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DEPARTMENT OF AERONAUTICAL ENGINEERING

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13. ABSTRACT

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SCIENTIFIC REPORT No. 5.

ON THE BUCKLING AND POSTBUCKLING OF CIRCULAR ARCHES AND RINGS.

by

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* On leave from Carnegie-Mellon University, Pittsburgh, Pa., U.S.A.

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ABSTRACT

This paper treats the buckling and postbuckling behavior of circular arches and rings under constant directional pressure. New exact and approximate solutions are given for the linearized eigenvalue problem. It is clearly demonstrated that the assumption of inextensibility is quite reasonable for the asymmetric buckling of steeper arches and of rings. Asymptotic analyses of early postbuckling behavior are given, based on the theory of Koiter and the formalism of Budiansky and Hutchinson. The postbuckling behavior is shown to be stable, and unaffected by the assumption of inextensibility.

NOMENCLATURE

a, b, a', b'	perturbation solution parameters defined by Eqs. (40) and (37)
A, A_i	displacement coefficients, dimensionless
A^*	displacement coefficient, dimensional
$\underline{A}$	displacement coefficient $(= \frac{A^* \varepsilon}{h})$
e, e_i	extensional strain
E	Young's modulus
h	arch thickness
H	thickness parameter
I	moment of inertia
K, K_i	curvature, dimensionless
L_i	differential operators
M, M_i	moment resultant, dimensionless
N, N_i	stress resultant, dimensionless
P^λ, P_m^λ	potential energy functionals
q	load per unit length of arch
Q	extensibility parameter defined by Eq. (16)
R	arch radius
S_i	nonlinear extensional strain
v, v_i	tangential displacement, dimensionless
w, w_i	tangential displacement, dimensionless

α	half angle of arch vertex
δ	first order variation
ε	porturbation parameter
ϕ	arch coordinate
λ	arch loading,dimensionless
χ, χ_i	rotation of normal to arch centerlines, dimensionless

1. INTRODUCTION

This paper treats the buckling and the postbuckling behavior of pinned arches, under constant directional ("dead") pressure, where the arch radius of curvature is a constant. A linearized treatment of this problem has been given by Chwalla and Kollbrunner [1]*, who assumed that the arch was inextensible during buckling. Kämmel [2] has shown that extensibility did not have a great effect on the buckling loads, a result confirmed by the present study. Also, Lévy [3] and Carrier [4] have given "closed form" solutions for the postbuckling of inextensional circular rings in terms of (implicit) elliptical integrals. It is also worth noting the recent contribution of Singer and Babcock [5], who discuss the relationship of semi-circular arches under dead pressure to that of complete rings, pointing out the need to account for rigid body displacements.

The present work treats the linearized buckling problem both exactly, within the framework of ring theory, accounting for extensibility, and approximately, extending Levy's approximation for rings to arches of various vertex angles. A qualitative measure of the midplane extension is also provided from the new exact solution.

Two asymptotic analyses of the early postbuckling behavior are presented, both based on the theory advanced by Koiter [6]. The first makes use of the formalism expounded by Budiansky and Hutchinson [7,8,9], and is carried

* Numbers in brackets refer to entries in the list of References.

out to include terms in the displacement expansion of order ε^2, where ε is the small buckling deflection parameter. It is shown that the post-buckling behavior of the arch following asymmetric buckling is stable, and it is not appreciately effected, qualitatively or quantitatively, by the inextensibility assumption. Further, it is shown that the inclusion of the second order terms in the displacements has no appreciable effect on the quantitative outcome.

Finally an anlysis based directly on Koiter's approach is presented, also accurate in the displacement expansion to order ε^2. The results are identical, for this problem, to those obtained using the Budiansky-Hutchinson formalism, and serve to confirm the idea that (at least for "simple" problems) accurate quantitative information can be obtained using only the buckling mode shapes in the postbuckling analysis. Koiter has previously considered the elastica [6], and a two-bar frame [10] using only the buckling mode, while Haftka and Nachbar [11] have compared results of a "one-term" solution with an exact solution, and the comparison was quite good.

2. GENERAL EQUATIONS

The equations used to describe the arch behavior are those of ring theory, which are also a one-dimensional subset of the nonlinear shell equations of Sander [12]. In dimensionless form they appear as follows. The stress and moment resultants are

$$N = \frac{1}{H}\left(e + \frac{1}{2}\chi^2\right) \tag{1a}$$

$$M = -\frac{d\chi}{d\phi} \tag{1b}$$

where e and χ are the linear in-plane strain and the rotation, respectively, which when expressed in terms of a tangential displacement v and a radial displacement w take the form

$$e = \frac{dv}{d\phi} - w, \qquad \chi = \frac{dw}{d\phi} + v \tag{2}$$

The displacements and the circumferential arch coordinate ϕ have been nondimensionalized with respect to the arch radius R, while the stress and moment resultants have been rendered dimensionless by dividing by (EI/R^2) and (EI/R), respectively. This results in the thickness ratio being introduced in equation (1a), i.e.,

$$H = \frac{1}{12}\left(\frac{h}{R}\right)^2 \tag{3}$$

Further, a dimensionless applied loading λ will be defined as

$$\lambda = \frac{qR^3}{EI} \tag{4}$$

where q is the load (uniform here) per unit length of arch. The geometry is pictured in Figure 1.

Now the (dimensionless) total potential energy of the transversely loaded arch may be written down as

$$P^{\lambda} = \frac{2R}{EI}(U + V)$$

$$= \frac{1}{H}\int_{-\alpha}^{\alpha}(e + \frac{1}{2}\chi^2)^2 d\phi + \int_{-\alpha}^{\alpha}(\frac{d\chi}{d\phi})^2 d\phi - 2\lambda\int_{-\alpha}^{\alpha} w d\phi \qquad (5)$$

In equation (5) the first integral represents the (linear and nonlinear) strain energy of middle surface extension, the second integral is the bending contribution to the energy, while the final term is the potential of the applied load.

From equation (5) one can deduce the potential energy corresponding to the linearized buckling problem (See Eqns. (47)of Section 5), denoted here as P_2^{λ} :

$$P_2^{\lambda} = \frac{1}{H}\int_{-\alpha}^{\alpha} e_1^2 d\phi + \int_{-\alpha}^{\alpha}(\frac{d\chi_1}{d\phi})^2 d\phi - \lambda\int_{-\alpha}^{\alpha}\chi_1^2 d\phi \ . \qquad (6)$$

The subscript one will, throughout this paper, denote a buckling displacement or stress quantity. The Euler-Lagrange equations of the quadratic functional (6) are straightforwardly obtained as

$$L_1 w_1 + L_2 v_1 = 0 \qquad (7a)$$

$$L_3 w_1 + L_1 v_1 = 0 \qquad (7b)$$

where the L_i are the differential operators

$$L_1 = \frac{d^3}{d\phi^3} - (1 - \lambda H) \frac{d}{d\phi}$$

$$L_2 = (1 + H) \frac{d^2}{d\phi^2} + \lambda H \qquad (8)$$

$$L_3 = H \frac{d^4}{d\phi^4} + \lambda H \frac{d^2}{d\phi^2} + 1$$

It is immediately of interest to note that equations (7) can be uncoupled to yield a single equation in v_1, or the identical equation in w_1 alone. Furthermore, that single equation is identically that given by Chwalla and Kollbrunner[1], which they derived by assuming inextensibility in the buckling state, i.e., $e_1 = 0$. That equation is, obtained here with no assumption regarding the extensibility;

$$(L_3L_2 - L_1^2)\, v_1 = v_1^{vi} + (2 + \lambda)v_1^{iv} + (1 + 2\lambda)v_1'' + \lambda v_1 = 0 \qquad (9)$$

The prime superscripts in equation (9) correspond to differentiation with respect to the arch coordinate. The question of inextensibility will be raised again in the sequel.

For completeness the natural (in terms of stress resultants) and geometric boundary conditions for the linearized problem are also noted:

$$\begin{array}{llll} \text{Either} & N_1 - M_1 = 0 & \text{or} & v_1 \text{ is prescribed} \\ \text{Either} & \dfrac{dM_1}{d\phi} - \lambda\chi_1 = 0 & \text{or} & w_1 \text{ is prescribed} \\ \text{Either} & M_1 = 0 & \text{or} & \dfrac{dw_1}{d\phi} \text{ is prescribed} \end{array} \qquad (10)$$

The boundary conditions are applied at $\phi = \pm\alpha$, where 2α is the arch vertex angle (Figure 1). Results for the complete ring are obtained by setting $\alpha = \frac{\pi}{2}$ [5].

3. LINEARIZED (BUCKLING) SOLUTIONS

A solution to the eigenvalue problem represented by the differential equation (9) has been given by Chwalla and Kollbrunner [1] for a simply supported arch, using the boundary conditions (reflecting the inextensibility condition)

$$v_1 = v_1' = v_1''' = 0 \quad \text{at} \quad \phi = \pm\alpha \tag{11}$$

Some of the resulting eigenvalues, corresponding to a solution that is asymmetric with respect to $\phi = 0$, are shown in Table 1.

An approximate solution for a semi-circular arch $(-\pi/2 \leq \phi \leq \pi/2)$ has been given by Lévy [3], and recently it has been generalized by Babcock and Singer [5] to properly account for the vanishing of the displacement v_1 at the arch ends. This approximation, which assumes an incompressible arch, is here generalized for arbitrary vertex angle 2α, to take the form:

$$w_1 = A \sin\frac{\pi\phi}{\alpha}, \quad v_1 = -\frac{\alpha A}{\pi}\left(1 + \cos\frac{\pi\phi}{\alpha}\right) \tag{12}$$

It is easily verified that the solution (12) satisfies the inextensible pin conditions at $\phi = \pm\alpha$. Then the linearized potential energy (6) can be easily converted into a Rayleigh Quotient to obtain (inextensible) approximate critical loads, i.e.,

$$\lambda_{cr} = \frac{\int_{-\alpha}^{\alpha}\left(\frac{d\chi_1}{d\phi}\right)^2 d\phi}{\int_{-\alpha}^{\alpha}\chi_1^2\, d\phi} \tag{13}$$

Substitution of the assumptions (12) into the quotient (13) yields

$$\lambda_{cr} = \frac{\frac{\pi^2}{\alpha^2}\left(\frac{\pi^2}{\alpha^2} - 1\right)^2}{2 + \left(\frac{\pi^2}{\alpha^2} - 1\right)^2} \tag{14}$$

Some numerical values of λ_{cr} are given in Table 1. The agreement is obviously quite good. Also, it is noted here that for the case $\alpha = \pi/2$, i.e., a complete ring, the mode shape represented by Eqs. (12) has been shown to be essentially identical to that obtained by Chwalla and Kollbrunner in their analysis [5].

Finally, in an attempt to assess the effect of the inextensibility assumption on the buckling load (see the work of Kämmel [2]) and to lay the groundwork for the study of the effects of inextensibility on the post-buckling behavior, an exact solution to the differential equations (7) and the boundary conditions (10) is now given. The asymmetric solution to equations (7) is*

$$\begin{aligned} w_1^E &= A_1 \sin\sqrt{\lambda}\,\phi + A_2 \sin\phi + A_3\phi \cos\phi \\ v_1^E &= -\frac{A_1}{\sqrt{\lambda}}\cos\sqrt{\lambda}\,\phi + (QA_3 - A_2)\cos\phi + A_3\phi \sin\phi \end{aligned} \tag{15}$$

where

$$Q = \frac{1 + H(\lambda - 1)}{1 - H(\lambda - 1)} \tag{16}$$

and the A_i are constants, determined to within an arbitrary factor by satisfaction of boundary conditions.

---

* This solution is not valid for $\lambda = 1$. However for $0 \le \phi \le \pi/2$, this is of no concern here, as $\lambda \ge 3$.

Using the results (15), the exact in-plane strain and rotation formulae (for buckling) can be written as:

$$e_1^E = (1 - Q)\ A_3 \sin\phi \tag{17a}$$

$$\chi_1^E = \left(\sqrt{\lambda} - \frac{1}{\sqrt{\lambda}}\right) A_1 \cos\sqrt{\lambda}\,\phi + (1 + Q) A_3 \cos\phi \tag{17b}$$

The eigenvalues (buckling loads) are obtained after satisfying the following boundary conditions:

$$w_1^E = v_1^E = 0, \quad M_1^E = -\frac{d\chi_1^E}{d\phi} = 0 \quad \text{at} \quad \phi = \pm\alpha. \tag{18}$$

The resulting transcendental equation from which the buckling load is calculated is found to be:

$$\sqrt{\lambda}\tan\sqrt{\lambda}\,\alpha = \frac{(1 + Q)\sin^2\alpha}{(1 + 2Q - \lambda Q)\sin\alpha\cos\alpha + \alpha(1 - \lambda)} \tag{19}$$

For the sake of comparison, the eigenvalue equation obtained by Chwalla and Kollbrunner is also given.

$$\sqrt{\lambda}\tan\sqrt{\lambda}\,\alpha = \frac{2\sin^2\alpha}{(3 - \lambda)\sin\alpha\cos\alpha + \alpha(1 - \lambda)} \tag{20}$$

It is seen that Eqs. (19) and (20) are identical when $Q = 1$, which corresponds to inextensional deformation (Eq. (17a)). Numerical values for the buckling pressures given by Eq. (19) are presented in Table 1, and again the agreement is excellent.

It is worth noting at this juncture a few features of the extensional strain e_1^E resulting from the exact solution. First, considering the coefficient on the right hand side of Eq. (17a), and the definition of Q (Eq. (16)), it can be seen that

$$1 - Q = - \frac{2H(\lambda - 1)}{1 - H(\lambda - 1)} \sim H << 1. \tag{21}$$

proportional to the square of the thickness-to-radius of the arch, and is thus quite small.

Secondly, it can be observed from Eq. (17a) that the extensional strain is an odd function of the arch coordinate, i.e.,

$$e_1^E(-\phi) = - e_1^E(\phi) \tag{22}$$

Also, it turns out that there exists another uncoupling of the exact differential equations (7) that yields the result

$$\frac{d^2 e_1^E}{d\phi^2} + e_1^E = 0 \tag{23}$$

Thus, in addition to the odd function represented by the sinusoid already obtained, there is another solution to the equations for which $w_1(\phi)$ and $e_1^E(\phi)$ are even functions of ϕ, the latter being a cosinusoid. This solution is not considered at all in the present work as attention is restricted only to arches steep enough for asymmetric (bifurcation) buckling to take place.

4. POSTBUCKLING: THE BUDIANSKY-HUTCHINSON APPROACH

In this section the early postbuckling behavior of the arch will be obtained, following the asymptotic analysis outlined in the very lucid exposition of Budiansky [9]. As with the buckling analysis, only asymmetric buckling of steeper arches is considered here. For very shallow arches symmetric buckling can take place at lower pressures [13]. However this snap type of buckling cannot be analysed with the Koiter theory. The starting point is the virtual work (or variational) statement of the full nonlinear problem:

$$\int_{-\alpha}^{\alpha} [N\ \delta e + N\chi\ \delta\chi + M\ \delta K - \lambda\delta w]d\phi = 0 \tag{24}$$

In addition to the nomenclature defined in the previous section, the curvature K has been introduced in Eq. (24). Then

$$M = K = -\frac{d\chi}{d\phi} \tag{25}$$

The variational equation (24) must hold for any kinematically consistent set of displacement variations δv and δw, with corresponding strain field variations δe, $\delta\chi$ and δK.

The following expansions are now introduced:

$$\begin{aligned}
v &= \lambda v_o + \varepsilon v_1 + \varepsilon^2 v_2 + \dots \\
w &= \lambda w_o + \varepsilon w_1 + \varepsilon^2 w_2 + \dots \\
e &= \lambda e_o + \varepsilon e_1 + \varepsilon^2 e_2 + \dots \\
\chi &= \lambda \chi_o + \varepsilon \chi_1 + \varepsilon^2 \chi_2 + \dots \\
K &= \qquad\ \ \varepsilon K_1 + \varepsilon^2 K_2 + \dots
\end{aligned}$$

$$N = -N_o + \varepsilon N_1 + \varepsilon^2 N_2 + \ldots .$$
$$M = \varepsilon M_1 + \varepsilon^2 M_2 \qquad (26)$$

Here ε is a small parameter such that when $\varepsilon \to 0$, then $\lambda \to \lambda_{cr}$, the biturcation buckling pressure. The curvature and moment expansions begin with terms of order ε to denote that the pre-buckling state (zeroth order terms in Eqs. (26)) is a membrane state. In fact, as it is easily demonstrated that a nonlinear membrane state cannot exist for this problem, the quantity χ_o shall be henceforth taken as zero, and thus only a linear membrane pre-buckling state is considered.

For the "unvaried" quantities in Eq. (24) the expansions (26) are substituted. After rearrangement by increasing powers of ε, the following new variational equation results:

$$\int_{-\alpha}^{\alpha} [-N_o \delta e - \lambda \delta w] d\phi$$
$$+ \varepsilon \int_{-\alpha}^{\alpha} [N_1 \delta e + M_1 \delta K - N_o \chi_1 \delta\chi] d\phi$$
$$+ \varepsilon^2 \int_{-\alpha}^{\alpha} [N_2 \delta e + M_2 \delta K + (N_1 \chi_1 - N_o \chi_2) \delta\chi] d\phi \qquad (27)$$
$$+ \varepsilon^3 \int_{-\alpha}^{\alpha} [N_3 \delta e + M_3 \delta K + (N_1 \chi_2 + N_2 \chi_1 - N_o \chi_3) \delta\chi] d\phi$$
$$+ \ldots\ldots = 0.$$

The pre-buckling state is defined by the vanishing of the zeroth order variation. After a simple calculation that state is found to be

$$N_o = \lambda, \quad e_o = -H, \quad w_o = H, \quad v_o = 0 \tag{25}$$

With the vanishing of the zeroth order term in Eq. (27), an ϵ may be divided through, and thus the buckling state itself is now found to be, recalling that $\epsilon \to 0$ implies $\lambda \to \lambda_{cr}$:

$$\int_{-\alpha}^{\alpha} [N_1 \delta e + M_1 \delta K - \lambda_{cr} \chi_1 \delta \chi] d\phi = 0 \tag{29}$$

The Euler-Lagrange equations of this variational equation are:

$$\frac{dN_1}{d\phi} + \lambda_{cr} \chi_1 - \frac{dM_1}{d\phi} = 0 \tag{30a}$$

$$N_1 - \lambda_{cr} \frac{d\chi_1}{d\phi} + \frac{d^2 M_1}{d\phi^2} = 0 \tag{30b}$$

Expressed in terms of displacements, Eqs. (30) are identical to the linearized equations (7) given earlier.

It was pointed out earlier that the set of displacement variations constantly alluded to need only be kinematically consistent, and since the buckling displacements most surely are such a set, they can be appropriately inserted into Eq. (29) to yield the Rayleigh Quotient (Budiansky's "energy equation") for the critical load,

$$\lambda_{cr} = \frac{\int_{-\alpha}^{\alpha} (N_1 e_1 + M_1 K_1) d\phi}{\int_{-\alpha}^{\alpha} (\chi_1)^2 d\phi} \tag{31}$$

The same substitution in the more complete variational statement (27) must be equally valid, and noting the quotient (31), it is found that

$$(\lambda_{cr} - \lambda) \int_{-\alpha}^{\alpha} \chi_1^2 \, d\phi$$

$$+ \varepsilon \int_{-\alpha}^{\alpha} [N_2 e_1 + M_2 K_1 + N_1 \chi_1^2 - \lambda \chi_1 \chi_2] d\phi \tag{32}$$

$$+ \varepsilon^2 \int_{-\alpha}^{\alpha} [N_3 e_1 + M_3 K_1 + N_1 \chi_1 \chi_2 + N_2 \chi_1^2 - \lambda \chi_1 \chi_3] d\phi = 0$$

At this point the relationship between $(\lambda - \lambda_{cr})$ appears to be taking shape. However, it appears that equations of the third order will have to be solved in order to obtain the coefficient of ε^2 above. In order to avoid this difficulty, an orthogonality condition is introduced as follows*:

$$\int_{-\alpha}^{\alpha} [N_1 e_m + M_1 K_m] d\phi = 0; \quad m=2,3,4..... \tag{33}$$

It follows from Eqs. (33) and (29), since the m^{th} order mode shapes must be kinematically admissible, that

$$\int_{-\alpha}^{\alpha} \chi_1 \chi_m \, d\phi = 0; \quad m = 2,3,4..... \tag{34}$$

In order to apply the orthogonality condition (33) and its consequence (34) successfully, the following consequences of constitutive linearity and geometric nonlinearity must be noted:

---

* The precise meaning of this orthogonality condition has not been established and is the subject of some discussion; see Reissner [14].

$$N_2e_1 = N_2S_1 = N_1S_2 = N_1(e_2 + \frac{1}{2}X_1^2);$$

$$N_3e_1 = N_3S_1 = N_1S_3 = N_1(e_3 + X_1X_2); \tag{35}$$

$$M_2K_1 = M_1K_2; \quad M_3K_1 = M_1K_3;$$

In Eqs. (35), S_i has been used to denote the total nonlinear strain of order ε^i, which is made up in part of lower order terms due to the geometric nonlinearity. Then notings Eqs. (35), (33) and (34), the variational equation (32) can be put into the very simple form:

$$\lambda = \lambda_{cr} + a'\varepsilon + b'\varepsilon^2 + \dots \tag{36}$$

where

$$a' = \frac{2}{3}\frac{\int_{-\alpha}^{\alpha} N_1X_1^2\, d\phi}{\int_{-\alpha}^{\alpha} X_1^2\, d\phi} \tag{37a}$$

$$b' = \frac{\int_{-\alpha}^{\alpha} [2N_1X_1X_2 + N_2X_1^2]\, d\phi}{\int_{-\alpha}^{\alpha} X_1^2\, d\phi} \tag{37b}$$

Since

$$N_1 = \frac{1}{H}S_1 = \frac{1}{H}e_1 \; ; \quad N_2 = \frac{1}{H}S_2 = \frac{1}{H}(e_2 + \frac{1}{2}X_1^2) \tag{38}$$

the result (36) can be written in its final form:

$$\frac{\lambda}{\lambda_{cr}} = 1 + a\varepsilon + b\varepsilon^2 + \dots \tag{39}$$

where

$$a = \frac{a'}{\lambda_{cr}} = \frac{3}{2H\lambda_{cr}}\frac{\int_{-\alpha}^{\alpha} e_1X_1^2\, d\phi}{\int_{-\alpha}^{\alpha} X_1^2\, d\phi} \tag{40a}$$

$$b = \frac{b'}{\lambda_{cr}} = \frac{1}{H\lambda_{cr}} \frac{\int_{-\alpha}^{\alpha}[2e_1\chi_1\chi_2 + e_2\chi_1^2 + \frac{1}{2}\chi_1^4]d\phi}{\int_{-\alpha}^{\alpha}\chi_1^2\, d\phi} \tag{40b}$$

Recall that according to the theory, if $a \neq 0$, then the structure will be in an unstable state in the postbuckling range, and it will also be imperfection sensitive. If $a = 0$, $b \neq 0$, then the postbuckling stability and the imperfection sensitivity vary as the arithmetic sign of b.

In the present result it appears at first glance that the existence of a stable or unstable equilibrium state in the postbuckling range depends on whether or not inextensibility is assumed. For from equation (40a), if $e_1(\phi)$ is non-zero, then the coefficient a might be. However, recalling Eq. (22), it is clear that the coefficient in question will always vanish for asymmetric buckling! Thus the exclusion (or inclusion) of the arch extensibility can at best only influence the quantitative nature of the post-buckling equilibrium state, and not its qualitative nature.

It has already been shown that for asymmetric buckling the buckling extensibility e_1 is quite small in magnitude, with no appreciable effect on the buckling load. With this in mind, as well as the discussion of the previous paragraph, it seems quite reasonable to stipulate that

$$e_1 = 0 \quad \text{and} \quad e_2 = 0 \tag{41}$$

In this instance the coefficient b reduces to

$$b = \frac{1}{2H\lambda_{cr}} \cdot \frac{\int_{-\alpha}^{\alpha} \chi_1^4\, d\phi}{\int_{-\alpha}^{\alpha} \chi_1^2\, d\phi} \tag{42}$$

Using the excellent approximate solution (12), with its critical load defined by Eq. (14), it is then a straightforward matter to calculate the postbuckling relationship:

$$\lambda = \lambda_{cr} + (12)\left(\frac{\alpha^2}{\pi^2}\right)\frac{\left[1 + 3\left(\frac{\pi^2}{\alpha^2} - 1\right) + \frac{3}{8}\left(\frac{\pi^2}{\alpha^2} - 1\right)^2\right]}{\left[2 + \left(\frac{\pi^2}{\alpha^2} - 1\right)^2\right]}\left(\frac{\varepsilon A^*}{h}\right)^2 \qquad (43)$$

(In writing this result cognizance has been taken of the fact that the coefficient A in Eq. (12) was nondimensionalized with respect to the radius R, i.e. $A = \frac{A^*}{R}$, and of the definition (3)).

It is clear from Eq. (43) that $\lambda \geq \lambda_{cr}$, and thus the postbuckling behavior is always stable, and that the arch (or ring if $\alpha = \frac{\pi}{2}$) will not be sensitive to imperfections in its behavior.

5. POSTBUCKLING: KOITER'S APPROACH

In order to examine the stability of the bifurcation point and of the postbuckling behavior using Koiter's approach directly, the total potential energy is first divided into the prebuckling contributions, and into the terms due to the buckling and prebuckling behavior. If the prebuckling behavior is again linearized, the "expansions" (44) are now introduced, i.e., let

$$\begin{aligned} e &= \lambda e_o + e \\ w &= \lambda w_o + w \\ \chi &= \chi \end{aligned} \tag{44}$$

Substituting these expansions into the energy functional (5) yields

$$\begin{aligned} \frac{2R}{EI}(U+V) &= \frac{1}{H}\int_{-\alpha}^{\alpha}(\lambda^2 e_o^2 - 2\lambda^2 w_o)\,d\phi \\ &+ \frac{1}{H}\int_{-\alpha}^{\alpha}(2\lambda e_o e - 2\lambda H w)\,d\phi \\ &+ \frac{1}{H}\int_{-\alpha}^{\alpha}[e^2 + \lambda e_o \chi^2 + e\chi^2 + \frac{1}{4}\chi^4]\,d\phi \\ &+ \int_{-\alpha}^{\alpha}\left(\frac{d\chi}{d\phi}\right)^2 d\phi \end{aligned} \tag{45}$$

In view of the prebuckling results obtained earlier, the total potential energy due to deformation beyond the prebuckling state can be written as

$$P^{\lambda} = \frac{2R}{EI}(U + V) + 2\alpha H\lambda^2 = P_1^{\lambda} + P_2^{\lambda} + P_3^{\lambda} + P_4^{\lambda} \tag{46}$$

where

$$P_1^{\lambda} = -2\lambda \int_{-\alpha}^{\alpha} (e + w)d\phi \tag{47a}$$

$$P_2^{\lambda} = \frac{1}{H}\int_{-\alpha}^{\alpha} e^2 d\phi - \lambda \int_{-\alpha}^{\alpha} \chi^2 d\phi + \int_{-\alpha}^{\alpha} \left(\frac{d\chi}{d\phi}\right)^2 d\phi \tag{47b}$$

$$P_3^{\lambda} = \frac{1}{H}\int_{-\alpha}^{\alpha} e\chi^2 d\phi \tag{47c}$$

$$P_4^{\lambda} = \frac{1}{4H}\int_{-\alpha}^{\alpha} \chi^4 d\phi \tag{47d}$$

Now, by the definition of $e(\phi)$ it is clear that

$$P_1^{\lambda} = -2\lambda \int_{-\alpha}^{\alpha} \left(\frac{dv}{d\phi} - w + w\right)d\phi = -2\lambda\left[v(\alpha) - v(-\alpha)\right] \tag{48}$$

Hence, for a supported arch, $P_1^{\lambda} \equiv 0$. Thus the potential energy change during and after buckling is simply

$$P^{\lambda} = P_2^{\lambda} + P_3^{\lambda} + P_4^{\lambda} \tag{49}$$

where P_2^{λ}, P_3^{λ} and P_4^{λ} are given by equations (47b, c,d).

Recall that the behavior of interest takes place at loads very close to $\lambda = \lambda_{cr}$, the buckling load. Thus, to a first approximation the energy functionals (47) can be expanded in Taylor series about the point $\lambda = \lambda_{cr}$. As P_3^{λ}, P_4^{λ} do not (for this problem) explicitly depend on λ, they appear only as constants, while

$$P_2^{\lambda} = P_2^{\lambda}\Big|_{\lambda=\lambda_{cr}} + (\lambda - \lambda_{cr})\frac{dP_2^{\lambda}}{d\lambda}\Big|_{\lambda=\lambda_{cr}} + \ldots$$

$$= P_2 + (\lambda - \lambda_{cr})P_2' + \ldots \tag{50}$$

where

$$P_2 = \frac{1}{H}\int_{-\alpha}^{\alpha} e^2 d\phi - \lambda_{cr}\int_{-\alpha}^{\alpha}\chi^2 d\phi + \int_{-\alpha}^{\alpha}\left(\frac{d\chi}{d\phi}\right)^2 d\phi \qquad (51a)$$

$$P_2' = -\int_{-\alpha}^{\alpha}\chi^2 d\phi \qquad (51b)$$

Then the total energy change is

$$P^{\lambda} = P_2 + (\lambda - \lambda_{cr})\,P_2' + P_3 + P_4 \qquad (52)$$

where $P_3 = P_3^{\lambda}$ and $P_4 = P_4^{\lambda}$.

Now the following displacement and strain expansions are introduced:

$$\begin{aligned} w &= \varepsilon w_1 + \varepsilon^2 w_2 + \ldots \\ v &= \varepsilon v_1 + \varepsilon^2 v_2 + \ldots \\ e &= \varepsilon e_1 + \varepsilon^2 e_2 + \ldots \\ \chi &= \varepsilon \chi_1 + \varepsilon^2 \chi_2 + \ldots \end{aligned} \qquad (53)$$

where ε is the same small parameter used previously. Introduction of the expansions (53) into the energy functionals yields, to order ε^4:

$$\begin{aligned} P_2 = \varepsilon^2 &\left\{\frac{1}{H}\int_{-\alpha}^{\alpha} e_1^2 d\phi - \lambda_{cr}\int_{-\alpha}^{\alpha}\chi_1^2 d\phi + \int_{-\alpha}^{\alpha}\left(\frac{d\chi_1}{d\phi}\right)^2 d\phi\right\} \\ &+ 2\varepsilon^3\left\{\frac{1}{H}\int_{-\alpha}^{\alpha} e_1 e_2 d\phi - \lambda_{cr}\int_{-\alpha}^{\alpha}\chi_1\chi_2 d\phi + \int_{-\alpha}^{\alpha}\frac{d\chi_1}{d\phi}\,\frac{d\chi_2}{d\phi}\,d\phi\right\} \\ &+ \varepsilon^4\left\{\frac{1}{H}\int_{-\alpha}^{\alpha}(e_2^2 + 2e_1e_3)d\phi - \lambda v\int_{-\alpha}^{\alpha}(\chi_2^2 + 2\chi_1\chi_3)d\phi\right. \\ &+ \left.\int_{-\alpha}^{\alpha}\left[\left(\frac{d\chi_2}{d\phi}\right)^2 + 2\,\frac{d\chi_1}{d\phi}\,\frac{d\chi_3}{d\phi}\right]d\phi\right\} \end{aligned} \qquad (54a)$$

$$P_2' = -\varepsilon^2 \int_{-\alpha}^{\alpha} \chi_1^2 d\phi - 2\varepsilon^3 \int_{-\alpha}^{\alpha} \chi_1\chi_2 d\phi - \varepsilon^4 \int_{-\alpha}^{\alpha} [\chi_2^2 + 2\chi_1\chi_3] d\phi \tag{54b}$$

$$P_3 = \varepsilon^3 \frac{1}{H} \int_{-\alpha}^{\alpha} e_1\chi_1^2 d\phi + \varepsilon^4 \frac{1}{H} \int_{-\alpha}^{\alpha} [e_2\chi_1^2 + 2e_1\chi_1\chi_2] d\phi \tag{54c}$$

$$P_4 = \varepsilon^4 \frac{1}{4H} \int_{-\alpha}^{\alpha} \chi_1^4 d\phi \tag{54d}$$

Then the same orthogonality conditions applied in the Budiansky-Hutchinson formalism are applied again (written entirely in terms of kinematic quantities) to the functionals (54). These functionals are then greatly simplified and take the form:

$$\begin{aligned} P_2 &= \varepsilon^2 \{ \frac{1}{H} \int_{-\alpha}^{\alpha} e_1^2 d\phi - \lambda_{cr} \int_{-\alpha}^{\alpha} \chi_1^2 d\phi + \int_{-\alpha}^{\alpha} (\frac{d\chi_1}{d\phi})^2 d\phi \} \\ &+ \varepsilon^4 \{ \frac{1}{H} \int_{-\alpha}^{\alpha} e_2^2 d\phi - \lambda_{cr} \int_{-\alpha}^{\alpha} \chi_2^2 d\phi + \int_{-\alpha}^{\alpha} (\frac{d\chi_2}{d\phi})^2 d\phi \}. \end{aligned} \tag{55a}$$

$$P_2' = -\varepsilon^2 \int_{-\alpha}^{\alpha} \chi_1^2 d\phi - \varepsilon^4 \int_{-\alpha}^{\alpha} \chi_2^2 d\phi \tag{55b}$$

$$\begin{aligned} P_3 &= \varepsilon^3 \frac{1}{H} \int_{-\alpha}^{\alpha} e_1\chi_1^2 \, d\phi \\ &+ \varepsilon^4 \frac{1}{H} \int_{-\alpha}^{\alpha} [e_2\chi_1^2 + 2e_1\chi_1\chi_2] d\phi \end{aligned} \tag{55c}$$

$$P_4 = \varepsilon^4 \frac{1}{4H} \int_{-\alpha}^{\alpha} \chi_1^4 \, d\phi \tag{55d}$$

Note that the term of order ε^2 in P_2 is simply the Rayleigh quotient, and thus it vanishes identically. Further, the term of order ε^4 in P_2' will be multiplied by $(\lambda - \lambda_{cr})$ according to equation (52).

Since it is expected that for this problem that $(\lambda - \lambda_{cr})$ will be of order ε^2, the term of order ε^4 in P_2' may be deleted as being of higher order. Thus the final form of the energy change is:

$$P = -\varepsilon^2(\lambda - \lambda_{cr}) \int_{-\alpha}^{\alpha} X_1^2 d\phi + \varepsilon^3 \frac{1}{H} \int_{-\alpha}^{\alpha} e_1 X_1^2\, d\phi$$

$$+ \varepsilon^4 \{\frac{1}{H} \int_{-\alpha}^{\alpha} [e_2 X_1^2 + 2e_1 X_1 X_2 + \frac{1}{4} X_1^4]\, d\phi \tag{56}$$

$$+ \frac{1}{H} \int_{-\alpha}^{\alpha} e_2^2 d\phi - \lambda_{cr} \int_{-\alpha}^{\alpha} X_2^2 d\phi + \int_{-\alpha}^{\alpha} (\frac{dX_2}{d\phi})^2 d\phi \}$$

The resemblance between the present approach and the Budiansky-Hutchinson approach is now more visible. However, it appears that there will be some difference due to the term

$$P_2\, [w_2, v_2] = \frac{1}{H} \int_{-\alpha}^{\alpha} e_2^2 d\phi - \lambda_{cr} \int_{-\alpha}^{\alpha} X_2^2 d\phi + \int_{-\alpha}^{\alpha} (\frac{dX_2}{d\phi})^2 d\phi, \tag{57}$$

in the expression of order ε^4 in Eq. (56). Terms of this type do appear consistently in Koiter's original analysis (see, for example, Section 3? of Ref. [6].), although they do not appear in analyses using the Budiansky-Hutchinson formalism (see, for example, Ref. [9].). For the present work, under the well established assumption of inextensibility, the terms $P_2[w_2, v_2]$ will be shown to yield no contribution to the result, and the results will be identical to those already obtained.

The second order displacements, v_2 and w_2, are obtained as the solution of the variational equation

$$\delta \left\{ \frac{1}{H} \int_{-\alpha}^{\alpha} e_2^2 d\phi - \lambda_{cr} \int_{-\alpha}^{\alpha} \chi_2^2 d\phi + \int_{-\alpha}^{\alpha} \left(\frac{d\chi_2}{d\phi}\right)^2 d\phi \right.$$

$$\left. + \frac{1}{H} \int_{-\alpha}^{\alpha} [e_2 \chi_1^2 + 2e_1 \chi_1 \chi_2 + \frac{1}{4} \chi_1^4] d\phi \right\} = 0 \tag{58}$$

In carrying out this variation all terms of first order are of course considered as known or given quantities. If inextensibility of the arch is assumed, $e_1 = e_2 = 0$, then the variational equation is just

$$\delta\left\{\int_{-\alpha}^{\alpha} \left(\frac{d\chi_2}{d\phi}\right)^2 d\phi - \lambda_{cr} \int_{-\alpha}^{\alpha} \chi_2^2 d\phi\right\} = 0 \tag{59}$$

Thus the variational equation leads to set of homogeneous equations that can have only a trivial solution*. Hence, for the inextensible arch, it is possible to take as solutions for the second order displacements the trivial ones, i.e.,

$$w_2 = v_2 = 0 \tag{60}$$

Hence the energy functional reduces to (recalling $e_1 = 0$ again):

$$P^{\lambda} = -\epsilon^2 (\lambda - \lambda_{cr}) \int_{-\alpha}^{\alpha} \chi_d^2 \, d\phi + \frac{\epsilon^4}{4H} \int_{-\alpha}^{\alpha} \chi_1^4 d\phi \tag{61}$$

Using the approximate solution (12), with $A = A^*/R$, it is a straight-forward matter to compute the energy change P^{λ} as

$$P^{\lambda} = \left(\frac{\epsilon A^*}{R}\right)^2 \frac{\alpha^3}{\pi^2}\left[2 + \left(\frac{\pi^2}{\alpha^2} - 1\right)^2\right](\lambda_{cr} - \lambda) + \left(\frac{\epsilon A^*}{R}\right)^4 \frac{\alpha^5}{2\pi^4 H}\left[\frac{3}{8}\left(\frac{\pi^2}{\alpha^2} - 1\right)^2 + 3\left(\frac{\pi^2}{\alpha^2} - 1\right) + 1\right] \tag{62}$$

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* A solution linearly dependent on the buckling solution is ruled out by the orthogonality condition.

At the critical load, $\lambda = \lambda_{cr}$, it is seen that P^{λ} [illegible] 0, and thus the bifurcation (critical) state is itself stable. The initial postbuckling load-deflection relationship is determined by the condition*

$$\frac{dP^{\lambda}}{d(\varepsilon A^{*})} = 0 \tag{63}$$

from which it follows that

$$\lambda = \lambda_{cr} + (12)\,\frac{\alpha^2}{\pi^2}\;\frac{\left[1 + 3\left(\frac{r}{\alpha} - 1\right) + \frac{3}{8}\left(\frac{r}{\alpha^2} - 1\right)^2\right]}{2 + \left(\frac{r^2}{\alpha^2} - 1\right)^2}\left(\frac{A^{*}\varepsilon}{h}\right)^2 \tag{64}$$

This is exactly the result (43) obtained previously.

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* This is an approximation to $\delta P^{\lambda} = 0$

6. DISCUSSION AND CONCLUSIONS

The present paper has considered the buckling and postbuckling behavior of circular arches (and rings) under constant directional loading. For arches steep enough so that asymmetric bifurcation buckling is critical, it has been demonstrated that the arch inextensibility is a negligible quantity in magnitude, in its effect on the buckling load, and in its effect on initial postbuckling behavior. In addition an approximate solution for the linearized eigenvalue problem was presented that duplicated almost precisely the buckling pressures obtained from the exact solution.

The numerical results for the eigenvalue problem as solved by Chwalla and Kollbrunner and as given by the present approximate and exact solutions have been given in Table 1. It is worth pointing out that the approximate (inextensible) eigenvalues are always greater than or equal to Chwalla and Kollbrunner's (inextensible) results, in accord with Rayleigh's Principle. The results when the extensibility is accounted for are always <u>less than</u> the inextensible results. This is in direct contrast to the analysis of Kämmel [2] which indicated that the buckling pressures would increase if extensibility were accounted for*. As it has already been pointed out, the

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* Singer and Babcock [5] have already pointed out that Kämmel's result was also erroneous in arriving at the coefficient $\lambda_{cr} = 4$ for constant directional pressure, assuming inextensible deformations.

differential equation solved by Chwalla and Kollbrunner, assuming inextensibility, is identical to that solved here, without that assumption. The difference is in the boundary conditions, where the inextensibility assumption provides less freedom (greater stiffness). Hence it is quite proper that the inextensible eigenvalues be upper bounds to the (freer) extensible eigenvalues.

It is also worth pointing out that the results given in Table 1 for the Chwalla-Kollbrunner eigenvalues are not precisely those given in the original paper [1]. Rather they were calculated (using a digital computer) directly from the transcendental equation (20). There are some discrepancies, although they are very small. For example:

	Chwalla and Kollbrunner [1]	Chwalla and Kollbrunner Eq. (20) - Digital Computer
$\alpha = \frac{\pi}{3}$	8.725	8.727
$\alpha = \frac{\pi}{2}$	3.265	3.271

It is easily verified that the original Chwalla and Kollbrunner results satisfy Eq. (20) to within the precision of limited-accuracy trigonometric tables and slide rule accuracy that must have governed calculations in the early 1930's.

The postbuckling behavior for the constant-directionally arches and rings has been shown to be quite stable, and thus imperfection sensitivity is not an issue here. Also it has been shown, under the reasonable assumption of inextensible buckling and postbuckling deformation, that the second order displacement solution does not change the results that would be obtained if only first order (buckling) displacements were used in the calculations.

Finally it has been shown that, for the special circumstances of this problem, the approach of Koiter as originally given and the variation proposed by Budiansky and Hutchinson lead to identical results.

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TABLE 1 : ASYMMETRIC BUCKLING PRESSURES FOR CIRCULAR ARCHES; $\lambda = qR^3/EI$.

	Solution of Chwalla and Kollbrunner Eq. (20)	Present Approximation Eq. (14)	Present Exact Result Eq. (19) (h/R = 1/10)
$\frac{\pi}{10}$	99.979598	99.979598	99.979541
$\frac{\pi}{8}$	63.967766	63.967766	63.967676
$\frac{\pi}{6}$	35.941318	35.941320	35.941151
$\frac{\pi}{4}$	15.859006	15.859031	15.858590
$\frac{\pi}{2}$	3.271245	3.272727	3.269233

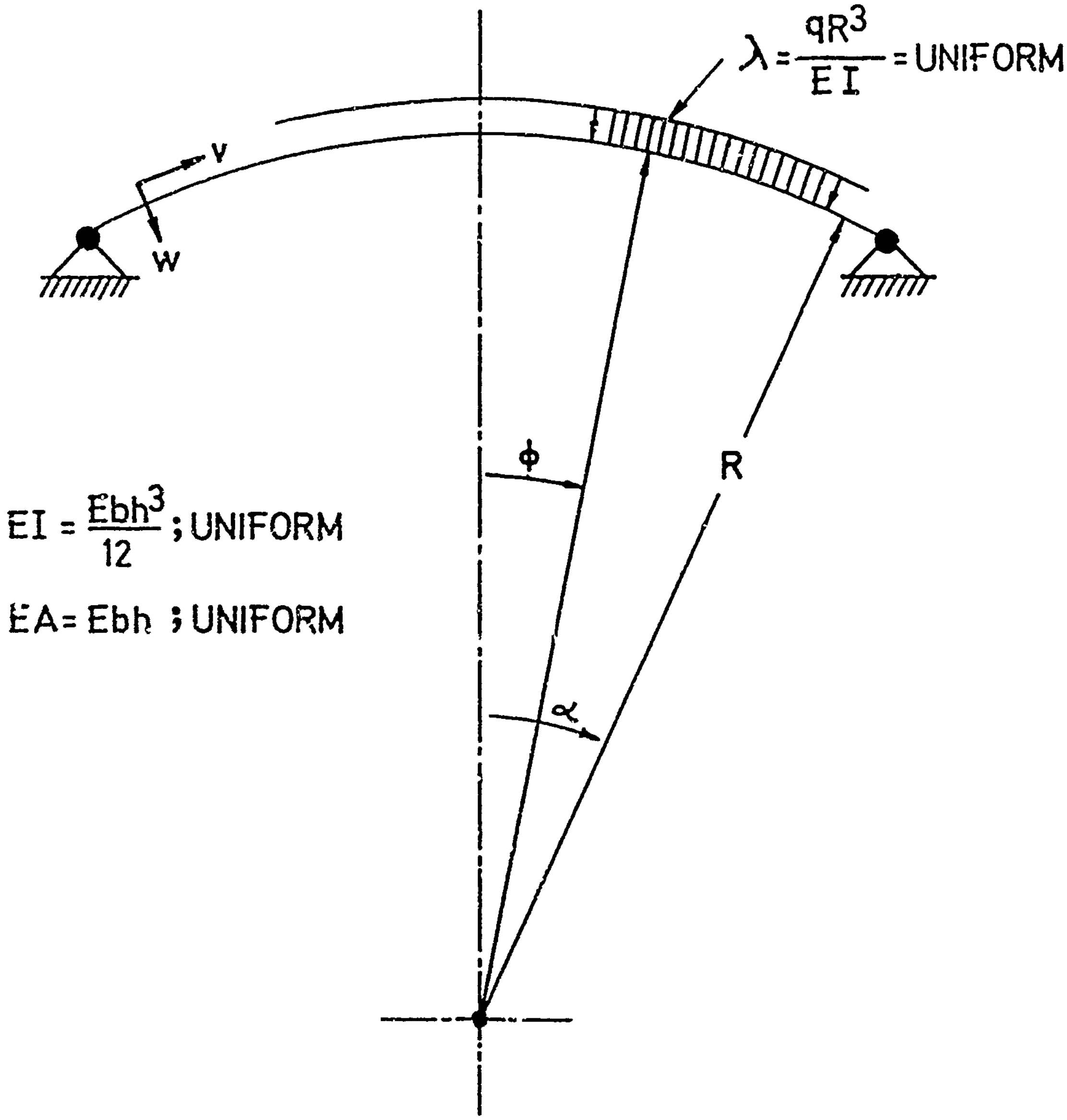

FIGURE. 1 ARCH GEOMETRY AND LOADING