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OPTIMAL PARAMETERS OF A SATELLITE GRAVITATIONAL STABILIZATION SYSTEM WITH GYRO-DAMPING

by

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SUMMARY

This paper investigates the transient attitude performance of a gravity gradient satellite. Damping is provided by a pair of two-degree-of-freedom gyroscopes. First the dynamical equations of motion are presented and used to obtain the characteristic equation. The parameters which optimize the spacecraft's transient attitude performance are determined from the solution of the characteristic equation. The results are then specialized to the case of an axisymmetric satellite and to the case of an external disturbance due to orbital eccentricity.

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13. ABSTRACT

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OPTIMAL PARAMETERS OF A SATELLITE
GRAVITATIONAL STABILIZATION SYSTEM WITH GYRO-DAMPING*)

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V. A. Sarychev and K. V. Lukanin

Contents

Introduction

The dynamics of a satellite gravity-gradient stabilization system with gyro-damping are examined in this paper. The gyro-damper consists of a pair of two-degree-of-freedom gyroscopes; the axes of rotation of these gyroscopes in the satellite equilibrium position are oriented symmetrically relative to the normal to the orbital plane. The natural satellite oscillations bring about precession of the gyroscope rotors coupled to the damping device, leading to dissipation of the energy of the system.

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References [1-8] have been devoted to the study of such systems. Insofar as the range of problems examined, the present paper is closest to [1]. Thus, with respect to determination of the maximum speed of response, the results of the present paper are in a particular case a refinement of the results of [1].

Greatest attention is paid in this paper to determination of the parameters with which maximum system response time is achieved. The cases of an axisymmetric satellite and of a satellite with arbitrary distribution of mass are investigated. In the study the values of the optimal parameters were found by the random search method with subsequent refinement, based on a hypothesis regarding the nature of the location of the roots of the characteristic system equation at the point of extremum. In the neighborhood of the optimal parameters, level lines of the degree of system stability are constructed as a function of two parameters, the remaining parameters fixed. The dependence of the degree of system stability and of the optimal parameters on the amplitude of the satellite eccentricity oscillations is constructed for an axisymmetric satellite.

1. Equation of Motion

The configuration of a satellite gravitational stabilization system with gyro-damping is described in detail in [1-3]. The satellite body contains two two-degree-of-freedom gyroscopes, which are used to provide for damping of the natural oscillations of the system and to obtain additional restoring moments.

Let us define the right rectangular coordinate systems necessary for the solution of the problem.

OXYZ is the orbital coordinate system. The OZ-axis is directed along the radius-vector connecting the centers of mass of the earth and satellite-gyroscope system. The OX-axis coincides with the transversal to the plane of orbit, the OY-axis with the normal.

Oxyz is a coordinate system linked to the satellite body. The Ox-, Oy-, Oz-axes are the principal axes of inertia of the satellite-gyroscope system. O is the center of mass of the system.

O_1xyz and O_2xyz are coordinate systems whose axes are parallel to the axes $Oxyz$; O_1 and O_2 are the centers of mass of the gyroscopes.

The axes of the gyroscope gimbal suspensions, a′b′ and a″b″ (Fig. 1), are parallel to each other, are located in the planes xO_1z and xO_2z, resp., and form the angle λ_0 with the axes O_1z and O_2z. The axes of rotation of the gyroscopes in the equilibrium position in circular orbit form the angle δ_0 with the axes O_1y and O_2y.

The location of the satellite-linked coordinate system relative to the orbital coordinate system is defined by means of the angles α (pitch), β (yaw) and γ (roll) (Fig. 2). The elements of the direction cosine table

	x	y	z
X	a_{11}	a_{12}	a_{13}
Y	a_{21}	a_{22}	a_{23}
Z	a_{31}	a_{32}	a_{33}

are expressed in terms of these angles in the following manner:

$$
\begin{aligned}
a_{11} &= \cos\alpha\cos\beta,\\
a_{12} &= \sin\alpha\sin\gamma - \cos\alpha\sin\beta\cos\gamma,\\
a_{13} &= \sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma,\\
a_{21} &= \sin\beta\\
a_{22} &= \cos\beta\cos\gamma,\\
a_{23} &= -\cos\beta\sin\gamma.\\
a_{31} &= -\sin\alpha\cos\beta.\\
a_{32} &= \cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma,\\
a_{33} &= \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma.
\end{aligned}
\qquad (1)
$$

In [2] the equations of motion of the satellite-gyroscope system were derived in an exact formulation without resorting to the precession theory. Assuming that the characteristic rate of gyroscope rotation is high and, consequently, that

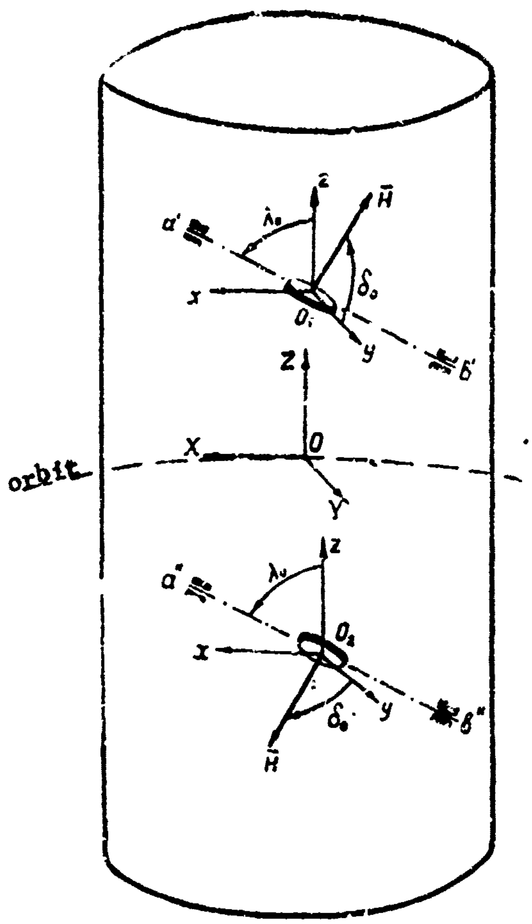

Fig. 1. Diagram of the satellite-gyroscope system.

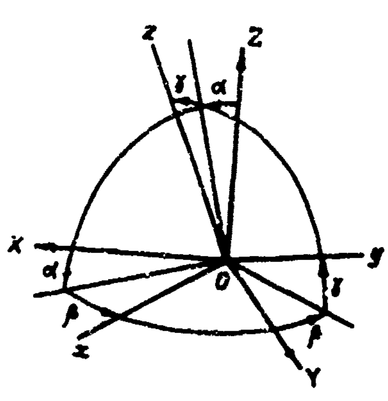

Fig. 2. Relationship between the trihedron Oxyz and the orbital coordinate system.

the moments of inertia of the gyroscopes for a finite value of the kinetic moments of the gyroscopes are small compared to the moments of inertia of the satellite, the moments of inertia of the gyroscopes can be neglected in the equations of motion. Such a simplification is equivalent to resorting to the precession theory of gyroscopes. The equations of motion will be written below using this simplification.

We introduce the following notation: A, B, C are the principal central moments of inertia of the satellite about the Ox-, Oy-, Oz-axes; $\bar{H}$ is the kinetic moment of each gyroscope; $\bar{K}_1$ is the damping coefficient of the gyrodamper; and $\bar{K}_2$ is the spring coefficient of the gyrodamper. In writing the equations of motion it is convenient to change over to the dimensionless parameters

$$\theta_A = \frac{A}{B}, \quad \theta_C = \frac{C}{B}, \quad H = \frac{\bar{H}}{B\omega_0}, \quad \kappa_1 = \frac{\bar{\kappa}_1}{B\omega_0}, \quad \kappa_2 = \frac{\bar{\kappa}_2}{B\omega_0^2}$$

and the dimensionless time $\tau = \omega_0 t$. Here $\omega_0 = \sqrt{\bar{\mu}/\bar{p}^3}$, where $\bar{\mu}$ is the gravitational constant and $\bar{p}$ is the orbit semi-major axis.

Then the equations of motion of a satellite with gyroscopes has the form [2]

$$
\begin{aligned}
\theta_A \dot{p} &= H(\beta'_{12}\dot{\delta}_1 + \beta''_{12}\dot{\delta}_2) + r[q + H(\beta'_{23} + \beta''_{23})] - \\
&\quad - q[\theta_C r + H(\beta'_{33} + \beta''_{33})] - 3(1+e\cos v)^3(1-\theta_C)\alpha_{32}\alpha_{33}, \\
\dot{q} &= H(\beta'_{22}\dot{\delta}_1 + \beta''_{22}\dot{\delta}_2) + p[\theta_C r + H(\beta'_{33} + \beta''_{33})] - \\
&\quad - r[\theta_A p + H(\beta'_{13} + \beta''_{13})] + 3(1+e\cos v)^3(\theta_A - \theta_C)\alpha_{33}\alpha_{31}, \\
\theta_C \dot{r} &= H(\beta'_{32}\dot{\delta}_1 + \beta''_{32}\dot{\delta}_2) + q[\theta_A p + H(\beta'_{13} + \beta''_{13})] - \\
&\quad - p[q + H(\beta'_{23} + \beta''_{23})] + 3(1+e\cos v)^3(1-\theta_A)\alpha_{31}\alpha_{32}. \\
K_1 \dot{\delta}_1 &= -K_2\delta_1 + H\sin\delta_0 - H(p\beta'_{12} + q\beta'_{22} + r\beta'_{32}), \\
K_1 \dot{\delta}_2 &= -K_2\delta_2 - H\sin\delta_0 - H(p\beta''_{12} + q\beta''_{22} + r\beta''_{32}).
\end{aligned}
\tag{2}
$$

Here the dot denotes differentiation with respect to dimensionless time τ,

$$
\begin{aligned}
&\beta'_{12} = \cos\lambda_0 \cos(\delta_0 + \delta_1), && \beta''_{12} = \cos\lambda_0 \cos(-\delta_0 + \delta_2), \\
&\beta'_{13} = -\cos\lambda_0 \sin(\delta_0 + \delta_1), && \beta''_{13} = -\cos\lambda_0 \sin(-\delta_0 + \delta_2) \\
&\beta'_{22} = \sin(\delta_0 + \delta_1), && \beta''_{22} = \sin(-\delta_0 + \delta_2). \\
&\beta'_{23} = \cos(\delta_0 + \delta_1), && \beta''_{23} = \cos(-\delta_0 + \delta_2). \\
&\beta'_{32} = -\sin\lambda_0 \cos(\delta_0 + \delta_1), && \beta''_{32} = -\sin\lambda_0 \cos(-\delta_0 + \delta_2). \\
&\beta'_{33} = \sin\lambda_0 \sin(\delta_0 + \delta_1), && \beta''_{33} = \sin\lambda_0 \sin(-\delta_0 + \delta_1),
\end{aligned}
\tag{3}
$$

p, q, r are the projections of the absolute angular velocity of the satellite onto the axes of the coordinate system Oxyz, δ_1 and δ_2 are the angles of deviation of the gyroscope rotational axes relative to their equilibrium positions in a circular orbit, v is the true anomaly and e is the orbital eccentricity.

The relationship between the projections of the absolute angular velocity of the satellite (p,q,r) and the angles α, β, γ is given by the formulas [2]

$$p = \dot{\alpha} a_{21} + \dot{\gamma} + \dot{\nu} a_{21} \, ,$$
$$q = \dot{\alpha} a_{12} + \dot{\beta} \sin\gamma + \dot{\nu} a_{22} \, ,$$
$$r = \dot{\alpha} a_{21} + \dot{\beta} \cos\gamma + \dot{\nu} a_{23} \, . \qquad (4)$$

where

$$\dot{\nu} = (1 + e\cos\nu)^2 \, , \qquad (5)$$

In the circular-orbit case the system of Eqs.(2)-(4) has the solution $\alpha = \beta = \gamma = \delta_1 = \delta_2 = p = r = 0$, $q = 1$, which corresponds to a satellite which has no motion with respect to the orbital coordinate system. The system equations of motion in a circular orbit, linearized in the neighborhood of this equilibrium position, are described in the following manner, after introducing the notation $H_1 = H \sin \delta_0$, $H_2 = H \cos \delta_0$, $K = K_0 + H_2$:

$$\ddot{\alpha} + 3(\theta_A - \theta_C)\alpha - H_1(\dot{\delta}_1 - \dot{\delta}_2) = 0 \, ,$$
$$2H_1\dot{\alpha} + K_1(\dot{\delta}_1 - \dot{\delta}_2) + K(\delta_1 - \delta_2) = 0 \, , \qquad (6)$$

$$\theta_C \ddot{\beta} + [2H_2 - (\theta_A + \theta_C - 1)]\dot{\gamma} + [2H_2 + (1 - \theta_A)]\beta +$$
$$+ H_2(\dot{\delta}_1 + \dot{\delta}_2)\sin\lambda_0 + H_2(\delta_1 + \delta_2)\cos\lambda_0 = 0 \, ,$$

$$\theta_A \ddot{\gamma} - [2H_2 - (\theta_A + \theta_C - 1)]\dot{\beta} + [2H_2 + 4(1 - \theta_C)]\gamma -$$
$$- H_2(\dot{\delta}_1 + \dot{\delta}_2)\cos\lambda_0 - H_2(\delta_1 + \delta_2)\sin\lambda_0 = 0 \, ,$$
$$- 2H_2\dot{\beta}\sin\lambda_0 + 2H_2\beta\cos\lambda_0 + 2H_2\dot{\gamma}\cos\lambda_0 +$$
$$+ 2H_2\gamma\sin\lambda_0 + K_1(\dot{\delta}_1 + \dot{\delta}_2) + K(\delta_1 + \delta_2) = 0 \, . \qquad (7)$$

2. Method of Determining the Optimal Parameters

One of the criteria for the quality of stabilization systems is the time required for transient processes to take place in them. In the present paper it was assumed that the transient response time is defined by the variable ξ, which is the distance between the rightmost root of the characteristic equation of the system

(6)-(7) and the imaginary axis. The characteristic equation of system (6)-(7) is written in the form of the product of the third- and fifth-degree equations

$$\lambda^3+\left(\frac{K}{K_1}+2\frac{H_1^2}{K_1}\right)\lambda^2+3(\theta_A-\theta_C)\lambda+3(\theta_A-\theta_C)\frac{K}{K_1}=0; \tag{8}$$

$$\lambda^5+\left(\frac{K}{K_1}+a_0\frac{2H_2^2}{K_1}\right)\lambda^4+a_1\lambda^3+\left(a_1\frac{K}{K_1}+a_2\frac{2H_2^2}{K_1}\right)\lambda^2+$$
$$+a_3\lambda+\left(a_3\frac{K}{K_1}-a_4\frac{2H_2^2}{K_1}\right)=0. \tag{9}$$

Here

$$a_0=\frac{\sin^2\lambda_0}{\theta_C}+\frac{\cos^2\lambda_0}{\theta_A},$$

$$a_1=\frac{\theta_A[2H_2+(1-\theta_A)]+\theta_C[2H_2-4(1-\theta_C)]+[2H_2-(\theta_A+\theta_C-1)]^2}{\theta_A\theta_C},$$

$$a_2=\frac{2(\theta_A+\theta_C-1)-2H_2-(4-5\theta_C)\sin^2\lambda_0+(1-2\theta_A)\cos^2\lambda_0}{\theta_A\theta_C}.$$

$$a_3=\frac{[2H_2+(1-\theta_A)][2H_2+4(1-\theta_C)]}{\theta_A\theta_C}.$$

$$a_4=\frac{2H_2+(1-\theta_A)\sin^2\lambda_0+4(1-\theta_C)\cos^2\lambda_0}{\theta_A\theta_C}. \tag{10}$$

The variable ξ depends on the seven dimensionless parameters θ_A, θ_C, H, K_1, K_2, δ_0, λ_0. The problem consists in determining the values of these parameters at which ξ reaches maximum value ξ_m. These parameter values will be said to be optimal. the present paper we are studying not only a satellite with arbitrary moments of inertia, but also the case of an axisymmetric satellite ($\theta_A = 1$).

The investigation was carried out in two steps. The first step consisted in determining ξ_m and the corresponding optimal parameters by the random search method. In the second step, the parameters we found were determined more exactly on the basis of a hypothesis as to the form of the roots of the characteristic equation at the point of extremum.

The essence of the random search method is as follows. We begin with a point p_0 in the parameter space of the system. From this point we take the random step Δp_1. If $\xi(p_0 + \Delta p_1) > \xi(p_0)$, then the point $p_0 + \Delta p_1$ is taken as the initial point for the next random step. If $\xi(p_0 + \Delta p_1) \leq \xi(p_0)$, then we take the random step Δp_2 from the point p_0. The point p_0 was considered to be the extremal point ξ if the variable ξ did not increase in m random steps from this point.

The value of ξ was calculated with the prescribed accuracy by the method of successive approximation. The zero approximation, the interval containing the rightmost root of Eqs. (8)-(9), was determined by the formula proposed in [1]. The subsequent refinements were carried out by the Routh scheme. Routh's scheme is constructed for equations obtained from (8)-(9) by right-displacement of all of their roots by a value equal to the distance between the centerpoint of the interval containing ξ and the imaginary axis. A detailed description of the process of calculating ξ is given in [1].

In the process of determining ξ_m by the random search method, it was discovered that if the values of ξ_m obtained at the end of optimization from various initial points differ by several percent, the scatter over the parameters themselves turns out to be considerably greater.

The difficulties involved in precise determination of the optimal parameters are due primarily to the great size of the parameter space and to the high degree of the characteristic equation. Moreover, as the investigation showed, a not insignificant part is played by the nature of the extremum itself, near which the derivatives of ξ with respect to several directions in parameter space differ substantially in value. This is explained by the fact that the real parts of the roots of the characteristic equation become equal near the extremum; as a consequence, the function ξ turns out to be non-analytic at the extremal point.

3. Optimal Parameters of a Satellite with Arbitrary Moments of Inertia

The satellite parameters were optimized by the random search method for fixed values of the angle λ_0. As mentioned above, the values of ξ_m obtained at the end of optimization from various initial points differ by several percent, while the scatter over the parameters turns out to be much greater. Given in the table are the maximum values obtained for ξ_m for each of the fixed values of λ_0, as well as the corresponding roots of characteristic equations (8)-(9).

λ_0	ξ_m	y_1	$y_{2,3}$	x_1	$x_{2,3}$	$x_{4,5}$
$0°$	[illegible]	[illegible]	-0.280 $\pm 1.208i$	-0.310	-0.331 $\pm 1.581i$	-0.2[illegible] $\pm$[illegible]
$30°$	[illegible]	[illegible]	[illegible]	-0.332	-0.327 ± 1.46[illegible]	-0.3[illegible] [illegible]
[illegible]	[illegible]	[illegible]	-0.404 $\pm 0.839i$	[illegible]	-0.452 [illegible]	-0.444 [illegible]
90	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]

In this table y_1, y_2, y_3 are the roots of Eq. (8), x_1, x_2, x_3, x_4, x_5 are the roots of Eq. (9).

From the table it can be seen that the real parts of the roots y_2, y_3, x_2, x_3, x_4, x_5 become approximately equal in the neighborhood of the extremum ξ, and with $\lambda_0 = 0°$, $30°$, and $60°$ the root x_1 also becomes equal to the real parts of these roots. The investigation was begun with the most complex case $\lambda_0 = 90°$.

Let us advance the following hypothesis as to the nature of the roots at the point of the real extremum ξ with $\lambda_0 = 90°$ the roots of the characteristic equations (8)-(9) have the form

$$y_1 = -d_1, \quad y_{2,3} = -d \pm i\omega_1, \quad x_1 = -d_2 \quad x_{2,3} = -d \pm i\omega_2$$
$$x_{4,5} = -d \pm i\omega_3, \quad d_1 \neq d, \quad d_2 \neq d. \tag{11}$$

The variables d, d_1, d_2, ω_1, ω_2, ω_3 are subject to determination. Naturally, of greatest interest is the parameter d, since, by construction, $d = \xi_m$.

The characteristic equations with roots (11) have the form

$$\lambda^3+(2d+d_1)\lambda^2+[(d^2+\omega_1^2)+2dd_1]\lambda+d_1(d^2+d_1^2)=0. \tag{12}$$

$$\begin{aligned}&\lambda^5+(4d+d_2)\lambda^4+[(\omega_2^2+\omega_3^2)+4dd_2+6d^2]\lambda^3+\\&+[(2d+d_2)(\omega_2^2+\omega_3^2)+6d^2d_2+4d^3]\lambda^2+[d(d+2d_2)(\omega_2^2+\omega_3^2)+\\&+\omega_2^2\omega_3^2+4d^3d_2+d^4]\lambda+d_2[d^2(\omega_2^2+\omega_3^2)+\omega_2^2\omega_3^2+d^4]=0.\end{aligned} \tag{13}$$

Let us impose constraints on the parameters of Eqs. (8)-(9) so that the roots of these equations will always have the form of (11). To do so, we equate the coefficients of Eqs. (8)-(9) to the corresponding coefficients of Eqs. (12)-(13), and we also rewrite the last two inequalities of (11). Then,

$$\begin{aligned}&\frac{K}{K_1}+\frac{2H_1^2}{K_1}=2d+d_1,\\&3(\theta_A-\theta_C)=d^2+\omega_1^2+2dd_1,\\&\frac{K}{K_1}3(\theta_A-\theta_C)=d_1(d^2+\omega_1^2),\\&\frac{K}{K_1}+a_0\frac{2H_2^2}{K_1}=4d+d_2,\\&a_1=(\omega_2^2+\omega_3^2)+4dd_2+6d^2,\\&a_1\frac{K}{K_1}+a_2\frac{2H_2^2}{K_1}=(2d+d_2)(\omega_2^2+\omega_3^2)+6d^2d_2+4d^3,\\&a_3=d(d+2d_2)(\omega_2^2+\omega_3^2)+\omega_2^2\omega_3^2+4d^3d_2+d^4,\\&a_3\frac{K}{K_1}-a_4\frac{2H_2^2}{K_1}=d_2[d^2(\omega_2^2+\omega_3^2)+\omega_2^2\omega_3^2+d^4],\\&d_1>d,\quad d_2>d.\end{aligned} \tag{14}$$

From (14) we get the following expressions:

$$\begin{aligned}&\omega_1^2=3(\theta_A-\theta_C)-d^2-2dd_1,\\&\frac{K}{K_1}=\frac{d_1(d^2+\omega_1^2)}{3(\theta_A-\theta_C)},\\&\frac{2H_1^2}{K_1}=2d+d_1-\frac{K}{K_1},\\&\frac{2H_2^2}{K_1}=\frac{1}{a_0}\left(4d+d_2-\frac{K}{K_1}\right)\\&\omega_2^2+\omega_3^2=a_1-4dd_2-6d^2,\\&\omega_2^2\omega_3^2=a_3-d(d+2d_2)(\omega_2^2+\omega_3^2)-4d^3d_2-d^4\\&d_1>d,\quad d_2>d;\end{aligned} \tag{15}$$

$$d_1=\frac{\frac{3}{2}(\theta_A-\theta_C)\pm\sqrt{D}}{2d}; \tag{16}$$

$$\left(a_1 - \frac{a_3}{a_0}\right) z_1 = \left(a_3 + \frac{a_2}{a_0}\right) z_2 . \tag{17}$$

Here,

$$z_1 = (4d + d_2)\frac{a_2}{a_0} + d_2\left[d^2(\omega_2^2 + \omega_3^2) + \omega_2^2\omega_3^2 + d^4\right],$$

$$z_2 = -(4d + d_2)\frac{a_3}{a_0} + (2d + d_2)(\omega_2^2 + \omega_3^2) + 6d^2 d_2 + 4d^3,$$

$$D = \frac{9}{4}(\theta_A - \theta_C)^2 - 2d\,\frac{3(\theta_A - \theta_C) z_1}{a_3 + \frac{a_2}{a_0}} . \tag{18}$$

If relations (15)-(18) hold, the roots of the equations being investigated, (8)-(9), can be represented in the form of (11).

By imposing constraints (15)-(18), a set containing, by hypothesis, the optimal parameters has been singled out in the space of the parameters θ_A, θ_C, H, K_1, K_2, δ_0. These optimal parameters correspond to the maximum value of d at which all the variables in (15)-(18) satisfy the physical meaning. Therefore, in addition to the purely formal constraints specified by (15)-(18), it is also necessary to require fulfillment of the conditions $\omega_1^2 \geq 0$, $\omega_2^2 \geq 0$, $\omega_3^2 \geq 0$, $D \geq 0$, $H_1^2 \geq 0$, $H_2^2 \geq 0$.

It is also necessary to remember that the sum of two moments of inertia is always greater than the third, and therefore $\theta_A + \theta_C \geq 1$. Since $K_1 > 0$ in the case of asymptotic stability, the conditions $H_1^2 \geq 0$, $H_2^2 \geq 0$ reduce to the requirement that the quantities $\frac{2H_1^2}{K_1}$, $\frac{2H_2^2}{K_1}$ be non-negative. Moreover, $\omega_2^2 \geq 0$, $\omega_3^2 \geq 0$ and are real. Consequently,

$$\omega_2^2 + \omega_3^2 \geq 0, \quad \omega_2^2\omega_3^2 \geq 0, \quad (\omega_2^2 + \omega_3^2)^2 - 4\omega_2^2\omega_3^2 \geq 0.$$

Thus, in order for the solution of the characteristic equations to exist in form (11), it is necessary to require fulfillment not only of relations (15)-(18), but also of the inequalities

$$\omega_1^2 \geq 0, \quad \frac{2H_1^2}{K_1} \geq 0, \quad \frac{2H_2^2}{K_1} \geq 0, \quad D \geq 0, \quad \omega_2^2 + \omega_3^2 \geq 0,$$
$$\omega_2^2\omega_3^2 \geq 0, \quad (\omega_2^2 + \omega_3^2)^2 - 4\omega_2^2\omega_3^2 \geq 0, \quad \theta_A + \theta_C \geq 1. \tag{19}$$

At a fixed value of the angle λ_0, expressions (10) for a_0, a_1, a_2, a_3, a_4 depend on the variables θ_A, θ_C, H_2; r_1 and r_2 in (18) depend on the variables θ_A, θ_C, H_2, d_2, d. Therefore, relation (17) can be used to define d_2 in terms of θ_A, θ_C, H_2, d. Considering, moreover, that d_1 is calculated by formula (16), we get that it is possible to choose four parameters arbitrarily in relations (15)-(18): θ_A, θ_C, H_2, d. These parameters must be chosen such that, by satisfying inequalities (19), we get the maximum value of d, equal in this case to δ_m.

The domains in which inequalities (19) and the inequalities $d_1 \geq d$, $d_2 \geq d$ are fulfilled were constructed in the plane of the parameters θ_A, θ_C (Fig. 3) for fixed values of H_2. The variable d increased steadily until the corresponding domain contracted to a point. Plotted in Fig. 3 are the boundaries of the domains for d = 0.5, 0.52, 0.525. As can be seen from the figure, the domains of existence of a solution in form (11) for any H_2 are determined by the curves $(\mu_2^2 + \mu_3^2)^2 - 4\mu_2^2\mu_3^2 = 0$ (curve I), D > 0 (curve II), $d_2 = d$ (curve III) and the straight line

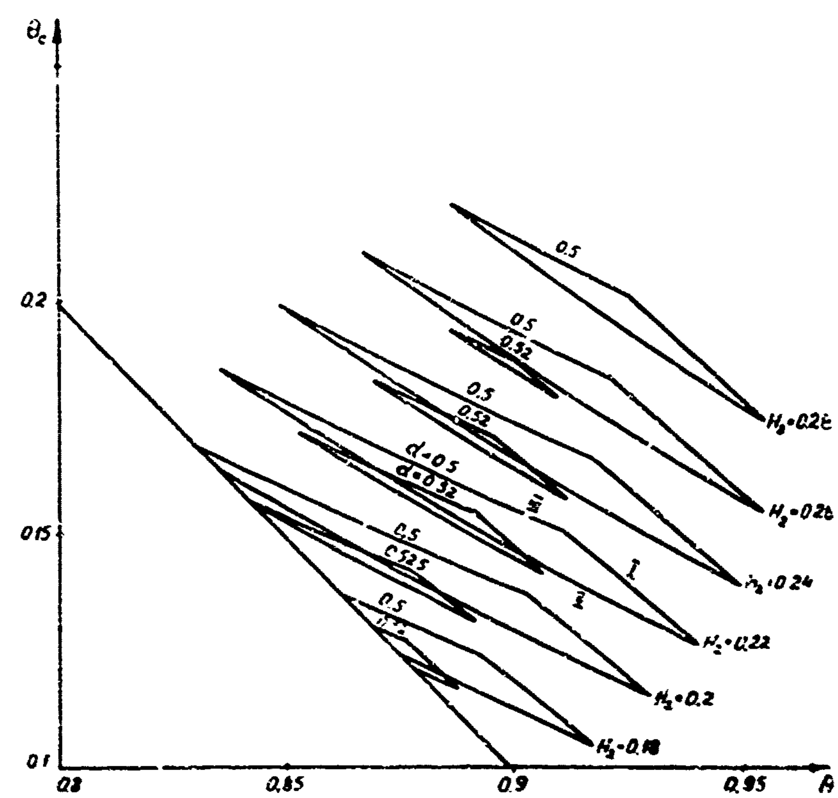

Fig. 3. Domains in which the inequalities are fulfilled for fixed values of H_2.

$\theta_A + \theta_C = 1$. The sign in front of the root in formula (16) for d_1 has no effect on the form of the domains; it merely brings about a double value in the definition of H_1. The maximum value $d \approx 0.525$ is reached when $H_2 \approx 0.19$.

A more important result, however, is the fact that for any H_2 there is a curve $d_2 = d$ (curve III) among the curves bounding the domains of existence of roots in form (11). Consequently, the equality $d_2 = d$ will hold at the real extremal point. For values of $\lambda_0 = 0^\circ$, 30°, 60° this was evident from the results obtained by the random search method. Thus, for any λ_0, the inequality $d_2 \geq d$ in (11) can be replaced by the equality $d_2 = d$. This makes it possible to choose a total of three parameters arbitrarily in formulas (15)-(18): θ_A, θ_C, d.

In (15)-(18) we set $d_2 = d$. Then relation (17), which contains θ_A, θ_C, d, H_2, can be used to determine H_2 for given values of θ_A, θ_C, d. From (15) we find in addition that

$$K_1 = [illegible] \quad [illegible]\, 2K_1, \quad K = \left(\frac{K}{K_1}\right) [illegible]$$

$$K_2 = K - H_2, \quad H = \sqrt{H_1^2 + H_2^2}, \quad \delta_0 = \arccos\left(\frac{H_1}{H}\right). \tag{20}$$

Formulas (20) associate definite values of the parameters H, δ_0, K_1, K_2 with any point in the domain of existence of the solution in the plane of the independent parameters θ_A, θ_C; at the same time, because of the two values in formula (16) the variables H and δ_0 will also take on two values. This double-value situation disappears on the curve D = 0. Formulas (20) make it possible to map the boundaries of the domains of existence of the solution from the plane of the independent parameters $\theta_A \theta_C$ onto the plane of one dependent and one independent parameter, e.g., θ_A. Such a mapping is convenient, in that it gives us some idea of the nature of the function f in the region of its extremum.

The results obtained for values of $\lambda_0 = 0^\circ$, 30°, 60°, 90° are presented in Figs. 4-13. As can be seen from these figures, the domains of existence of solutions of form (11) with $d_2 = d$ are determined by two curves:

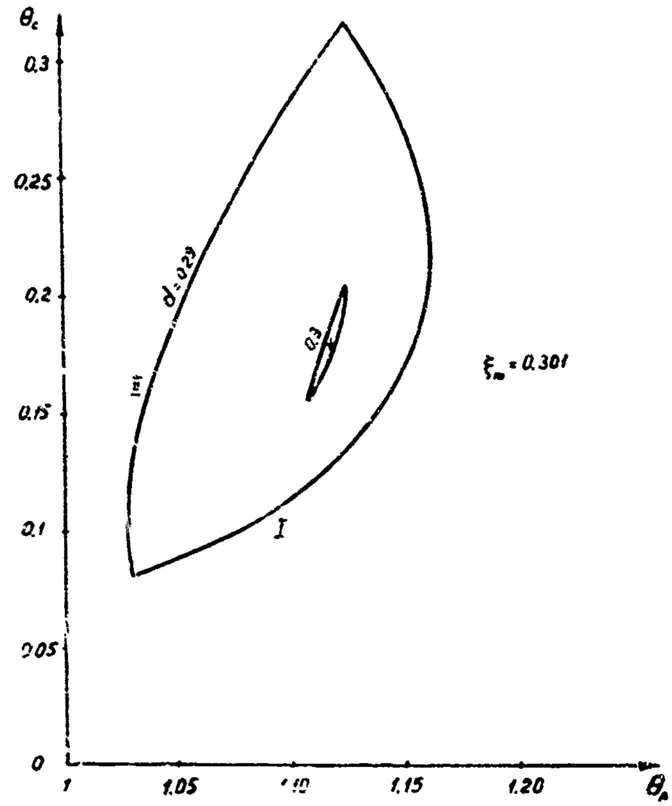

Fig. 4. Plane of the independent parameters θ_A, θ_C for $\lambda_0 = 0$.

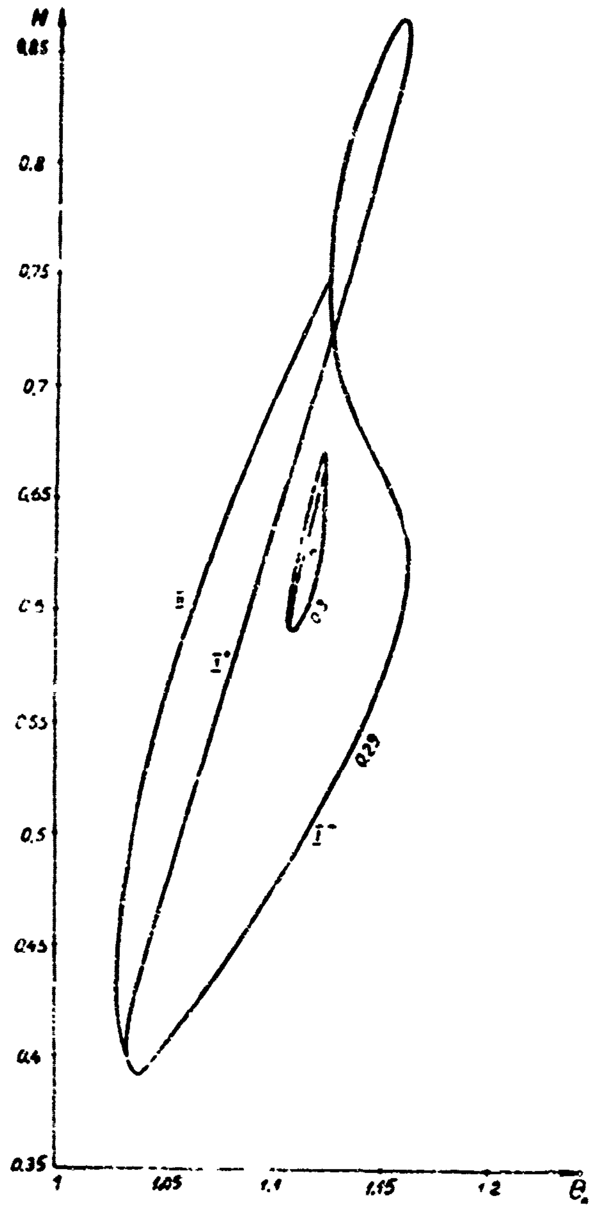

Fig. 5. Dependence of H on θ_A for $\lambda_0 = 0$.

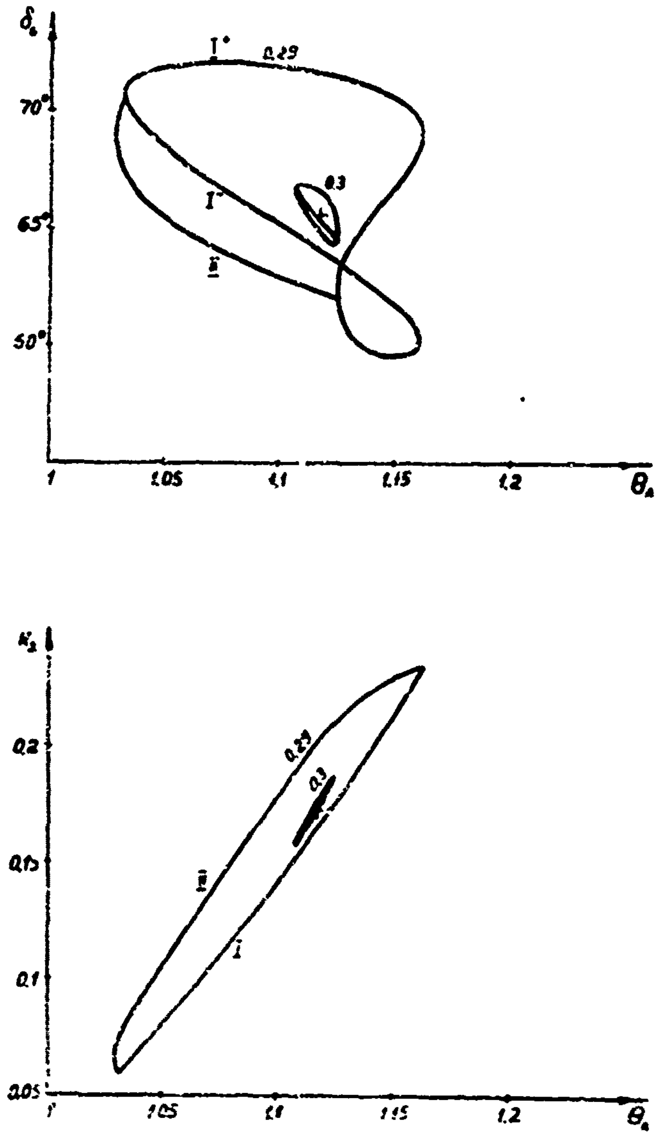

Fig. 6. Dependence of δ_η and $\varkappa_p$ on θ_A for $\lambda_0 = 0$.

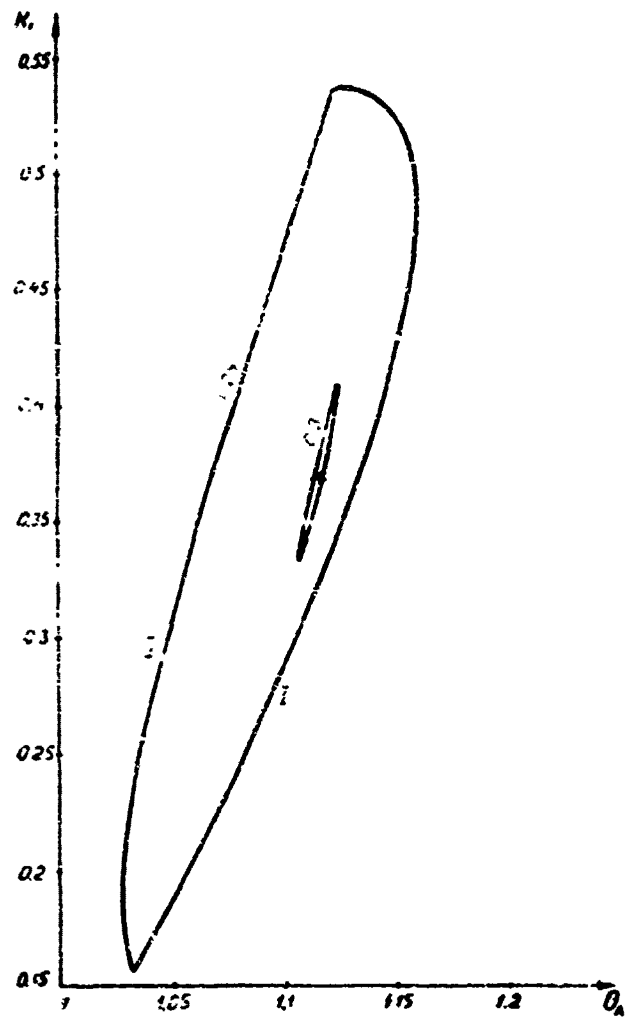

Fig. 7. Dependence of K_1 on θ_A with $\lambda_0 = 0$.

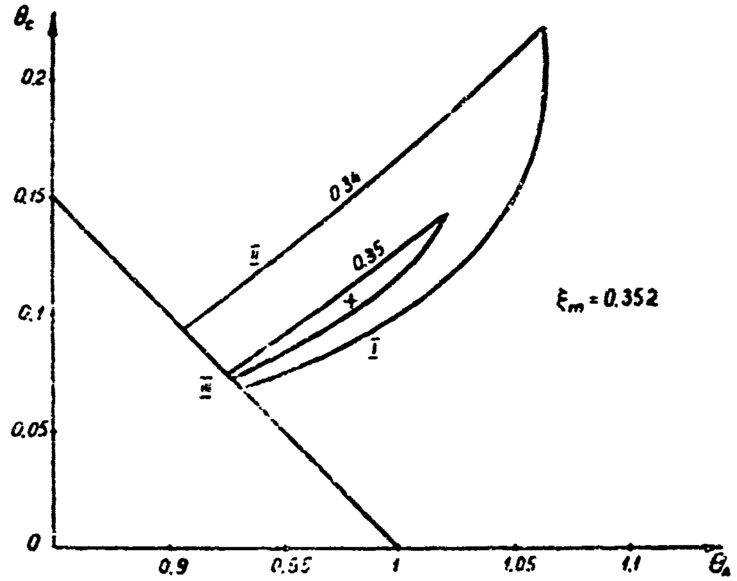

Fig. 8. Plane of the independent parameters θ_A, θ_C with $\lambda_0 = 30°$.

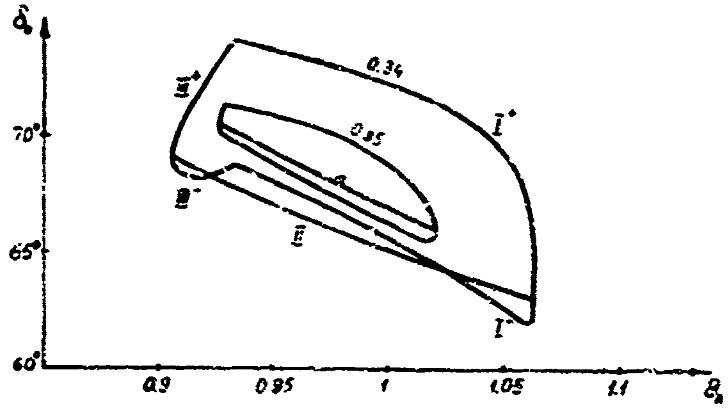

Fig. 9. Dependence of δ_0 on θ_A with $\lambda_0 = 30°$.

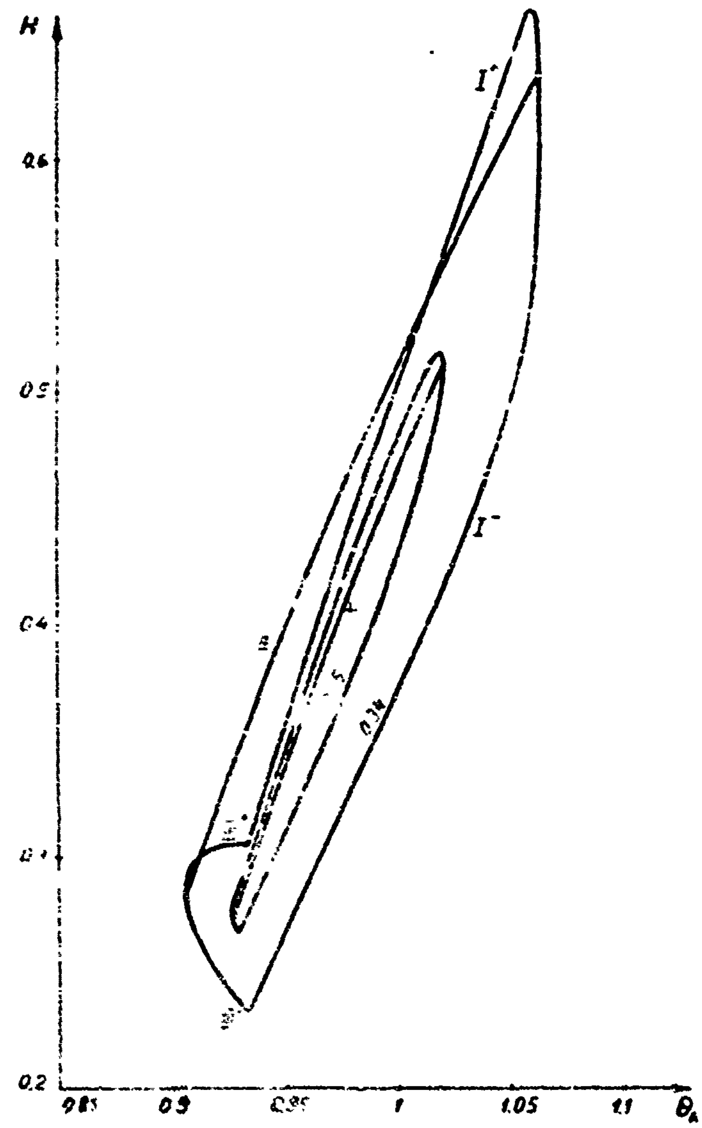

Fig. 10. Dependence of H on ϑ_A with $\lambda_0 = 30^\circ$.

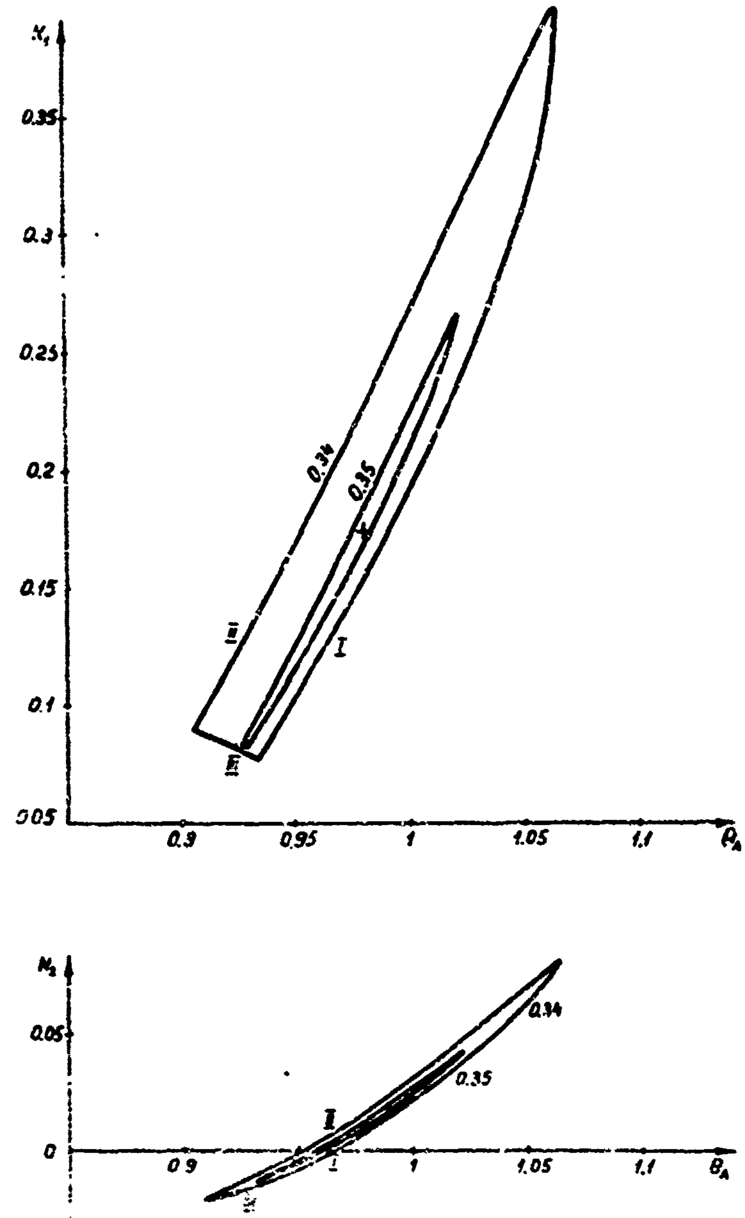

Fig. 11. Dependence of K_1 and K_2 on θ_A with $\lambda_0 = 30^\circ$.

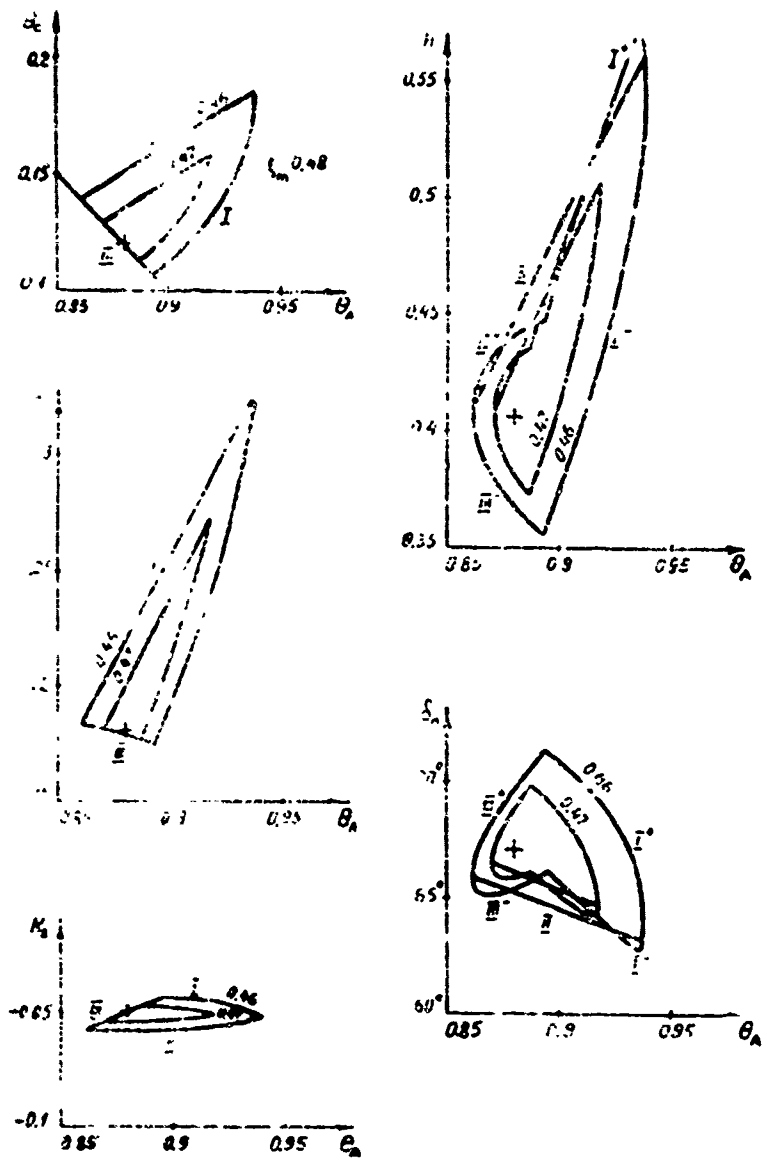

Fig. 12. Parameters ϑ_A, ϑ_C, H, K_1, K_2, δ_0 with $\lambda_0 = 60^\circ$.

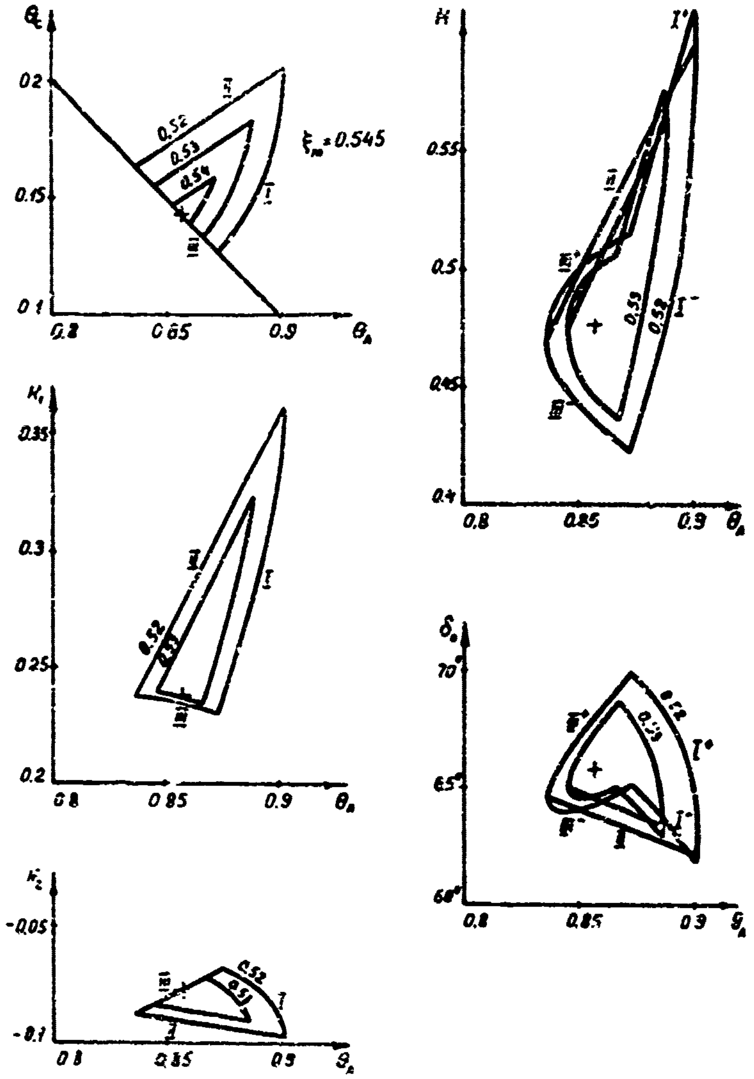

Fig. 13. Parameters ϑ_A, θ_C, H, K_1, K_2, δ_0 with $\lambda_0 = 90^o$.

$(\omega_2^2 + \omega_3^2)^2 - 4\,\omega_2^2\omega_3^2 = 0$ (curve I), $D = 0$ (II) and, beginning with a certain value of λ_0, the line $\theta_A + \theta_C = 1$ (III). Because d_1 is two-valued in its definition, curves I and III of the plane of θ_A, θ_C are mapped into the curves I^+, I^-, III^+, III^- on the planes of θ_A, H and θ_A, δ_0. $D = 0$ on curve II, and therefore the definition of d_1 is not two-valued. The cross in all the figures denotes points at which the maximum value $d = \xi_m$ is reached. At the optimal point, $D = 0$, and therefore H and δ_0 are uniquely determined. From the nature of the curves d = const it can be seen that in the region of the extremum the function ξ is less sloping when $\lambda_0 = 0$. With increasing λ_0 the distance between the curves d = const decreases. The considerable elongation of the figures described by the curves d = const confirms the fact found by the random search that the derivatives of ξ with respect to various directions in parameter space differ substantially.

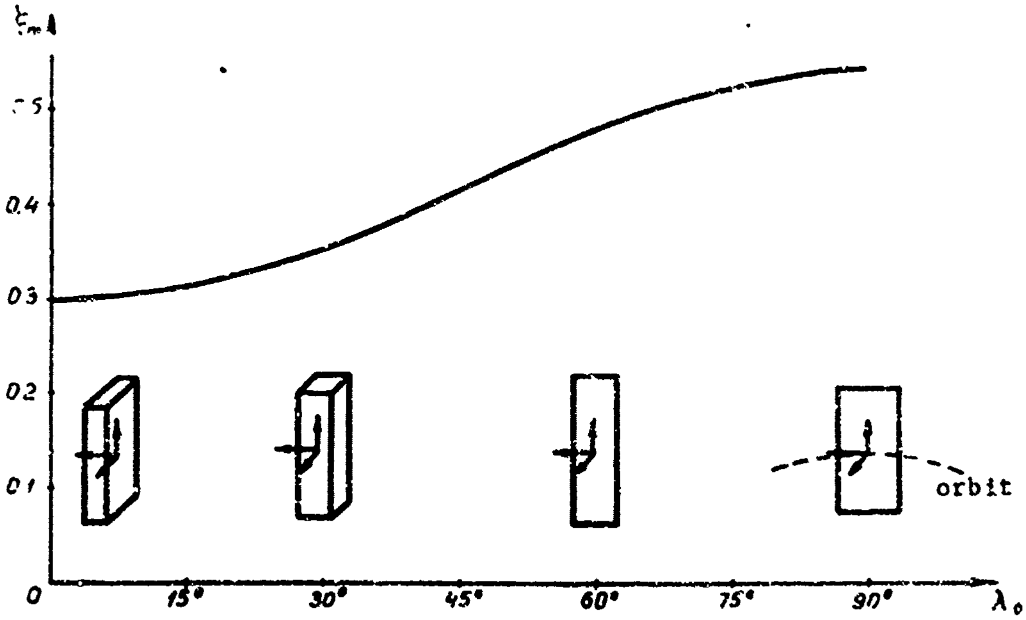

Fig. 14. The variable ξ_m and the shape of a satellite in the optimal configuration as a function of λ_0.

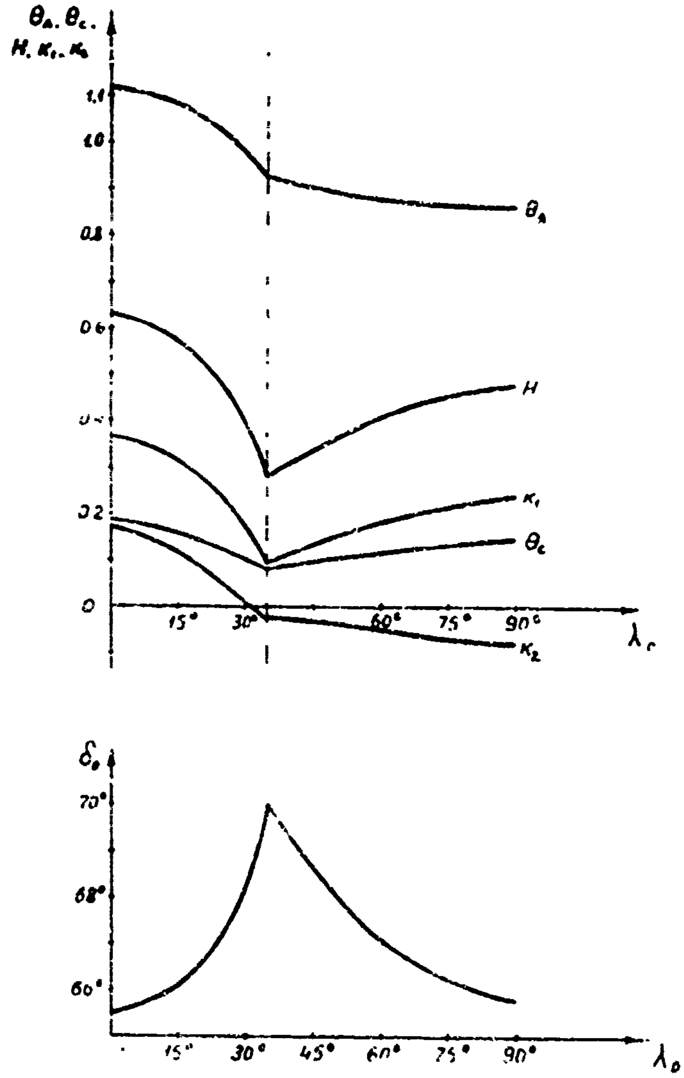

Fig. 15. Dependence of θ_A, θ_C, H, K_1, K_2, δ_0 on λ_0.

Shown in Fig. 14-15 is the dependence of ξ_m and of the optimal parameters θ_A, θ_C, H, K_1, K_2, δ_0 on λ_0. $\theta_A > 1$ when $\lambda_0 < 27^\circ$, and the optimal satellite configuration is gravitationally unstable, since the condition $1 > \theta_A > \theta_C$ should be fulfilled for stability. Stability is achieved by the kinetic moments of the gyroscopes. The corner points on the graphs for $\lambda_0 = 35^\circ$ are explained by the fact that when $\lambda_0 \geq 35^0$ the optimal parameters are found on the boundary $\theta_A + \theta_C = 1$. In this case the satellite is a gravitationally stable plate. We note that when $\lambda_0 > 31^0$, the optimal value of the elasticity coefficient K_2 is less than zero. The maximum value $\xi_m = 0.545$ is reached when $\lambda_0 = 90^0$, $\theta_A = 0.857$, $\theta_C = 0.143$, $K_1 = 0.238$, $K_2 = -0.078$, $H = 0.476$, $\delta_0 = 65.78^0$. A variant of the configuration for $\lambda_0 = 90^0$ was also considered in [1], where $\xi_m = 0.48$ was obtained.

The optimal parameters θ_A, θ_C, H, K_1, K_2, δ_0 were sought on a set yielding roots of form (11) for $d_2 = d$. In order to prove that the parameter values we found do indeed yield a local maximum of ξ, it is necessary to study the neighborhood of the point of extremum for arbitrary parameter values. The following consideration turns out to be convenient.

We fix four of the six optimal parameters and in the plane of the remaining two we construct a region within which the degree of stability ξ is greater than a given value of ξ_0. To do so, we make the exchange $\lambda = \xi_0 + \rho$ in Eqs. (8)-(9). The region to be sought is one in which all the Hurwitz determinants of the characteristic equations obtained with respect to ρ are positive. By assigning several values to ξ_0, we get a number of regions imbedded in each other with boundaries yielding level lines of ξ as functions of the two chosen parameters for optimal values of the remaining parameters. The closure of the level lines of ξ will be a proof that the parameter values we found do indeed yield a local maximum for ξ.

Constructed in Figs. 16-19 are level lines corresponding to values of $\xi = 0.2, 0.3, 0.4$ for all parameter combinations with $\lambda_0 = 90^0$. The cross denotes the point of maximum. As can be seen from the figures, all the level lines of ξ are closed and encompass the point of maximum.

Figs. 16-19 give an idea of the behavior of the function ξ in the region of the maximum. They may turn out to be useful in a structural realization of the stabilization system, when it is difficult to get precise optimal parameter values.

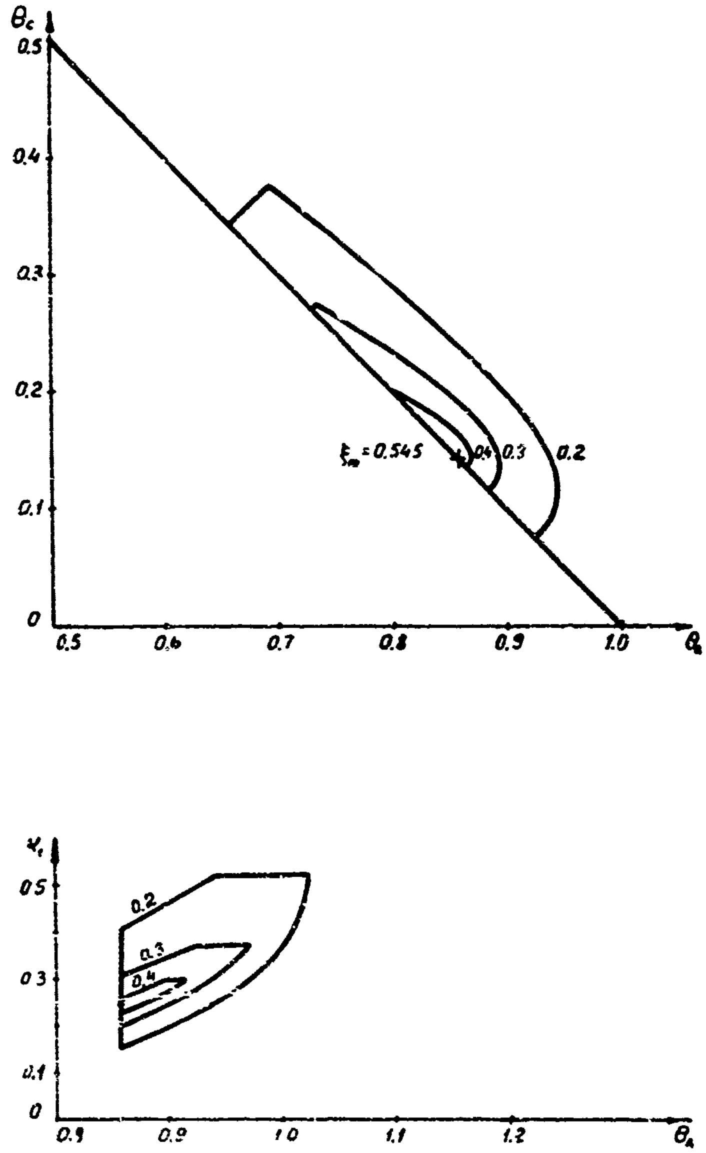

Fig. 16. Level lines of ξ for $\lambda_0 = 90^{\circ}$.

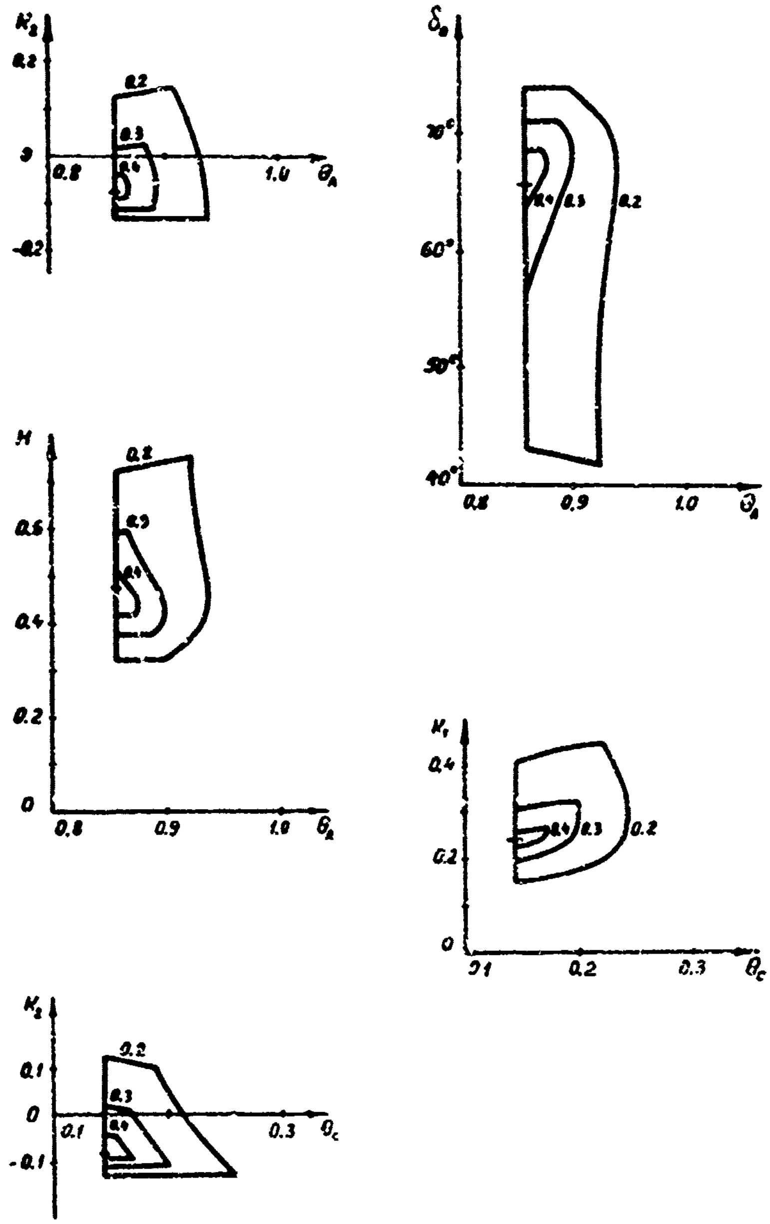

Fig. 17. Level lines of [illegible] for $\lambda_0 = 90°$.

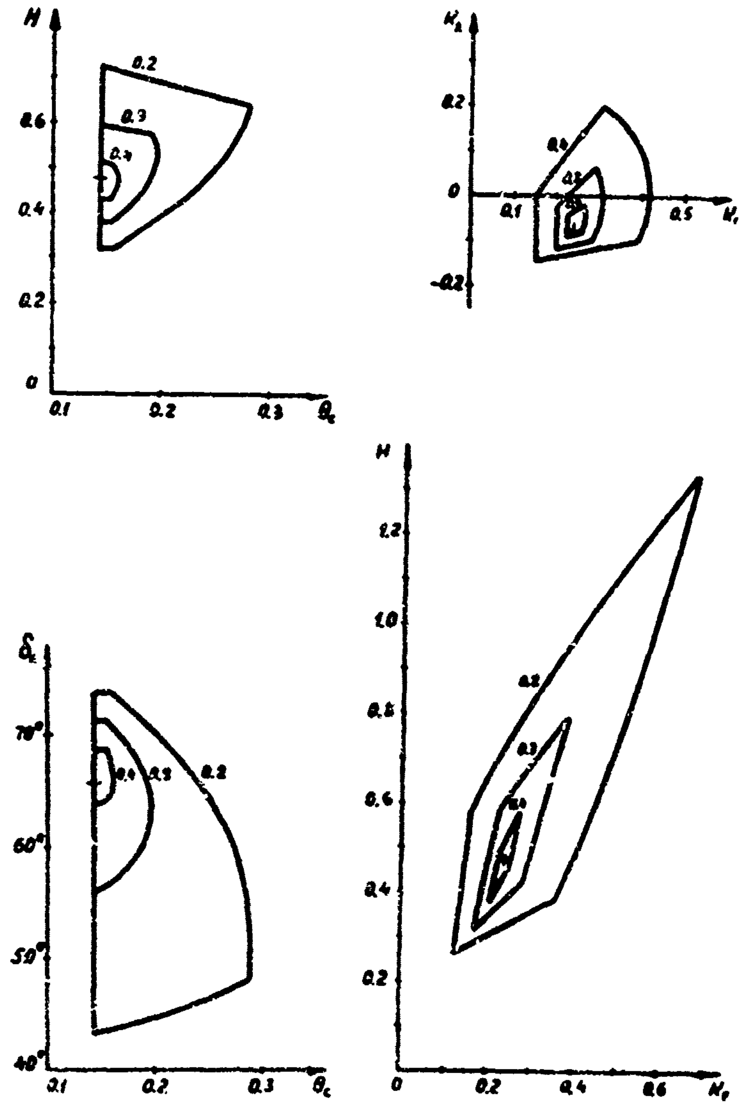

Fig. 18. Level lines of ξ for $\lambda_0 = 90^\circ$.

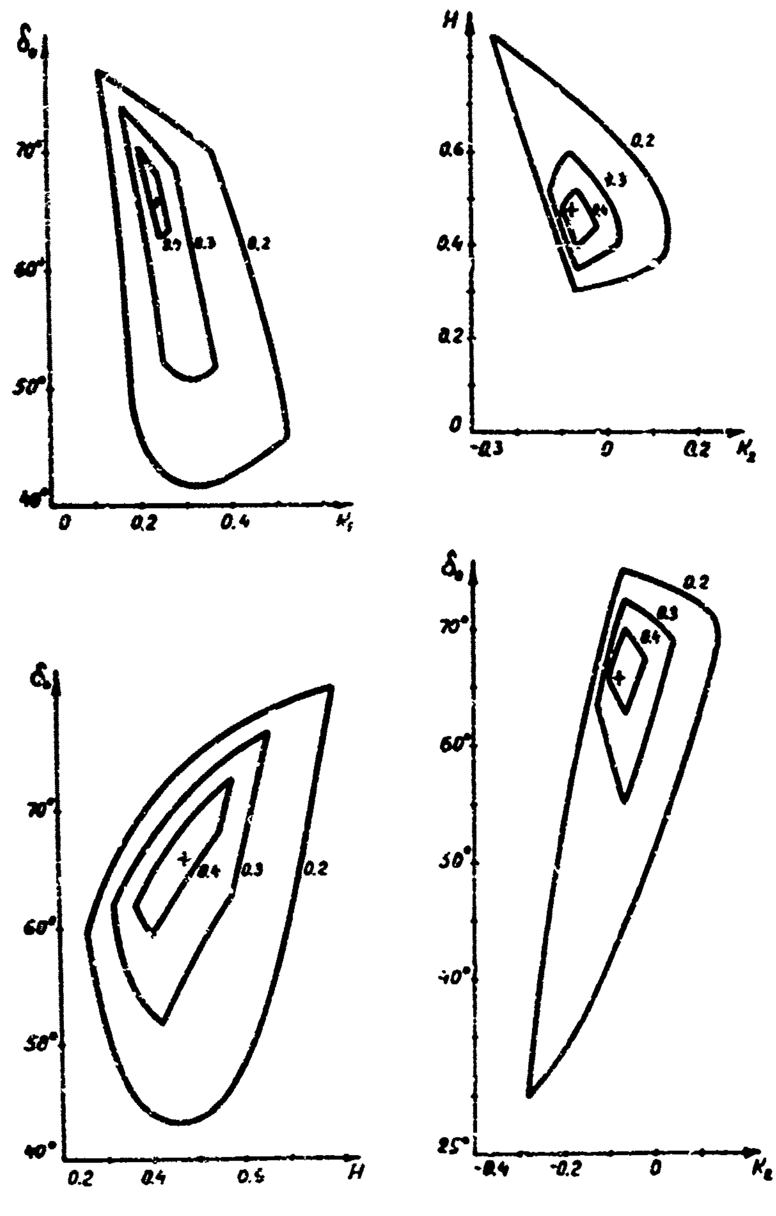

Fig. 19. Level lines of ξ for $\lambda_0 = 90^\circ$.

4. Optimal Parameters of an Axisymmetric Satellite

An axisymmetric satellite is defined by the condition $\theta_A = 1$. A study made by the random search method has shown that in this case the equality $d_2 = d$ does not hold for all values of λ_0. In particular, for $\lambda = 90^\circ$, this is clearly evident in Fig. 3. In fact, if for any H_2 we decrease d, then for a certain value of d the straight line $\theta_A = 1$ intersects curves I and II, whereas curve III ($d_2 = d$) will remain on the left. This reasoning forces us to seek the optimal parameters on a set specified by conditions (11). In (15)-(18) we set $\theta_A = 1$. Relation (17), which contains θ_C, H_2, d, d_2, can be used once again to determine d_2. Then, by formulas (20), we find all the remaining parameters. Although in this case the independent parameters are θ_C and H_2, it is only natural not to construct regions defined in the plane of these parameters by the inequalities (19) and $d_1 \geq d$, $d_2 \geq d$, but to proceed immediately to construct, by formulas (20), a mapping of the boundaries of these regions onto the planes of one of the dependent parameters H, K_1, K_2, δ_0 and of the independent parameter θ_C.

The results obtained for values of $\lambda_0 = 0^\circ$, 30°, 60° and 90° are shown in Figs. 20-24. When $\theta_A = 1$, the domains of existence of the solution in form (11) are defined by three curves: $(\omega_2^2 + \omega_3^2)^2 - 4\,\omega_2^2\omega_3^2 = 0$ (curve I), $D = 0$ (curve II), and $d_2 = d$ (curve III). The nature of the domains changes substantially for various λ_0. If, however, for $\lambda_0 = 0^\circ$, 30° it still turns out that $d_2 = d = \xi_m$ at the point of optimum, then for $\lambda_0 = 60^\circ$ the curve $d_2 = d$ occurs only when $d = 0.42$, while the point of optimum is defined by curves I and II, and $d_2 > d = \xi_m$ at this point. As in the case of arbitrary values of θ_A, when $\theta_A = 1$ the figures described by the curves d = const are seen to be quite elongated.

The dependence of ξ_m and of the optimal parameters θ_C, H, K_1, K_2, δ_0 on λ_0 is plotted in Fig. 25. The maximum value $\xi_m = 0.483$ is reached when $\lambda_0 = 90^\circ$, $\theta_C = 0.172$, $K_1 = 0.501$, $K_2 = 0.018$, $H = 0.703$, $\delta_0 = 64.6^\circ$. The level lines of ξ for this case are shown in Figs. 26-29.

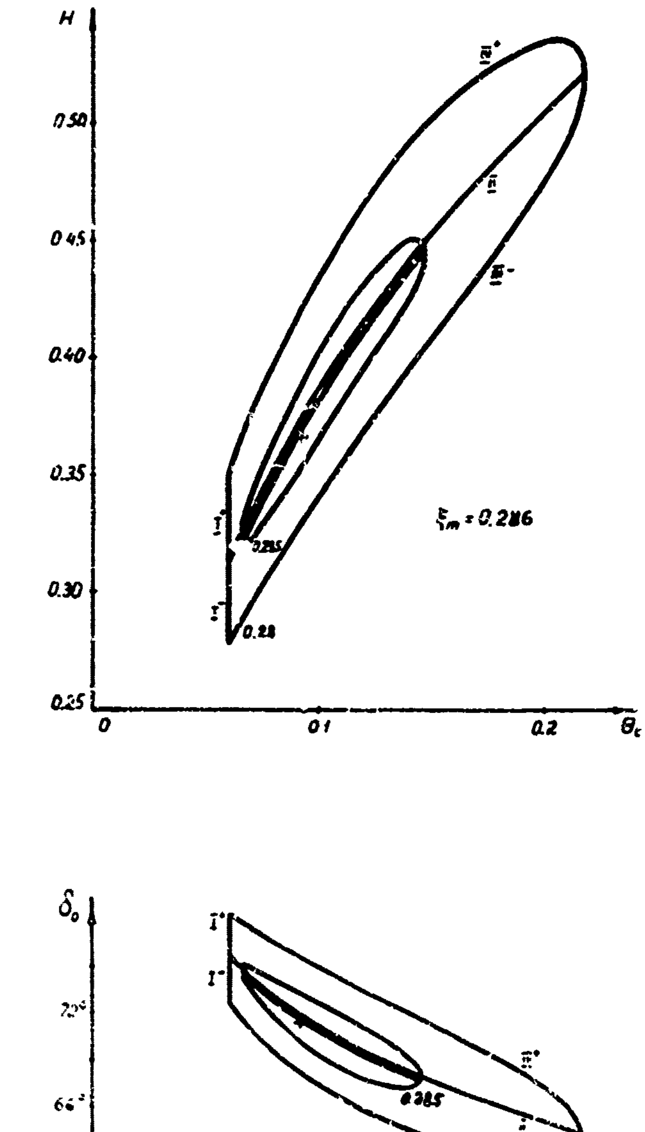

Fig. 20. Dependence of H and δ_0 on θ_C for $\theta_A = 1$, $\lambda_0 = 0$.

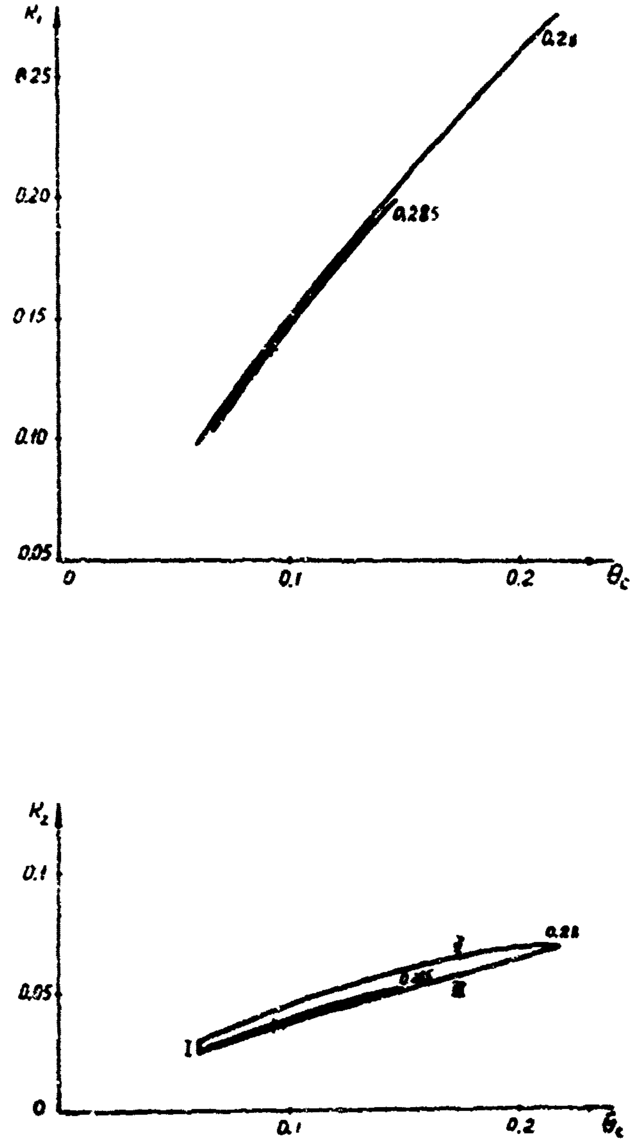

Fig. 21. Dependence of K_1, K_2 on θ_C for $\theta_A = 1$, $\lambda_0 = 0$.

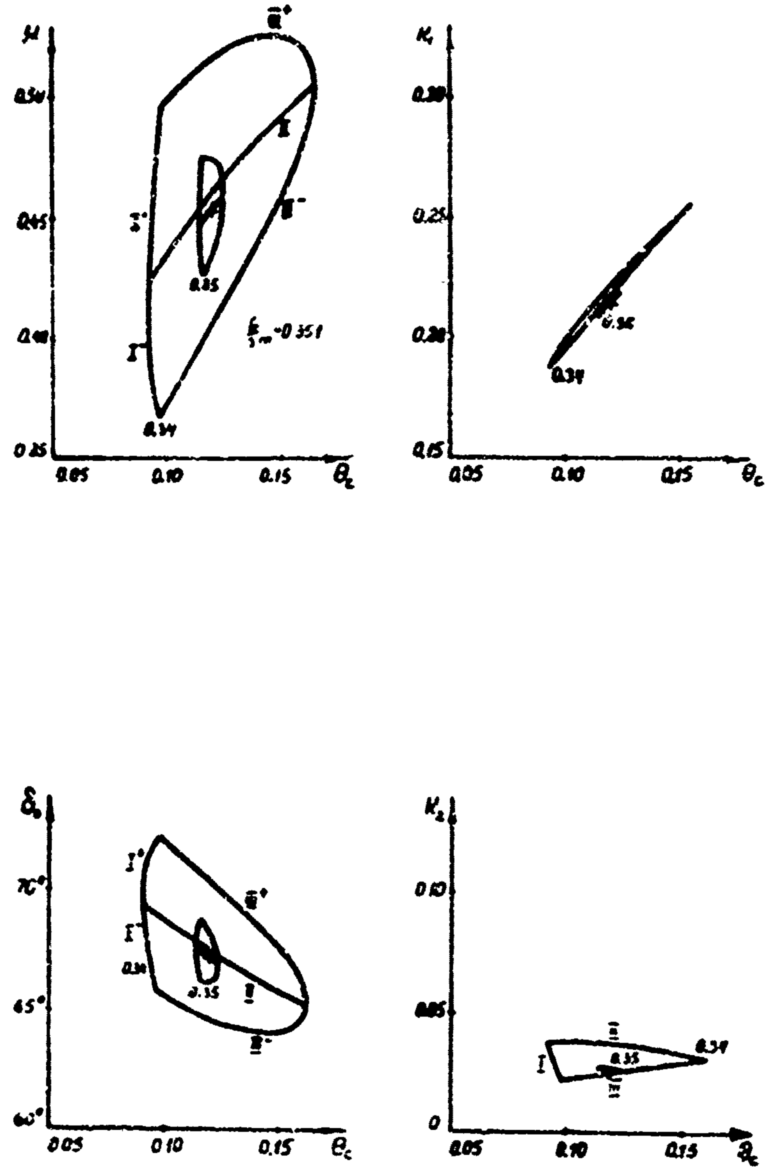

Fig. 22. Dependence of H, K_1, K_2, δ_0 on θ_C for $\theta_A = 1$, $\lambda_0 = 30^\circ$.

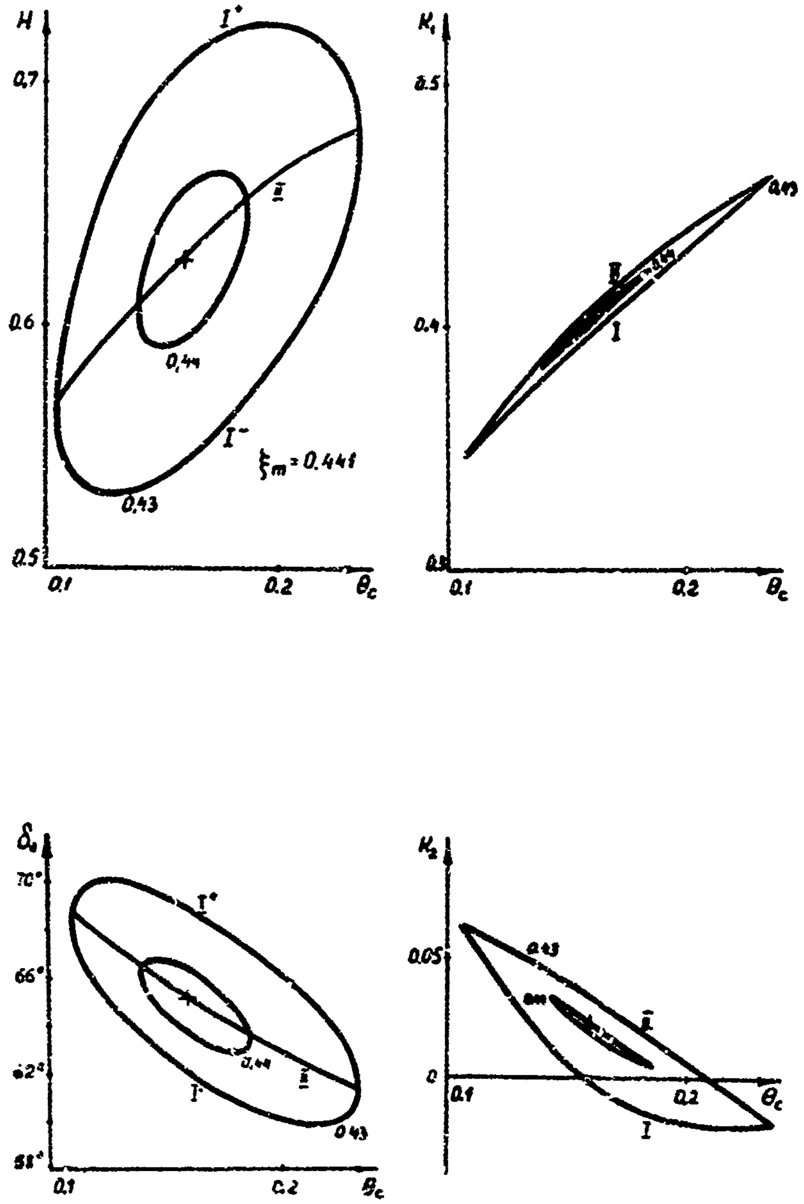

Fig. 23. Dependence of H, K_1, K_2, δ_0 on θ_C for $\theta_A = 1$, $\lambda_0 = 60°$.

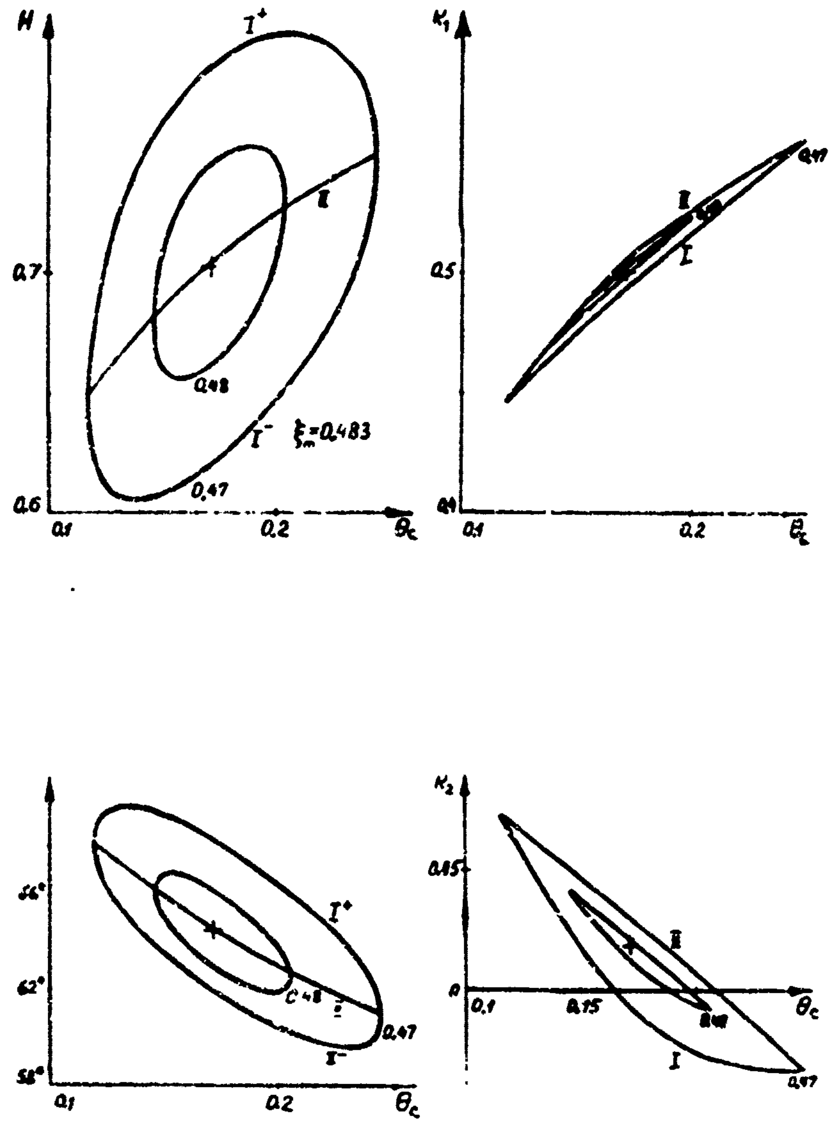

Fig. 24. Dependence of H, K_1, K_2, δ_0 on θ_C for θ_A = 1, λ_0 = 90°.

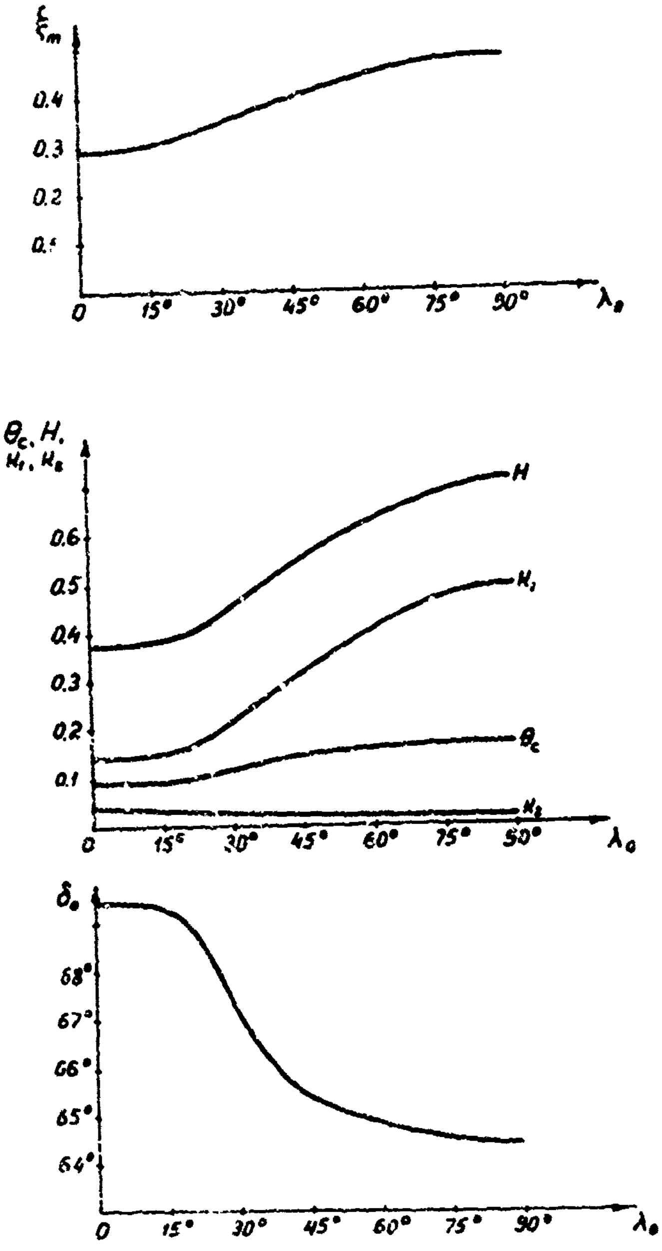

Fig. 25. Optimal parameters of an axisymmetric satellite.

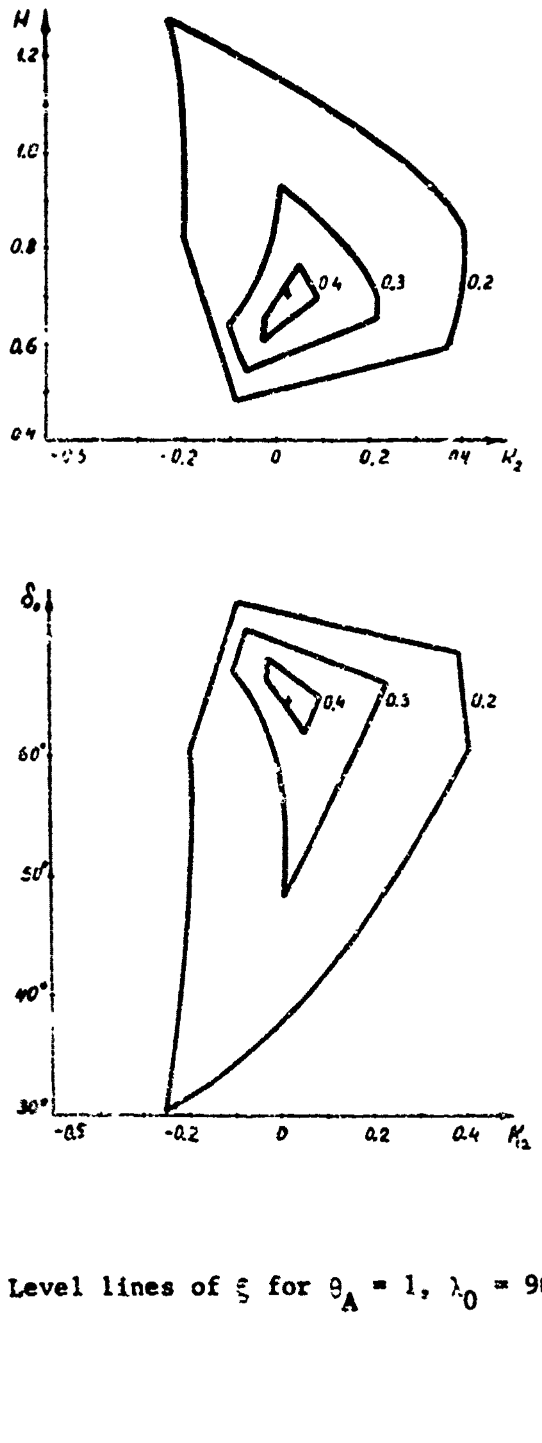

Fig. 26. Level lines of ξ for $\theta_A = 1$, $\lambda_0 = 90^{\circ}$.

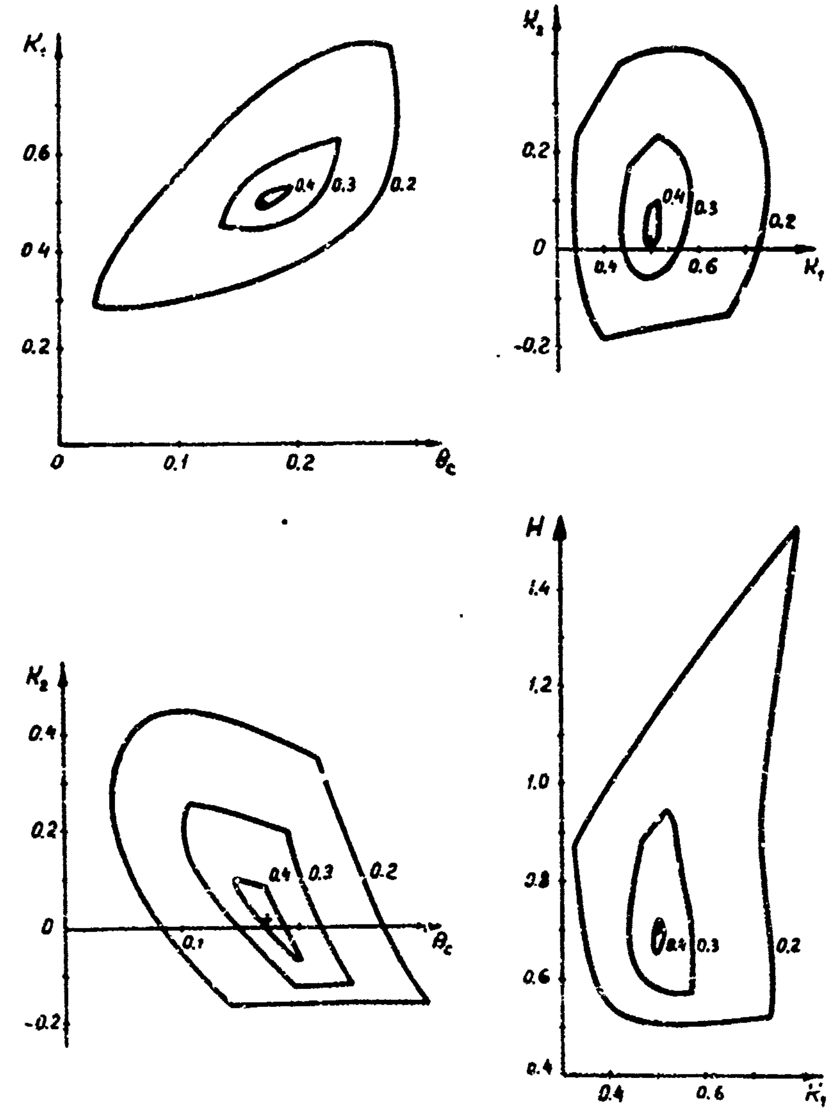

Fig. 27. Level lines of ξ for $\vartheta_A = 1$, $\lambda_0 = 90°$.

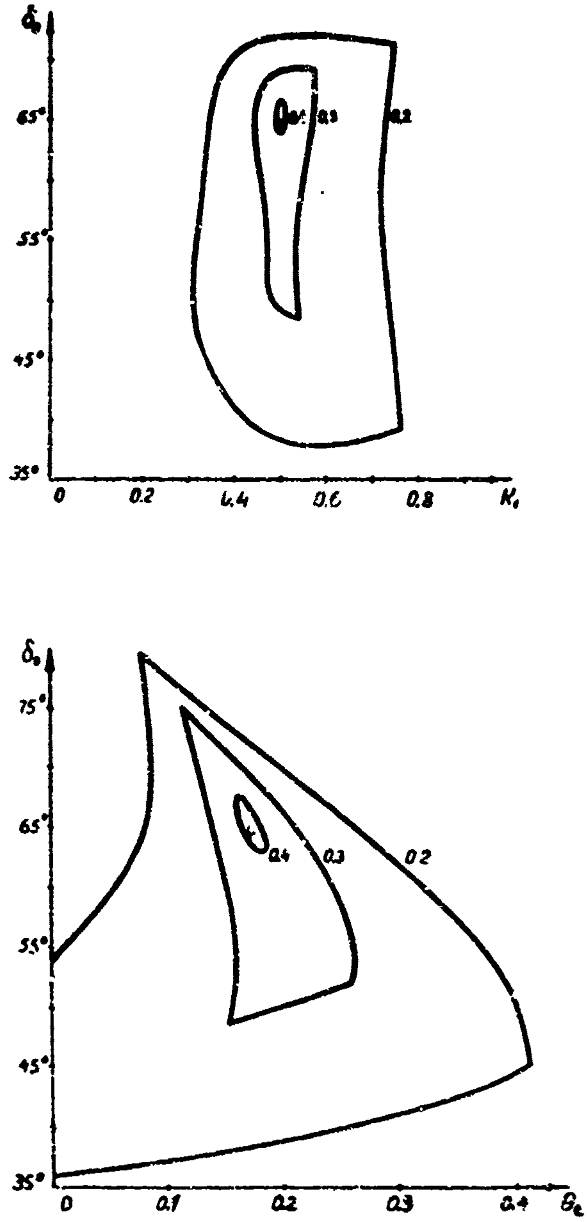

Fig. 28. Level lines of ξ for $\theta_A = 1$, $\lambda_0 = 90°$.

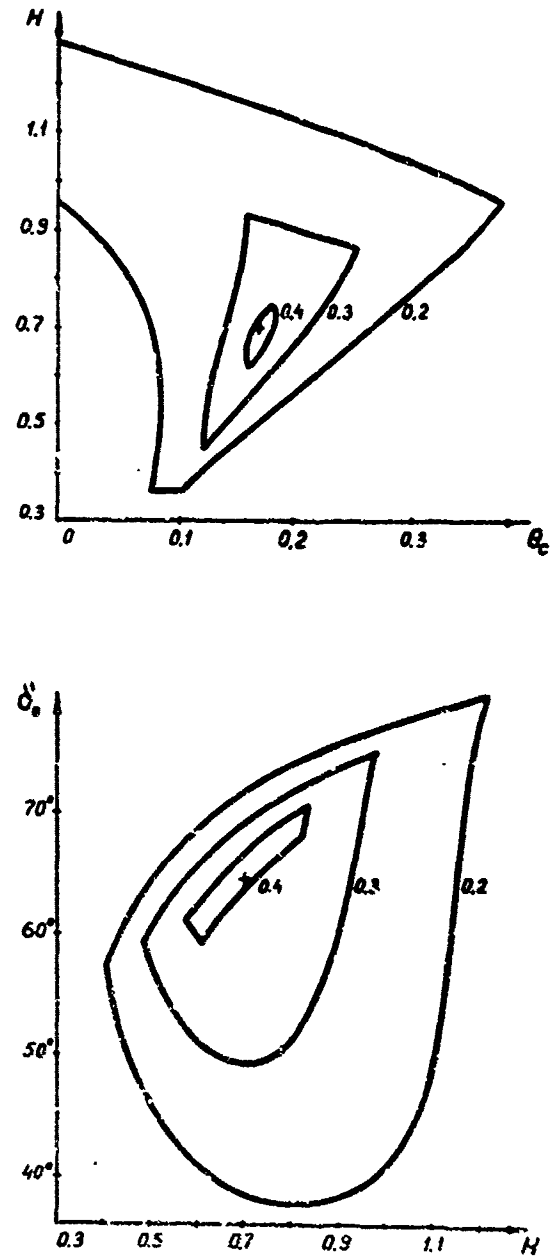

Fig. 29. Level lines of ξ for $\theta_A = 1$, $\lambda_0 = 90°$.

5. Eccentricity Oscillations

In an elliptical orbit, due to the non-uniformity of rotation of the radius-vector connecting the centers of mass of the earth and the satellite, the stable equilibrium position being studied goes over into periodic oscillations, called eccentricity oscillations. The period of the eccentricity oscillations is equal to the period of revolution of the satellite center of mass along the elliptical orbit; the oscillation amplitude is proportional to the magnitude of the orbital eccentricity.

The eccentricity oscillations are a particular solution $\beta = \gamma \equiv 0$, $\delta_1 + \delta_2 \equiv 0$, $\alpha = \alpha(t)$, $\delta_1 - \delta_2 = f(t)$ of system (2). This particular solution must be found in the form of a series in powers of the eccentricity e. Limiting ourselves to first terms in the expansion in e, we find that the amplitude R_1 of the satellite eccentricity oscillations is determined by the formula [1,8]

$$R_1^2 = 4e^2 \frac{\kappa_1^2 + [(\kappa_2 + H\cos\delta_0) + 2H^2\sin^2\delta_0]^2}{\kappa_1^2[3(\theta_A - \theta_c) - 1]^2 + \{(\kappa_2 + H\cos\delta_0)[3(\theta_A - \theta_c) - 1] - 2H^2\sin^2\delta_0\}^2} . \qquad (21)$$

The transient response of the satellite-gyroscope system in the case of a circular orbit is constructed in Fig. 30. The satellite is axisymmetric ($\theta_A = 1$) with parameters optimal when $\lambda_0 = 90°$. The initial values of the angles and velocities were chosen as follows: $\alpha = \beta = \gamma = 10°$, $\delta_1 = \delta_2 = \dot{\delta}_1 = \dot{\delta}_2 = p = r = 0$, $q = 1$. From the figure it can be seen that in two satellite orbital revolutions the oscillation amplitude with respect to all angles becomes less than one degree.

The transient response of the satellite under the same initial conditions in the case of an elliptical orbit with eccentricity $e = 0.02$ is given in Fig. 31. The initial value of the true anomaly v was chosen to be zero. Steady-state periodic motion in pitch with an amplitude of $3.6°$ can be seen in the figure. This value is in good agreement with formula (21), which yields $R = \frac{R_1}{e} = 3.06$, from which we get $R_1 = R\, e = 0.0612$ rad $= 3.5°$ for $e = 0.02$.

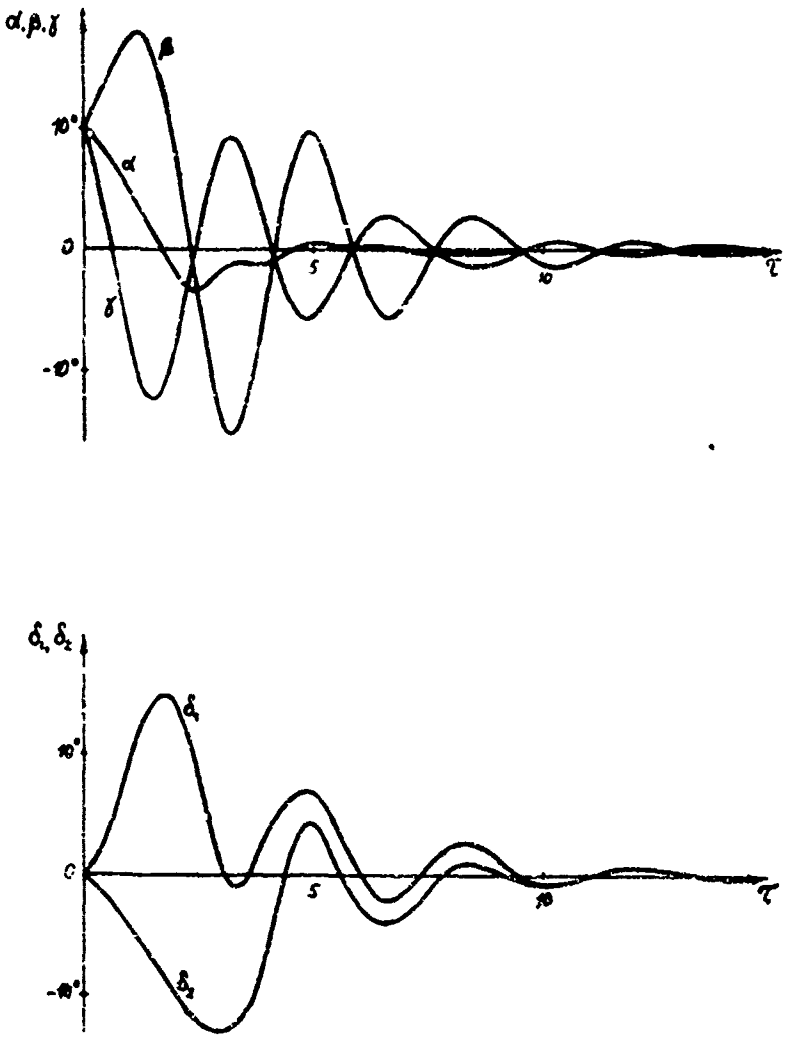

Fig. 30. Optimal transient process for $\varsigma_A = 1$, $\lambda_0 = 90^\circ$, $e = 0$.

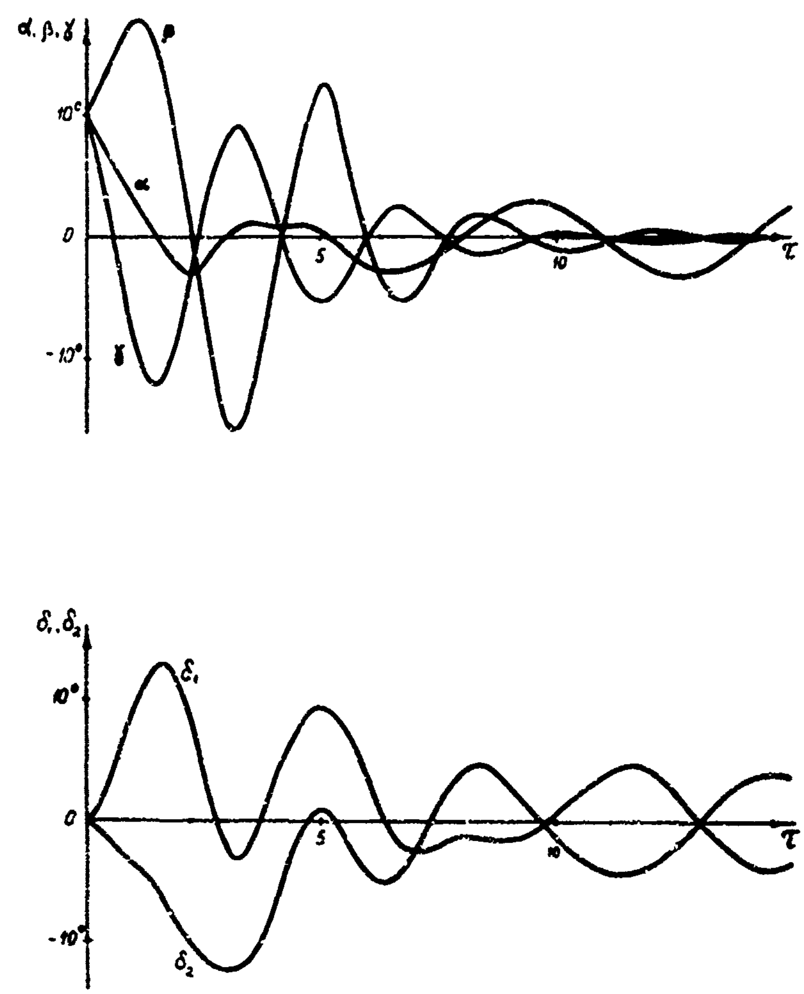

Fig. 31. Optimal transient process for $\theta_A = 1$, $\lambda_0 = 90^\circ$, $e = 0.02$.

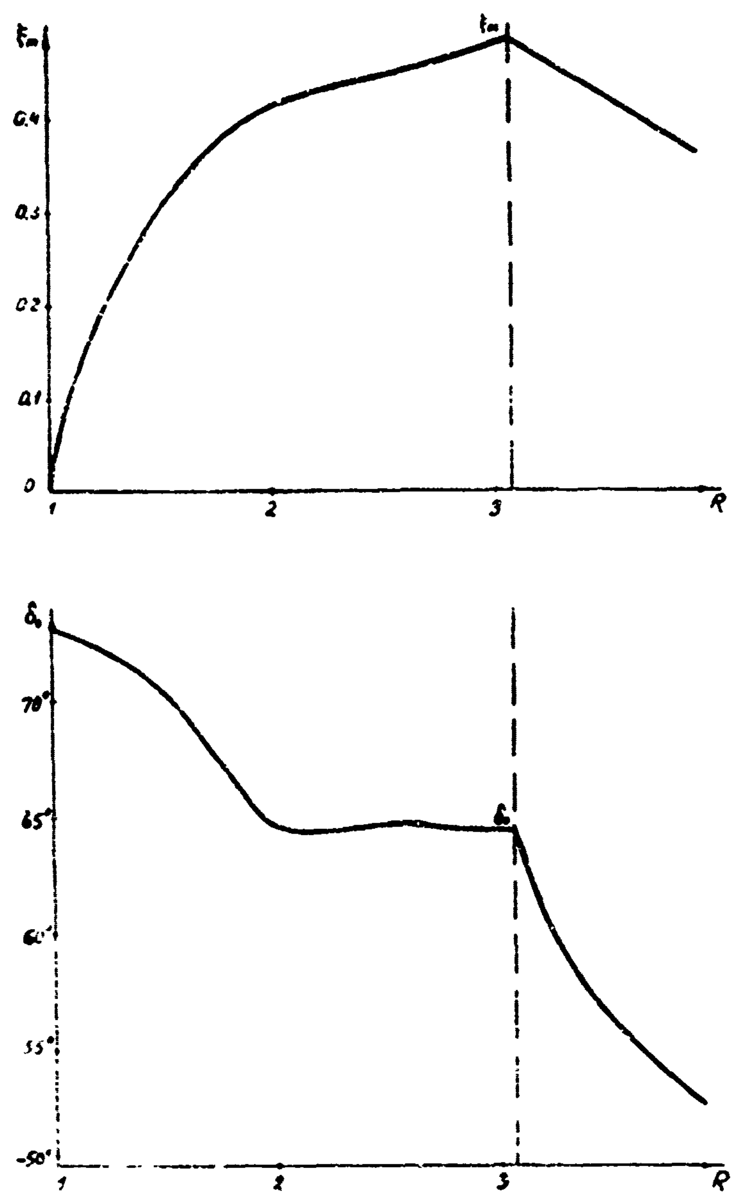

Fig. 32. Dependence of ξ_m and δ_0 on R for $\alpha_A = 1$, $\lambda_0 = 90^{\circ}$.

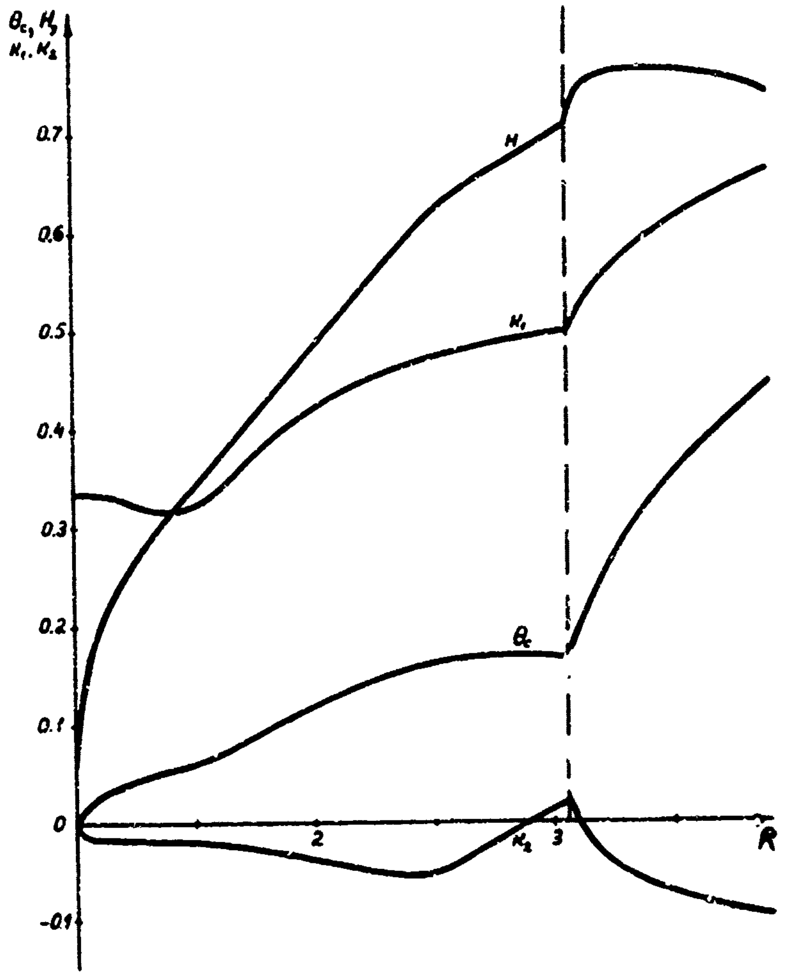

Fig. 33. Dependence of θ_C, H, K_1, K_2 on R for θ_A = 1, λ_0 = 90°.

We select the parameters of the axisymmetric satellite being considered such that maximum ξ is obtained for a fixed value of $R = \frac{R_1}{e}$. The results obtained by the random search method are given in Figs. 32-33. From the figures it can be seen that the value of R can be reduced from three units to two without decreasing the degree of stability ξ_m appreciably. When R = 1, the degree of stability $\xi_m = 0$; here it is easy to show that this will remain valid for a satellite with any desired value of λ_0 and arbitrary distribution of mass. In fact, in seeking minimum R by formula (21), we get that the minimum value R = 1 is reached when

$$\frac{H^2 \sin^2 \delta_0}{K_1} = \frac{H_1^2}{K_1} = 0 .$$

In this case, on the basis of the Routh-Hurwitz criterion, for characteristic equation (8) we find that $\xi_m = 0$. The point R = 3.06 corresponds to the optimal satellite parameters obtained in section 4. The break in the graphs for R = 3.06 is explained by the differing nature of the roots at which the extremum ξ is realized, to the right and left of this point.

Conclusion

The degree of system stability ξ is not an analytic function at the point of maximum. This is due to the fact that the real parts of the roots of the characteristic equation being examined become equal at the point of maximum ξ. Even though it has negative aspects, this fact permits us to make the optimal parameter values obtained by various rough numerical methods more precise by introducing relations between the parameters obtainable by equating the real parts of the roots of the characteristic equation being studied.

As applied to the satellite configuration with gyro-damping being considered here, the following results were obtained by this method. The maximum value of the degree of system stability $\xi_m = 0.545$ is reached when $\lambda_0 = 90°$, which corresponds to location of the gyroscope gimbal axes along the Ox-axis; in the optimal configuration, the satellite is a gravitationally stable plate. When $\lambda_0 < 27°$, the optimal satellite configuration is gravitationally unstable, and system stability is achieved by using the kinetic moments of the gyroscopes. When $\lambda_0 > 31^\circ$, the elasticity coefficient $K_a < 0$. The maximum degree of stability of an axisymmetric satellite is also reached when $\lambda_0 = 90^\circ$. In this case $\xi_m = 0.483$.

The level lines of ξ, plotted in two parameters in Figs. 16-19 for an arbitrary satellite and in Figs. 26-29 for an axisymmetric satellite, allow representation of the function ξ in the region of the maximum. These figures may prove to be useful in the design stage of the system, when it is usually difficult to get exact parameter values.

The study we made of the dependence of ξ_m on the magnitude of the amplitude of the eccentricity oscillations R_1 for an axisymmetric satellite yielded the following results for $\lambda_0 = 90^\circ$. With optimal satellite parameter values, $R_1 = 3.06e$. It is possible to specify parameters for which $R_1 = 2e$, and the degree of stability does not change appreciably. When $R_1 = e$ we find that $\xi_m = 0$, and it is not possible to get $R_1 < e$. This is also true for a satellite with arbitrary mass distribution and any λ_0.

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