AD743945

NUSC Technical Report 4343

Estimation of the Magnitude-Squared Coherence Function (Spectrum)

G. CLIFFORD CARTER
Sonar Technology Department

DDC
RECEIVED
JUN 26 1972
A

19 May 1972

NAVAL UNDERWATER SYSTEMS CENTER

Approved for public release; distribution unlimited.

Reproduced by
NATIONAL TECHNICAL
INFORMATION SERVICE
U S Department of Commerce
Springfield VA 22151

105

ADMINISTRATIVE INFORMATION

This study was originally prepared as a thesis in partial fulfillment of the requirements for the degree of Master of Science in Electrical Engineering at The University of Connecticut.

REVIEWED AND APPROVED: 19 May 1972

W. A. Von Winkle
Director of Science and Technology

WHITE SECTION
BUFF SECTION
UNANNOUNCED
JUSTIFICATION
BY
DISTRIBUTION/AVAILABILITY CODES
DIST. AVAIL. and/or SPECIAL

Inquiries concerning this report may be addressed to the authors,
New London Laboratory, Naval Underwater Systems Center,
New London, Connecticut 06320

UNCLASSIFIED
Security Classification

DOCUMENT CONTROL DATA - R & D

Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) Naval Underwater Systems Center Newport, Rhode Island 02840	2a. REPORT SECURITY CLASSIFICATION UNCLASSIFIED 2b. GROUP

3. REPORT TITLE

ESTIMATION OF THE MAGNITUDE-SQUARED COHERENCE FUNCTION (SPECTRUM)

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

Research Report

5. AUTHOR(S) (First name, middle initial, last name)

G. Clifford Carter

6. REPORT DATE 19 May 1972	7a. TOTAL NO. OF PAGES 102	7b. NO. OF REFS 32
8a. CONTRACT OR GRANT NO. b. PROJECT NO c. d.	9a. ORIGINATOR'S REPORT NUMBER(S) TR 4343 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)	

10. DISTRIBUTION STATEMENT

Approved for public release; distribution unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY Department of the Navy

13. ABSTRACT

A method of estimating the magnitude-squared coherence function (spectrum) for zero-mean processes that are wide-sense stationary and random is presented. The estimation technique utilizes the weighted overlapped segmentation fast Fourier transform (FFT) approach. Analytical and empirical results for statistics of the estimator are presented for the processes. Analytical expressions are derived in the nonoverlapped case. Empirical results show a decrease in bias and variance of the estimator with increasing overlap and suggest that a 50-percent overlap is highly desirable when cosine (Hanning) weighting is used.

DD FORM 1 NOV 65 1473 (PAGE 1)
0102-014-6600

UNCLASSIFIED
Security Classification

UNCLASSIFIED
Security Classification

14. KEY WORDS	LINK A ROLE	LINK A WT	LINK B ROLE	LINK B WT	LINK C ROLE	LINK C WT
Coherence						
Complex Coherence Function						
Fast Fourier Transform (FFT)						
Magnitude-Squared Coherence Function (Spectrum)						
Weighted Overlapped Segmentation Fast Fourier Transform (FFT)						
Zero-Mean Processes (Wide-Sense Stationary and Random)						

DD FORM 1 NOV 65 1473 (BACK)
(PAGE 2)

Ia

UNCLASSIFIED
Security Classification

ABSTRACT

A method of estimating the magnitude-squared coherence function (spectrum) for zero-mean processes that are wide-sense stationary and random is presented. The estimation technique utilizes the weighted overlapped segmentation fast Fourier transform (FFT) approach. Analytical and empirical results for statistics of the estimator are presented for the processes. Analytical expressions are derived in the non-overlapped case. Empirical results show a decrease in bias and variance of the estimator with increasing overlap and suggest that a 50-percent overlap is highly desirable when cosine (Hanning) weighting is used.

Ib

ACKNOWLEDGMENT

My involvement with coherence estimation began in June 1970 during a consultation between Messrs. R. M. Kennedy, R. R. Kneipfer, C. R. Arnold, and J. F. Ferrie, of the Naval Underwater Systems Center (NUSC). In February 1971, at an NUSC symposium, I presented some of the practical aspects of coherence estimation. The paper presented at that symposium served as a foundation for this thesis, which was begun in May 1971 under the technical direction of Drs. C. H. Knapp, of The University of Connecticut, and A. H. Nuttall, of NUSC.

I acknowledge the assistance of the following individuals: From the University of Connecticut — Drs. H. M. Lucal and R. J. Kochenburger for reviewing portions of the manuscripts; from the Naval Ship Systems Command — Mr. S. Blazek for support of this work; from NUSC — Drs. P. G. Cable and F. J. Kingsbury for support, Dr. E. S. Eby and Mr. C. P. Hunter for reviewing portions of the manuscript, Mmes. A. Meloney and M. Patton for typing the manuscript, Mr. J. C. Sikorski for digitizing the data, and Mr. B. P. Hoddersen for preparing the figures.

TABLE OF CONTENTS

	Page
ABSTRACT	i
ACKNOWLEDGMENT	ii
LIST OF ILLUSTRATIONS	v
LIST OF TABLES	vii
GLOSSARY	ix
Chapter	
I. INTRODUCTION	1
II. COHERENCE FUNCTION (SPECTRUM) AND ITS USES	5
II. A. Coherence Function	5
II. A. 1. Correlation matrix and wide-sense stationarity	5
II. A. 2. Ergodicity	7
II. A. 3. Gaussian assumption	8
II. A. 4. Power spectral density matrix	8
II. A. 5. Definition	9
II. B. Uses of Coherence Function	11
II. B. 1. A measure of system linearity	11
II. B. 2. A measure of correlation	13
II. B. 3. A measure of signal-to-noise ratio	13
III. COHERENCE-ESTIMATION PROCEDURE	18
III. A. Quantized Sequence Obtained from Weighted Segment of Data	19
III. B. Coherence Estimator	22
IV. STATISTICS OF ESTIMATE OF COHERENCE	26
IV. A. Probability Density and Cumulative Distribution Functions	26
IV. A. 1. Probability density function	27
IV. A. 2. Cumulative distribution function	28
IV. A. 3. Computer evaluation	29
IV. B. mth Moment of Density Function	40

TABLE OF CONTENTS (Cont'd)

IV. C. Bias and Variance 41
IV. C. 1. Bias 42
IV. C. 2. Variance 44
IV. C. 3. Digital computer evaluation of bias and variance 47

V. EXPERIMENTAL INVESTIGATION OF OVERLAP EFFECTS. 66

V. A. Method 66
V. A. 1. Data generation 67
V. A. 2. Analysis program 72
V. B. Results 74

VI. CONCLUSIONS 82

Appendixes
A. STATISTICS OF MAGNITUDE-COHERENCE ESTIMATOR . 84

B. DERIVATION OF A SIMPLIFIED EXPRESSION FOR THE EXPECTATION OF THE ESTIMATE OF MAGNITUDE-SQUARED COHERENCE 88

LIST OF REFERENCES 91

LIST OF ILLUSTRATIONS

Figure		Page
1.	Linear System with Impulse Response $h(\tau)$	11
2.	Signal $s(t)$ as Received at Two Sensors	14
3.	Two Time Series from Processes $x(t)$ and $y(t)$ . . .	20
4.	Overlapped Weighting Functions (Modified from Knapp[22]) .	20
5.	Summary Flow Chart for Density and Cumulative Distribution	30
6.	Probability Density and Cumulative Distribution Functions for $\lvert\hat{\gamma}\rvert^2$ Given $n = 32$ and $\lvert\gamma\rvert^2 = 0.0$.	32
7.	Probability Density and Cumulative Distribution Functions for $\lvert\hat{\gamma}\rvert^2$ Given $n = 32$ and $\lvert\gamma\rvert^2 = 0.3$.	33
8.	Probability Density and Cumulative Distribution Functions for $\lvert\hat{\gamma}\rvert^2$ Given $n = 32$ and $\lvert\gamma\rvert^2 = 0.6$.	34
9.	Probability Density and Cumulative Distribution Functions for $\lvert\hat{\gamma}\rvert^2$ Given $n = 32$ and $\lvert\gamma\rvert^2 = 0.9$.	35
10.	Probability Density and Cumulative Distribution Functions for $\lvert\hat{\gamma}\rvert^2$ Given $n = 64$ and $\lvert\gamma\rvert^2 = 0.0$.	36
11.	Probability Density and Cumulative Distribution Functions for $\lvert\hat{\gamma}\rvert^2$ Given $n = 64$ and $\lvert\gamma\rvert^2 = 0.3$.	37
12.	Probability Density and Cumulative Distribution Functions for $\lvert\hat{\gamma}\rvert^2$ Given $n = 64$ and $\lvert\gamma\rvert^2 = 0.6$.	38
13.	Probability Density and Cumulative Distribution Functions for $\lvert\hat{\gamma}\rvert^2$ Given $n = 64$ and $\lvert\gamma\rvert^2 = 0.9$.	39
14.	Summary Flow Chart for Bias and Variance Computations .	40

LIST OF ILLUSTRATIONS (Cont'd)

15. Bias of $|\hat{\gamma}|^2$ versus $|\gamma|^2$ and n 54

16. Variance of $|\hat{\gamma}|^2$ versus $|\gamma|^2$ and n . . . 55

17. Bias and Variance versus $|\gamma|^2$ for n = 32 . . . 56

18. Bias and Variance versus $|\gamma|^2$ for n = 40 . . . 57

19. Bias and Variance versus $|\gamma|^2$ for n = 48 . . . 58

20. Bias and Variance versus $|\gamma|^2$ for n = 56 . . . 59

21. Bias and Variance versus $|\gamma|^2$ for n = 64 . . . 60

22. Bias and Variance versus n for $|\gamma|^2 = 0.0$ . . 61

23. Bias and Variance versus n for $|\gamma|^2 = 0.04$ . . 62

24. Bias and Variance versus n for $|\gamma|^2 = 0.28$ . . 63

25. Bias and Variance versus n for $|\gamma|^2 = 0.32$ . . 64

26. Bias and Variance versus n for $|\gamma|^2 = 0.96$ . . 65

27. Summary Flow Chart for Thesis Version of FFT Spectral Density Estimation Program . . 73

28. Bias of $|\hat{\gamma}|^2$ When $|\gamma|^2 = 0.0$ and n = 32 . . . 76

29. Variance of $|\hat{\gamma}|^2$ When $|\gamma|^2 = 0.0$ and n = 32 . . 77

30. Bias of $|\hat{\gamma}|^2$ When $|\gamma|^2 = 0.3$ and n = 32 . . . 78

31. Variance of $|\hat{\gamma}|^2$ When $|\gamma|^2 = 0.3$ and n = 32 . . 79

32. Number of FFTs Required for Overlapped Processing . 81

LIST OF TABLES

Table		Page
1.	Bias and Variance of $\lvert\hat{\gamma}\rvert^2$ for $n = 32$	49
2.	Bias and Variance of $\lvert\hat{\gamma}\rvert^2$ for $n = 40$	50
3.	Bias and Variance of $\lvert\hat{\gamma}\rvert^2$ for $n = 48$	51
4.	Bias and Variance of $\lvert\hat{\gamma}\rvert^2$ for $n = 56$	52
5.	Bias and Variance of $\lvert\hat{\gamma}\rvert^2$ for $n = 64$	53
6.	Empirical Results for $\lvert\gamma\rvert^2 = 0.0$ and $n = 32$	75
7.	Empirical Results for $\lvert\gamma\rvert^2 = 0.3$ and $n = 32$	75

GLOSSARY

$C_{xy}(f)$	Real part of $\Phi_{xy}(f)$
$\hat{C}_{xy_s}(f_k)$	Real part of $\hat{\Phi}_{xy_s}(f_k)$
$E[\ ]$	Expectation
$\mathbf{M}_{xy}(f)$	Power spectral density matrix
$Q_{xy}(f)$	Imaginary part of $\Phi_{xy}(f)$
$\hat{Q}_{xy_s}(f_k)$	Imaginary part of $\hat{\Phi}_{xy_s}(f_k)$
$R_{xx}(\tau)$	Autocorrelation function of $x(t)$ process
$R_{xy}(\tau)$	Cross-correlation function of $x(t)$ and $y(t)$
$\mathbf{R}_{xy}(\tau)$	Cross-correlation matrix
$R_{yy}(\tau)$	Autocorrelation function of $y(t)$ process
$X_s(f_k)$	kth discrete Fourier coefficient obtained from sth weighted segment of $x(t)$ data
$Y_s(f_k)$	kth discrete Fourier coefficient obtained from sth weighted segment of $y(t)$ data
$\Gamma(\ )$	Gamma function
$\lvert\gamma_{xy}(f)\rvert^2$	Magnitude-squared coherence (MSC) function (spectrum)
$\lvert\hat{\gamma}_{xy}(f_k)\rvert^2$	Estimate of MSC at kth frequency

$\lvert\gamma_{xy}(f)\rvert$	Magnitude coherence (MC) function (spectrum)
$\lvert\hat{\gamma}_{xy}(f_k)\rvert$	Estimate of MC at kth frequency
$\Phi_{xx}(f)$	Auto power spectral density function of x(t) process
$\hat{\Phi}_{xx_s}(f_k)$	Estimate of $\Phi_{xx}(f)$ at kth frequency obtained from sth weighted segment of data
$\Phi_{xy}(f)$	Cross-power spectral density function of x(t) and y(t)
$\hat{\Phi}_{xy_s}(f_k)$	Estimate of $\Phi_{xy}(f)$ at kth frequency obtained from sth weighted segment of data
$\Phi_{yy}(f)$	Auto-power spectral density function of y(t) process
$\hat{\Phi}_{yy_s}(f_k)$	Estimate of $\Phi_{yy}(f)$ at kth frequency obtained from sth weighted segment of data
*	Complex conjugation
$\forall$	For all

ESTIMATION OF THE
MAGNITUDE-SQUARED COHERENCE FUNCTION (SPECTRUM)

I. INTRODUCTION

The complete probability structure of the zero-mean processes $x(t)$ and $y(t)$, which are wide-sense stationary and jointly Gaussian, is specified by the spectral density matrix,

$$\mathbf{M}_{xy}(f) = \begin{bmatrix} \Phi_{xx}(f) & \Phi_{xy}(f) \\ \Phi_{yx}(f) & \Phi_{yy}(f) \end{bmatrix} , \tag{1.1}$$

where

$\Phi_{xx}(f)$ is the (real) auto power spectral density function of $x(t)$,

$\Phi_{yy}(f)$ is the (real) auto power spectral density function of $y(t)$, and

$\Phi_{xy}(f)$ is the (complex) cross power spectral density function of $x(t)$ and $y(t)$ and consists of a real or coincidental (CO) spectrum and an imaginary or quadrature (quad) spectrum.[1]

A simplifying ratio is the complex coherence function (spectrum),

$$\gamma_{xy}(f) = \frac{\Phi_{xy}(f)}{\sqrt{\Phi_{xx}(f)\,\Phi_{yy}(f)}} , \tag{1.2}$$

or, more commonly, the magnitude-squared coherence function,

$$\left| \gamma_{xy}(f) \right|^2 = \frac{\left| \Phi_{xy}(f) \right|^2}{\Phi_{xx}(f)\, \Phi_{yy}(f)} \quad . \tag{1.3}$$

The term "coherence" can imply Eqs. (1.2), (1.3), or the positive square root of Eq. (1.3).

Equation (1.3) possesses a number of useful attributes: First, it always falls between zero and one. Second, it is zero if the processes $x(t)$ and $y(t)$ are uncorrelated. Third, it is equal to unity if and only if there exists a linear relation between $x(t)$ and $y(t)$.[2]

These attributes are of particular significance in sonar systems where a waveform received at two spatially separated elements of a hydrophone array may be corrupted by additive noise uncorrelated from the first to the second element.

Unfortunately, the difficulty in estimating the true coherence has plagued modern statisticians.[3] An analytical expression was derived by Goodman[1] for the probability density function of the estimate of magnitude coherence $|\hat{\gamma}|$ when several independent observations (or segments) of the processes are available. A closed-form solution for the cumulative distribution function, as a finite sum of hypergeometric functions, can be found by proper identification of variables in the work of Fisher.[4] The application of Fisher's work to this problem is believed original in this thesis. Earlier, statistics for coherence estimation were found in tables, and graphs,[5-7] and transformations to be performed on the coherence estimator were suggested so as to "normalize" (make Gaussian) the density function.[8,9]

Certain empirical studies have also been conducted. Haubrich suggested that the total time series under investigation be segmented into a number of shorter segments overlapping one another by 50 percent and that a triangular weighting function be applied to each segment.[7] Tick showed empirical examples of the types of estimates to anticipate when the true coherence is 0.2 and mean lagged product techniques are used.[3] Benignus empirically showed the bias and confidence intervals to expect when n independent segments are processed using a rectangular weighting function.[10]

Current techniques for coherence estimation involve applying the fast Fourier transform (FFT).[11] Some of the latest published results on coherence estimation are limited in scope to processes that have relatively flat spectra.[10] The problems associated with nonflat spectra can be avoided through judicious choice of a time-weighting (or windowing) function.[12-14] The use of a weighting function is necessary for data not spectrally flat and should be prudently selected for unknown data. In coherence estimation, the application of a weighting function results in wasted data (loss of stability and increase of bias) unless overlapped processing[14] is employed. In underwater acoustic environments, which require weighting functions and good spectral resolution, but which remain stationary only for limited amounts of time, such wastage can not be permitted.

This thesis empirically determines the effect of overlap processing on the estimated magnitude-squared coherence function when cosine (or Hanning, after Julius von Hann) weighting has been applied.

The empirical method for determining the effect of overlap has been limited in scope to a cosine weighting function, a finite time history, and a desired

frequency resolution (half-power) bandwidth. Under these conditions, estimates of bias and variance of the estimate of magnitude-squared coherence have been made for two values of coherence. The behavior of these statistics as a function of increasing overlap is presented and is believed original.

II. COHERENCE FUNCTION (SPECTRUM) AND ITS USES

This chapter defines the coherence function (spectrum). Additionally, it reviews those terms necessary for its definition or helpful in its estimation. Finally, this chapter presents some examples of the uses of coherence to lay a background for why this particular function is meaningful.

II. A. COHERENCE FUNCTION

The essense of the coherence function is a collapsed power spectral density matrix. To fully appreciate the intricacies of its definition, it is first necessary to review some basic concepts. They include the correlation matrix, wide-sense stationarity, ergodicity, Gaussian assumption, and power spectral density matrix.

II. A. 1. Correlation Matrix and Wide-Sense Stationarity

The general correlation function between zero-mean processes $x(t)$ and $y(t)$, which are real and nonstationary, is defined by Davenport and Root,[15] as follows:

$$R_{xy}(t_1, t_2) \triangleq E\left[x(t_1)\, y(t_2)\right] , \tag{2.1}$$

which depends on the absolute time instants t_1 and t_2. If the cross-correlation function depends only on the time difference $\tau = t_2 - t_1$ and does not depend on the time origin, that is, if

$$R_{xy}(t, t+\tau) = R_{xy}(\tau) \triangleq E\left[x(t)\, y(t+\tau)\right], \tag{2.2}$$

then the processes are called wide-sense stationary. It is not necessary for $R_{xy}(\tau)$ to be an even or an odd function.

Similarly, the autocorrelation function in the wide-sense stationary case becomes

$$R_{xx}(\tau) \triangleq E\left[x(t)\, x(t+\tau)\right], \tag{2.3}$$

which is an even function. The autocorrelation function of the process $y(t)$ is similarly defined.

The correlation matrix for the wide-sense stationary processes $x(t)$ and $y(t)$ may now be defined by

$$\mathbf{R}_{xy}(\tau) \triangleq \begin{bmatrix} R_{xx}(\tau) & R_{xy}(\tau) \\ R_{yx}(\tau) & R_{yy}(\tau) \end{bmatrix}. \tag{2.4}$$

When two zero-mean random processes have a correlation matrix that depends only on the time difference, it is meaningful to talk about the Fourier transformation of the correlation matrix.[16]

II. A. 2. Ergodicity

Random processes can be characterized by an infinite number of waveforms. Each of these waveforms is referred to as a sample or member function of the random process and is itself infinite in duration.[17] Statistics of an order higher than the correlation function can be computed by averaging over the ensemble of all sample functions. These statistics can also be computed from any one of the sample functions.

If all the higher order statistics, when computed from any one of the sample functions, are the same as the ensemble average over all the sample functions, then the processes are called ergodic. In particular, the correlation matrix computed over any one sample function is the same as the correlation matrix computed over an ensemble of sample functions. It should be noted that it is possible for the correlation matrix to be the same when computed over different sample functions and yet for some higher order statistics to differ when computed over different sample functions.

It is important for the results presented in this thesis that the correlation matrix be the same when computed over different sample functions. This, in essence, allows the correlation matrix (or its linear transformations) to be specified with probability one from one sample function. If the correlation matrix does not differ when computed over different sample functions, the processes are still called ergodic, but now some qualifying adjective must be applied to denote the strength of the ergodicity.[18] This author chooses to use the adjective "wide-sense" to specify the strength of the ergodicity. Processes that are wide-sense ergodic are also wide-sense stationary.

II. A. 3. Gaussian Assumption

If two zero-mean processes are jointly Gaussian and wide-sense stationary, then their correlation matrix dictates all higher order statistics.[16]

II. A. 4. Power Spectral Density Matrix

Several important concepts have preceded this section: First, in order to mathematically determine the power spectral density matrix, the processes must be stationary (in the wide sense). In practical estimation situations ergodicity is presumed, and only one time-limited sample function is collected for each process under investigation. It is desirable, but not necessary, that the two processes be jointly Gaussian. When the two processes are both stationary and jointly Gaussian, then knowledge of their correlation matrix completely specifies the statistics of the processes.

Given the zero-mean processes $x(t)$ and $y(t)$, which are real, stationary, and jointly Gaussian, a complete characterization for the probability structure of the processes is specified in terms of the power spectral density matrix $\mathbf{M}_{xy}(f)$:[1]

$$\mathbf{M}_{xy}(f) \triangleq \begin{bmatrix} \Phi_{xx}(f) & \Phi_{xy}(f) \\ \Phi_{yx}(f) & \Phi_{yy}(f) \end{bmatrix} . \tag{2.5}$$

The power spectral density functions composing the elements of the power spectral density matrix are the Fourier transforms of the associated correlation functions. The cross power spectral density function is

$$\Phi_{xy}(f) \triangleq \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j2\pi f \tau} d\tau \quad . \tag{2.6}$$

In general, this function is complex since $R_{xy}(\tau)$ is not necessarily odd or even. Similarly,

$$\Phi_{xx}(f) \triangleq \int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j2\pi f \tau} d\tau \quad , \tag{2.7}$$

which is purely real since $R_{xx}(\tau)$ is even.

II.A.5. Definition

The complex coherence function for two wide-sense stationary processes is a normalized complex cross power spectral density function given by

$$\gamma_{xy}(f) \triangleq \frac{\Phi_{xy}(f)}{\sqrt{\Phi_{xx}(f)\ \Phi_{yy}(f)}} \quad . \tag{2.8}$$

Since $\Phi_{xy}(f)$ is complex,

$$\Phi_{xy}(f) = C_{xy}(f) + j\, Q_{xy}(f) \quad . \tag{2.9}$$

Further, $\Phi_{xx}(f)$ and $\Phi_{yy}(f)$ are nonnegative, real functions of f,

$$\Phi_{xx}(f) \geq 0 \tag{2.10}$$

and

$$\Phi_{yy}(f) \geq 0 \quad . \tag{2.11}$$

The magnitude of the complex coherence function (or, simply, the magnitude coherence) is

$$\left| \gamma_{xy}(f) \right| = \frac{\left| \Phi_{xy}(f) \right|}{\sqrt{\Phi_{xx}(f)\ \Phi_{yy}(f)}} \quad . \tag{2.12}$$

It follows directly that the square of the magnitude of the complex coherence function (or, simply, the magnitude-squared coherence) is

$$\left| \gamma_{xy}(f) \right|^2 = \frac{\left| \Phi_{xy}(f) \right|^2}{\Phi_{xx}(f)\ \Phi_{yy}(f)} \quad , \tag{2.13a}$$

$$= \frac{C_{xy}^2(f) + Q_{xy}^2(f)}{\Phi_{xx}(f)\ \Phi_{yy}(f)} \quad . \tag{2.13b}$$

Although the term "coherence" can imply Eqs. (2.8), (2.12), or (2.13), it usually refers to Eq. (2.13).

For ease of notation, the dependence on f is often not specified; for example,

$$\left| \gamma_{xy} \right|^2 = \frac{C_{xy}^2 + Q_{xy}^2}{\Phi_{xx}\ \Phi_{yy}} \quad . \tag{2.14}$$

II. B. USES OF COHERENCE FUNCTION

The magnitude-squared coherence function for the zero-mean, wide-sense stationary processes $x(t)$ and $y(t)$ is useful in several ways, which will be proved in the following sections. First, for two processes that are linearly related, the magnitude-squared coherence function is unity. Second, for two independent processes, the magnitude-squared coherence function is zero. Third, under the assumptions to be presented, the magnitude-squared coherence function serves as a signal-to-noise measure.

II. B. 1. A Measure of System Linearity

The magnitude-squared coherence function can be used to measure system linearity.[12] In Fig. 1 consider the linear system with input $x(t)$, impulse response $h(\tau)$, and output $y(t)$. The output $y(t)$ is expressed by the convolution integral

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau \quad . \tag{2.15}$$

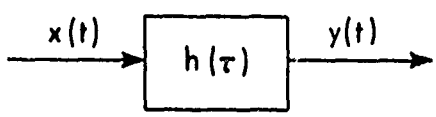

Fig. 1. Linear System with Impulse Response $h(\tau)$

The frequency-domain equivalent is a multiplication obtained via the Fourier transformation:

$$Y(f) = H(f)\, X(f) \tag{2.16}$$

If $x(t)$ is a sample function of a stationary random process,[2] then

$$\Phi_{xy}(f) = H(f)\, \Phi_{xx}(f) \tag{2.17}$$

and

$$\Phi_{yy}(f) = H(f)\, H^*(f)\, \Phi_{xx}(f) = H(f)\, \Phi^*_{xy}(f) \quad . \tag{2.18}$$

Since the magnitude-squared coherence defined by Eq. (2.13) can be written as

$$\left| \gamma_{xy}(f) \right|^2 = \frac{\Phi_{xy}(f)\, \Phi^*_{xy}(f)}{\Phi_{xx}(f)\, \Phi_{yy}(f)} \quad , \tag{2.19}$$

application of Eqs. (2.17) and (2.18) yields

$$\left| \gamma_{xy}(f) \right|^2 = H(f)\, \frac{1}{H(f)} = 1 \ , \ \forall\, f \ . \tag{2.20}$$

Consequently, the magnitude-squared coherence between the input and output of a linear system is unity.

II. B. 2. A Measure of Correlation

If the zero-mean processes $x(t)$ and $y(t)$ are independent, they are also uncorrelated and orthogonal; that is

$$R_{xy}(\tau) = E\left[x(t)y(t+\tau)\right] = E\left[x(t)\right] E\left[y(t+\tau)\right] = 0 \quad , \tag{2.21}$$

$$\Phi_{xy}(f) = \int_{-\infty}^{\infty} R_{xy}(\tau)\, e^{-j2\pi f\tau}\, d\tau \quad = 0 \quad , \tag{2.22}$$

and

$$\left| \gamma_{xy}(f) \right|^2 = 0 \quad , \quad \forall\ f \ . \tag{2.23}$$

Hence, if the two processes are independent or uncorrelated with zero mean, the magnitude-squared coherence between them is zero.

II. B. 3. A Measure of Signal-to-Noise Ratio

Consider a signal, $s(t)$, passed through two linear filters and received at two sensors where it is corrupted by uncorrelated additive noises. The received waveform at each sensor is then passed through two linear filters, as shown in Fig. 2.

Assume that $s(t)$, $n_1(t)$, and $n_2(t)$ are uncorrelated; that is,

$$E\left[n_1(t)\, n_2(t+\tau)\right] = 0 \quad , \tag{2.24}$$

$$E\left[n_1(t)\ s(t+\tau)\right] = 0 \quad , \tag{2.25}$$

and

$$E\left[n_2(t)\, s(t+\tau)\right] = 0 \quad . \tag{2.26}$$

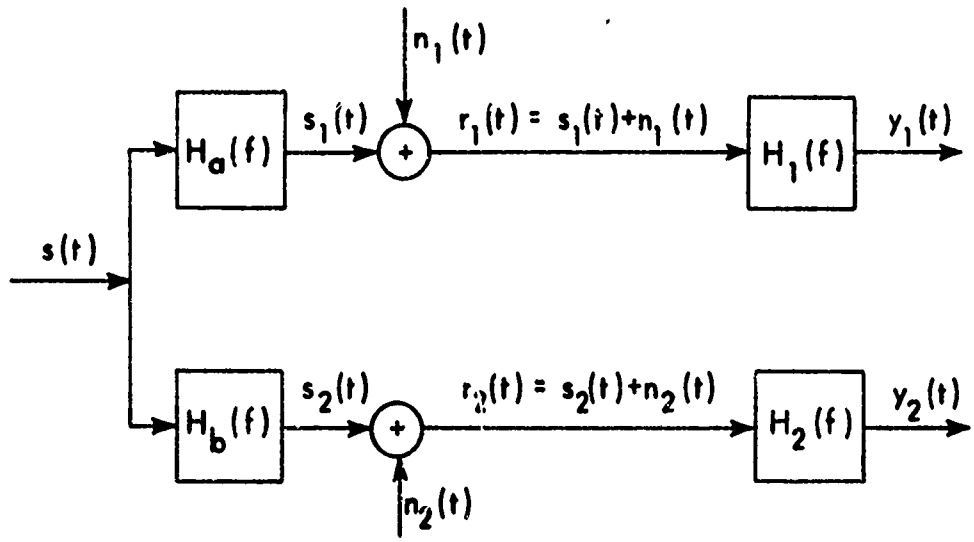

Fig. 2. Signal s(t) as Received at Two Sensors

Then,

$$\Phi_{y_1 y_1}(f) = \Phi_{r_1 r_1}(f)\, \left|H_1(f)\right|^2 \quad , \tag{2.27a}$$

$$= \left[\Phi_{s_1 s_1}(f) + \Phi_{n_1 n_1}(f)\right] \left|H_1(f)\right|^2 \quad , \tag{2.27b}$$

and

$$= \left[\Phi_{ss}(f)\, \left|H_a(f)\right|^2 + \Phi_{n_1 n_1}(f)\right] \left|H_1(f)\right|^2 \quad ; \tag{2.27c}$$

$$\Phi_{y_2 y_2}(f) = \Phi_{r_2 r_2}(f)\, \left|H_2(f)\right|^2 \quad , \tag{2.28a}$$

$$= \left[\Phi_{s_2 s_2}(f) + \Phi_{n_2 n_2}(f)\right] \left|H_2(f)\right|^2 \quad , \tag{2.28b}$$

and

$$= \left[\Phi_{ss}(f) \left|H_b(f)\right|^2 + \Phi_{n_2n_2}(f)\right] \left|H_2(f)\right|^2 ; \qquad (2.28c)$$

$$\Phi_{y_1y_2}(f) = H_1(f)\, H_2^*(f)\, \Phi_{r_1r_2}(f) , \qquad (2.29a)$$

$$= H_1(f)\, H_2^*(f)\, \Phi_{s_1s_2}(f) , \qquad (2.29b)$$

and

$$= H_1(f)\, H_2^*(f)\, H_a(f)\, H_b^*(f)\, \Phi_{ss}(f) ; \qquad (2.29c)$$

$$\left|\gamma_{y_1y_2}(f)\right|^2 = \frac{\Phi_{ss}^2(f) \left| H_a(f)\, H_b^*(f)\right|^2}{\left[\Phi_{ss}(f) \left| H_a(f)\right|^2 + \Phi_{n_1n_1}(f)\right]\left[\Phi_{ss}(f) \left| H_b(f)\right|^2 + \Phi_{n_2n_2}(f)\right]} . \qquad (2.30a)$$

and

$$= \left|\gamma_{r_1r_2}(f)\right|^2 \qquad (2.30b)$$

Equation (2.30b) is independent of both $H_1(f)$ and $H_2(f)$; that is, the coherence between the two received waveforms is not changed by linear filtering.

There are two special cases of Eq. (2.30a) that are of interest: First, when

$$\Phi_{n_1n_1}(f) = \Phi_{n_2n_2}(f) = \Phi_{nn}(f) \qquad (2.31)$$

and

$$\left| H_a(f)\right|^2 = \left| H_b(f)\right|^2 = 1 \qquad (2.32)$$

(as, for example, if

$$H_a(f) = e^{-j2\pi f \tau_a} \tag{2.33}$$

and

$$H_b(f) = e^{-j2\pi f \tau_b} \tag{2.34}$$

corresponding to time delays τ_a and τ_b of a directional signal), then

$$\left| \gamma_{y_1 y_2}(f) \right|^2 = \frac{\Phi_{ss}^2(f)}{\left[\Phi_{ss}(f) + \Phi_{nn}(f)\right]^2} \tag{2.35}$$

and

$$\frac{\Phi_{ss}(f)}{\Phi_{nn}(f)} = \frac{\left| \gamma_{y_1 y_2}(f) \right|}{1 - \left| \gamma_{y_1 y_2}(f) \right|} . \tag{2.36}$$

Second, when

$$n_1(t) = 0 \quad , \tag{2.37}$$

$$\Phi_{n_2 n_2}(f) = \Phi_{nn}(f) \quad , \tag{2.38}$$

and

$$\left| H_a(f) \right|^2 = \left| H_b(f) \right|^2 = 1 \quad , \tag{2.39}$$

then

$$\left| \gamma_{y_1 y_2}(f) \right|^2 = \frac{\Phi_{ss}^2(f)}{\left[\Phi_{ss}(f) + \Phi_{nn}(f)\right] \Phi_{ss}(f)} \tag{2.40}$$

and

$$\frac{\Phi_{ss}(f)}{\Phi_{nn}(f)} = \frac{\left| \gamma_{y_1 y_2}(f) \right|^2}{1 - \left| \gamma_{y_1 y_2}(f) \right|^2} \quad . \tag{2.41}$$

This is a generalization of work done by Roth,[19] Carter and Arnold,[12] and Knapp.[20]

III. COHERENCE-ESTIMATION PROCEDURE

The procedure for estimating the coherence or magnitude-squared coherence functions for wide-sense ergodic (and, hence, wide-sense stationary), zero-mean random processes $x(t)$ and $y(t)$ is discussed in this chapter. (References within this chapter to $x(t)$ and $y(t)$ apply to those specific processes with the noted characteristics, that is, zero-mean, wide-sense ergodic.) The basic objective is to obtain estimates of the elements of the spectral density matrix,

$$\mathbf{M}_{xy}(f) = \begin{bmatrix} \Phi_{xx}(f) & \Phi_{xy}(f) \\ \Phi_{yx}(f) & \Phi_{yy}(f) \end{bmatrix} , \tag{3.1}$$

in order to form the magnitude-squared coherence estimator.

The estimation procedure described is the direct method, which is discussed in part by Welch,[21] Knapp,[22] Bingham,[13] Benignus,[10] Nuttall,[14] and Carter and Arnold.[12] It includes cosine weighting and overlapped processing and is used because of the computational advantage of the FFT.[11]

Briefly, the method implemented consists of obtaining two finite-time series from the random processes being investigated. The time series are segmented into n segments, each having P-data points. For example, from each process there may be 32 segments, each segment having 4096 points. The segments may be overlapped or disjoint. Each segment is multiplied by a weighting function,

and the FFT of the weighted P-point sequence is performed. The Fourier coefficients for each weighted segment are then used to estimate the elements of the power spectral density matrix. The power spectral estimates thus obtained from each set of weighted sequences are then averaged over all the n segments. Next, the resultant estimates are used to form the magnitude-squared coherence.[12]

III. A. QUANTIZED SEQUENCE OBTAINED FROM WEIGHTED SEGMENT OF DATA

Consider the time-limited sample functions of processes x(t) and y(t) (specified in Chapter III.). Let the sample functions be further constrained so that they have the same bandwidth. This may come about as a result of (1) the physics of the experiment, (2) the bandpass characteristics of some recording device, or (3) the intentional introduction of bandpass or low-pass filters to prevent aliasing. Analog to digital (A/D) conversion of the signals is now accomplished by sampling the two analog signals at a frequency, f_s Hz, greater than twice the bandwidth of the signals. This technique yields two quantized sequences of numbers or time series. The quantization error decreases as the number of bits in the quantizer increases. (Errors as a result of quantization are beyond the scope of this work.)

Let these two time series from processes x(t) and y(t), which are drawn continuously for convenience, be depicted as in Fig. 3.

The method of overlapped weighted segmentation requires that before estimating the coherence between x(t) and y(t), both x(t) and y(t) be

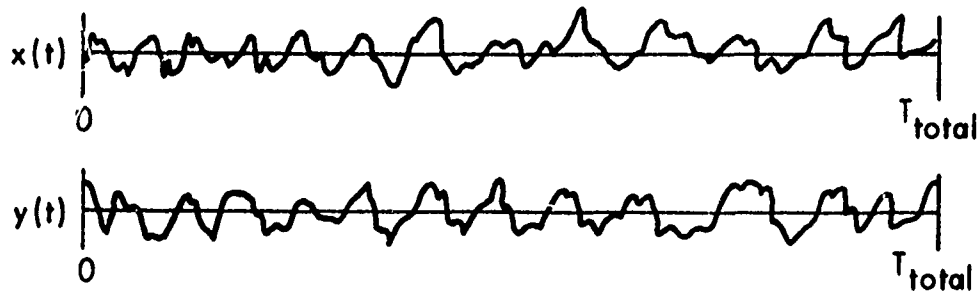

Fig. 3. Two Time Series from Processes x(t) and y(t) multiplied by a series of real weighting functions, $w_s(t)$, or sampled and quantized versions thereof, as in Fig. 4.

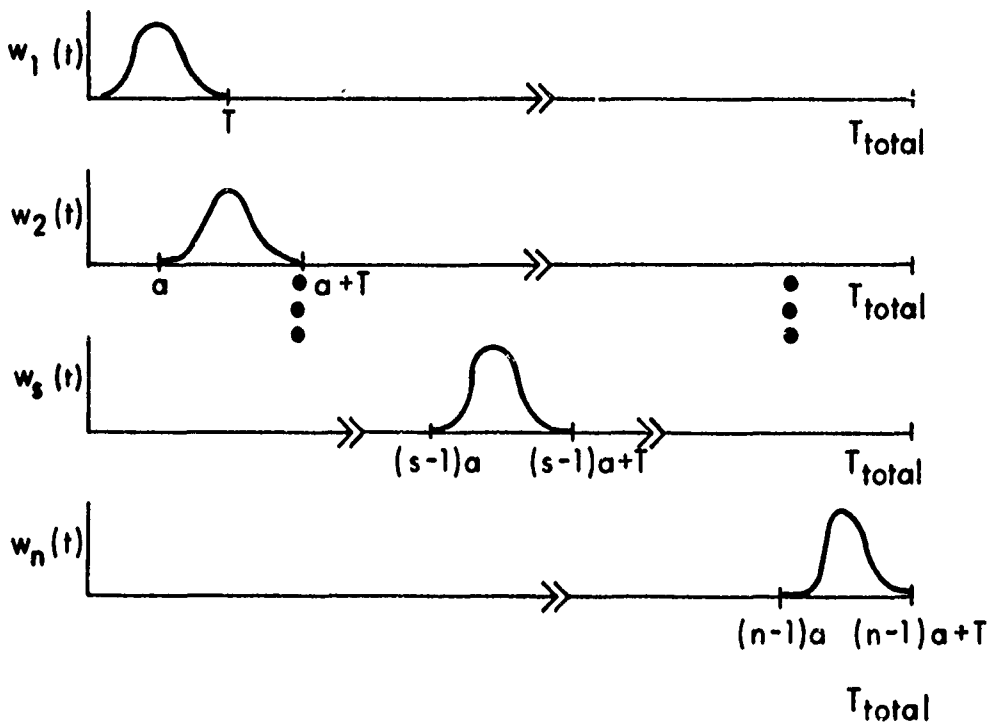

Fig. 4. Overlapped Weighting Functions (Modified from Knapp[22])

The method implemented computes a P-point discrete Fourier transform (DFT) for each of the weighted segments. The frequency-domain equivalent of multiplying each segment by a weighting function is a convolution of the true

spectrum with the Fourier transform of the weighting function. Hence, the weighting function should be judiciously selected in order that the true spectrum be least distorted.

The factors affecting the selection of the segment length and window shape are

1. $w_s(t)$ should be relatively easy to compute.

2. $\frac{T_{total}}{T}$ should be large in order that the amount of averaging be sufficient to reduce the bias and variance of the spectral estimates.

3. $\frac{d^n}{dt^n} w_s(t)$ should be continuous for $n = 0, 1, 2, \ldots,$ up to some reasonable limit, since this ensures that the sidelobes of the Fourier transform of $w_s(t)$ die off rapidly.

4. The Fourier transform of $w_s(t)$ should also be narrow in the main lobe (narrower than the finest detail of the true spectral density matrix of processes $x(t)$ and $y(t)$). Generally, this lobe is narrowed by increasing T.

The specific selection of a weighting function involves a number of tradeoffs. A commonly used weighting (or windowing) function is the cosine (Hanning) function defined[13] by

$$w_s(t) = \begin{cases} \frac{1}{2}\left(1 - \cos\left\{2\pi\left[\frac{t-(s-1)a}{T}\right]\right\}\right), & (s-1)a \le t \le T + (s-1)a, \\ 0, & \text{elsewhere}. \end{cases} \tag{3.2}$$

The percentage overlap from Fig. 4 is, simply,

$$p_o = \begin{cases} \left(\frac{T-a}{T}\right) 100, & a \le T, \\ 0, & a > T. \end{cases} \tag{3.3}$$

Therefore, for example, if $T = 1$ and $a = \frac{1}{2}$, then $p_o = 50$ percent. Whereas if $T = 1$ and $a = 3/4$, then $p_o = 25$ percent.

Note that if $a \geq T$ there would be no overlap, and each segment would be virtually independent of the previous one (except for correlated edge effects). All theoretical results here are concerned with the case of independent segments, that is, no overlap. A detailed analysis of the effect of overlapped weighted segmentation for estimating auto power spectral density functions is given by Nuttall.[14]

III. B. COHERENCE ESTIMATOR

Let x_{sp}, where $p = 0, 1, 2, \ldots, P-1$ denote the P-point sequence obtained from the sth weighted segment of process $x(t)$. In estimating the coherence function, it is necessary to evaluate a transformation of this weighted sequence. The FFT is a fast algorithm for evaluating a special case of the Z-transform of a finite sequence of numbers. The two sided Z-transform of an infinite sequence is defined by

$$X_s(z) = \sum_{p=-\infty}^{\infty} x_{sp} z^{-p} , \tag{3.4}$$

where z equals any complex variable.[23]

Evaluation of the Z-transform at P equally spaced points around the unit circle for a P-point sequence yields the P-point DFT:[23]

$$X_s(f_k) = \sum_{p=0}^{P-1} x_{sp} e^{-j2\pi f_k p/P} , \tag{3.5}$$

where x_{sp} is the finite weighted sequence, $p = 0, 1, \ldots, P-1$, and $s = 1, 2, \ldots, n$. Equation (3.5) can be evaluated for $k = 0, 1, \ldots, P-1$, with a fast algorithm requiring on the order of $P \log_2 P$ complex multiplications and additions.[11]

Similarly, a vector, $Y_s(f_k)$, is formed for each segment (that is, $s = 1, 2, \ldots, n$).

The estimate of the auto power spectral density function of $x(t)$ at the kth frequency, obtained from the sth weighted segment, is given by

$$\hat{\Phi}_{xx_s}(f_k) = \frac{\Delta t}{P}\left[X_s(f_k)\, X_s^*(f_k)\right], \text{ where } \Delta t = 1/f_s \quad . \tag{3.6}$$

Similarly,

$$\hat{\Phi}_{yy_s}(f_k) = \frac{\Delta t}{P}\left[Y_s(f_k)\, Y_s^*(f_k)\right], \tag{3.7}$$

and the estimate of the cross power spectral density function is

$$\hat{\Phi}_{xy_s}(f_k) = \frac{\Delta t}{P}\left[X_s(f_k)\, Y_s^*(f_k)\right] . \tag{3.8}$$

Equation (3.8) can be rewritten in terms of the real and imaginary parts,

$$\hat{C}_{xy_s}(f_k) = \frac{\Delta t}{P} \operatorname{Re}\left[X_s(f_k)\, Y_s^*(f_k)\right] \tag{3.9}$$

and

$$\hat{Q}_{xy_s}(f_k) = \frac{\Delta t}{P} \operatorname{Im}\left[X_s(f_k)\, Y_s^*(f_k)\right] . \tag{3.10}$$

Next, the estimates of the elements of the power spectral density matrix are obtained by averaging over the number of segments, n . The estimate of the magnitude-squared coherence follows directly:

$$\left|\hat{\gamma}(f_k)\right|^2 \triangleq \frac{\left[\frac{1}{n}\sum_{s=1}^{n}\hat{C}_{xy_s}(f_k)\right]^2 + \left[\frac{1}{n}\sum_{s=1}^{n}\hat{Q}_{xy_s}(f_k)\right]^2}{\left[\frac{1}{n}\sum_{s=1}^{n}\hat{\Phi}_{xx_s}(f_k)\right]\left[\frac{1}{n}\sum_{s=1}^{n}\hat{\Phi}_{yy_s}(f_k)\right]} , \tag{3.11}$$

where k indexes the discrete frequency of interest and n is the number of overlapped segments.

The estimate of magnitude coherence is

$$\left|\hat{\gamma}(f_k)\right| = +\sqrt{\left|\hat{\gamma}(f_k)\right|^2} . \tag{3.12}$$

It is of practical interest to note (as pointed out by Jenkins and Watts[8]) that an alternate and seemingly reasonable form of the estimate yields

$$\left|\hat{\gamma}_w(f_k)\right|^2 = \frac{1}{n}\sum_{s=1}^{n}\frac{\left|X_s(f_k)\,Y_s^*(f_k)\right|^2}{X_s(f_k)\,X_s^*(f_k)\,Y_s(f_k)\,Y_s^*(f_k)} \tag{3.13a}$$

and

$$= \frac{1}{n}\sum_{s=1}^{n}\frac{X_s(f_k)\,Y_s^*(f_k)\,X_s^*(f_k)\,Y_s(f_k)}{X_s(f_k)\,X_s^*(f_k)\,Y_s(f_k)\,Y_s^*(f_k)} = 1 . \tag{3.13b}$$

This fact is so basic that it is often not discussed. However, it points out that regardless of the value of the true magnitude-squared coherence, $\left|\hat{\gamma}\right|^2 = 1.0$

when $n = 1$. Consequently, the estimate is, in general, biased; the actual bias depends on $|\gamma|^2$ and κ. In practice, n should be large, as will be shown.

IV. STATISTICS OF ESTIMATE OF COHERENCE

Goodman, in his Eqs. (4.51) and 4.60),[1] derived an analytical expression for the probability density function of the magnitude-coherence estimate, $|\hat{\gamma}|$, based on Eqs. (3.11) and (3.12). His results were based on two zero-mean processes that were stationary, Gaussian, and random and had been segmented into n independent observations (that is, nonoverlapped segments). Each segment was assumed large enough to ensure adequate spectral resolution. Further, each segment was assumed perfectly weighted (windowed), in the sense that the Fourier coefficient at some kth frequency was to have "leaked" no power from other bins. However, Hannan[24] points out that the statistics do not hold at the zeroth or folding frequencies.

The material in this chapter relating to magnitude-squared coherence is believed to be new (Carter and Nuttall[25]) and is a direct extension of Goodman's work.[1] All of Goodman's original assumptions hold. Statistics of the magnitude-coherence estimator are given in Appendix A.

IV.A. PROBABILITY DENSITY AND CUMULATIVE DISTRIBUTION FUNCTIONS

The first-order probability density and cumulative distribution functions for the estimate of magnitude-squared coherence, given the true value of magnitude-squared coherence and the number, n, of independent segments processed,

are presented in closed-form. The expressions are evaluated and plotted.

IV.A.1. Probability Density Function

The conditional probability density function for the estimate of magnitude-squared coherence, $|\hat{\gamma}|^2$, between two processes, given $|\gamma|^2$ and n, is[9]

$$p\left(|\hat{\gamma}|^2 \,\middle|\, n, |\gamma|^2\right) = (n-1)\left(1-|\gamma|^2\right)^n \left(1-|\hat{\gamma}|^2\right)^{n-2}$$

$$\cdot\, {}_2F_1\left(n, n; 1; |\gamma|^2 \, |\hat{\gamma}|^2\right), \quad 0 \le |\gamma|^2 |\hat{\gamma}|^2 < 1 \tag{4.1}$$

It then follows, knowing $|\hat{\gamma}| = \left[|\hat{\gamma}|^2\right]^{\frac{1}{2}}$, that

$$p\left(|\hat{\gamma}| \,\middle|\, n, |\gamma|\right) = p\left(|\hat{\gamma}|^2 \,\middle|\, n, |\gamma|^2\right) 2|\hat{\gamma}| \,. \tag{4.2}$$

Equation (4.2) can be shown (Appendix A) to be Goodman's result.[1] The density function, Eq. (4.1), can be rewritten using Eq. (15.35) of Abramowitz[26] in the following alternate forms:

$$p\left(|\hat{\gamma}|^2 \,\middle|\, n, |\gamma|^2\right) = (n-1)\left(1-|\gamma|^2\right)^n \left(1-|\hat{\gamma}|^2\right)^{n-2}$$

$$\cdot \left(1-|\gamma|^2 |\hat{\gamma}|^2\right)^{1-2n} \; {}_2F_1\left(1-n, 1-n; 1; |\gamma|^2 |\hat{\gamma}|^2\right) \tag{4.3}$$

and

$$p\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right) = (n-1)\left[\frac{\left(1-\left|\gamma\right|^2\right)\left(1-\left|\hat{\gamma}\right|^2\right)}{\left(1-\left|\gamma\right|^2\left|\hat{\gamma}\right|^2\right)^2}\right]^n$$

$$\cdot \frac{\left(1-\left|\gamma\right|^2\left|\hat{\gamma}\right|^2\right)}{\left(1-\left|\hat{\gamma}\right|^2\right)^2} \; {}_2F_1\left(1-n,\ 1-n;\ 1;\left|\gamma\right|^2\left|\hat{\gamma}\right|^2\right). \quad (4.4)$$

Equations (4.3) and (4.4) are desirable because ${}_2F_1\left(1-n,\ 1-n;\ 1;\left|\gamma\right|^2\left|\hat{\gamma}\right|^2\right)$ can be expressed as an $(n-1)$st order polynomial (Abramowitz, Eq. (15.4.1)[26]).

A special case of the density function occurs when $\left|\gamma\right|^2 = 0.0$. In that event,

$$p\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2 = 0.0\right) = (n-1)\left(1-\left|\hat{\gamma}\right|^2\right)^{n-2} . \quad (4.5)$$

IV.A.2. Cumulative Distribution Function

Fisher,[4] working on statistics of the estimate of the squared correlation coefficient, derived the probability density for that random variable. He integrated the result and achieved a closed-form solution for the cumulative distribution function; specifically the solution was a finite sum of ${}_2F_1$ functions, each one a finite-order polynomial. Although these statistics are for a different problem, proper identification of variables yields exactly the integration formula needed to find the cumulative distribution of the estimate of magnitude-squared coherence, namely,

$$P\left(|\hat{\gamma}|^2 \middle| n, |\gamma|^2\right) = |\hat{\gamma}|^2 \left[\left(\frac{1-|\gamma|^2}{1-|\gamma|^2|\hat{\gamma}|^2}\right)^n \sum_{k=0}^{n-2} \left(\frac{1-|\hat{\gamma}|^2}{1-|\gamma|^2|\hat{\gamma}|^2}\right)^k\right]$$

$$\cdot\, {}_2F_1\left(-k,\ 1-n;\ 1;\ |\gamma|^2|\hat{\gamma}|^2\right). \tag{4.6}$$

In the special case when $|\gamma|^2 = 0$, the cumulative distribution function becomes

$$P\left(|\hat{\gamma}|^2 \middle| n,\ |\gamma|^2 = 0.0\right) = |\hat{\gamma}|^2 \sum_{k=0}^{n-2} \left(1-|\hat{\gamma}|^2\right)^k, \tag{4.7}$$

which can be simplified to give

$$P\left(|\hat{\gamma}|^2 \middle| n,\ |\gamma|^2 = 0.0\right) = 1 - \left(1-|\hat{\gamma}|^2\right)^{n-1}. \tag{4.8}$$

Equation (4.8), when differentiated, yields the probability density function, Eq. (4.5).

IV.A.3. Computer Evaluation

The probability density function, Eq. (4.4), can be evaluated readily on a large digital computer in floating-point arithmetic. Evaluation for 100 values of $|\hat{\gamma}|^2$ between 0.0 and 0.99 requires computing 100 (n - 1)st order polynomials for each value of $|\gamma|^2$ and n. The

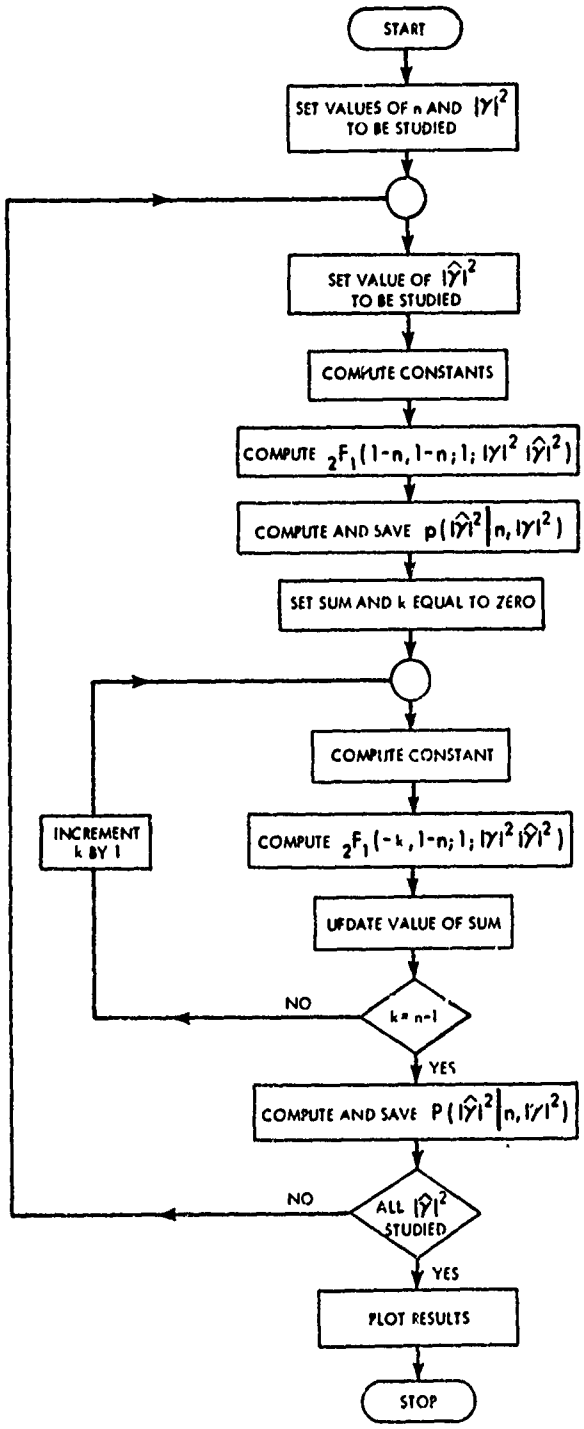

Fig. 5. Summary Flow Chart for Density and Cumulative Distribution

cumulative distribution, Eq. (4.6), can be similarly evaluated. The density function and the cumulative-distribution function were computed as illustrated in Fig. 5 for ten values of $|\gamma|^2$ and for $n = 32$, 40, 48, 56, and 64. (The computations and 100 plots were done on the UNIVAC 1108 in less than 5 minutes.) Example plots are included in Figs. 6 through 13.

One example of how these plots can be used is as follows: Magnitude-squared coherence, $|\hat{\gamma}|^2$, is estimated by averaging over 32 disjoint segments of data (that is, $n = 32$). Suppose the estimated value is approximately 0.3, then from Fig. 7

$$\mathrm{Prob}\left(L < |\hat{\gamma}|^2 \,\middle|\, n = 32, |\gamma|^2 = 0.3\right)$$

$$= \int_L^{\infty} p\left(|\hat{\gamma}|^2 \,\middle|\, n = 32, |\gamma|^2 = 0.3\right) d|\hat{\gamma}|^2 \qquad (4.9a)$$

and

$$= 1 - \int_{-\infty}^{L} p\left(|\hat{\gamma}|^2 \,\middle|\, n = 32, |\gamma|^2 = 0.3\right) d|\hat{\gamma}|^2 , \qquad (4.9b)$$

Equation (4.9b) could be set equal to, for example, 0.9, and the value of L, from Fig. 6, is 0.2.

The upper limit is found from

$$\mathrm{Prob}\left(|\hat{\gamma}|^2 < U \,\middle|\, n = 32, |\gamma|^2 = 0.3\right) =$$

$$\int_{-\infty}^{U} p\left(|\hat{\gamma}|^2 \,\middle|\, n = 32, |\gamma|^2 = 0.3\right) d|\hat{\gamma}|^2 , \qquad (4.10)$$

which could be set equal to, for example, 0.9, and the value of A, from Fig. 7, is 0.43. Hence,

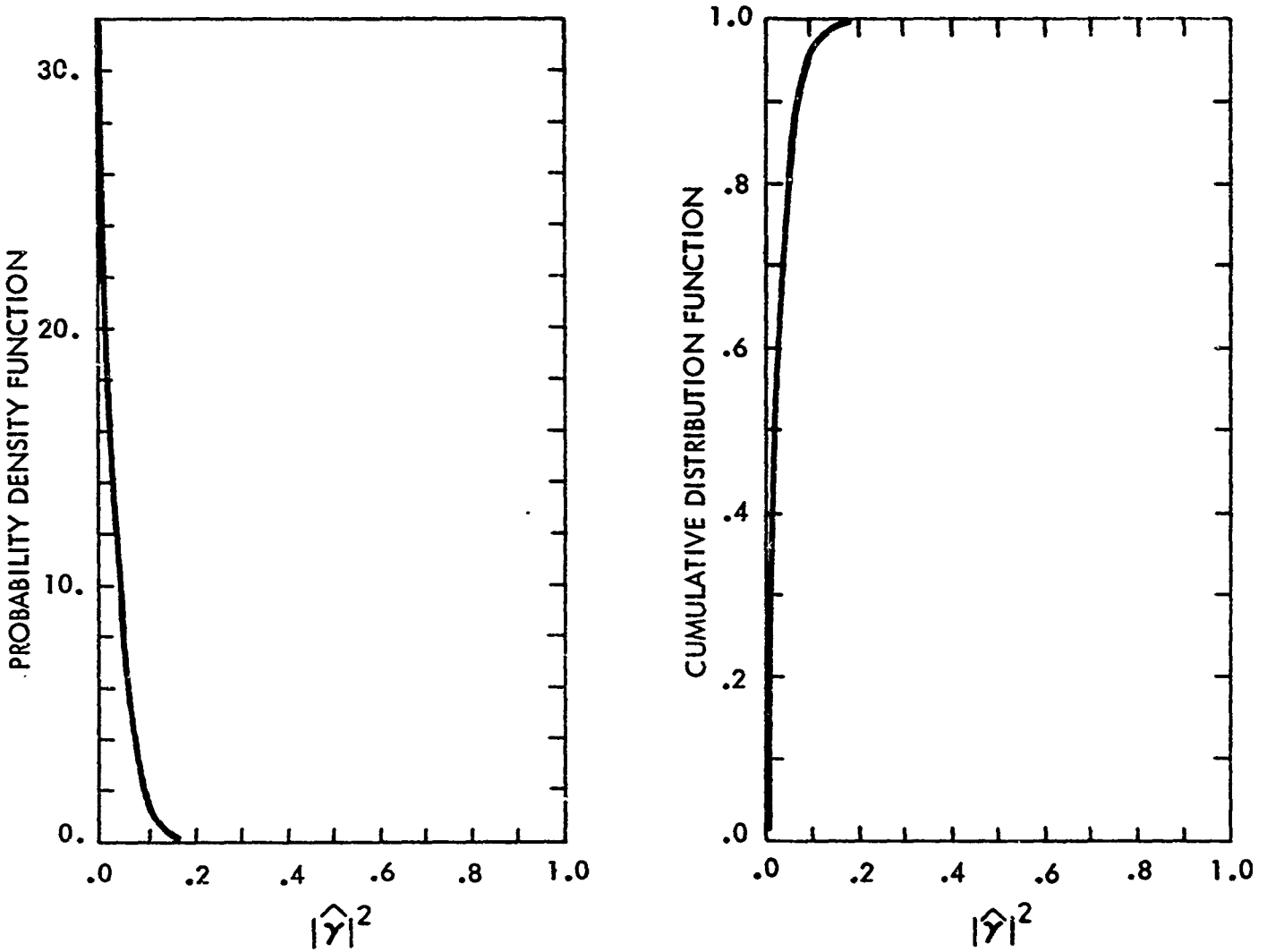

Fig. 6. Probability Density and Cumulative Distribution Functions for $|\hat{\gamma}|^2$ Given $n = 32$ and $|\gamma|^2 = 0.0$

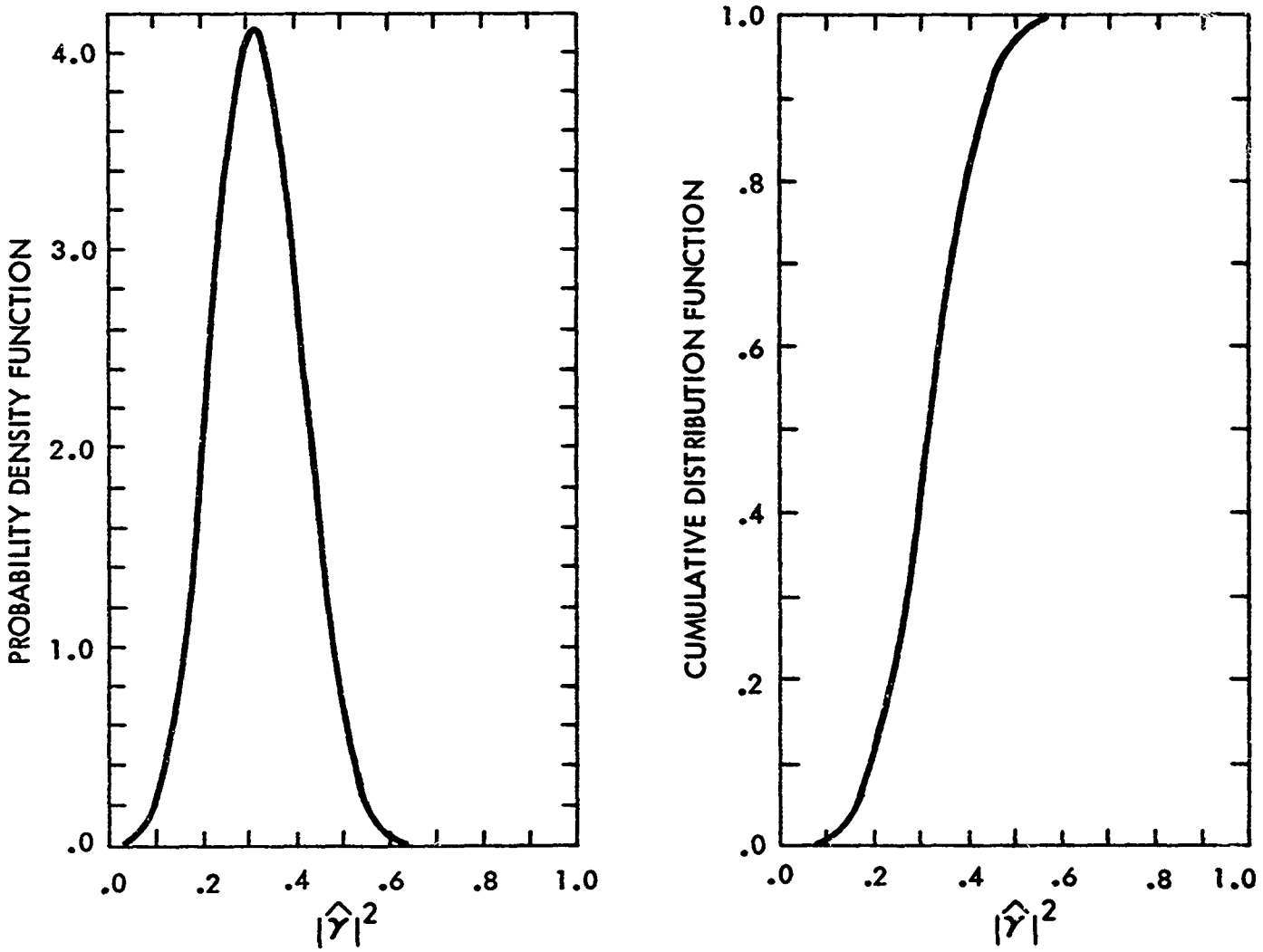

Fig. 7. Probability Density and Cumulative Distribution Functions for $|\hat{\gamma}|^2$ Given $n = 32$ and $|\gamma|^2 = 0.3$

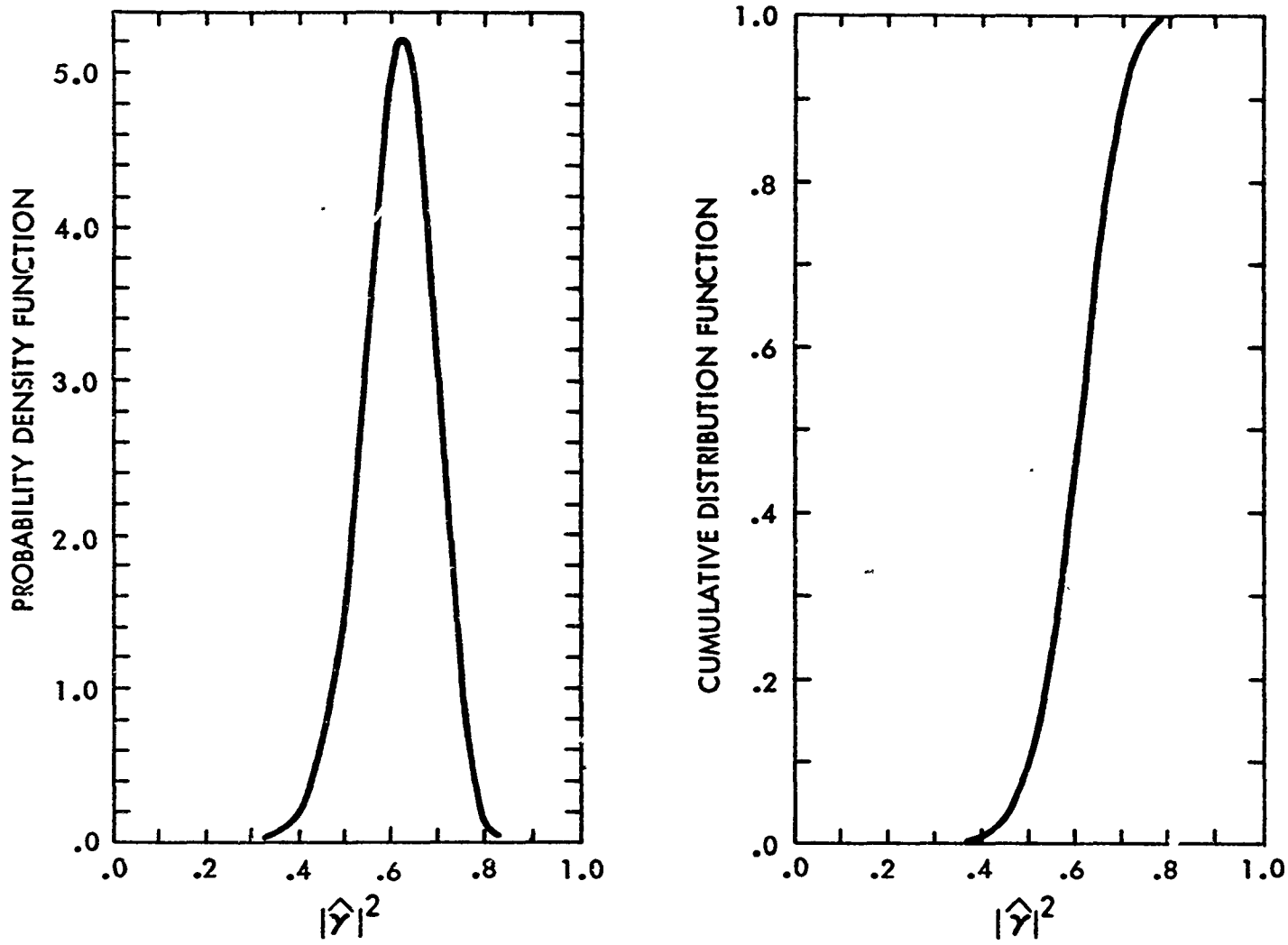

Fig. 8. Probability Density and Cumulative Distribution Functions for $|\hat{\gamma}|^2$ Given $n=32$ and $|\gamma|^2 = 0.6$

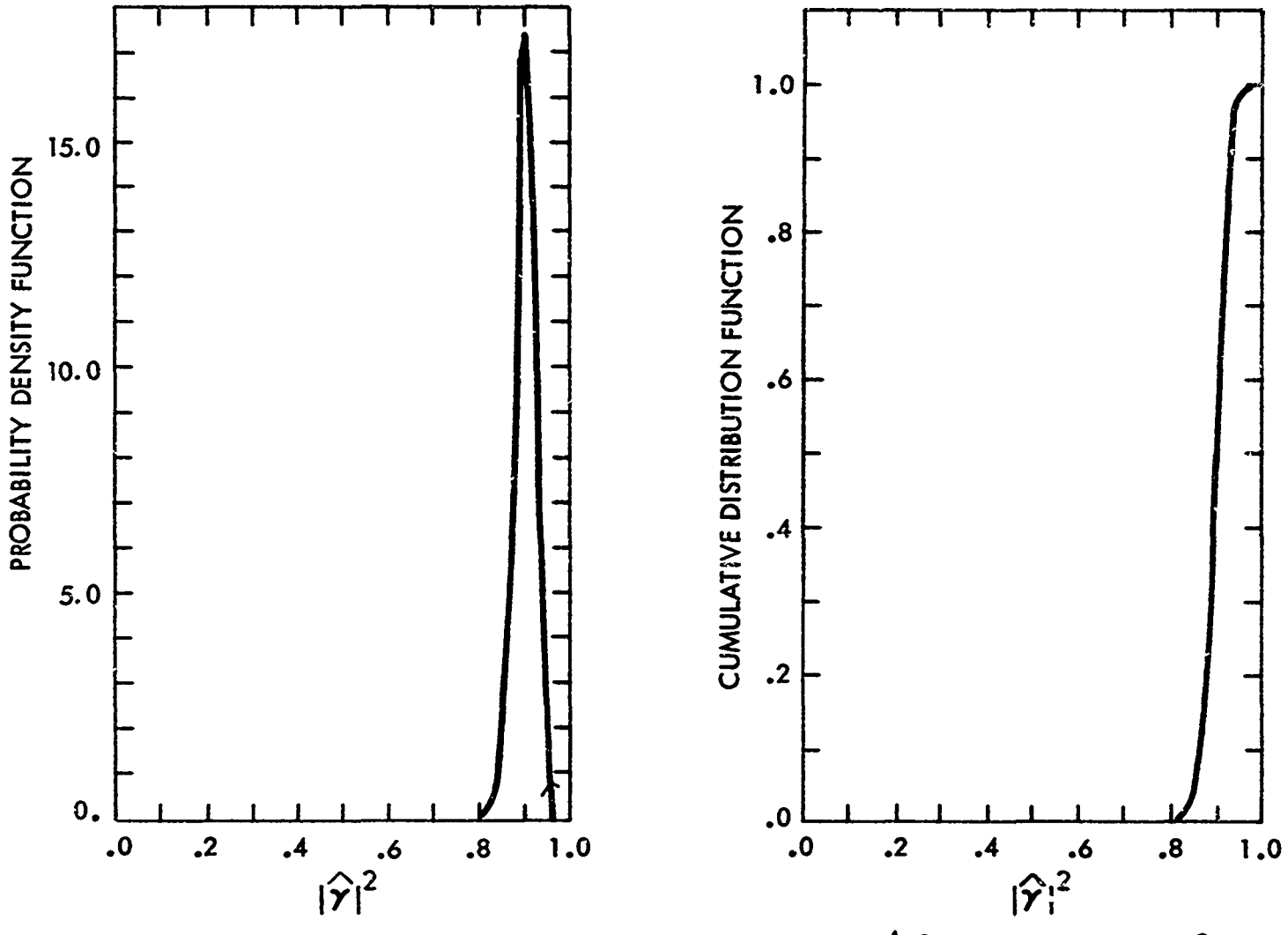

Fig. 9. Probability Density and Cumulative Distribution Functions for $|\hat{\gamma}|^2$ Given $n=32$ and $|\gamma|^2=0.9$

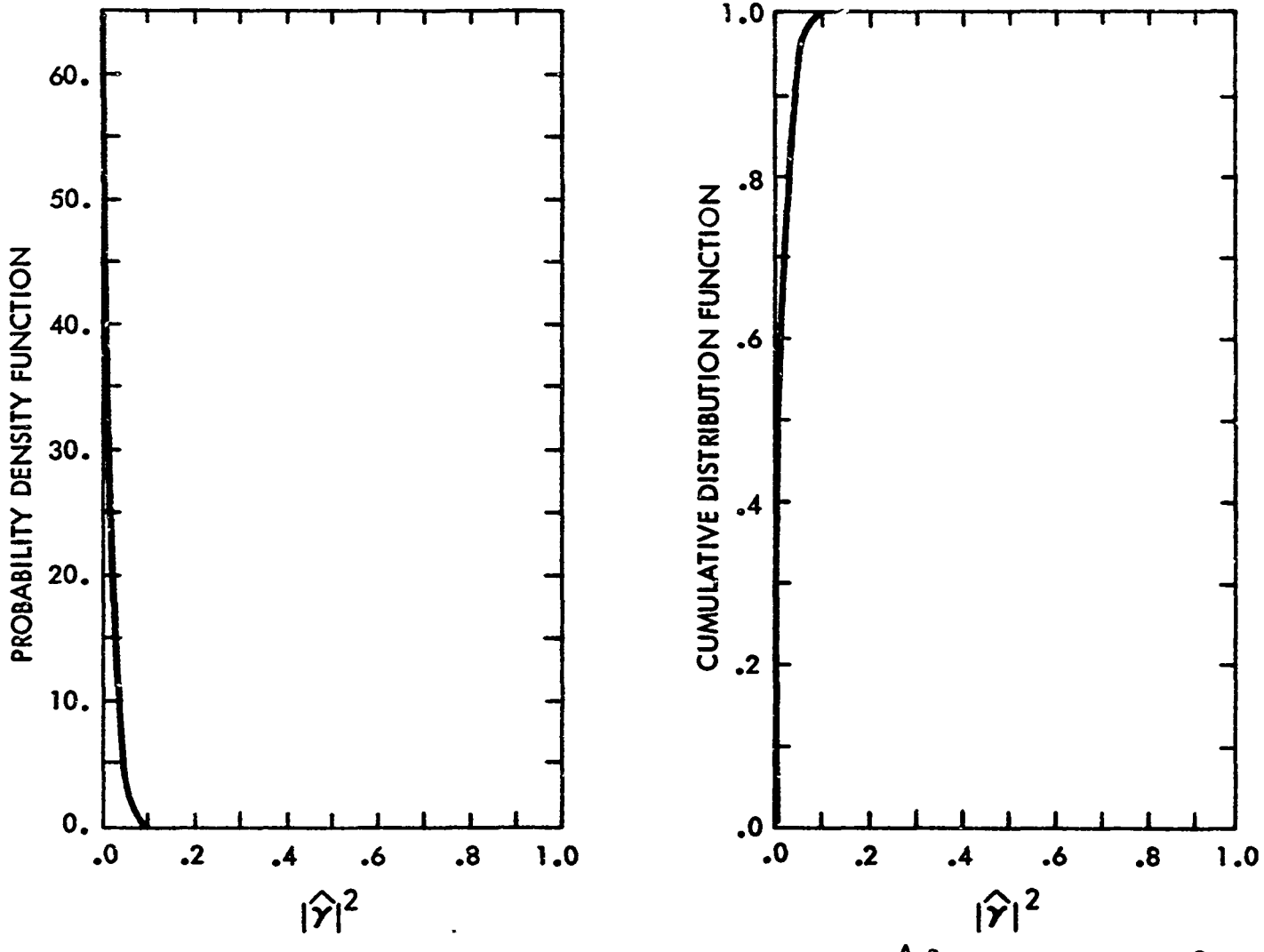

Fig. 10. Probability Density and Cumulative Distribution Functions for $|\hat{\gamma}|^2$ Given $n = 64$ and $|\gamma|^2 = 0.0$

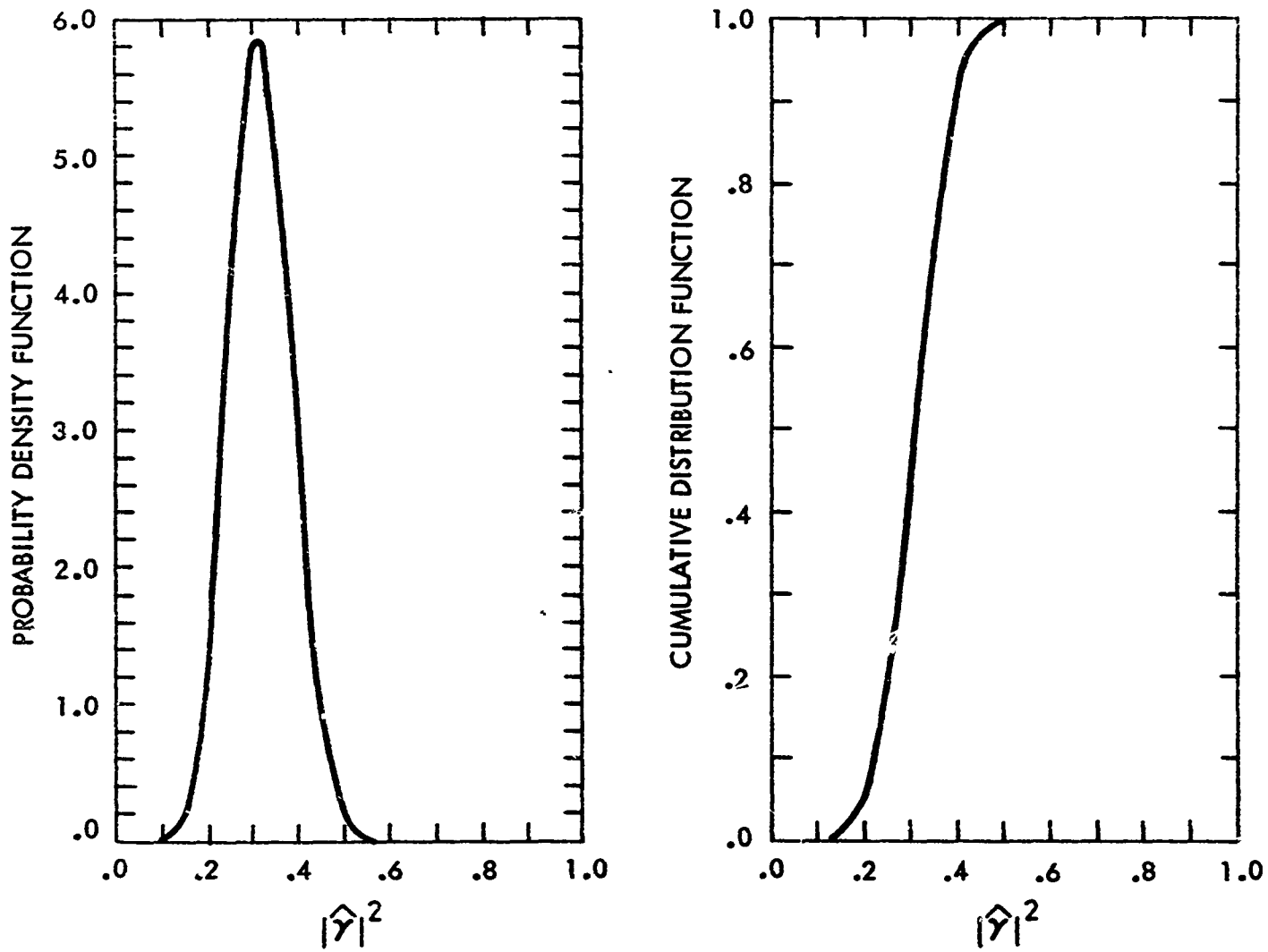

Fig. 11. Probability Density and Cumulative Distribution Functions for $|\hat{\gamma}|^2$ Given $n = 64$ and $|\gamma|^2 = 0.3$

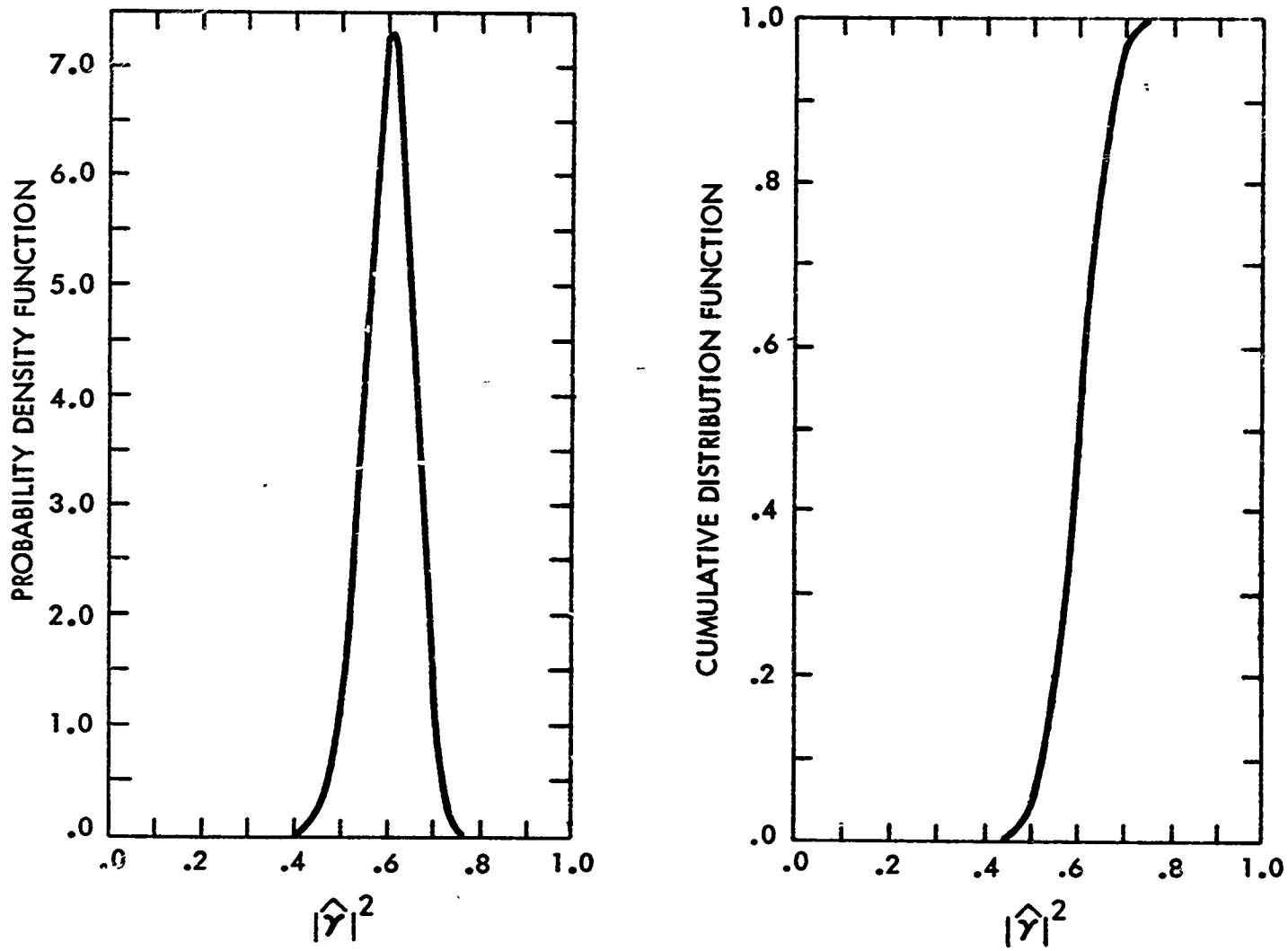

Fig. 12. Probability Density and Cumulative Distribution Functions for $|\hat{\gamma}|^2$ Given $n = 64$ and $|\gamma|^2 = 0.6$

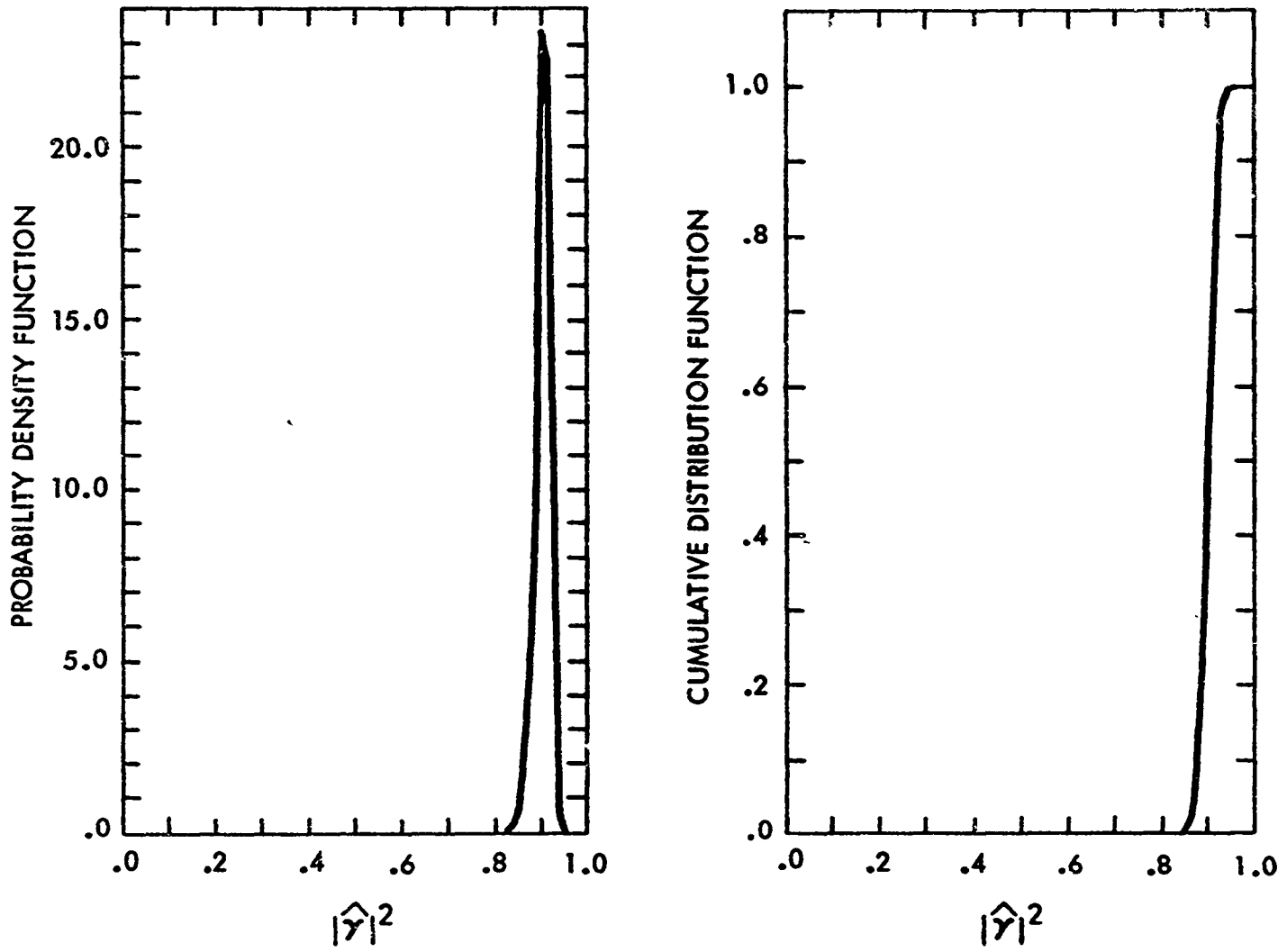

Fig. 13. Probability Density and Cumulative Distribution Functions for $|\hat{\gamma}|^2$ Given n = 64 and $|\gamma|^2 = 0.9$

$$\int_{0.2}^{0.43} p\left(\left|\hat{\gamma}\right|^2 \middle| n = 32, \left|\gamma\right|^2 = 0.3\right) d\left|\hat{\gamma}\right|^2 = 0.8 \quad . \tag{4.11}$$

On the basis of Eq. (4.11), the probability that the estimator falls in the range (0.2, 0.43) is 0.8, given that the exact value of the unknown parameter was 0.3 and that 32 disjoint segments were used.

Proper use of the cumulative distribution function yields confidence intervals for the estimate of magnitude-squared coherence or any "one for one" transformation of it, such as the positive square root or $10 \log_{10} \left(\frac{\left|\hat{\gamma}\right|^2}{1 - \left|\hat{\gamma}\right|^2}\right)$. (See, for example, Cramer[27] or Carter and Nuttall.[25])

IV. B. mth MOMENT OF DENSITY FUNCTION

The mth moment of the magnitude-squared coherence is given by

$$E\left[\left(\left|\hat{\gamma}\right|^2\right)^m \middle| n, \left|\gamma\right|^2\right] = \int_{-\infty}^{+\infty} p\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right)\left(\left|\hat{\gamma}\right|^2\right)^m d\left|\hat{\gamma}\right|^2 =$$

$$(n-1)\left(1 - \left|\gamma\right|^2\right)^n \int_0^1 \left(1 - \left|\hat{\gamma}\right|^2\right)^{n-2} {}_2F_1\left(n, n; 1; \left|\gamma\right|^2 \left|\hat{\gamma}\right|^2\right)$$

$$\left|\hat{\gamma}\right|^{2m} d\left|\hat{\gamma}\right|^2 \quad , \tag{4.12}$$

where use has been made of the density function, Eq. (4.1).

Application of Eq. 7.512(12) by Gradshteyn[28] yields

$$E\left[\left(\left|\hat{\gamma}\right|^2\right)^m \middle| n, \left|\gamma\right|^2\right] = \left(1 - \left|\gamma\right|^2\right)^n \frac{\Gamma(n)\ \Gamma(m+1)}{\Gamma(n+m)}$$

$$\cdot\, {}_3F_2\left(m+1,\ n,\ n;\ m+n,\ 1;\ \left|\gamma\right|^2\right). \quad (4.13)$$

The three-two hypergeometric functions denoting three numerator terms and two denominator terms are given by

$${}_3F_2\,(a, b, c;\ d, e; z) \doteq \sum_{k=0}^{\infty} \frac{(a)_k (b)_k (c)_k}{(d)_k (e)_k} \frac{z^k}{k!}, \quad (4.14)$$

where the $(a)_k$ notation is Pochhammer's symbol[26] defined by

$$(a)_k \triangleq \frac{\Gamma(a+k)}{\Gamma(a)}. \quad (4.15)$$

The mth moment for the estimate of magnitude-coherence is given in Appendix A.

These results can be verified through proper identification of variables in the work of Anderson,[29] who extended Fisher's original work[4] on the squared correlation coefficient.

IV. C. BIAS AND VARIANCE

This section deals with the bias and variance of the estimator $\left|\hat{\gamma}\right|^2$. Exact and approximate expressions are presented. In addition, computer evaluation of the exact expressions is presented to lend meaning to these results.

IV.C.1. Bias

Consider now the first moment of the probability density function for the estimate of magnitude-squared coherence. This moment can be written

$$E\left[\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right] = \frac{\left(1-\left|\gamma\right|^2\right)^n}{n} \; {}_3F_2\left(2, n, n; \; n+1, 1; \left|\gamma\right|^2\right) . \tag{4.16}$$

If $\left|\gamma\right|^2 = 1.0$, ${}_3F_2 = \infty$; therefore, the evaluation of Eq. (4.16) is not meaningful. When $\left|\gamma\right|^2 = 0.0$; ${}_3F_2 = 1.0$, which yields

$$E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2 = 0.0\right) = \frac{1}{n} . \tag{4.17}$$

Tedious manipulation of Eq. (4.16) (Appendix B) yields the simpler result:

$$E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right) = \frac{1}{n} + \frac{n-1}{n+1}\left|\gamma\right|^2 \; {}_2F_1\left(1, 1; n+2; \left|\gamma\right|^2\right) . \tag{4.18}$$

An extremely useful approximation can be made by expanding Eq. (4.18) to obtain

$$E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right) \cong \frac{1}{n} + \frac{n-1}{n+1}\left|\gamma\right|^2 + \frac{(n-1)\,1!}{(n+1)(n+2)}\left(\left|\gamma\right|^2\right)^2$$

$$+ \frac{(n-1)\,2!}{(n+1)(n+2)(n+3)}\left(\left|\gamma\right|^2\right)^3 . \tag{4.19}$$

Computation of higher order approximating polynomials is also easily performed and is based on an analytical expression for $E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right)$.

The bias or expected estimation error is defined as

$$\text{Bias} \triangleq E\left(|\hat{\gamma}|^2 \middle| n, |\gamma|^2\right) - |\gamma|^2 \quad . \tag{4.20}$$

From Eq. (4.18), an exact expression for the bias is

$$\text{Bias} = \frac{1}{n} + \frac{n-1}{n+1} |\gamma|^2 \; {}_2F_1\left(1, 1; n+2; |\gamma|^2\right) - |\gamma|^2 \quad . \tag{4.21}$$

Expanding Eq. (4.21) gives the approximation

$$B_o \cong \frac{1}{n} - \frac{2}{n+1} |\gamma|^2 + \frac{1!\,(n-1)}{(n+1)(n+2)} \left(|\gamma|^2\right)^2 + \frac{(n-1)\,2!}{(n+1)(n+2)(n+3)} \left(|\gamma|^2\right)^3 , \tag{4.22a}$$

$$\text{Bias} \cong \begin{cases} B_o & , \quad B_o \geq 0 \\ 0 & , \quad B_o < 0 \end{cases} \quad . \tag{4.22b}$$

As an example of using this approximation for $n = 8$, the exact bias lies in the range (0.0, 0.125), depending on $|\gamma|^2$; and the maximum difference between Eqs. (4.21) and 4.22) is 0.0027 at $|\gamma|^2 = 0.86$. For large n, Eq. (4.22a and b) reduces to

$$\text{Bias} \cong \frac{1}{n} \left(1 - |\gamma|^2\right)^2 \quad . \tag{4.22c}$$

It should be noted (see, for example, Eqs. (4.22a) and (4.22b)) that

$$\lim_{n \to \infty} (\text{Bias}) = 0 \quad ; \tag{4.23}$$

therefore, the estimator may be referred to as asymptotically unbiased.

An empirically determined bias was found by Benignus[10] to be

$$\text{Bias} = \frac{1}{n}\left(1 - |\gamma|^2\right) , \tag{4.24}$$

which fits the true curve for $|\gamma|^2 = 0.0$ and $|\gamma|^2 = 1.0$.

It is suggested that the higher order polynomial expression for bias, Eq. (4.22c), analytically derived, be used (as opposed to Benignus' result,[10] Eq. (4.24)), especially for small n. However, it can be shown that Benignus' result is an upper bound on the bias for any n.

A formula for the bias of $|\hat{\gamma}|^2$ owing to insufficient spectral resolving power(for example, FFT too small) is given by Jenkins and Watts,[8] but is beyond the scope of this thesis. The formula for bias derived above assumes sufficient resolving power.

IV.C.2. Variance

The variance of the estimator, namely, the second moment about the mean, is given by

$$\text{Variance} \equiv V = E\left[\left(|\hat{\gamma}|^2\right)^2\right] - \left[E\left(|\hat{\gamma}|^2\right)\right]^2 . \tag{4.25}$$

The second moment of the density function is, as a consequence of Eq. (4.16),

$$E\left[\left(|\hat{\gamma}|^2\right)^2 \,\middle|\, n, |\gamma|^2\right] = \frac{2\left(1 - |\gamma|^2\right)^n}{n(n+1)} \, {}_3F_2\left(3, n, n; n+2, 1; |\gamma|^2\right) . \tag{4.26}$$

When $|\gamma|^2 = 0.0$, Eq. (4.26) yields the result

$$E\left[\left(|\hat{\gamma}|^2\right)^2 \,\middle|\, n, |\gamma|^2 = 0.0\right] = \frac{2}{n(n+1)} \quad . \tag{4.27}$$

An exact expression for the variance of $|\hat{\gamma}|^2$ is obtained from Eqs. (4.16), (4.25), and (4.26). The result is

$$V = \frac{2\left(1-|\gamma|^2\right)^n}{n(n+1)} \; {}_3F_2\left(3, n, n; n+2, 1; |\gamma|^2\right)$$

$$- \left[\frac{\left(1-|\gamma|^2\right)^n}{n} \; {}_3F_2\left(2, n, n; n+1, 1; |\gamma|^2\right)\right]^2 . \tag{4.28}$$

For the special case of $|\gamma|^2 = 0.0$,

$$V = \frac{2}{n(n+1)} - \left(\frac{1}{n}\right)^2 = \frac{n-1}{n^2(n+1)} \; , \; |\gamma|^2 = 0.0 \tag{4.29a}$$

and

$$\cong \frac{1}{n^2} \quad , \text{ for large } n \text{ and } |\gamma|^2 = 0.0 \; . \tag{4.29b}$$

In order to avoid tedious and error-prone hand manipulation, computer-aided formula manipulation[30] of Eq. (4.26) was used to yield an approximation for the variance of $|\hat{\gamma}|^2$. The result is

$$V_o \cong \frac{(n-1)}{n(n+1)}\left[\frac{1}{n} + 2\frac{n-2}{n+2}\left|\gamma\right|^2 - 2\frac{2n^3 - n^2 - 2n + 3}{(n+1)(n+2)(n+3)}\left(\left|\gamma\right|^2\right)^2\right.$$

$$+ 2\frac{n^4 - 6n^3 - n^2 + 10n - 8}{(n+1)(n+2)\ (n+3)(n+4)}\left(\left|\gamma\right|^2\right)^3$$

$$\left. + \frac{13n^5 - 15n^4 - 113n^3 + 27n^2 + 136n - 120}{(n+1)(n+2)^2(n+3)(n+4)(n+5)}\left(\left|\gamma\right|^2\right)^4\right] \quad (4.30a)$$

or

$$V \cong \begin{cases} V_o\,, & V_o \geq 0 \\ 0\,, & V_o < 0 \end{cases} \quad . \quad (4.30b)$$

As an example of using the approximation given by Eq. (4.30) for $n = 8$, the exact variance lies in the range $(0.0,\ 0.031)$, and the maximum error due to the approximation in Eq. (4.30) is 0.0067 at $\left|\gamma\right|^2 = 0.83$. This result is a generalization of the third-order approximation by Jenkins and Watts,[8] which has no zeroth order term; that is, it assumes no variance of $\left|\hat{\gamma}\right|^2$ when $\left|\gamma\right|^2 = 0.0$.

In particular, for large n and $\left|\gamma\right|^2 \neq 0$, Eq. (4.30) reduces to

$$V \cong \frac{2}{n}\left|\gamma\right|^2\left(1 - \left|\gamma\right|^2\right)^2 \quad , \quad (4.31)$$

which has a maximum value of $8/27n$ at $\left|\gamma\right|^2 = 1/3$.

Hence, the variance of the estimator in the case where $|\gamma|^2$ is unknown (but nonzero) decreases inversely proportional to n.

IV.C.3. Digital Computer Evaluation Of Bias and Variance

Practical experience in estimating the magnitude-squared coherence function leads one to anticipate certain bias and variance problems. For a given number of segments, n, when $|\gamma|^2 = 1.0$, neither a bias nor a variance problem exists; however, when $|\gamma|^2 = 0.0$, the average value estimator always appears greater than 0.0. Further, when $|\gamma|^2$ is about 0.3 to 0.4, the variance of the estimator appears much greater than when $|\gamma|^2 = 0.0$. Primarily because this and the behavior of the estimator with increasing n can not be readily sensed from Eqs. (4.21) and (4.28), a computer program has been written to evaluate and plot those functions (see Fig. 14). The results, Tables 1 through 5 and Figs. 15 through 26, dramatically portray the behavior of these complicated (but readily evaluated) functions. Tables and graphs of this type have been prepared in the past by Amos and Koopmans.[5]

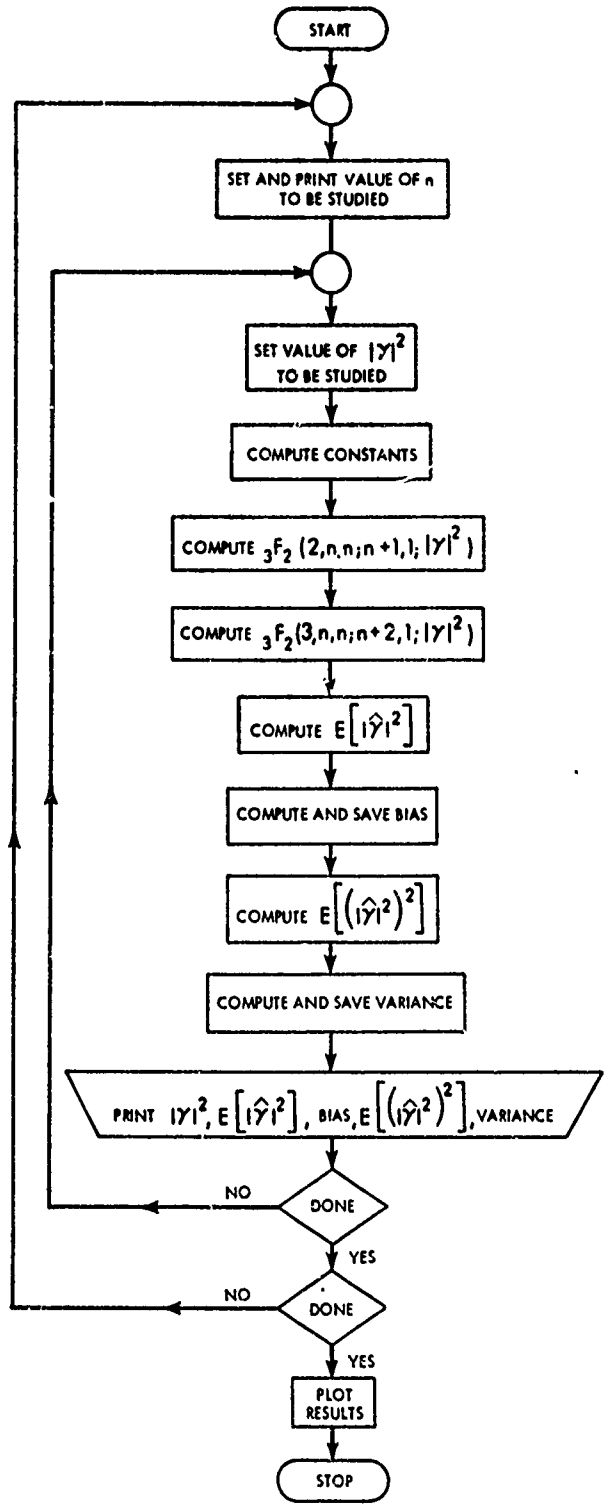

Fig. 14. Summary Flow Chart for Bias and Variance Computations

TABLE 1

BIAS AND VARIANCE OF $|\hat{\gamma}|^2$ FOR n = 32

$\lvert\gamma\rvert^2$	$E(\lvert\hat{\gamma}\rvert^2)$	Bias	$E[(\lvert\hat{\gamma}\rvert^2)]^2$	Variance
.00000	.31250-01	.31250-01	.18939-02	.91738-03
.40000-01	.68870-01	.28870-01	.75808-02	.28377-02
.80000-01	.10658+00	.26579-01	.15823-01	.44634-02
.12000+00	.14438+00	.24378-01	.26654-01	.58089-02
.16000+00	.18227+00	.22267-01	.40110-01	.68888-02
.20000+00	.22025+00	.20247-01	.56227-01	.77182-02
.24000+00	.25832+00	.18318-01	.75041-01	.83127-02
.28000+00	.29648+00	.16482-01	.96590-01	.86886-02
.32000+00	.33474+00	.14738-01	.12091+00	.88624-02
.36000+00	.37309+00	.13088-01	.14805+00	.88513-02
.40000+00	.41153+00	.11533-01	.17803+00	.86732-02
.44000+00	.45007+00	.10072-01	.21091+00	.83462-02
.48000+00	.48871+00	.87068-02	.24672+00	.78895-02
.52000+00	.52744+00	.74379-02	.28551+00	.73225-02
.56000+00	.56627+00	.62661-02	.32732+00	.66654-02
.60000+00	.60519+00	.51920-02	.37220+00	.59391-02
.64000+00	.64422+00	.42165-02	.42018+00	.51651-02
.68000+00	.68334+00	.33402-02	.47132+00	.43659-02
.72000+00	.72256+00	.25640-02	.52566+00	.35644-02
.76000+00	.76189+00	.18886-02	.58326+00	.27845-02
.80000+00	.80131+00	.13147-02	.64416+00	.20509-02
.84000+00	.84084+00	.84326-03	.70841+00	.13891-02
.88000+00	.88047+00	.47491-03	.77606+00	.82562-03
.92000+00	.92021+00	.21019-03	.84717+00	.38801-03
.96000+00	.96005+00	.48847-04	.92180+00	.10573-03
.10000+01	.10000+01	.00000	.10000+01	.00000

TABLE 2

BIAS AND VARIANCE OF $|\hat{\gamma}|^2$ FOR $n = 40$

$\lvert\gamma\rvert^2$	$E(\lvert\hat{\gamma}\rvert^2)$	Bias	$E[(\lvert\hat{\gamma}\rvert^2)]^2$	Variance
.00000	.25000-01	.25000-01	.12195-02	.59451-03
.40000-01	.63085-01	.23085-01	.61677-02	.21880-02
.80000-01	.10124+00	.21243-01	.13784-01	.35342-02
.12000+00	.13947+00	.19474-01	.24099-01	.46458-02
.16000+00	.17778+00	.17779-01	.37141-01	.55357-02
.20000+00	.21616+00	.16158-01	.52942-01	.62174-02
.24000+00	.25461+00	.14612-01	.71532-01	.67043-02
.28000+00	.29314+00	.13141-01	.92942-01	.70106-02
.32000+00	.33174+00	.11745-01	.11721+00	.71507-02
.36000+00	.37042+00	.10425-01	.14435+00	.71391-02
.40000+00	.40918+00	.91808-02	.17442+00	.69912-02
.44000+00	.44801+00	.80137-02	.20744+00	.67223-02
.48000+00	.48692+00	.69239-02	.24344+00	.63484-02
.52000+00	.52591+00	.59117-02	.28247+00	.58859-02
.56000+00	.56498+00	.49776-02	.32455+00	.53515-02
.60000+00	.60412+00	.41222-02	.36973+00	.47625-02
.64000+00	.64335+00	.33459-02	.41803+00	.41364-02
.68000+00	.68265+00	.26490-02	.46950+00	.34916-02
.72000+00	.72203+00	.20322-02	.52418+00	.28465-02
.76000+00	.76150+00	.14960-02	.58210+00	.22204-02
.80000+00	.80104+00	.10408-02	.64330+00	.16329-02
.84000+00	.84067+00	.66704-03	.70783+00	.11044-02
.88000+00	.88038+00	.37527-03	.77572+00	.65547-03
.92000+00	.92017+00	.16560-03	.84701+00	.30789-03
.96000+00	.96004+00	.37304-04	.92176+00	.84845-04
.10000+01	.10000+01	.00000	.10000+01	.00000

TABLE 3

BIAS AND VARIANCE OF $|\hat{\gamma}|^2$ FOR n = 48

$\lvert\gamma\rvert^2$	$E(\lvert\hat{\gamma}\rvert^2)$	Bias	$E[(\lvert\hat{\gamma}\rvert^2)]^2$	Variance
.00000	.20833-01	.20833-01	.85034-03	.41631-03
.40000-01	.59231-01	.19231-01	.52853-02	.17769-02
.80000-01	.97691-01	.17691-01	.12468-01	.29247-02
.12000+00	.13621+00	.16213-01	.22425-01	.38710-02
.16000+00	.17480+00	.14797-01	.35181-01	.46273-02
.20000+00	.21344+00	.13443-01	.50764-01	.52054-02
.24000+00	.25215+00	.12153-01	.69199-01	.56175-02
.28000+00	.29093+00	.10926-01	.90513-01	.58758-02
.32000+00	.32976+00	.97616-02	.11474+00	.59929-02
.36000+00	.36866+00	.86615-02	.14189+00	.59816-02
.40000+00	.40763+00	.76254-02	.17201+00	.58552-02
.44000+00	.44665+00	.66538-02	.20513+00	.56269-02
.48000+00	.48575+00	.57469-02	.24126+00	.53106-02
.52000+00	.52491+00	.49051-02	.28045+00	.49201-02
.56000+00	.56413+00	.41286-02	.32271+00	.44700-02
.60000+00	.60342+00	.34179-02	.36809+00	.39746-02
.64000+00	.64277+00	.27732-02	.41661+00	.34491-02
.68000+00	.68219+00	.21948-02	.46830+00	.29088-02
.72000+00	.72168+00	.16832-02	.52320+00	.23691-02
.76000+00	.76124+00	.12386-02	.58133+00	.18462-02
.80000+00	.80086+00	.86127-03	.64274+00	.13565-02
.84000+00	.84055+00	.55169-03	.70744+00	.91652-03
.88000+00	.88031+00	.31001-03	.77549+00	.54359-03
.92000+00	.92014+00	.13637-03	.84691+00	.25541-03
.96000+00	.96003+00	.29739-04	.92173+00	.71213-04
.10000+01	.10000+01	.00000	.10000+01	.00000

TABLE 4

BIAS AND VARIANCE OF $|\hat{\gamma}|^2$ FOR n = 56

$\lvert\gamma\rvert^2$	$E(\lvert\hat{\gamma}\rvert^2)$	Bias	$E[(\lvert\hat{\gamma}\rvert^2)]^2$	Variance
.00000	.17857-01	.17857-01	.62657-03	.30769-03
.40000-01	.56480-01	.16480-01	.46844-02	.14944-02
.80000-01	.95157-01	.15157-01	.11549-01	.24943-02
.12000+00	.13389+00	.13887-01	.21244-01	.33178-02
.16000+00	.17267+00	.12671-01	.33791-01	.39751-02
.20000+00	.21151+00	.11510-01	.49213-01	.44769-02
.24000+00	.25040+00	.10402-01	.67535-01	.48339-02
.28000+00	.28935+00	.93494-02	.88780-01	.50571-02
.32000+00	.32835+00	.83514-02	.11297+00	.51577-02
.36000+00	.36741+00	.74084-02	.14014+00	.51470-02
.40000+00	.40652+00	.65206-02	.17030+00	.50366-02
.44000+00	.44569+00	.56884-02	.20348+00	.48583-02
.48000+00	.48491+00	.49119-02	.23970+00	.45642-02
.52000+00	.52419+00	.41913-02	.27900+00	.42264-02
.56000+00	.56353+00	.35270-02	.32140+00	.38376-02
.60000+00	.60292+00	.29190-02	.36692+00	.34103-02
.64000+00	.64237+00	.23678-02	.41559+00	.29575-02
.68000+00	.68187+00	.18735-02	.46744+00	.24926-02
.72000+00	.72144+00	.14364-02	.52250+00	.20288-02
.76000+00	.76106+00	.10566-02	.58079+00	.15800-02
.80000+00	.80073+00	.73449-03	.64234+00	.11601-02
.84000+00	.84047+00	.47025-03	.70717+00	.78333-03
.88000+00	.88026+00	.26403-03	.77533+00	.46438-03
.92000+00	.92012+00	.11579-03	.84683+00	.21832-03
.96000+00	.96002+00	.24268-04	.92171+00	.61738-04
.10000+01	.10000+01	.00000	.10000+01	.00000

TABLE 5

BIAS AND VARIANCE OF $|\hat{\gamma}|^2$ FOR n = 64

$\lvert\gamma\rvert^2$	$E(\lvert\hat{\gamma}\rvert^2)$	Bias	$E[(\lvert\hat{\gamma}\rvert^2)]^2$	Variance
.00000	.15625-01	.15625-01	.48077-03	.23663-03
.40000-01	.54418-01	.14418-01	.42499-02	.12886-02
.80000-01	.93258-01	.13258-01	.10871-01	.21743-02
.12000+00	.15214+00	.12145-01	.20365-01	.29030-02
.16000+00	.17108+00	.11080-01	.32752-01	.34842-02
.20000+00	.21006+00	.10062-01	.48053-01	.39273-02
.24000+00	.24909+00	.90924-02	.66289-01	.42422-02
.28000+00	.28817+00	.81707-02	.87481-01	.44386-02
.32000+00	.32730+00	.72971-02	.11165+00	.45267-02
.36000+00	.36647+00	.64720-02	.13882+00	.45166-02
.40000+00	.40570+00	.56955-02	.16901+00	.44187-02
.44000+00	.44497+00	.49676-02	.20224+00	.42434-02
.48000+00	.48429+00	.42887-02	.23854+00	.40016-02
.52000+00	.52366+00	.36589-02	.27792+00	.37041-02
.56000+00	.56308+00	.30783-02	.32042+00	.33619-02
.60000+00	.60255+00	.25473-02	.36605+00	.29862-02
.64000+00	.64207+00	.20658-02	.41484+00	.25886-02
.68000+00	.68163+00	.16342-02	.46681+00	.21806-02
.72000+00	.72125+00	.12527-02	.52198+00	.17740-02
.76000+00	.76092+00	.92125-03	.58038+00	.13809-02
.80000+00	.80064+00	.64025-03	.64204+00	.10133-02
.84000+00	.84041+00	.40970-03	.70697+00	.68399-03
.88000+00	.88023+00	.22980-03	.77521+00	.40543-03
.92000+00	.92010+00	.10040-03	.84678+00	.19082-03
.96000+00	.96002+00	.20237-04	.92169+00	.54669-04
.10000+01	.10000+01	.00000	.10000+01	.00000

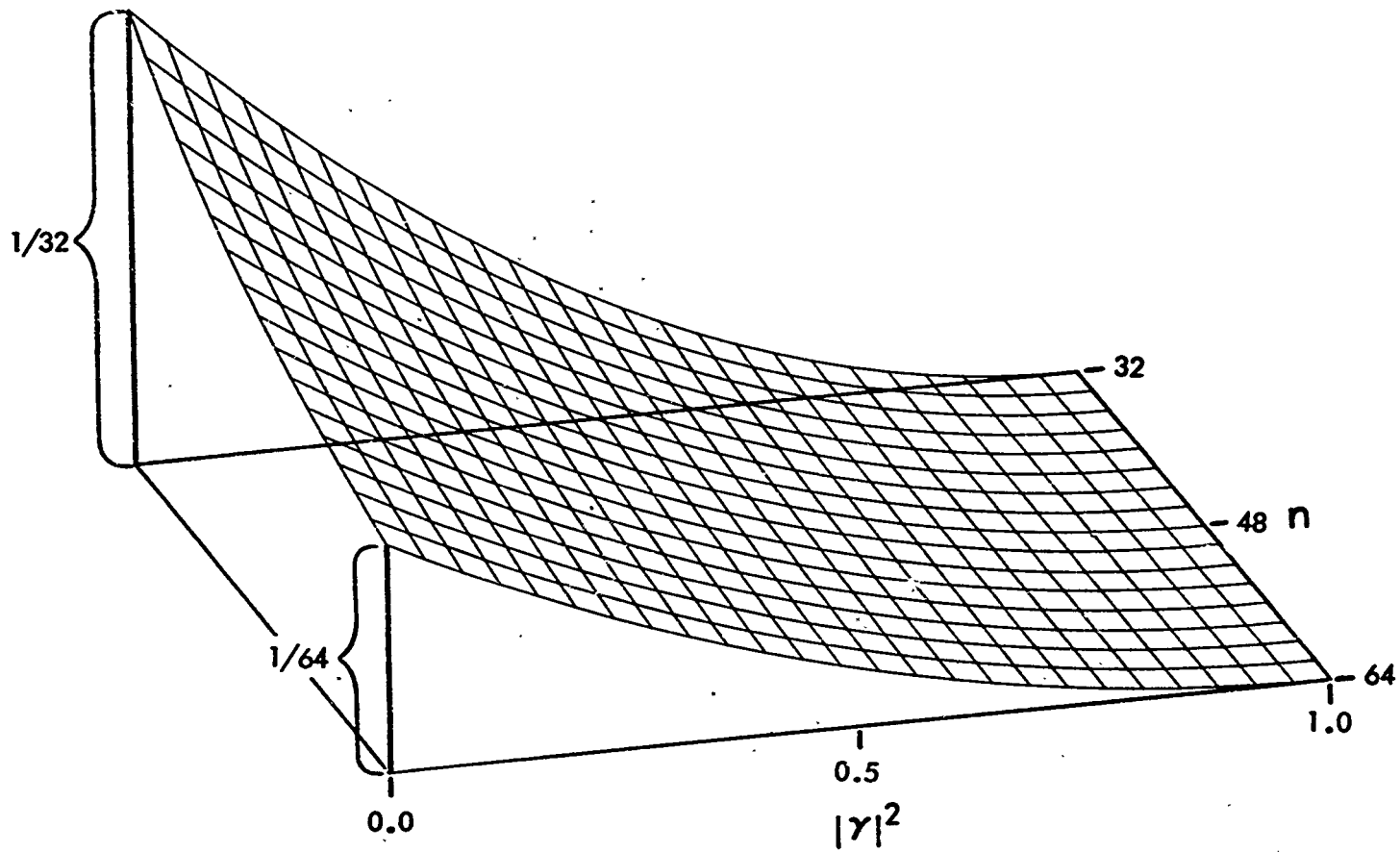

Fig. 15. Bias of $|\hat{\gamma}|^2$ versus $|\gamma|^2$ and n

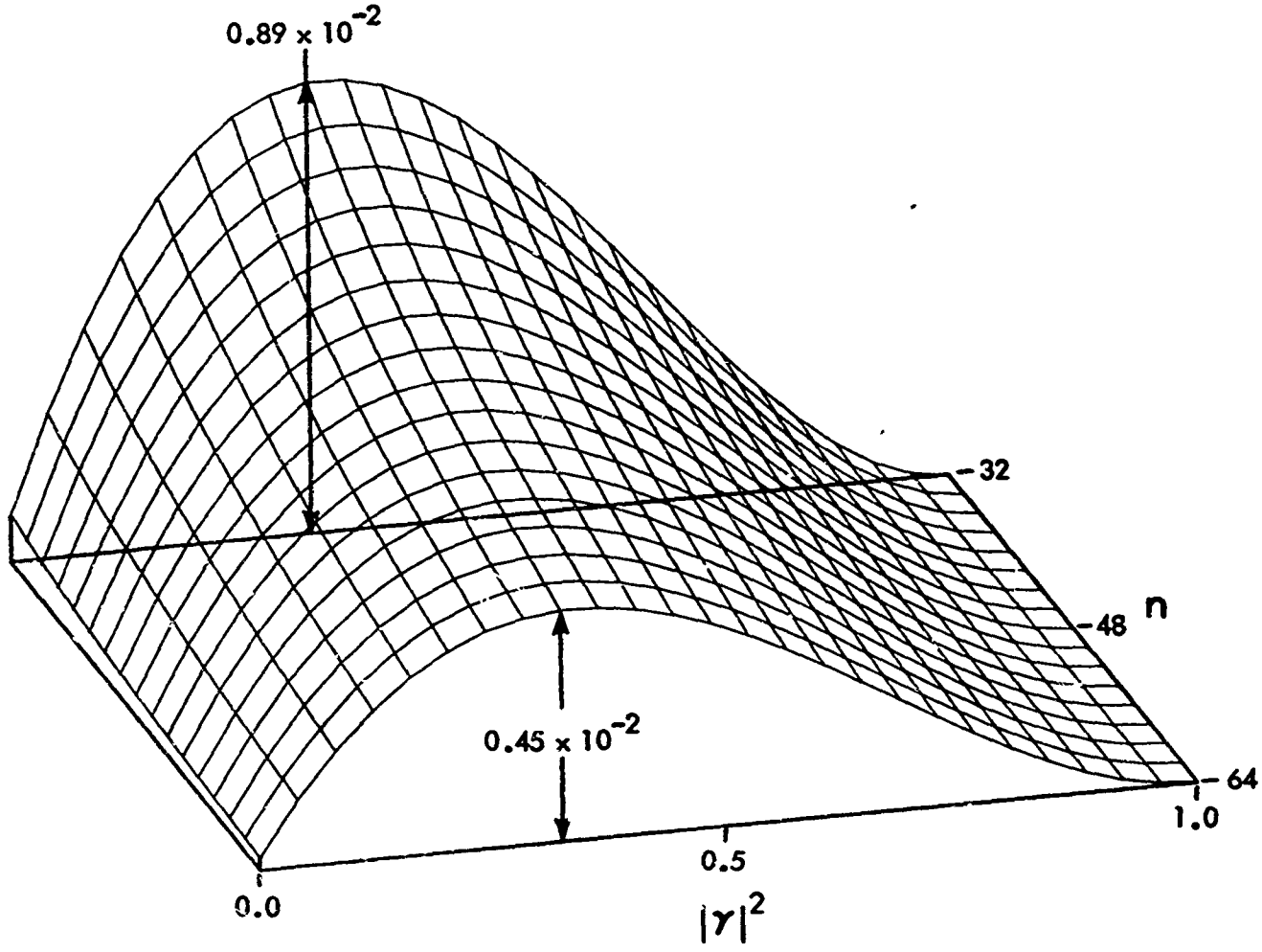

Fig. 16. Variance of $|\hat{\gamma}|^2$ versus $|\gamma|^2$ and n

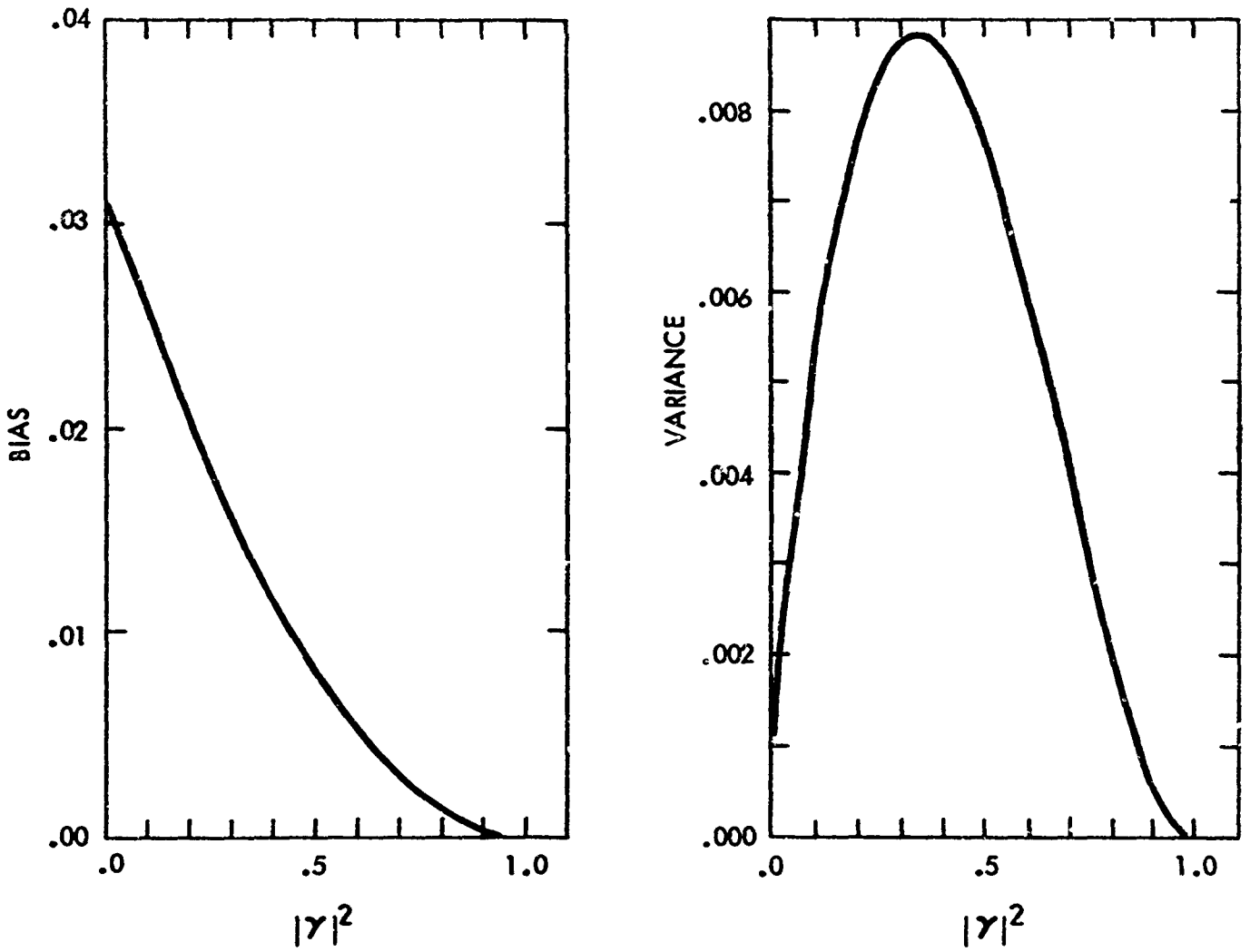

Fig. 17. Bias and Variance versus $|\gamma|^2$ for n = 32

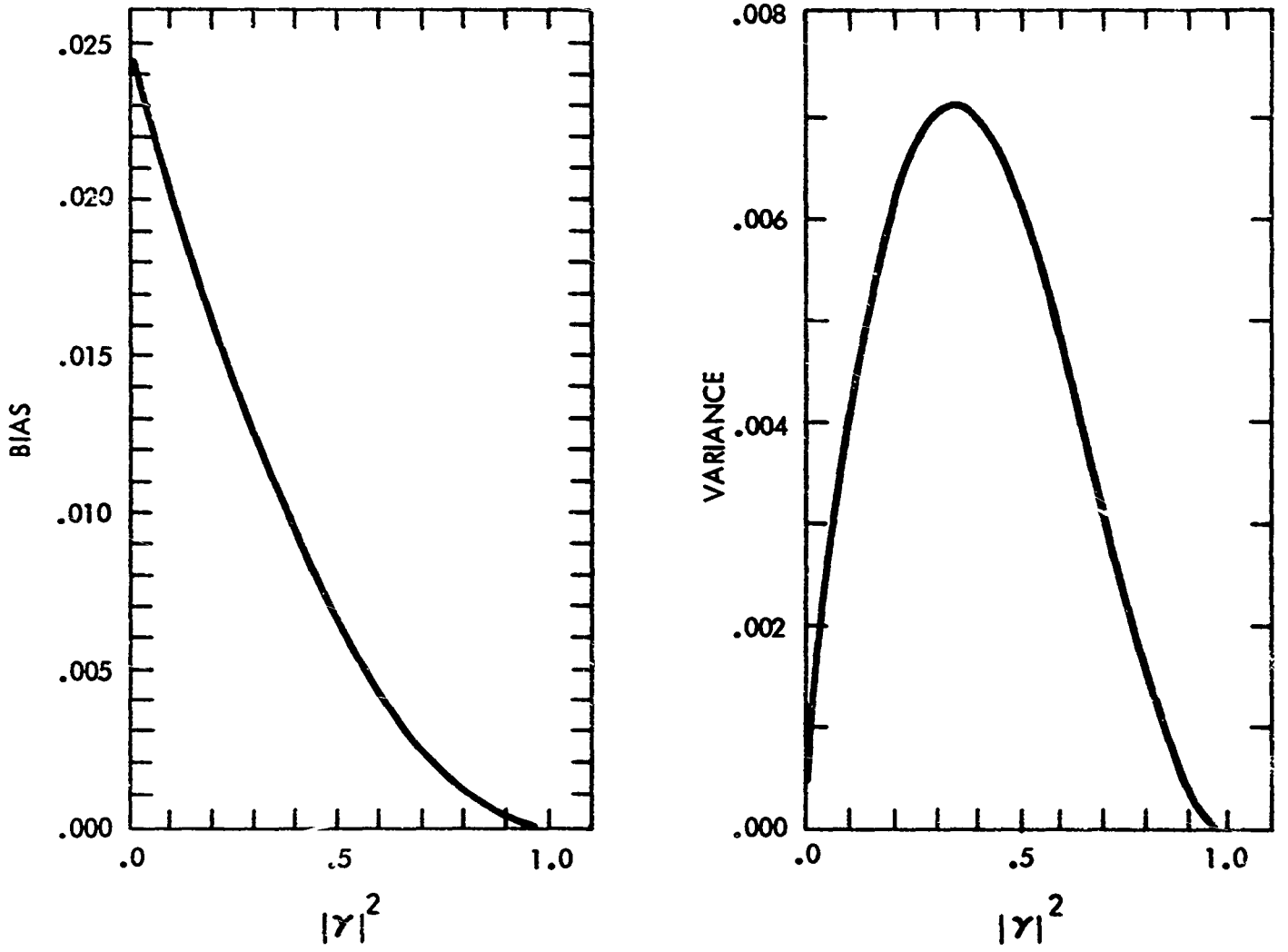

Fig. 18. Bias and Variance versus $|\gamma|^2$ for $n = 40$

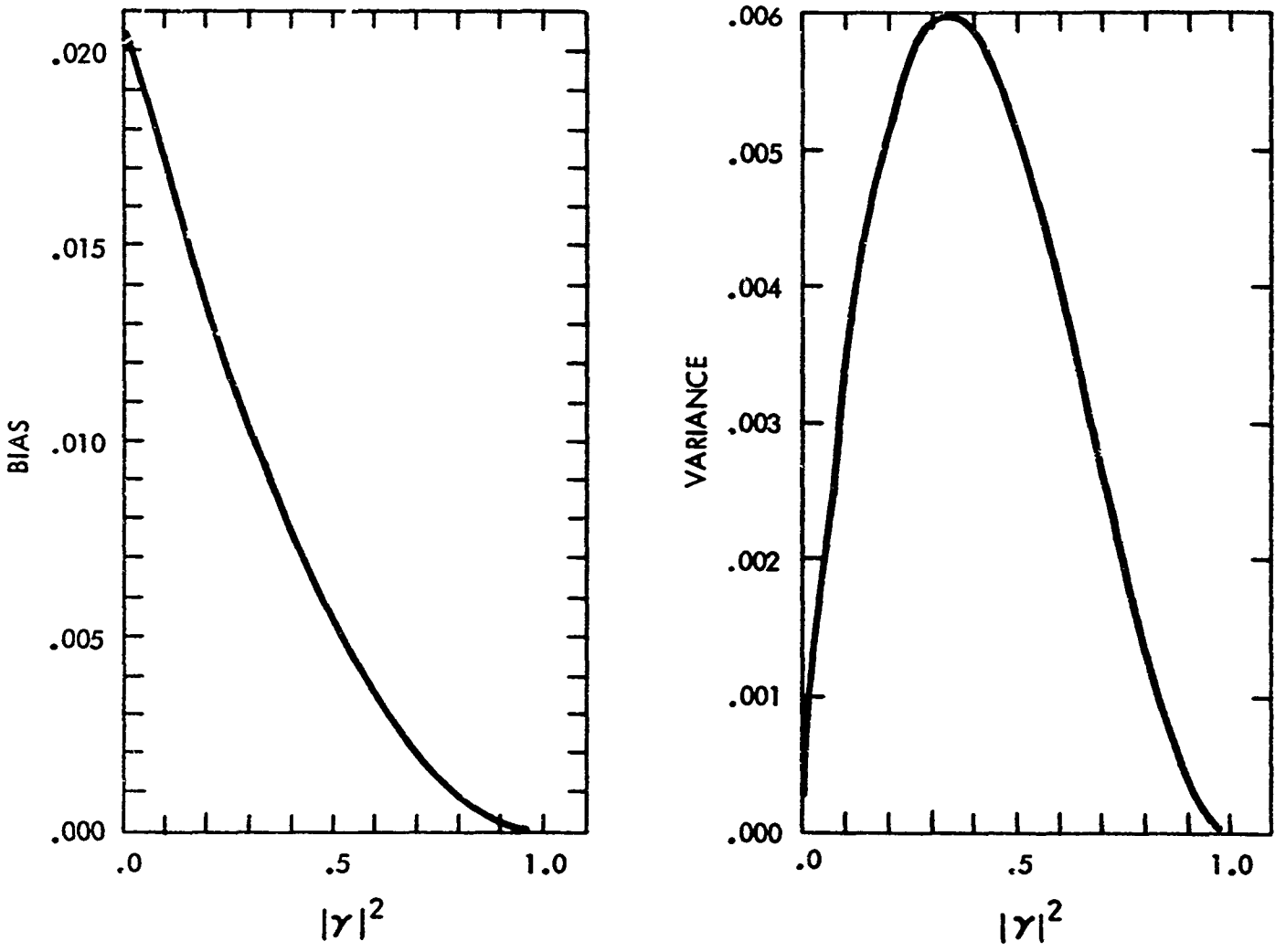

Fig. 19. Bias and Variance versus $|\gamma|^2$ for n = 48

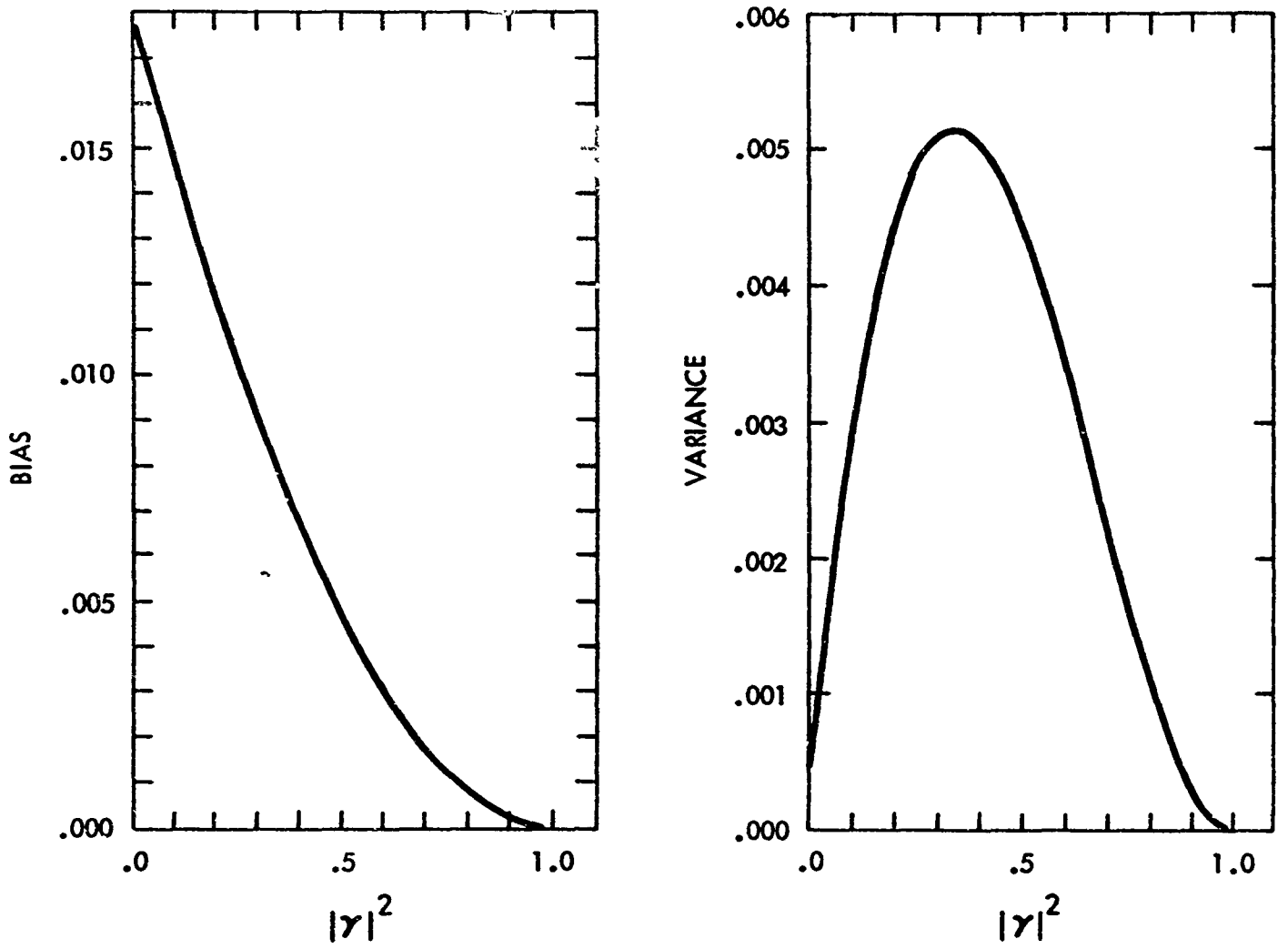

Fig. 20. Bias and Variance versus $|\gamma|^2$ for n = 56

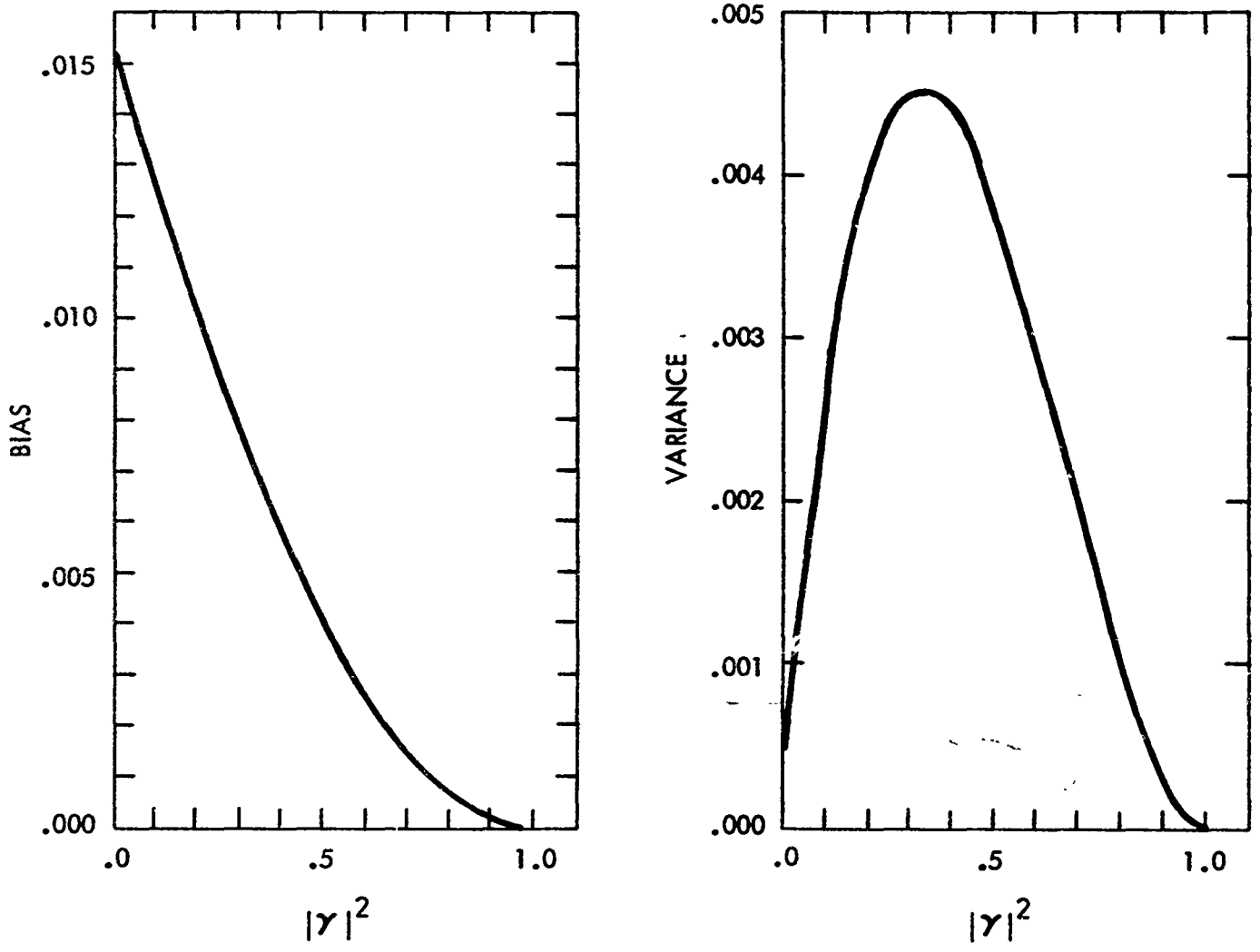

Fig. 21. Bias and Variance versus $|\gamma|^2$ for n = 64

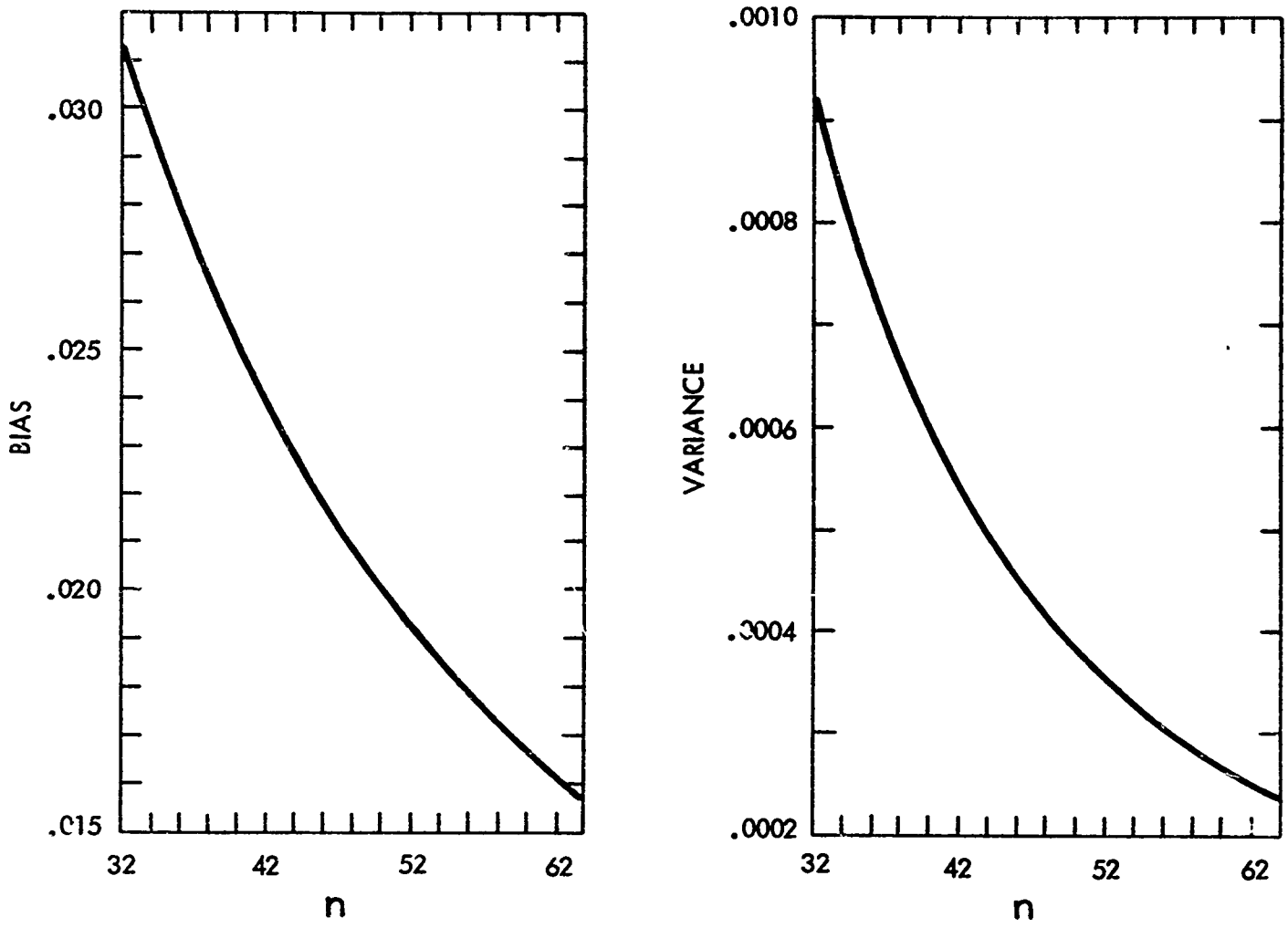

Fig. 22. Bias and Variance versus n for $|\gamma|^2 = 0.0$

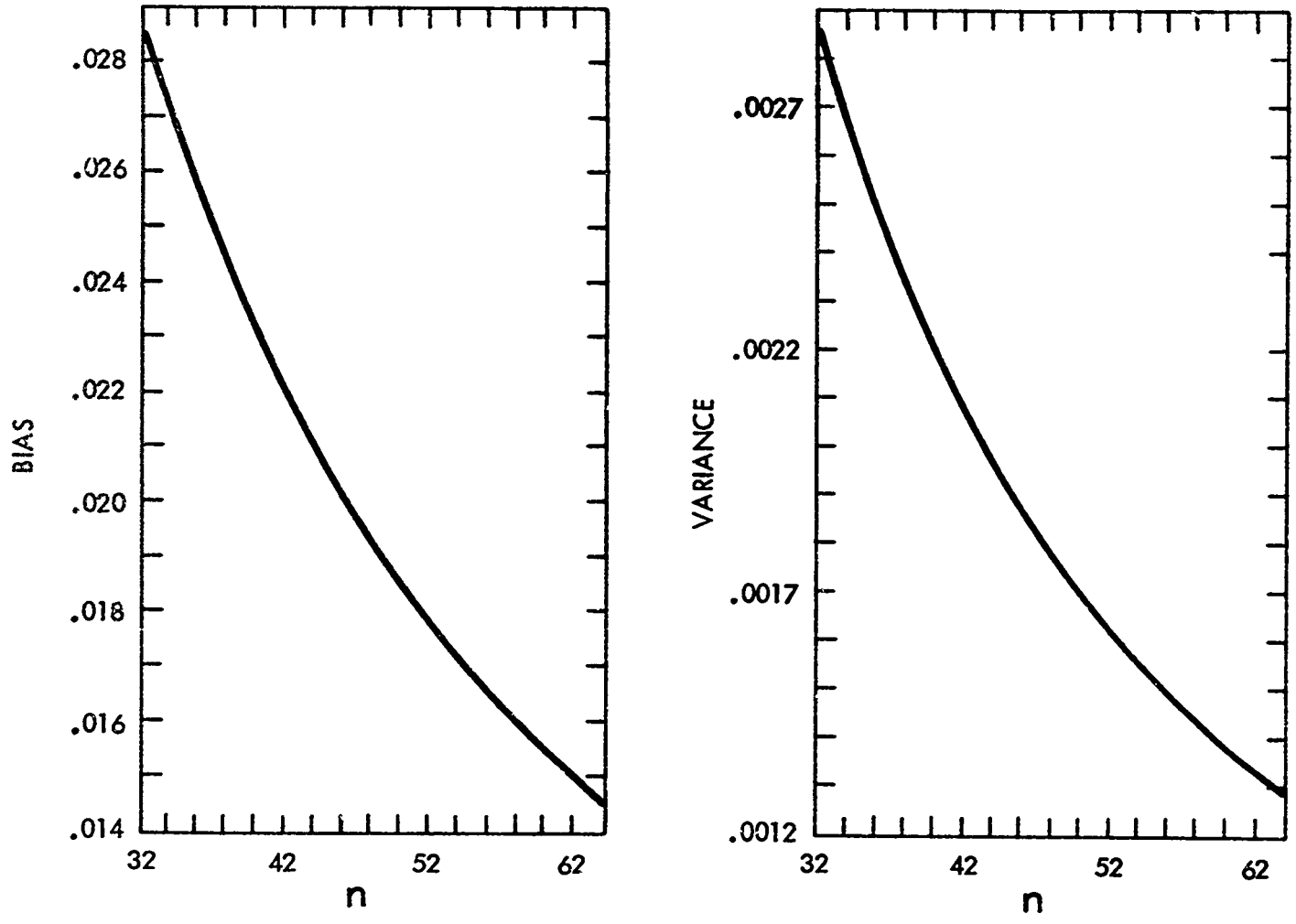

Fig. 23. Bias and Variance versus n for $|\gamma|^2 = 0.04$

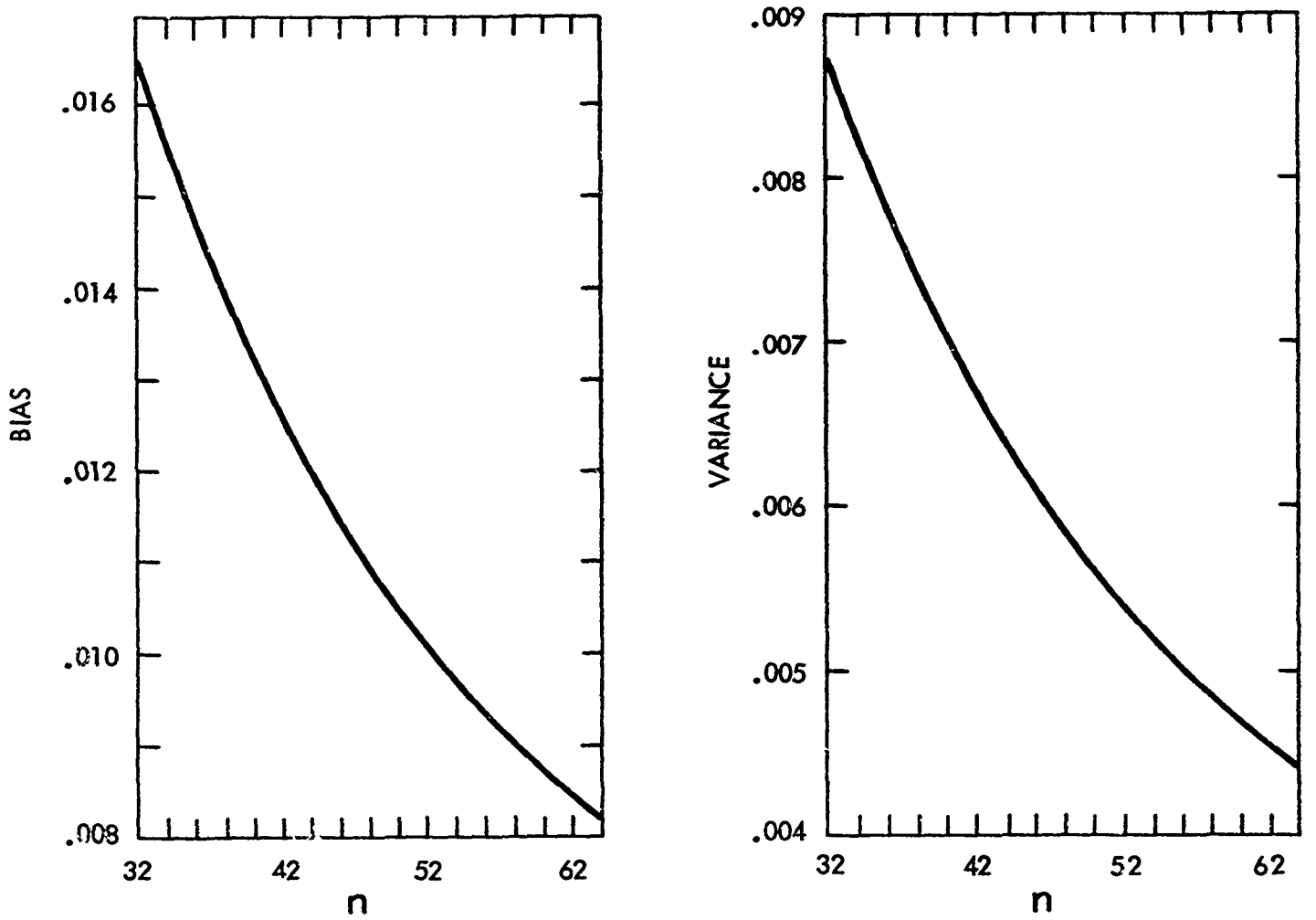

Fig. 24. Bias and Variance versus n for $|\gamma|^2 = 0.28$

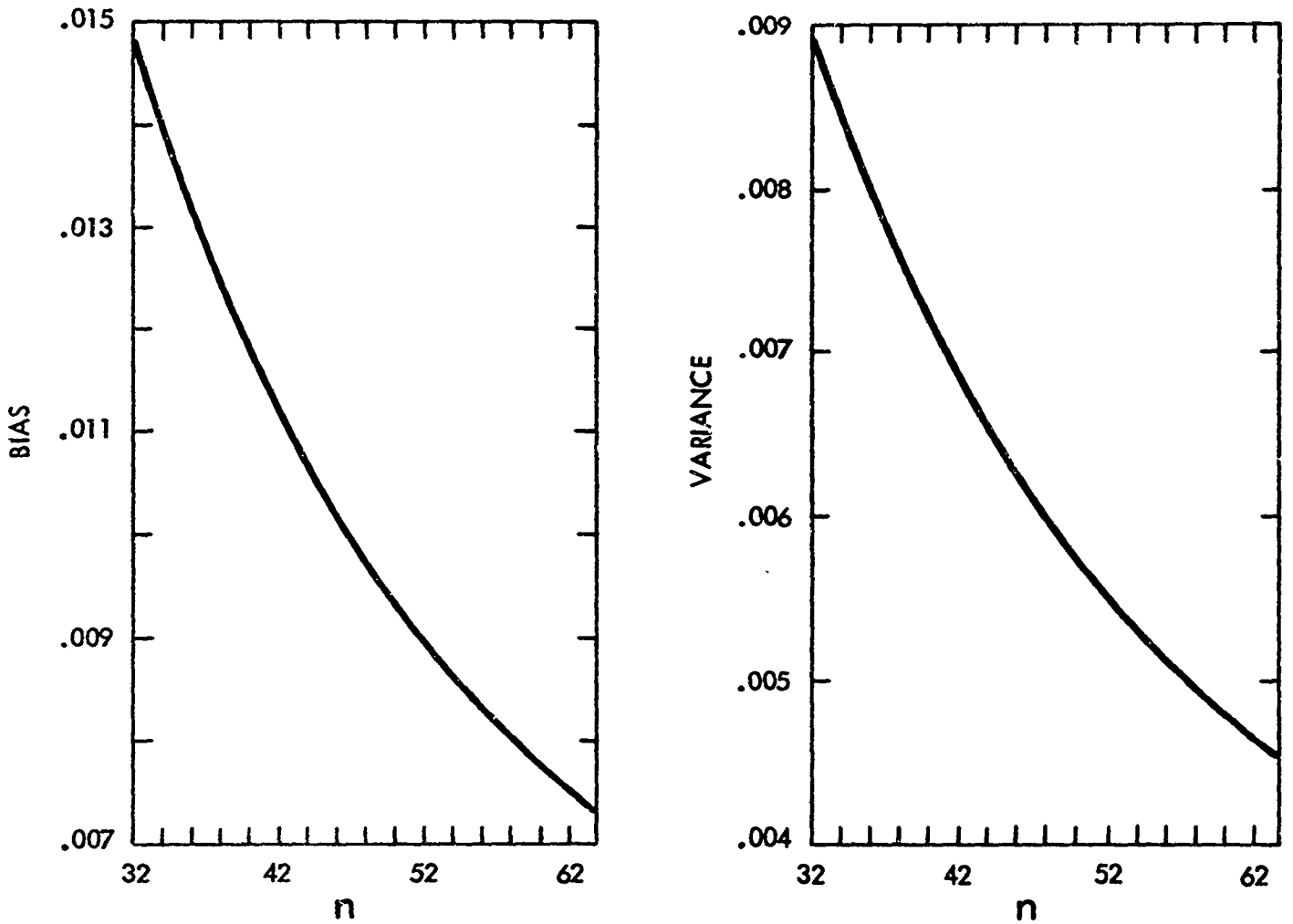

Fig. 25. Bias and Variance versus n for $|\gamma|^2 = 0.32$

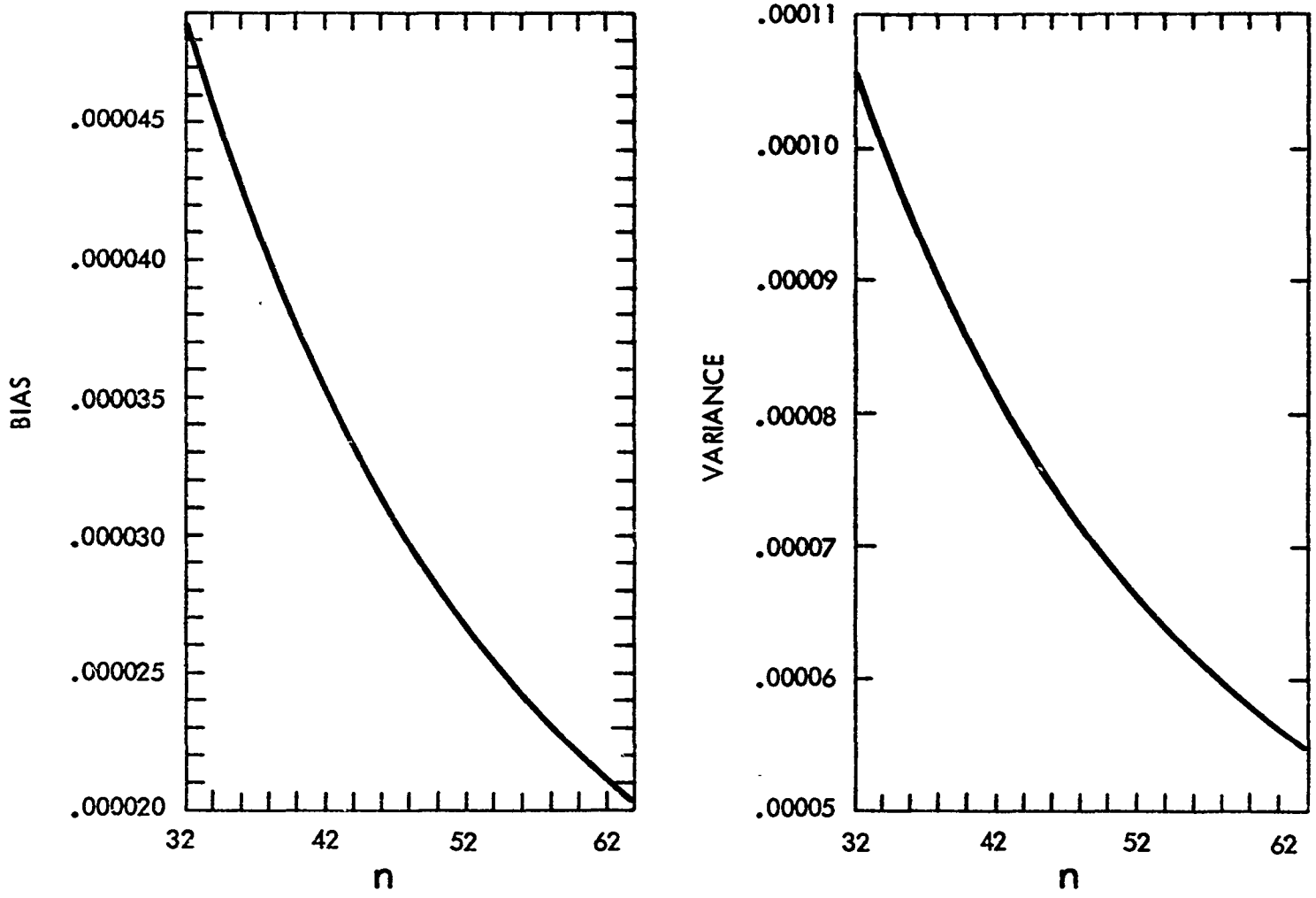

Fig. 26. Bias and Variance versus n for $|\gamma|^2 = 0.96$

V. EXPERIMENTAL INVESTIGATION OF OVERLAP EFFECTS

An experiment has been conducted to study the effect of overlap of data on the estimate $|\hat{\gamma}|^2$. The analytical results presented earlier relate only to the case of independent segments (that is, the case of zero percent overlap). This experiment examines the effect of different amounts of overlap on bias and variance of $|\hat{\gamma}|^2$.

Intuitively, it seems that the application of nonoverlapping weighting function does not make the best use of the data when forming the estimator $|\hat{\gamma}(f_k)|^2$. This inefficiency is similar to the wastage in forming auto power spectral density functions shown by Nuttall.[14] When $|\hat{\gamma}(f_k)|^2$ is formed without overlap, larger bias and larger variance result than when $|\hat{\gamma}(f_k)|^2$ is formed from the same data with overlap. Because this inefficiency can not be permitted in many practical situations of interest (for example, underwater acoustic environments), it is desirable to know how much the bias and variance can be reduced and at what expense this reduction can be achieved.

V.A. METHOD

The method of achieving the desired objective is straightforward in concept. Data are generated with an accurately prespecified value of magnitude-squared coherence, $|\gamma_g|^2$, which is independent of frequency, f. Since the data

have been generated so that the magnitude-squared coherence is independent of frequency, the sample mean and variance of $|\hat{\gamma}|^2$ can be empirically determined for the given overlap by averaging over frequency. These data can then be reprocessed at several different overlaps to form estimates of bias and variance.

V.A.1. Data Generation

Consider the zero-mean, wide-sense stationary, Gaussian waveforms $n_1(t)$ and $n_2(t)$ that are statistically independent and have power spectral density functions $\Phi_{n_1 n_1}(f)$ and $\Phi_{n_2 n_2}(f)$, respectively. Statistical independence dictates that they be uncorrelated; that is,

$$R_{n_1 n_2}(\tau) = E\left[n_1(t)\, n_2(t+\tau)\right] = 0 \quad . \tag{5.1}$$

In order to generate two processes with magnitude-squared coherence independent of frequency, let (Nuttall and Carter[31])

$$x(t) = n_1(t) + G n_2(t) \tag{5.2}$$

and

$$y(t) = n_2(t) + G n_1(t) \quad . \tag{5.3}$$

The cross-correlation of $x(t)$ and $y(t)$ is

$$R_{xy}(\tau) = E\left\{\left[(n_1(t) + G n_2(t)\right]\left[(n_2(t+\tau) + G n_1(t+\tau)\right]\right\} \quad . \tag{5.4}$$

Expanding, dropping terms that go to zero, and taking the Fourier transform of Eq. (5.4) yields

$$\Phi_{xy}(f) = G\Phi_{n_1n_1}(f) + G\Phi_{n_2n_2}(f) \quad . \tag{5.5}$$

The autocorrelation of $x(t)$ is

$$R_{xx}(\tau) = E\left\{\left[n_1(t) + Gn_2(t)\right]\left[n_1(t+\tau) + Gn_2(t+\tau)\right]\right\} \quad . \tag{5.6}$$

Expanding, dropping terms that go to zero, and taking the Fourier transform of Eq. (5.6) yields

$$\Phi_{xx}(f) = \Phi_{n_1n_1}(f) + G^2\Phi_{n_2n_2}(f) \quad . \tag{5.7}$$

Similarly,

$$\Phi_{yy}(f) = \Phi_{n_2n_2}(f) + G^2\Phi_{n_1n_1}(f) \quad . \tag{5.8}$$

Thus, the magnitude-squared coherence between $x(t)$ and $y(t)$ is

$$\left|\gamma_{xy}(f)\right|^2 = \frac{\left|G\Phi_{n_1n_1}(f) + G\Phi_{n_2n_2}(f)\right|^2}{\left[\Phi_{n_1n_1}(f) + G^2\Phi_{n_2n_2}(f)\right]\left[\Phi_{n_2n_2}(f) + G^2\Phi_{n_1n_1}(f)\right]} \quad . \tag{5.9}$$

Now introducing the assumption that $\Phi_{n_1n_1}(f) = \Phi_{n_2n_2}(f) \equiv \Phi_{nn}(f)$,

$$\left|\gamma_{xy}(f)\right|^2 = \frac{4G^2\,\Phi_{nn}^2(f)}{\left(1+G^2\right)^2\;\Phi_{nn}^2(f)} = \frac{4G^2}{\left(1+G^2\right)^2} \quad , \tag{5.10}$$

which is independent of frequency.

In order to prespecify a desired magnitude-squared coherence, $\left|\gamma_d\right|^2$, between $x(t)$ and $y(t)$, the gain G of Eqs. (5.2) and (5.3) must be selected by solving Eq. (5.10):

$$G = \begin{cases} \dfrac{1-\sqrt{1-\left|\gamma_d\right|^2}}{\left|\gamma_d\right|} & , \quad 0 < \left|\gamma_d\right| \leq 1 \\ 0 & , \quad \left|\gamma_d\right|^2 = 0.0 \end{cases} \quad . \tag{5.11}$$

Under the assumptions made, a prespecified desired value for magnitude-squared coherence can be generated. Because the generated processes will later be used to empirically determine a very small quantity (bias), it is important that the generated value of coherence is indeed the desired value. In the actual generation of two processes, the assumption $\Phi_{n_1n_1}(f) = \Phi_{n_2n_2}(f)$ may be violated; therefore it becomes important to determine how sensitive Eq. (5.10) is to this assumption. Consider then

$$\frac{\Phi_{n_2n_2}(f)}{\Phi_{n_1n_1}(f)} = 1 + \Delta(f) \cong 1 \quad . \tag{5.12}$$

It is easily shown[31] by substituting Eq.(5.12) into Eq. (5.9) that the value of magnitude-squared coherence generated, $|\gamma_g|^2$, is

$$|\gamma_g|^2 = |\gamma_d|^2 \frac{1 + \Delta + \frac{1}{4}\Delta^2}{1 + \Delta + \frac{1}{4}\Delta^2 |\gamma_d|^2} , \tag{5.13}$$

where $|\gamma_d|^2$ = desired value of $|\gamma|^2$, and the dependence of f is dropped for convenience.

The error in the generated value is

$$|\gamma_g|^2 - |\gamma_d|^2 = |\gamma_d|^2 \left(1 - |\gamma_d|^2\right) \frac{\frac{1}{4}\Delta^2}{1 + \Delta + \frac{1}{4}|\gamma_d|^2 \Delta^2} . \tag{5.14}$$

Evaluation of Eq. (5.14) to third order in Δ yields

$$|\gamma_g|^2 - |\gamma_d|^2 \cong |\gamma_d|^2 \left(1 - |\gamma_d|^2\right) \tfrac{1}{4} \Delta^2 (1 - \Delta) . \tag{5.15}$$

This quantity is maximum at $|\gamma_d|^2 = \frac{1}{2}$ and, hence, the maximum error is approximately

$$\text{Max error} \cong \left(\tfrac{1}{4}\Delta\right)^2 (1 - \Delta) . \tag{5.16}$$

Therefore, for example, when $\Delta = 0.01$, the maximum error is approximately 6×10^{-6}; for $\Delta = 0.05$, the maximum error is approximately 1.5×10^{-4}. (A table of errors versus $|\gamma_d|^2$ and Δ,[31] as computed from Eq. (5.14), yields results similar to the given approximations.)

The processes generated according to Eqs. (5.2) and (5.3) have been shown to be relatively insensitive to minor differences in the power spectral density functions of the original uncorrelated waveforms $n_1(t)$ and $n_2(t)$.

The procedure for generating variable-coherence time series can briefly be summarized as follows: One Gaussian noise source uncorrelated from point to point was used to generate a time-limited sample function of $n_1(t)$ and, later, of $n_2(t)$. (This method eliminates the need for two identical filters.) The waveforms were band-limited using a low-pass filter and digitized. The digital data were then stored on magnetic tape in a format compatible with overlapped processing. Digital versions of $x(t) = n_1(t) + Gn_2(t)$ and $y(t) = n_2(t) + Gn_1(t)$ were generated from digital versions of $n_1(t)$ and $n_2(t)$ for two values of $\left|\gamma_{xy}\right|^2$. (Investigation for other values of true magnitude-squared coherence appeared to be unnecessary.)

A Hewlett Packard Noise Generator, Model No. HP3722A, was used for data generation with the following settings:

Sequence Length:	Infinite
Bandwidth:	5 kHz
Gaussian rms:	0.6 x 3.16 volts (open circuit).

The output power density function is flat to within $\pm$ 0.3 dB, provided the input power voltage fluctuates no more than $\pm$ 10 percent. This corresponds to a Δ, in Eq. (5.12), of 7.152×10^{-2} and a maximum error in the generated magnitude-squared coherence, from Eq. (5.16), of 3×10^{-4}. For example, if $\left|\gamma_d\right|^2 = 0.5000$ one could expect $0.5004 > \left|\gamma_g\right|^2 > 0.4996$, making it

impossible to measure extremely small bias.

The data were low-pass filtered through a two-section Krohn-hite filter, which was 6 dB down at 1200 Hz and rolling off at 96 dB per octave. These band-limited data were A/D converted with a Control Data Corporation (CDC) 15-bit converter. Sampling was done at $f_s = 4096$ Hz.

V.A.2. Analysis Program

The FORTRAN program (coded by G. C. Carter, C. R. Arnold, and J. F. Ferrie, of NUSC)(1) implements Eq. (3.11); (2) generates data with known coherence from two incoherent sources according to Section V. A. 1; and (3) computes the sample mean, bias, and variance of the estimator, as described below. A summary flowchart of the program is presented in Fig. 27. The cosine weighting function was coded by A. H. Nuttall, of NUSC. Singleton[32] coded the mixed radix FFT. The FFT size used was 4096 data points (1 sec), which yields 2048 positive frequencies and direct current. Frequencies beyond 1000 Hz were discounted in making estimates of bias and variance to protect against (1) unknown noise in the digitizing system and (2) difference in the two auto power spectral density functions.

Estimates of the bias are performed according to

$$\widehat{\text{Bias}} = \left[\frac{1}{1000} \sum_{k=1}^{1000} \left| \hat{\gamma}(f_k) \right|^2 \right] - \left| \gamma \right|^2 , \tag{5.17}$$

and estimates of the sample variance according to

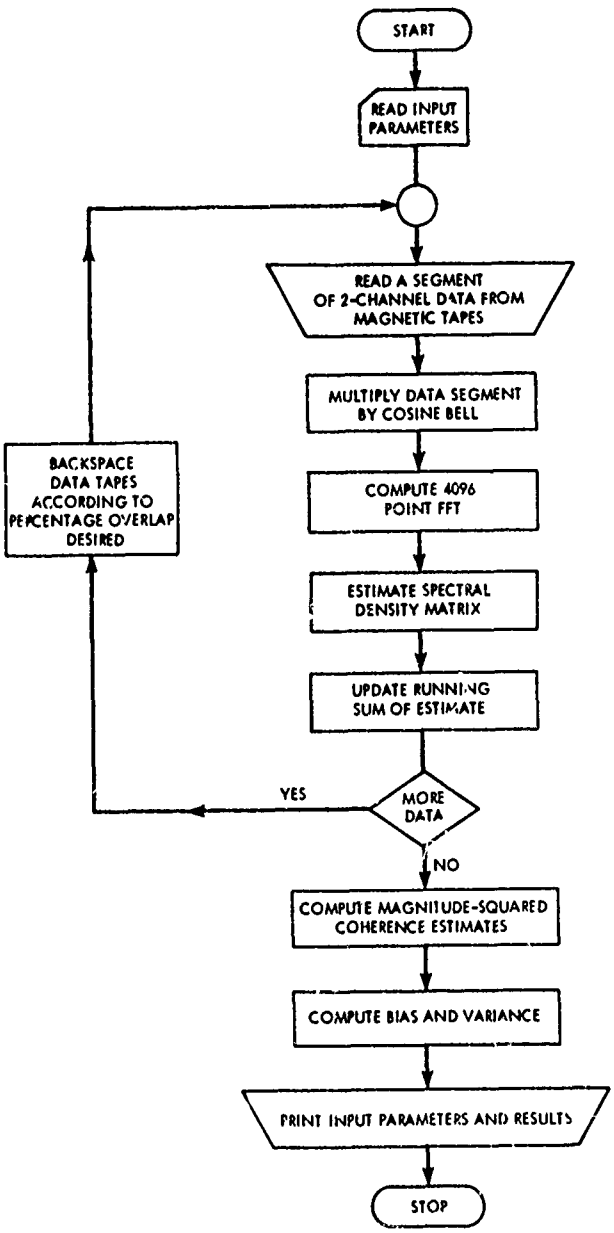

Fig. 27. Summary Flow Chart for Thesis Version of FFT Spectral Density Estimation Program

$$\widehat{\mathrm{Var}} = \frac{1}{999} \sum_{k=1}^{1000} \left[\left| \hat{\gamma}(f_k) \right|^2 - \widehat{\mathrm{Bias}} \right]^2 . \tag{5.18}$$

Results of the experiment are described in the next section.

V. B. RESULTS

Results of the experiment for a joint set of data, each (32 x 4096) samples long, are included in tabular and graphic form. Confidence bands for the estimates of bias can be determined from the estimates of variance. However, it must be realized that 1000 samples (frequencies) were used to determine the average, and that each sample is correlated to the extent of approximately 0.5 with neighboring estimates (empirical results). This agrees with analytical results provided for auto spectral estimates.[14]

It is apparent from the results (Tables 6 and 7 and Figs. 28 through 31) that the bias and variance of $\left|\hat{\gamma}\right|^2$ can be reduced through the use of overlapped processing. For example, when $\left|\gamma\right|^2 = 0.0$, the variance of the estimator with 50-percent overlap equals 31 percent of the variance of the estimator with 0-percent overlap. With 50-percent overlap, the bias is 55 percent as large as with 0-percent overlap. Similarly, when $\left|\gamma\right|^2 = 0.3$, the variance is 55 percent of the 0-percent overlap estimator, and the bias is 50 percent as large. It also can be seen from the results that 62.5-percent overlap is similar to having processed twice as much data with 0-percent overlap. There is one possible exception: The bias for $\left|\gamma\right|^2 = 0.3$ is 36 percent as large as the 0-percent overlap estimator. This is better than 50 percent,

TABLE 6

EMPIRICAL RESULTS FOR $|\gamma|^2 = 0.0$ AND $n = 32$

Percent Overlap	No. FFTs	Bias	Variance
0.0	32	$.3156 \times 10^{-1}$	$.9463 \times 10^{-3}$
12.5	36	$.2834 \times 10^{-1}$	$.7130 \times 10^{-3}$
25.0	42	$.2370 \times 10^{-1}$	$.5197 \times 10^{-3}$
37.5	50	$.2043 \times 10^{-1}$	$.4069 \times 10^{-3}$
50.0	63	$.1749 \times 10^{-1}$	$.2929 \times 10^{-3}$
62.5	83	$.1582 \times 10^{-1}$	$.2480 \times 10^{-3}$
75.0	125	$.1571 \times 10^{-1}$	$.2463 \times 10^{-3}$

TABLE 7

EMPIRICAL RESULTS FOR $|\gamma|^2 = 0.3$ AND $n = 32$

Percent Overlap	No. FFTs	Bias	Variance
0.0	32	1.442×10^{-2}	$.8007 \times 10^{-2}$
12.5	36	1.093×10^{-2}	$.7770 \times 10^{-2}$
25.0	42	$.959 \times 10^{-2}$	$.5965 \times 10^{-2}$
37.5	50	$.717 \times 10^{-2}$	$.5067 \times 10^{-2}$
50.0	63	$.597 \times 10^{-2}$	$.4441 \times 10^{-2}$
62.5	83	$.515 \times 10^{-2}$	$.4063 \times 10^{-2}$
75.0	125	$.494 \times 10^{-2}$	$.4020 \times 10^{-2}$

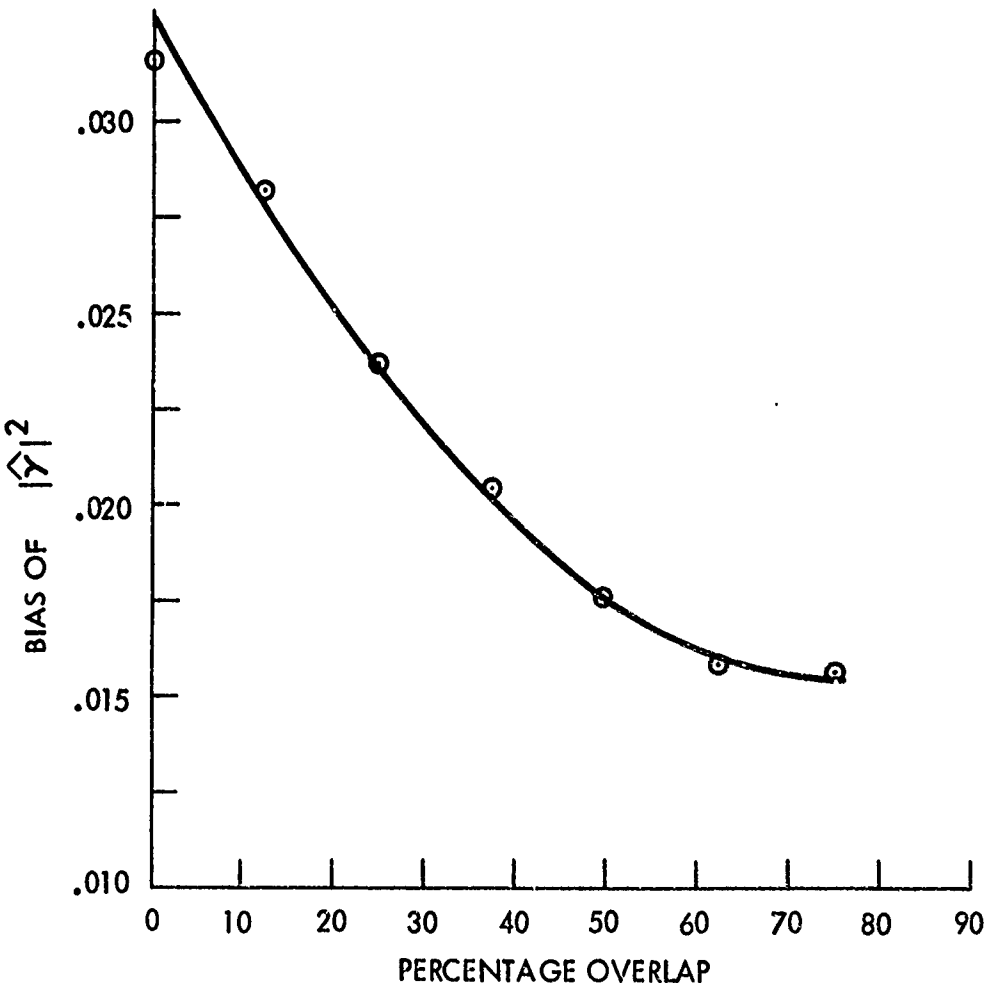

Fig. 28. Bias of $|\hat{\gamma}|^2$ When $|\gamma|^2 = 0.0$ and $n = 32$

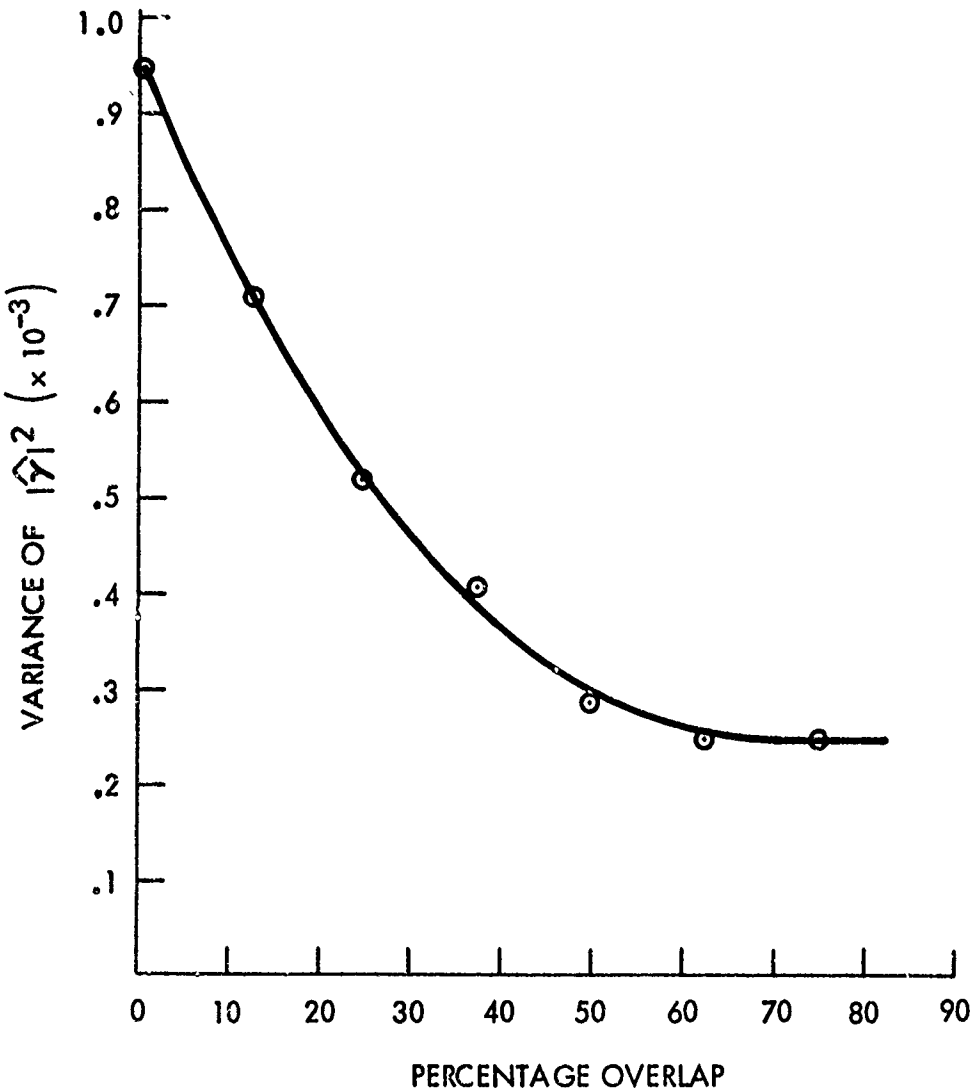

Fig. 29. Variance of $|\hat{\gamma}|^2$ When $|\gamma|^2 = 0.0$ and $n = 32$

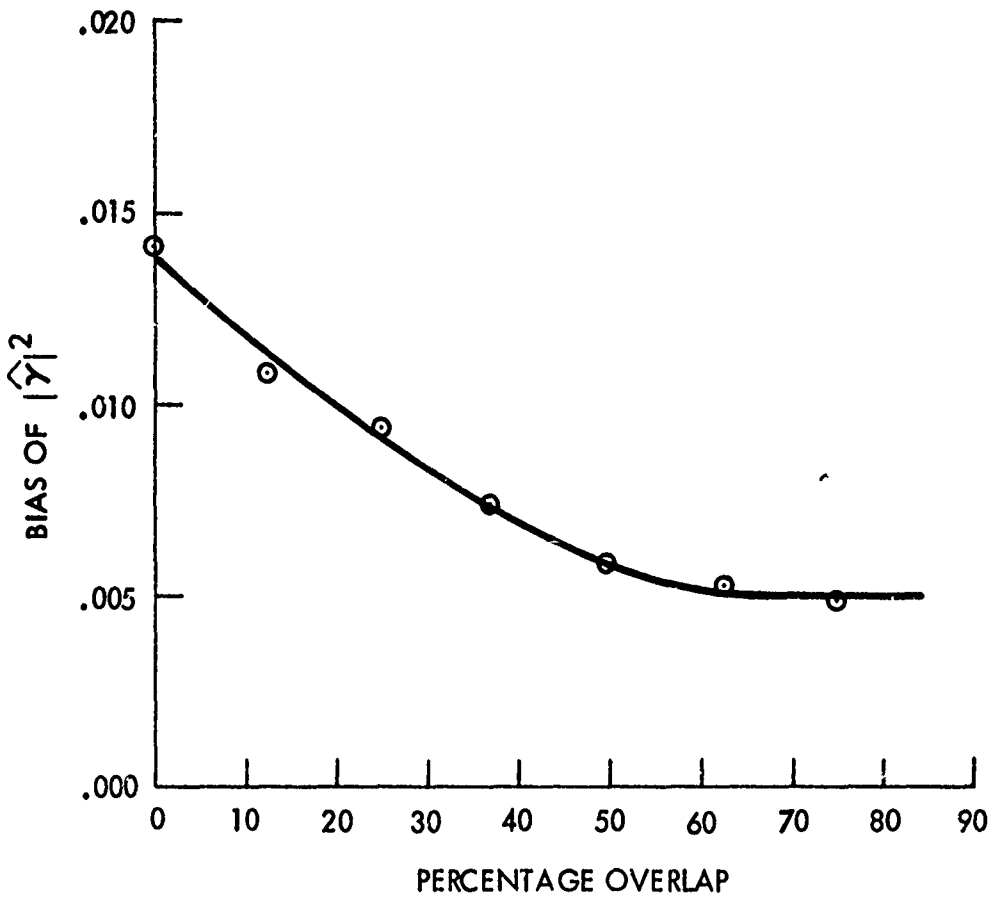

Fig. 30. Bias of $|\hat{\gamma}|^2$ When $|\gamma|^2 = 0.3$ and $n = 32$

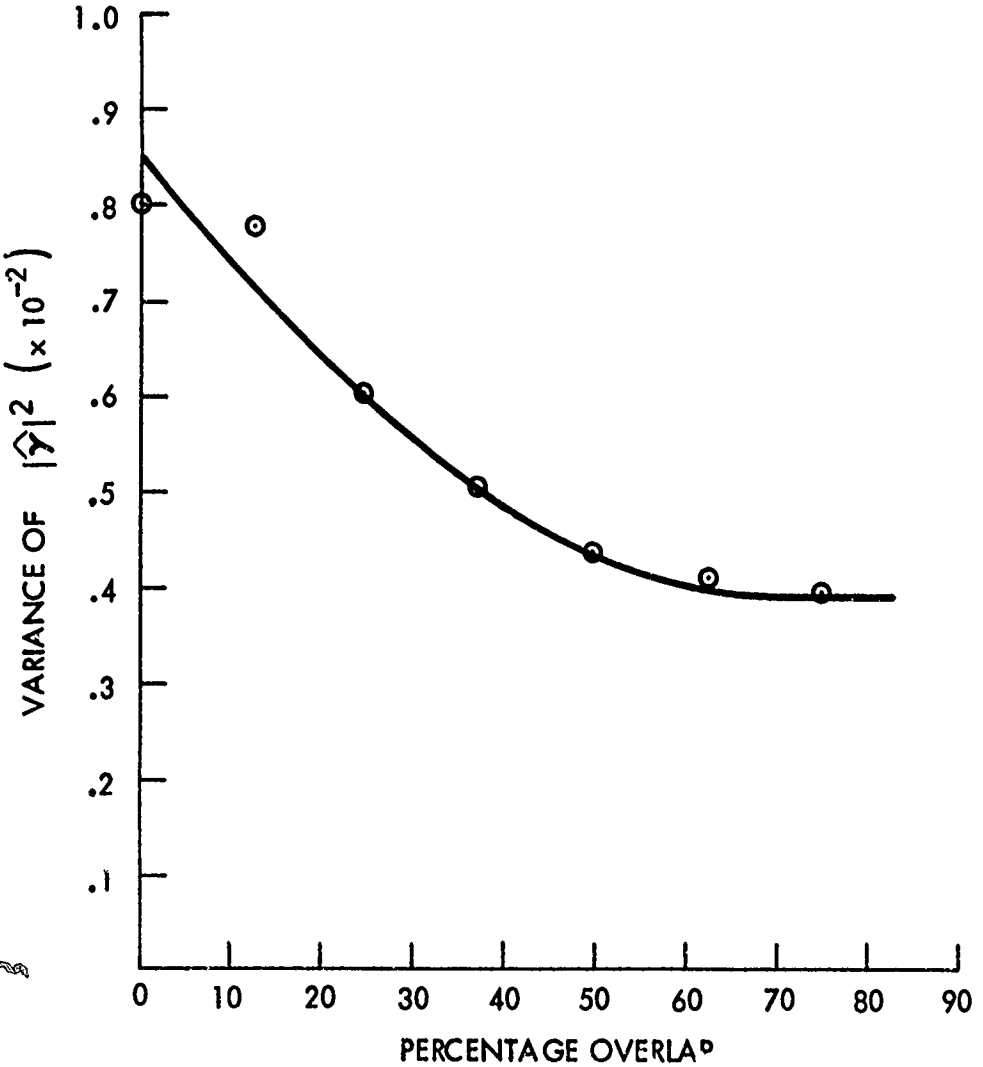

Fig. 31. Variance of $|\hat{\gamma}|^2$ When $|\gamma|^2 = 0.3$ and $n = 32$

which would be expected from twice as much data.

Quite naturally, there is an increase in computational cost associated with overlapped processing. Specifically, the number of FFTs to be performed (a measure of the computational cost) increases with the percent overlap specified (Fig. 32). The number of FFTs required for 50-percent overlap is approximately twice the number for 0-percent overlap.

Increasing the overlap from 50 percent to 62.5 percent, requires 32-percent more FFTs, but the variance of the estimator becomes only 80-95 percent of its value at 50-percent overlap. In most cases, the improvement to be derived from using 62.5-percent overlap, as opposed to 50-percent overlap, will not warrant the increased computational costs, and should be used only when stringent variance and bias reduction requirements are demanded.

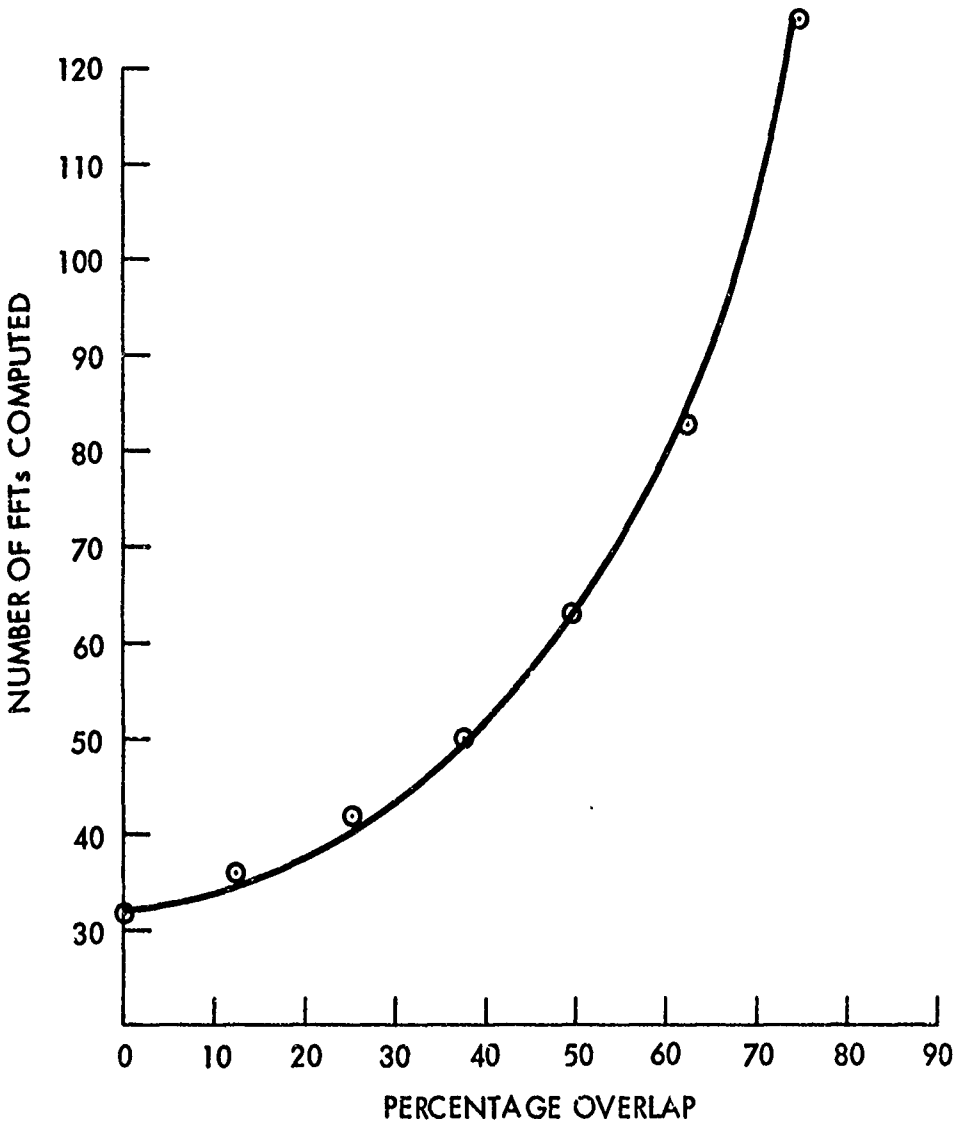

Fig. 32. Number of FFTs Required for Overlapped Processing

VI. CONCLUSIONS

A detailed analytical analysis of the statistics for estimating the magnitude-squared coherence function (spectrum) has been made. When such estimates are made, time-limited sample functions of long duration, which are stationary (in the wide sense) over the period of observation, must be available. Expressions for the probability density, the cumulative distribution, and the bias and variance of $|\hat{\gamma}|^2$ have been presented for the case where no overlap processing is used. Evaluation of these expressions, which are dependent on both the true value of coherence and the number of observed segments, n, dramatically portrays the requirement that n be large.

The application of a cosine-weighting function in order to reduce errors due to sidelobe leakage wastes the available data. As shown empirically, proper use of the data in terms of reduced bias and variance of the estimator can be achieved through overlapped processing. It appears that a 62.5-percent overlap is roughly equivalent to having twice as much data available. The reduced bias and variance of the estimator achieved through 62.5-percent overlapped processing can be realized almost entirely through a 50-percent overlap. The computational cost associated with 50-percent overlap is not unreasonable. With 50-percent overlap, variance and bias reductions are achieved that are similar to reductions resulting from processing twice as much data with 0-percent overlap. This significant gain to be obtained from 50-percent overlap processing

should not be overlooked in estimating the magnitude-squared coherence function (spectrum) when cosine (Hanning) weighting is used and data are limited.

APPENDIX A

STATISTICS OF MAGNITUDE-COHERENCE ESTIMATOR

Goodman, in his Eqs. (4.51) and (4.60), derived an analytical expression for the probability density function of the coherence estimate, $|\hat{\gamma}|$.[1] His results are based on two zero-mean processes that are stationary, Gaussian, and random and are segmented into n independent observations (that is, n nonoverlapped segments). Each segment is assumed perfectly windowed, as defined in Chapter IV. The probability that the estimate of coherence would take on some value, $|\hat{\gamma}|$, conditioned on the true coherence being equal to $|\gamma|$ and upon n independent observations, was given by Goodman as

$$p\left(|\hat{\gamma}| \,\middle|\, |\gamma|, n\right) = \frac{2|\hat{\gamma}|\left(1 - |\gamma|^2\right)^n}{\Gamma(n)\ \Gamma(n-1)} \left(1 - |\hat{\gamma}|^2\right)^{n-2}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma^2(n+k)}{\Gamma^2(k+1)} \left(|\gamma|^2 |\hat{\gamma}|^2\right)^k , \qquad \text{(A.1)}$$

where $\Gamma(n)$ is the Gamma function, namely,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx \quad \text{and} \quad \Gamma(n) = (n-1)\,\Gamma(n-1) \ .$$

This probability density function is also given by Hannan.[24] However, this form of the density function is rather cumbersome. A work by Enochson and Goodman[9] suggests that it may be written in terms of the hypergeometric function.

It is first necessary to observe, from Abramowitz and Stegun,[26] that

$$ {}_2F_1\left(n, n; 1; |\gamma|^2 |\hat{\gamma}|^2\right) = \frac{1}{\Gamma^2(n)} \sum_{k=0}^{\infty} \frac{\Gamma^2(n+k)}{\Gamma(k+1)} \frac{\left(|\gamma|^2 |\hat{\gamma}|^2\right)^k}{k!} \ . \tag{A.2}$$

Further, when k is an integer,

$$k! = \Gamma(k+1) \ . \tag{A.3}$$

(See, for example, Abramowitz and Stegun.[26]) Thus, Eq. (A.1) is

$$p\left(|\hat{\gamma}| \,\middle|\, |\gamma|, n\right) = 2|\hat{\gamma}|\left(1 - |\gamma|^2\right)^n \left(1 - |\hat{\gamma}|^2\right)^{n-2} \frac{(n-1)}{\Gamma(n)\,\Gamma(n)}$$

$$\cdot \sum_{k=0}^{\infty} \frac{\Gamma^2(n+k)}{\Gamma(k+1)} \frac{\left(|\gamma|^2 |\hat{\gamma}|^2\right)^k}{k!} \tag{A.4}$$

or, more simply, substituting Eq. (A.2) into (A.4),

$$p\left(|\hat{\gamma}| \,\middle|\, |\gamma|, n\right) = 2|\hat{\gamma}|\left(1 - |\gamma|^2\right)^n \left(1 - |\hat{\gamma}|^2\right)^{n-2} (n-1)$$

$$\cdot\, {}_2F_1\left(n, n; 1; |\gamma|^2 |\hat{\gamma}|^2\right) \ . \tag{A.5}$$

This form of the density function, Eq. (A.5), is more favorable than the form in Eq. (A.1) because the hypergeometric function is well documented.

The mth moment of magnitude coherence can be shown, as in Eq. (4.13), to be

$$E\left(|\hat{\gamma}|^m \middle| n, |\gamma|\right) = \left(1 - |\gamma|^2\right)^n \frac{\Gamma(n)\,\Gamma\left(\frac{m}{2}+1\right)}{\Gamma\left(n+\frac{m}{2}\right)}$$

$$\bullet\, {}_3F_2\left(\frac{m}{2}+1,\, n,\, n;\, \frac{m}{2}+n,\, 1; |\gamma|^2\right). \qquad \text{(A.6)}$$

Exact expressions for the bias and variance follow directly in a manner parallel to that in part IV. C.

It is instructive to show the relation between the biases of estimates of the magnitude coherence and the magnitude-squared coherence. Define the biases as follows:

$$B_1 \triangleq E\left(|\hat{\gamma}|\right) - |\gamma| \qquad \text{(A.7)}$$

and

$$B_2 \triangleq E\left(|\hat{\gamma}|^2\right) - |\gamma|^2 . \qquad \text{(A.8)}$$

Because the variance of $|\hat{\gamma}|$ must be nonnegative,

$$E\left(|\hat{\gamma}|^2\right) \geq \left[E\left(|\hat{\gamma}|\right)\right]^2 . \qquad \text{(A.9)}$$

Using Eqs. (A.7) and (A.8) yields

$$B_2 + |\gamma|^2 \geq \left(B_1 + |\gamma|\right)^2 = B_1^2 + 2B_1|\gamma| + |\gamma|^2 \ . \tag{A.10}$$

Thus,

$$B_2 \geq 2B_1|\gamma| + B_1^2 \ . \tag{A.11}$$

For example, consider the case $|\gamma| = 1.0$. Now $B_1 = 0.0$, $B_2 = 0.0$, and Eq. (A.11) hold with equality. Consider also $|\gamma| = 0.0$. Then

$$E\left(|\hat{\gamma}| \,\middle|\, n, |\gamma| = 0.0\right) = \frac{\Gamma(n)\,\Gamma(3/2)}{\Gamma(n + 1/2)} \ . \tag{A.12}$$

Using Eq. (6.1.47) of Abramowitz and Stegun,[26] Eq. (A.12) yields for large n

$$E\left(|\hat{\gamma}| \,\middle|\, n, |\gamma| = 0.0\right) \simeq \frac{\Gamma(3/2)}{\sqrt{n}} = \tfrac{1}{2}\sqrt{\pi/n} \ . \tag{A.13}$$

For $|\gamma|^2 = 0$, Eq. (4.21) gives

$$B_2 = \frac{1}{n} \ , \quad |\gamma|^2 = 0.0 \ . \tag{A.14}$$

Thus, the inequality holds and Eq. (A.11) becomes

$$\frac{1}{n} \geq \frac{\pi}{4}\left(\frac{1}{n}\right) \ , \quad |\gamma| = 0.0 \ . \tag{A.15}$$

APPENDIX B

DERIVATION OF A SIMPLIFIED EXPRESSION FOR THE EXPECTATION OF THE ESTIMATE OF MAGNITUDE-SQUARED COHERENCE

The major steps in deriving a simplified expression for $E\left(|\hat{\gamma}|^2 \middle| n, |\gamma|^2\right)$ are presented here.

According to Eq. (4.26),

$$E\left(|\hat{\gamma}|^2 \middle| n, |\gamma|^2\right) = \frac{\left(1 - |\gamma|^2\right)^n}{n} \; {}_3F_2\left(2, n, n; n+1, 1; |\gamma|^2\right), \tag{B.1}$$

which can be manipulated into the form

$$E\left(|\hat{\gamma}|^2 \middle| n, |\gamma|^2\right) = \left(1 - |\gamma|^2\right)^n \sum_{k=0}^{\infty} \frac{(n)_k (k+1)}{(n+k)} \frac{|\gamma|^{2k}}{k!} . \tag{B.2}$$

Adding and subtracting n from the numerator term in Eq. (B.2) yields

$$E\left(|\hat{\gamma}|^2 \middle| n, |\gamma|^2\right) = \left(1 - |\gamma|^2\right)^n \left[\sum_{k=0}^{\infty} \frac{(n)_k}{k!} |\gamma|^{2k} + \sum_{k=0}^{\infty} \frac{(1-n)(n)_k}{(k+n)\, k!} |\gamma|^{2k}\right] . \tag{B.3}$$

Recognizing that

$$\frac{1}{k+n} = \frac{(n)_k}{n(n+1)_k} , \tag{B.4}$$

it follows that

$$E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right) = \left(1 - \left|\gamma\right|^2\right)^n \left[\sum_{k=0}^{\infty} \frac{(n)_k (b)_k}{(b)_k} \frac{\left|\gamma\right|^2}{k!} + \frac{(1-n)}{n} \cdot \sum_{k=0}^{\infty} \frac{(n)_k (n)_k}{(n+1)_k} \frac{\left|\gamma\right|^{2k}}{k!}\right] . \tag{B.5}$$

In terms of ${}_2F_1$ functions, Eq. (B.5) becomes

$$E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right) = \left(1 - \left|\gamma\right|^2\right)^n \left[{}_2F_1\left(n, b; b; \left|\gamma\right|^2\right) + \frac{(1-n)}{n} {}_2F_1\left(n, n; n+1; \left|\gamma\right|^2\right)\right] . \tag{B.6}$$

By using Eq. (15.1.8) of Abramowitz and Stegun,[26] Eq. (B.6) reduces to

$$E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right) = \left(1 - \left|\gamma\right|^2\right)^n \left[\left(1 - \left|\gamma\right|^2\right)^{-n} + \frac{1-n}{n} {}_2F_1\left(n, n; n+1; \left|\gamma\right|^2\right)\right] . \tag{B.7}$$

Simplifying and applying Eq. (15.3.3) of Abramowitz and Stegun,[26] Eq. (B.7) can be further reduced to

$$E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right) = 1 + \frac{(1-n)}{n} \left(1 - \left|\gamma\right|^2\right) {}_2F_1\left(1, 1; n+1; \left|\gamma\right|^2\right) . \tag{B.8}$$

Finally, by applying Eq. (15.2.6) of Abramowitz and Stegun,[26] with $a = 1$, $b = 1$, and $c = n + 1$, Eq. (B.8) can be manipulated into the form

$$E\left(\left|\hat{\gamma}\right|^2 \middle| n, \left|\gamma\right|^2\right) = \frac{1}{n} + \frac{n-1}{n+1} \left|\gamma\right|^2 \, {}_2F_1\left(1, 1; n+2; \left|\gamma\right|^2\right). \quad \text{(B.9)}$$

LIST OF REFERENCES

[1]N. R. Goodman, "On the Joint Estimation of the Spectra, Cospectrum, and Quadrature Spectrum of a Two-Dimensional Stationary Gaussian Process," Scientific Paper 10, NYU, New York (March 1957).

[2]J. S. Bendat and A. G. Piersol, Measurement and Analysis of Random Data (New York: John Wiley & Sons, Inc., 1966).

[3]L. J. Tick, Spectral Analysis of Time Series, ed. by B. Harris (New York: John Wiley & Sons, Inc., 1967), 133-152.

[4]R. A. Fisher, Contributions to Mathematical Statistics (New York: John Wiley & Sons, Inc., 1950) (Chapter 14 originally published as "The General Sampling Distribution of the Multiple Correlation Coefficient," Proceedings of the Royal Society, Series A, Vol. 121 (1928), 654-673).

[5]D. E. Amos and L. H. Koopmans, "Tables of the Distribution of the Coefficient of Coherence for Stationary Bivariate Gaussian Processes," Sandia Corp. Monograph SCR-483 (1963).

[6]M. J. Alexander and C. A. Vok, "Tables of the Cumulative Distribution of Sample Multiple Coherence," Rocketdyne Division, North American Aviation, Inc., Research Report 63-67 (November, 1963).

[7]R. A. Haubrich, "Earth Noise, 5 to 500 Millicycles per Second, 1. Spectral Stationarity, Normality, and Nonlinearity, "Journal of Geophysical Research, Vol. 70, No. 6 (March, 1965), 1415-1427.

[8]G. M. Jenkins and D. G. Watts, Spectral Analysis and Its Applications (San Francisco: Holden-Day, Inc., 1968).

[9]L. D. Enochson and N. R. Goodman, "Gaussian Approximations to the Distribution of Sample Coherence," Air Force Flight Dynamics Lab., Research and Tech. Div., AF Systems Command, Wright-Patterson AFB, Ohio, Bull. AFFDL-TR-65-67 (June, 1965).

[10]V. A. Benignus, "Estimation of Coherence Spectrum and Its Confidence Interval Using the Fast Fourier Transform," IEEE Transactions of Audio and Electroacoustics, Vol. AU-17, No. 2, (June, 1969), 145-150.

[11]J. W. Cooley and J. W. Tukey, "An Algorithm for the Machine Calculation of Complex Fourier Series," Mathematics of Computation, Vol. 19 (April, 1965), 297-301.

[12]G. C. Carter and C. R. Arnold, "Some Practical Considerations of Coherence Estimation," NUSC Tech Memo TD 113-19-71, New London, Ct. (November, 1971)

[13]C. Bingham, M. D. Godfrey, and J. W. Tukey, "Modern Techniques of Power Spectrum Estimation," IEEE Transactions on Audio and Electroacoustics, Vol. AU-15, No. 2 (June, 1967), 56-66.

[14]A. H. Nuttall, "Spectral Estimation by Means of Overlapped FFT Processing of Windowed Data," NUSC Report No. 4169, New London, Ct. (October, 1971).

[15]W. B. Davenport and W. L. Root, An Introduction to the Theory of Random Signals and Noise (New York: McGraw-Hill Book Co., Inc., 1958).

[16]J. M. Wozencraft and I. M. Jacobs, Principles of Communication Engineering (New York: John Wiley & Sons, Inc., 1967).

[17]H. L. Van Trees, Detection, Estimation, and Modulation Theory, Part I (New York: John Wiley & Sons, Inc., 1968).

[18]R. C. Dubes, The Theory of Applied Probability (Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1968).

[19]P. R. Roth, "Effective Measurements Using Digital Signal Analysis," IEEE Spectrum, Vol. 8, No. 4 (April, 1971), 62-70.

[20]C. H. Knapp, personal communication.

[21]P. D. Welch, "The Use of Fast Fourier Transform for the Estimation of Power Spectra: A Method Based on Time Averaging Over Short, Modified Periodograms," IEEE Transactions on Audio and Electroacoustics, Vol. AU-15, No. 2 (June, 1967), 70-73.

[22]C. H. Knapp, "An Algorithm for Estimation of the Inverse Spectral Matrix," General Dynamics Electric Boat Research Project (April, 1969).

[23]B. Gold and C. M. Rader, Digital Processing of Signals (New York: McGraw-Hill Book Co., Inc., 1969).

[24] E. S. Hannan, Multiple Time Series (New York: John Wiley & Sons, Inc., 1970).

[25] G. C. Carter and A. H. Nuttall, "Evaluation of the Statistics of the Estimate of Magnitude-Squared Coherence," NUSC Tech Memo TC-193-71, New London, Ct. (28 September 1971).

[26] M. Arbramowitz and I. A. Stegun (Editors), Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables, U. S. Government Printing Office, Washington, D. C. (1964).

[27] H. Cramer, Mathematical Methods of Statistics (Princeton, N. J: Princeton University Press, 1946).

[28] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products (New York: Academic Press, 1965).

[29] T. W. Anderson, An Introduction to Multivariate Statistical Analysis (New York: John Wiley & Sons, Inc., 1958).

[30] A. H. Nuttall and D. H. Wood, personal communication.

[31] A. H. Nuttall and G. C. Carter, "On Generating Processes with Specified Coherence," NUSC Tech Memo TC-187-71, New London, Ct. (22 September 1971).

[32] R. C. Singleton, "An Algorithm for Computing the Mixed Radix Fast Fourier Transform," IEEE Transactions on Audio and Electroacoustics, Vol. AU-17, No. 2, (June, 1969), 93-102.