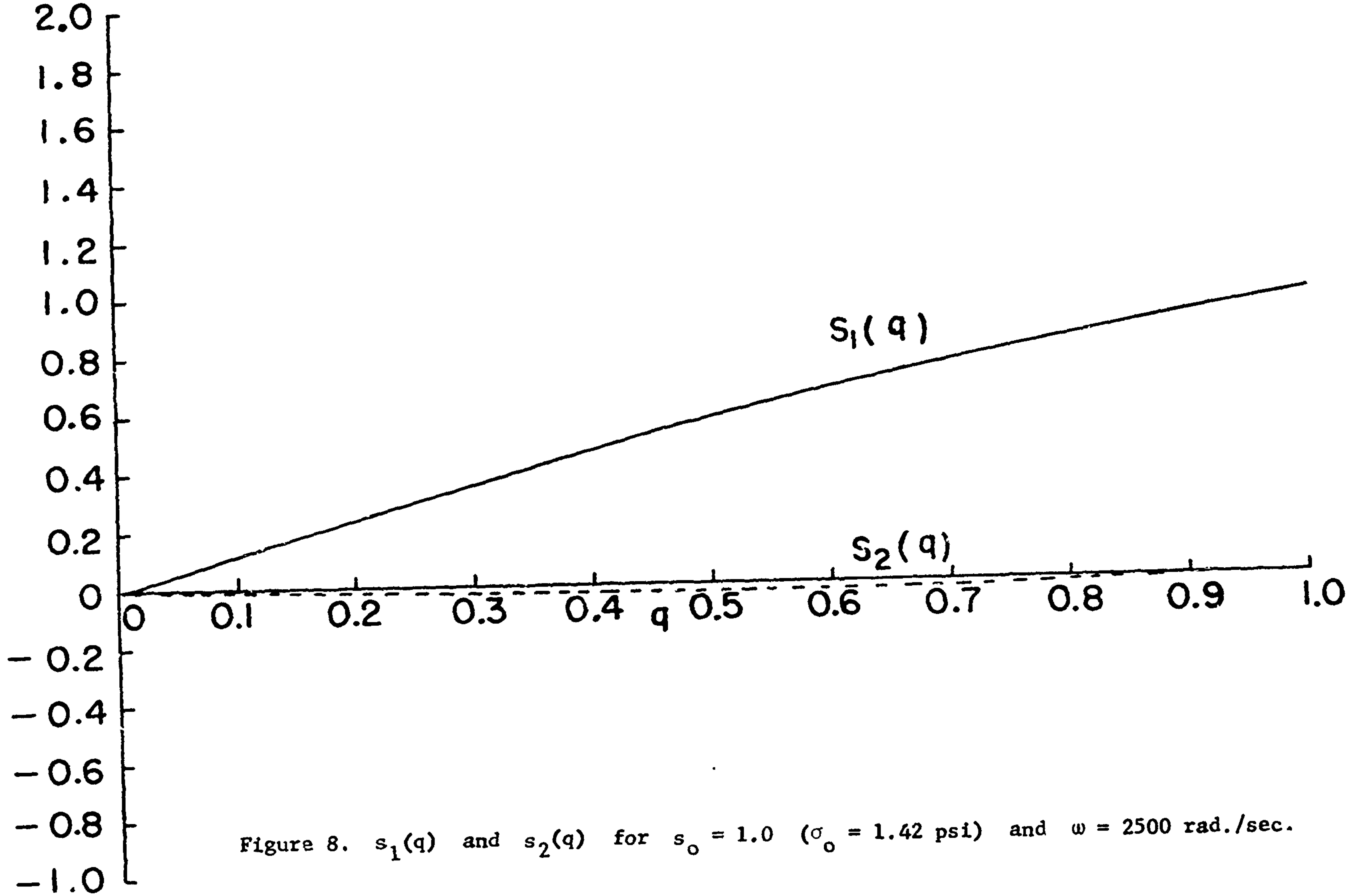

Figure 8. $s_1(q)$ and $s_2(q)$ for $s_o = 1.0$ ($\sigma_o = 1.42$ psi) and $\omega = 2500$ rad./sec.

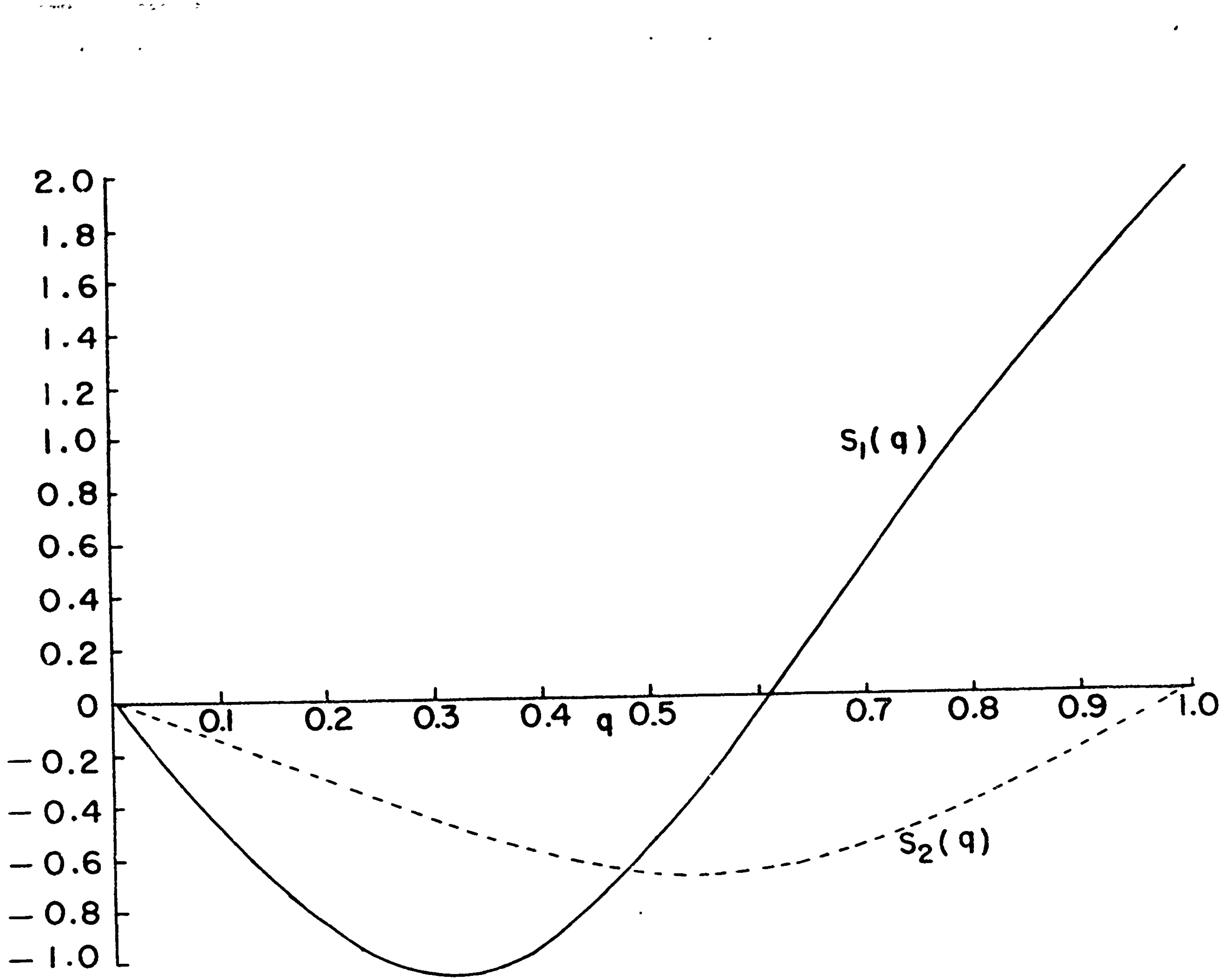

Figure 9. $s_1(q)$ and $s_2(q)$ for $s_o = 2.0$ ($\sigma_o = 2.84$ psi) and $\omega = 2500$ rad./sec.

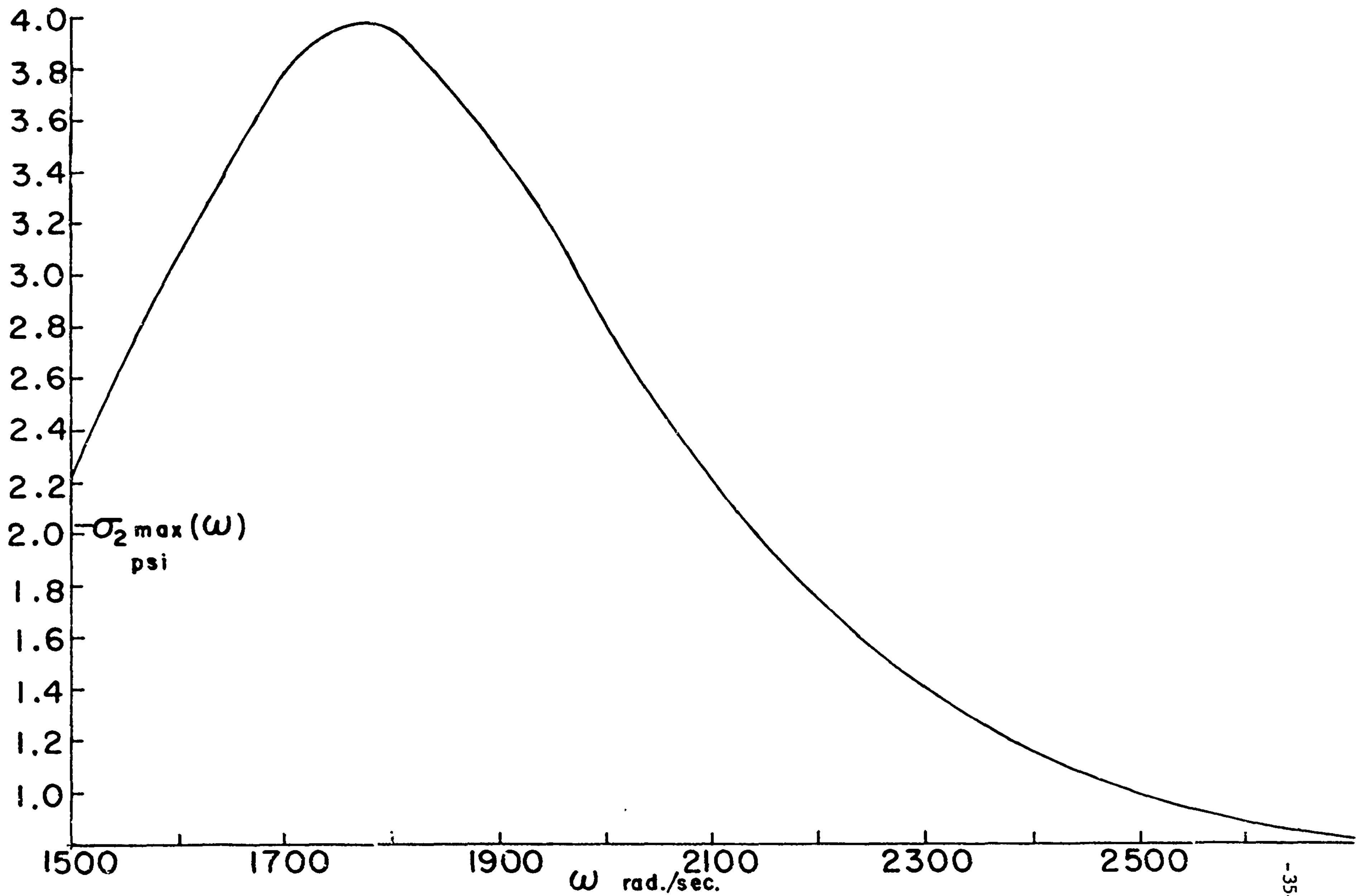

Figure 10. Peak stress $-\sigma_{2max}$ versus frequency ω given $\tau(q)$ for ω = 2500 rad./sec.
σ_o = 2.84 psi .