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AN INFORMAL SURVEY OF MULTI-ECHELON INVENTORY THEORY

by

Andrew J. Clark

Serial T-259
24 February 1972

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

Program in Logistics

Contract N00014-67-A-0214
Task 0001, Project NR 347 020
Office of Naval Research

50

NONE

Security Classification

DOCUMENT CONTROL DATA - R & D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author) THE GEORGE WASHINGTON UNIVERSITY PROGRAM IN LOGISTICS	2a. REPORT SECURITY CLASSIFICATION NONE
	2b. GROUP

3. REPORT TITLE

AN INFORMAL SURVEY OF MULTI-ECHELON INVENTORY THEORY

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

SCIENTIFIC

5. AUTHOR(S) (First name, middle initial, last name)

CLARK, ANDREW J.

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
24 February 1972	49	68

8a. CONTRACT OR GRANT NO. N00014-67-A-0214	9a. ORIGINATOR'S REPORT NUMBER(S) T-259
b. PROJECT NO. NR 347 020	
c.	9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)
d.	

10. DISTRIBUTION STATEMENT

This document has been approved for public release and sale; its distribution is unlimited.

11. SUPPLEMENTARY NOTES	12. SPONSORING MILITARY ACTIVITY
	Office of Naval Research

13. ABSTRACT

An informal nonmathematical survey of research in multi-echelon inventory theory is given, covering published results through 1971. An introductory section defines the term, "multi-echelon", and establishes the kinds of problems involving multi-echelon considerations. Subsequent sections provide surveys of research on deterministic and stochastic multi-echelon inventory control problems, allocation models, and multi-echelon planning and evaluation models. A final section discusses the present state of the art and suggests directions for future research. A bibliography of papers concerning multi-echelon inventory theory and applications is included.

DD FORM 1 NOV 65 1473

NONE

Security Classification

NONE

Security Classification

14 KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Inventory Theory						
Research Review						
Multi-echelon						
Deterministic						
Stochastic						

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CONTENTS

I. INTRODUCTION

In broad terms, multi-echelon inventory theory is concerned with a variety of inventory problems involving two or more interrelated supply or production facilities. Since most actual inventory problems of significance have multi-echelon aspects, it is natural that previous research on single-facility problems be extended along this additional dimension of complexity. To fully appreciate the directions of multi-echelon research, therefore, it is useful to have a basic familiarity with analytic approaches to the single-facility problems. Surveys of this work are given by Scarf (1963-47), Vienott (1966-57), Iglehart (1967-34), and Hochstaedter (1969-32).[1] The surveys by Vienott and Iglehart also emphasize early results of multi-product and multi-activity research.

1. Structure of Multi-echelon Inventory Systems.

The most common notion of a multi-echelon inventory system is one involving a number of retail outlets (stores, facilities, installations, bases) in business to satisfy customer demands for goods and which, in turn, act as customers of higher-level wholesale activities (warehouses, depots, factories). The wholesale activities themselves may be customers of still higher-level wholesale activities or production facilities. A grocery store chain, as illustrated in Figure 1, is a familiar example of such a system. It is important to note that the system in this example pertains to a given product such as a particular kind and brand of soup. Even for the same grocery store chain, another product may have a different structure (there may be a different factory, or no regional warehouses, or a different mix of retailers, etc.).

A multi-echelon inventory system can also be portrayed as a directed network wherein the nodes represent the various activities or facilities in the system and the linkages represent flows of goods. If the network has at most one incoming link for each node and flows are acyclic (no loops in the network), it is called an "arborescence" or inverted tree structure. Thus, the example in Figure 1 is an arborescence. When viewed as a directed network, it is apparent that much more complex interconnected systems of facilities can exist: a retailer may obtain resupply from more than one wholesaler, or a wholesaler may procure from more than one factory, or a retailer may sometimes supply another retailer; the number of such combinations is very large. However, most

[1] In each pair of reference numbers, the first is the year of publication and the second identifies the paper in the Reference section.

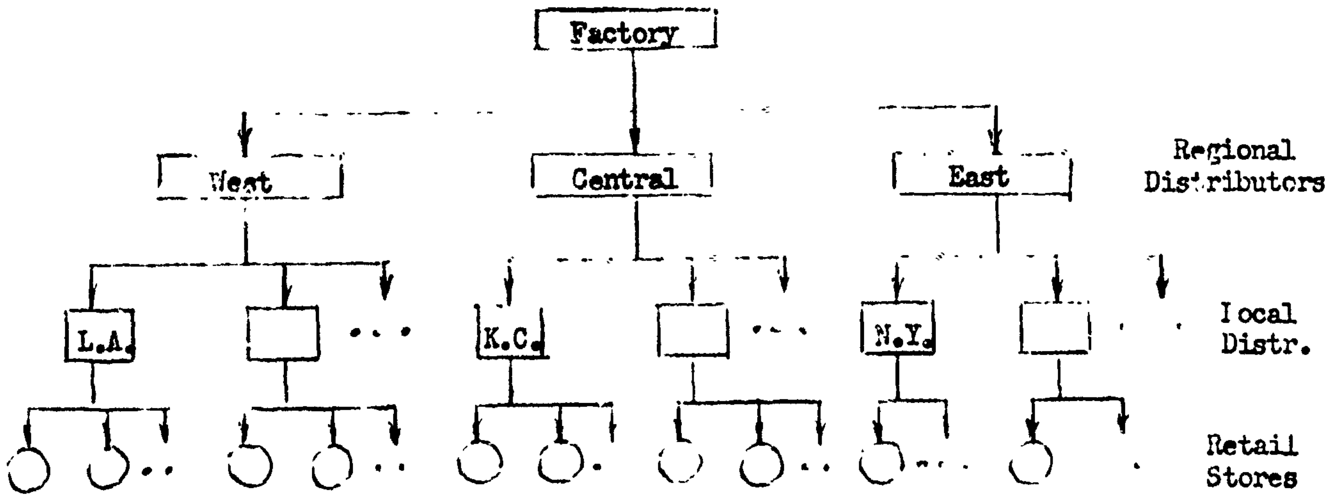

Figure 1. Distribution System for a Grocery Product

of the work in multi-echelon inventory theory has been confined to arborescence structures.

When considering arborescence-type activity structures, the various levels of the system are commonly identified as echelons and the problems investigated are multi-echelon ones. However, since this descriptor makes sense only for arborescence structures and has had even more restricting definitions in some studies, its use in the remainder of this review will be avoided except as defined and used in a particular study.

Finally, two special kinds of arborescence structures are commonly used in the literature. As illustrated in Figure 2, these are the series structure, consisting of two or more activities with each supplying only one other (lower-level) activity, and the parallel structure, consisting of a number of activities experiencing independent external demands; both cases have an implied infinite source of supply as indicated. In the series case, external demands may occur only at the lowest activity or at all activities according to the assumptions of a particular study.

2. Interpretations of Inventory "Activity".

The literature on multi-activity inventory theory contains a wide variety of labels and meanings of what is referred to here as an "activity". Most commonly, it is defined as an entity that maintains a physical stock of one or more products in order to meet specified demands and which, in turn, obtains resupply from some specified (or implied) source or sources. Other definitions,

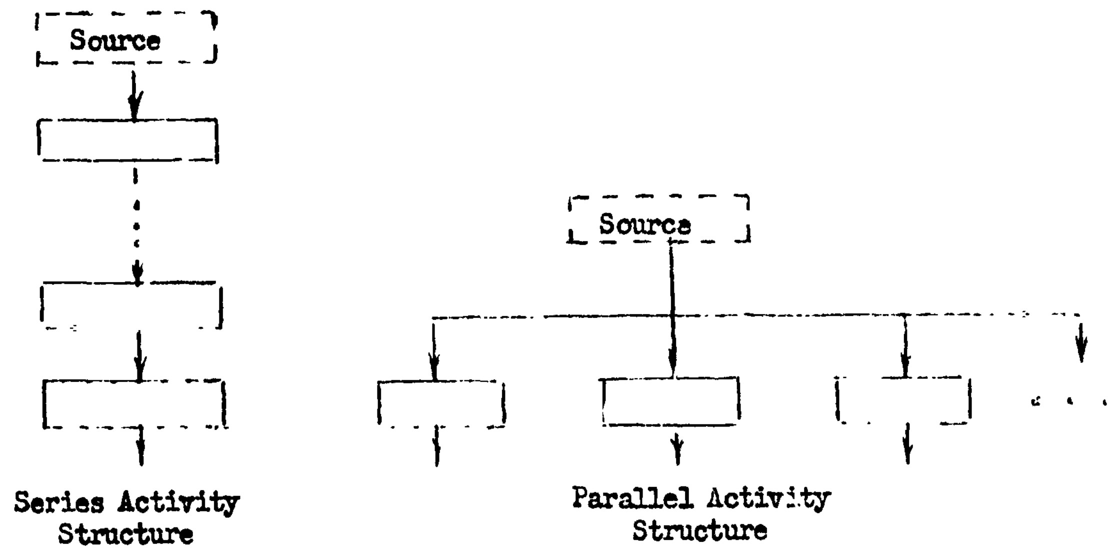

Figure 2. Special Types of Arborescence Structures

used in particular studies, expand the role of the entity to include production or repair capabilities as well as supply; thus, an activity may represent a stage of factory production, together with an in-process inventory, that satisfies requirements of other activities and which receives (orders) materials from specified source activities. Still other studies define activities as more abstract processes that are interrelated in some fashion. Due to these various meanings and nomenclatures, the more generic term "activity" will be used in this review unless otherwise explicitly defined in a particular study.

3. Multi-activity Inventory Control Problems.

Viewed in terms of a network of activities, with external demands occurring at some or all of them, the basic multi-activity inventory control problem for a given product is one of establishing rules or policies which, if followed, cause flows of the product through the network as functions of time and which satisfy a prescribed performance objective such as minimizing expected costs or meeting a prescribed level of customer service. The set of such policies, for any given system, usually contain ordering policies (resupply, procurement, production, repair), which prescribe amounts over time that each activity orders from its supplier(s), and supply policies (issuing, delivery, distribution, allocation) which control amounts over time that each activity ships to those activities designated as its customers. A common situation which warrants this distinction is one where there is insufficient stock (due to randomness of demand or a variety of other causes) at a particular supplier to fill all the orders it receives (as prescribed by the ordering policies of its customers) and some kind of rationing, or supply policy, is thereby required.

To solve this inventory control problem, a variety of models have been formulated which are characterized by environmental assumptions similar to those commonly applied in models of the single-activity problem. The more important of these distinguishing features are expressed by the following dichotomies:[1]

Deterministic - Stochastic. In a deterministic model, external demands at each designated activity are known in advance with certainty. For a stochastic model, the demands are assumed known to within a given probability distribution (or conditional distributions as used in Bayesian and other techniques).

Single-Product - Multi-Product. A single-product model deals with only one product at a time, ignoring possible interactions with other products. A multi-product model considers a number of products simultaneously in terms of at least one interrelating factor such as a budget or storage constraint.

Stationary - Nonstationary. Parameters used to define external demands are assumed to be independent of time in a stationary model whereas they may vary over time in a nonstationary model. (The same distinction may also pertain to cost factors and other elements of the problem).

Continuous Review - Periodic Review. In a continuous review model, opportunities to review the stock position of the system and to implement derived policies are assumed to occur continuously over time. In a periodic review model, such opportunities are assumed to exist only at discrete points in time, normally with a given periodicity.

Consumable Product - Reparable Product. In a reparable product model, some or all of the amounts issued to meet external demands are regenerated somewhere in the system in the form of items which, after undergoing repair or overhaul processes, can be reissued. In a consumable product model, all issues represent permanent losses to the system.

Backlog - No Backlog. A model assumes backlogging of demand if unsatisfied demands are retained and satisfied from later resupply. In the no-backlog (lost sales) assumption, unsatisfied demands are not retained.

In the review that follows, these descriptors are often used to identify and characterize various multi-activity inventory control and other models that have been developed. In particular, Section II is concerned with deterministic inventory control models whereas Section III covers stochastic inventory control models.

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[1] These are actually only convenient labels or catch-phrases used to roughly categorize models. The exact meanings must, in each case, be gleaned from the mathematical formulation of the model.

4. Allocation Models.

Decisions affecting the allocation or distribution of a given amount of stock to several activities in a multi-activity system represent supply policies in a restricted type of inventory control process. This class o problems also includes the one of redistributing existing stocks to provide better service. Problems of these kinds are reviewed in Section IV.

5. Other Multi-activity Problems.

In addition to inventory control and allocation problems, there are several other decision, planning, or evaluation problems that have been investigated in a multi-activity environment. Solutions to these problems tend to have "one-time" applications in contrast with the everyday applications of results of the inventory control models. In Section V, several such problems are identified and studies relating to their solutions are reviewed.

6. Informality of the Review.

This review of work in multi-activity inventory theory is informal in the sense that the use of mathematics is avoided and there is a considerable bias of authorship with respect to selections, interpretations, and comments. The latter is particularly applicable to Section VI, where impressions of the current state of the art and directions that future work might follow are presented.

II. DETERMINISTIC INVENTORY CONTROL MODELS.

In the deterministic multi-activity inventory control problem, schedules of external requirements to be satisfied by each activity in the system are given as functions of time for one or more products. In this problem, many of the activities are often viewed as being various kinds of production facilities or processes for which ordering policies are interpreted as production schedules. Inventories in such systems may be held in the form of semi-finished products at various production stages or as final products at terminal facilities. Various constraints are normally imposed, such as limitations on production or transportation capacities, or interproduct dependencies of several kinds. The usual objective of models for this problem is to minimize total production, transportation, and inventory costs.

One of the early investigations of the deterministic multi-activity problem was by Evans (1958-22). The stationary case was considered for a general activity network with external demands given in terms of deterministic rates at designated nodes and production occurring at other specified nodes. The formulation permits transportation and production losses and considers the production of interdependent products. Inventories are permitted, but only artificially in terms of known "stockpile rates" analogous to demand rates. Capacity limits on production and transportation are considered. With these assumptions and a corresponding cost structure, a model is developed to find flow rates through the network which minimize total expected costs per unit of time. Recognizing computation difficulties, a number of restrictions are then specified which reduce the model to a linear programming transportation problem. Finally, transport paths of the general model are restricted in such a way that results concerning multi-product systems may be derived from a two-product system and solutions to the two-product problem are given.

The first model to overcome severe limitations of early work on the problem was by Zangwill (1966-67). A periodic, multi-product, multi-activity deterministic model was developed that consisted of a linking together of single-activity models developed earlier (1966-66). The model assumes that the activities in the system are arranged in an acyclic network wherein the flow of products is uni-directional; that is, the output of any activity does not go to any other activity that directly or indirectly provides inputs to the given activity. The model is multi-product to the extent permitted by such network representations; for example, an activity with two inputs and one output can represent the combining (in a production stage) of two component products into a single output product. The model also assumes that production costs are concave, inventory costs are piecewise concave functions, demands are known in advance for a given number of time periods, excess demand is backlogged, and production/inventory times are fixed integral numbers of time periods.

Zangwill first shows that the total cost function is piecewise concave. He then shows that the function is concave on certain bounded polyhedral sets called basic sets. Using theory of concave functions, he shows that the total cost, considered as a function on a particular basic set, attains a minimum at an extreme point of the set. The union of all basic sets is proven to be the set of all feasible production schedules. The total cost function, now considered as a function of all feasible production schedules, must be minimized on some basic set, and hence, at an extreme point of some basic set. Defining the dominant set as the set of all extreme points of all basic sets, an optimal production schedule must therefore be in the dominant set. The principal result of the analysis consists of characterizations of the dominant set. For special cases of series and parallel networks, Zangwill develops dynamic programming algorithms that search the dominant set for optimal production schedules.

In a later study (1969-68), Zangwill shows that previous results for the case of a series activity structure without backlogging can be deduced immediately from the characterization of extreme flows in networks having exactly one source (sink). Furthermore, under the concavity assumptions on costs, there exists an optimal schedule which is an extreme flow in the associated single-source network. (An extreme flow is an extreme point of the convex set of feasible solutions for the problem and, as Zangwill previously demonstrated, has the property that any node in the network can have at most one positive input.) Exploiting these results, Zangwill presents an extremely efficient algorithm that is superior to the one in (67).

Recognizing the network interpretation of the deterministic problem, Veinott (1969-58) pointed out that such network models are equivalent to transshipment Leontief substitution systems and, as such, the characterization of the extreme flow follows alternatively from the characterization of the extreme points of the system's solution set. With this, Veinott expands upon the formulation of the problem as a Leontief substitution model with concave costs; in particular, the arrangement of activities in a general arborescence structure is considered. Veinott shows that the solution algorithm developed by Zangwill extends to this case but that the amount of computation, depending linearly upon the number of wholesale facilities but to the fourth power of the number of time periods, can still be extensive. With rather severe assumptions about the cost functions, Veinott presents a simpler and more efficient solution algorithm for the general arborescence model.

After summarizing and extending previous results of the single-activity deterministic problem, Love (1968-44) considered the case of a number of activities arranged in series. He shows that if, in addition to concavity assumptions, per-unit ordering costs are non-increasing over time for each activity and per-unit holding costs for each activity are always greater than or equal to those for the next activity in the series structure, then there exists a nested extreme optimal solution. He defines a nested production schedule as being one where if any activity produces in a given time period, then so does the next facility in the series structure. This result is exploited to develop a more efficient solution algorithm. Love next considers the stationary case where requirements and costs are the same in each time period. He shows that under mild conditions there is a periodic optimal schedule; that is, the amount produced at a given activity is repeated a fixed number of periods later. An algorithm is presented for computing the periodic optimal schedule for most cases of interest.

As a continuation of the Zangwill-Veinott approach, Kalymon (1970-39) developed a decomposition algorithm which is computationally feasible for arborescence structures that were previously too large to solve. The model assumes that holding costs are linear and that the production costs, at all but the final product facilities, are linear with a set-up charge. In the model, the amount of computation is shown to increase exponentially with the number of facilities having followers, but only linearly with final product facilities.

A deterministic, periodic, multi-product, multi-activity production/inventory problem was considered by Von Lanzenauer (1970-60); this work represents a generalization of previous studies by Elmaghraby-Ginsberg (1964-21) and Young (1967-63). In this study, all costs (production, holding, shortage) are assumed linear with set-up (production conversion) costs for production for each activity and product. For each product, a sequence of the activities is specified to indicate the technological ordering of production stages for the product. Each activity can process only one product in each time period. The problem is to determine the sequence of production at all activities as well as the lot size and number of lots for each product such that total costs are minimized. For this problem, a mixed bivalent linear programming model is constructed. It is noted that problems of realistic size cannot be solved by currently available integer programming algorithms.

III. STOCHASTIC INVENTORY CONTROL MODELS

In the single-activity inventory control problem with stochastic demands, a simple kind of ordering policy, identified as an (s, S)-type policy, has been subject to intensive analysis. For this policy, an order is placed to raise available stocks (on hand plus due in) up to S (stockage objective) whenever the available stock becomes less than or equal to s (reorder point); for dynamic models, the values for s and S may change over time. The policy parameters, s and S, are commonly referred to as "critical levels".

The (s, S)-type ordering policy has also played a major role in multi-activity research. Although a number of techniques previously used for single-activity problems have been extended to the multi-activity problem, the use of (s, S)-type policies represents a common denominator. Some of the multi-activity research has been concerned with finding conditions under which these policies, when applied to individual activities, are optimal from a system point of view. Other studies investigate system behavior when such policies are assumed and methods are sought for assigning values to s and S at each activity such that some system performance criterion is satisfied. In general, whenever the use of (s, S)-type policies can be justified, the multi-activity inventory control problem becomes more computationally tractable; otherwise, as several of the studies have demonstrated, the amount of computation required to find optimal solutions becomes prohibitive for problems of realistic size.

For convenience, the more important techniques used to analyze the multi-activity inventory control problem are labeled as follows, with the authors of initial or early papers being indicated in each case:[1]

Expected Cost Analysis (Berman-Clark)
Stationary Process Analysis (Love)
Dynamic Programming (Clark-Scarf)
Dynamic Process Analysis (Bessler-Veinott)
Network Theory (Connors-Zangwill)

In most of these approaches, several papers have appeared which extend or refine the initial development; these are reviewed below in approximate chronological order.

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[1] Other techniques that have been used, or are potentially applicable, include stochastic linear programming, servomechanism theory, and simulation.

1. Expected Cost Models.

In expected cost models, the form of the inventory policies is expressed in terms of a few variables; normally, (s, S)-type policies are assumed with s and S considered as unknowns. Expressions for expected costs per unit of time are then constructed as functions of the policy variables. Values for the policy variables that minimize the expected average costs are found by calculus or difference equation methods. Such models are further characterized as being stationary and usually stochastic; either continuous or periodic review may be assumed. The first analytic work on inventory problems used this approach; in fact, the famous Wilson lot size formula is the solution of such a model (with deterministic demand).

One of the early attempts to extend this approach to the multi-activity inventory problem was by Berman-Clark (1955-5). This work treated a single product in a two-level activity structure (several bases supplied by a depot) in context with three main procurement practices: life-of-type, where sufficient amounts are purchased all at once to satisfy all expected future demands; periodic procurement, where a system purchase is made at regular time intervals in sufficient quantities to last throughout the period; and open-end contracting, where an order on a manufacturer is placed each time the system-wide stocks reach a minimum (critical) level. For each type of procurement, expressions for expected average costs, as functions of (s, S)-type policy variables, are derived which include representations of the following features: random external demands at bases; repair or no repair (reparable or consumable items); two kinds of resupply time for bases, normal and expedite; transshipment among bases; obsolescence probabilities; random lead time and repair cycles; and the usual holding, shortage, and ordering costs, all linear. From these cost functions, expressions for minimizing values of the policy variables are obtained. For bases, the critical levels are functions of Lagrangian multipliers which are used to control distribution among the activities in the system.

Hadley-Whitin (1961-28) also used the expected cost approach to analyze the inventory problem for a low-demand item in a single-level system of parallel activities (depots). Demands at each depot are assumed to be independent, stationary Poisson distributions. Procurement lead time is assumed to be constant as well as the time required for either of two available modes of redistribution among the depots. Items are ordered one at a time from an external source as demands occur; therefore, the total system stock remains constant over time. It is assumed that redistribution to a depot will not occur until there is a back order. With these assumptions and linear functions for inventory costs, decision rules are developed for allocation of new procurement, redistribution of stocks among the depots, and for determining depot stockage objectives in order to minimize expected costs resulting from system and depot stockouts, cost of redistribution, and costs of transportation from the source. When more than a single unit is on order from the external source, the allocation of each unit ready to be delivered is determined by the solution of a dynamic programming problem. The assumption that demands are replaced one-for-one by procurement and the form of the redistribution policy confine the model to low demand items.

In a later study (1963-29), Hadley-Whitin used the same general approach to consider the case of higher demand items. This time, it is assumed that the system as a whole uses an (s, S)-type policy and that redistribution is considered whenever a depot's stock falls to critical levels that are set by external criteria. Furthermore, only that depot triggering the redistribution decision is considered as a receiver each time. All other assumptions for this model are essentially the same as for the previous one. Again, cost minimizing expressions are obtained for determining stockage objectives at the depots, values for the system procurement policy, and sources and amounts for redistribution decisions. A dynamic programming algorithm is presented for allocating system procurements to the depots upon receipt from the external source.

The same problem (a number of parallel activities in a single-level system subject to centralized control) was also investigated by Gross (1963-26). In this study, however, attention is confined to a single time period and no *a priori* assumptions are made concerning the form of stockage and redistribution policies. The objective is to find amounts to order and/or to redistribute at the beginning of the period such that total expected costs during the time period are minimized.[1] In the model, all costs (ordering, holding, shortage, transshipment) are assumed linear without set-up costs. Deliveries into the

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[1] It should be noted that confinement of attention to a single time period avoids implicit assumptions, which are not always valid, concerning the stationary properties of the inventory process in infinite horizon expected cost models. Thus, this model can be considered as being truly optimal for the stated problem, whereas the other expected cost models are generally only approximations.

system and transshipments occur at the start of the period (zero procurement and transshipment times). External demand may be given by any continuous density function which may differ by location. Under these assumptions, a total cost function is first formulated for the case of two locations. From this, minimizing values for ordering and transshipment amounts are derived which are dependent upon the initial stock posture at each activity and the system as a whole. An iterative procedure is then given to generalize the results to an arbitrary number of locations.

The problem formulated by Gross was also considered by Krishnan-Rao (1965-41), but with one important difference. Instead of allowing transshipments to occur at the start of the period in anticipation of expected shortages, it is assumed that shortages actually realized at the end of the period are satisfied from stocks at activities with ending surplus; thus, transshipment amounts are not considered as policy variables as in the Gross model. Because of this simplifying assumption, Krishnan-Rao obtain a simple solution to the problem of determining amounts to order that minimize the expected costs. Numeric examples are given to illustrate results of the model.

2. Stationary Process Analysis.

If nonstationary aspects of inventory problems are removed, then techniques based upon Markov processes and elements of renewal or queuing theory become useful. Here, a fixed ordering policy of a simple form (usually an (s, S)-type policy) is chosen for which the inventory level over time becomes a particular stochastic process. The principal problem then is to find a stationary distribution of the process which, if it exists, will be a function of the policy used and of the demand distribution, but not of any costs that might be involved. However, a cost structure can be imposed upon the process in expressions representing average expected cost per time unit. These may then be minimized with respect to the several parameters that characterize the policy.[1]

One of the first investigations of a multi-activity inventory control problem using a stationary analysis method was by Love (1967-43). In this study, a two-activity series system is considered, with both activities using continuous review (S-1, S) policies (such policies are also commonly called one-for-one ordering policies). External demand, occurring only at the lower

[1] Although the expected cost method is similiar to this procedure, it does not include a formal analysis of the stationary properties of the underlying stochastic process; instead, implicit assumptions are made in this regard which are not always valid.

activity, is assumed to be given by a Poisson distribution. Resupply time at both activities is exponentially distributed. Per unit shortage costs are assumed to be linear according to age of the back order. Holding costs are linear and reorder costs are not considered since ordering is one-for-one as demands occur. Under these assumptions, expressions are obtained for expected total waiting time (e.g., back order days) and stock on hand per unit of time. Love then demonstrates convexity properties of the expected total system cost as functions of the stockage objectives and, as a result, the existence of optimal stockage objectives (values for S at each activity). He then establishes bounds on the minimizing values and gives an algorithm for determining the optimal policies.

The inventory control problem for a repairable item in a two-level supply/repair system was investigated by Rosenman-Hockstra (1964-46) using a stationary process approach. In this study, it is assumed that items can be repaired locally (area facilities) or centrally (NICP - National Inventory Control Point) according to given rates and that losses to the system are negligible. It is further assumed that external demands are Poisson distributed, that lower-level facilities use continuous review, one-for-one (S-1, S) ordering policies, and that replenishment times and repair cycles are given constants. Under these assumptions, a cost-free model is developed with the objective of distributing a given system stock among the various activities in order to minimize total expected customer waiting time.

In the model, stationary distributions for number of items in the NICP repair cycle and net (serviceable) stock at the NICP are developed, from which an expression for average delay in satisfying demands from lower activities is derived. Similarly, distributions for repair cycle and stock on hand at the area facilities are given, from which the average number of back orders are obtained and summed over all areas to find the average customer waiting time. Using these results, a marginal value method is used to distribute system stock in order to minimize total customer waiting time.

In addition to the model itself, the paper presents actuarial and exponential smoothing techniques for demand forecasting, and discusses institutional methods and procedures for implementing the results.

Sherbrooke (1968-48) also used the stationary process approach to analyze the multi-product problem involving inventories of recoverable (repairable) items in a system of parallel activities (bases) supported by a higher level activity (depot).

In the model, a Bayesian procedure is used for estimating external demands (occurring only at base level), with the prior distribution being a gamma function and the underlying demand being a logarithmic Poisson process. The derivation of these distributions is discussed in some detail. Although the expression of demand as a Bayesian process is separable from the multi-activity and other structural aspects of the model, the use of the indicated distributions enables the derivation of a distribution of demand observed by the depot that provides a decomposition of the problem.

The model assumes (S-1, S) continuous review policies at lower activities and since all stocks are conserved (being always reparable), there is no system re-ordering after initial stocks are established. The primary objective of the model is to establish stockage objectives (values for S) at each activity which minimize the sum of expected back orders on all recoverable items at the lower installations for a given budgetary constraint. Each recoverable item may be repaired at base level and/or at depot level according to a specified ratio. Economies of scale are not considered and lateral resupply (redistribution) is assumed to not occur. Relative back order costs may be included in terms of essentialities by base and by item. Repair and shipping times are assumed random (implicitly in the assumed form of the demand distributions). Under these assumptions and objective function, a five-step procedure is given for finding optimal solutions. First, using a previously developed expression for expected number of base back orders, the average delay per demand against the depot is found for each item as a function of depot stock. Second, for each level of depot stock and each base, expected base back orders are calculated as a function of the base stock. Third, for each level of depot stock, an allocation to the bases is made which minimizes the total expected back orders; this is done by a marginal analysis method. Fourth, the minimum expected system back orders is found as a function of total system stock (bases plus depot). Finally, the multi-item aspect is considered by the use of a marginal value method to allocate a given investment across items: each additional increment of investment is assigned to that item for which the largest reduction in expected system back orders will result.

The mathematical justification for this computational procedure is then given wherein the objective function is also generalized to allow constraints upon the expected number of back orders at each base. For the generalized objective function, an integer programming problem results which can be reformulated with Lagrange multipliers and is separable by item. However, computational difficulties arise if the Lagrange multipliers differ by base; this occurs whenever separate back order constraints are given by base. If they are all assumed equal, the marginal analysis method used in the computation procedure provides a good approximation to the optimal solution, particularly if the number of items is large enough.

Although the model assumes stationarity, it is suggested that periodic recomputations be made whenever systemic changes in the support or demand structure become known. The paper concludes with two generalizations of the model: the use of a Bayesian procedure for estimating the ratio of base versus depot repair, and the inclusion of condemnation (system losses) in the model formulation.

Confining attention to stationary properties of the inventory process, Simon (1971-50) refined and extended corresponding portions of the Sherbrooke model. As before, a parallel structure of bases supported by a depot is considered where repair of a failed item can occur at base and/or depot level. The model allows system losses to occur at a specified rate; a totally reparable or totally consumable item can be considered according to the value of this condemnation rate. For bases, an (S-1, S) replenishment policy is assumed; at the depot, an (s, S) policy is used. Repair times, shipping times, and procurement lead time are all assumed to be deterministic and independent. External demands are assumed to be generated by independent Poisson processes with given rates. With these assumptions, Simon investigates the stationary properties of a given item in the multi-activity structure. For each base, he obtains exact expressions for the steady-state number of back orders and proves that the number of units in repair is stationary and Poisson distributed. Similar results are obtained for the depot. For a consumable item, exact and approximate expressions are derived for the stationary expected depot response time to a base demand; the approximate expression is analogous to one used by Sherbrooke.

3. Dynamic Programming.

The mathematical technique most often used in formal inventory theory is the functional equation approach of dynamic programming. In this method, periodic review is assumed and optimal policies for a given time period are found from a cost function that normally consists of three parts: (a) costs of ordering, if any, (b) expected (period) costs which are independent of the ordering policy and (c) expected future costs (from the next period on) which do depend upon amounts ordered. Under suitable restrictions upon cost and demand processes, simple types of cost minimizing policies exist and may be calculated by a recursive process which starts at some future time period and works backwards in time.

As a continuation of the classic dynamic programming approach used in single-activity problems, Clark (1958-13) and Clark-Scarf (1960-16) formulated and solved the problem of a single product carried at an arbitrary number of activities arranged in a series structure. In the model, external demand can be stochastic, dynamic, and can occur at any arbitrary mix of the included activities. The model permits economies of scale (fixed reorder costs) only at the highest activity in the series structure. At lower activities, ordering (shipping) costs are assumed linear as functions of amount ordered. The model assumes periodic review and expected period costs (holding and shortage) are assumed convex. Shipping times are constants but can be an arbitrary number of periods for each activity. Backlogging is assumed at all activities.

The essential innovation of this model is the interpretation of the inventory system as a nested set of echelons rather than as individual activities. The model associates, with each activity, an echelon consisting of all stock in the system at that activity and below, including all on hand and in transit (due in) amounts. With this interpretation, the multi-state variable problem for the system as a whole (i. e., the on hand and due in amounts at each activity are separate state variables in the total cost function) can be decomposed into a set of interconnected one-state variable problems, one for each echelon in the system. Each of these problems can then be solved, by the usual single-activity technique, for critical levels constituting the optimal ordering policy.

The set of one-state variable problems are interconnected by "implied shortage costs" generated at each echelon (except the highest one) and passed on (included in the cost function) to the next higher echelon. Thus, the optimal policy is first established for the lowest echelon, from which implied shortage costs are obtained. These costs are then included in the cost function for the next higher echelon for which the process is repeated. The procedure continues up the echelon structure until optimal policies are obtained for each echelon. The ordering policies determined in this fashion take the form of periodic (S-1, S) policies at lower echelons and an (s, S)-type policy at the highest echelon. The issuing policy is that each activity satisfies as much of an order (demand) as may be permitted by available stocks in each instance.

In (16), a brief analysis of the parallel echelon structure is given. It is shown that, in general, the problem cannot be broken down into a set of single-activity problems. However, a solution of this character is given under the assumption that the stock positions at the several lower installations is continually "in balance", i. e., that the stock position at each activity is not significantly over or under that of the other activities relative to expected costs. A concluding short section of this paper discusses extensions of the theory to serial repair and production processes.

In a subsequent paper (1962-17), Clark-Scarf considered the case of a fixed reorder cost at lower installations with all other elements of the problem the same as before. In this case, it was found that the optimal solution could not be broken down into a sequence of single-state variable problems, but that the cost function for the optimal solution could be bounded (above and below) by such calculations. Thus, critical levels for each activity and time period could be found with associated costs which bracket those of the optimal solution. Furthermore, it is demonstrated that if the policies (critical levels) for the upper bound were used, then the actual costs incurred would be less than or equal to those of the upper bound.

A more comprehensive multi-activity inventory control model based upon the Clark-Scarf approach was developed (in 1962) by Clark-Metcalfe; although the model itself was not documented (except in the form of a computer program), example results are given in (14). This model incorporated features which, although not proven optimal, provides more realistic representations of practical inventory problems. The model permits stochastic, dynamic external demands at any activity in a general arborescence structure, recovery through repair (he treatment of this aspect of the model was later shown to be optimal (9)), deterministic time lags (procurement, resupply, repair cycles) of arbitrary length, multi-stage production with inventories permitted between stages, and a form of continuous stock review. The cost structure of the model is analogous to that of the Clark-Scarf model. For portions of the activity structure consisting of activities in series, the implied shortage cost functions of the Clark-Scarf method are used to interconnect echelons. For portions consisting of parallel activities, a shortage cost function is built up from the cost functions of the lower activities to provide echelon interconnections (see (45) for a description of this procedure). Outputs of the model consist of values for a single critical level at each lower activity and values for an (s, S)-type policy at the highest activity; a supply policy is derived from the implied shortage cost functions. In conjunction with the decision model, a simulation model was constructed to provide realizations of system performance while operating under the ordering and supply policies generated by the analytic model.

Continuing the Clark-Scarf formulation, Hochstaedter (1970-31) considered the case of activities in parallel with a common supplier. The cost and demand structure was assumed the same as in the Clark-Scarf model, but with fixed reorder costs permitted at the lower activities as well as at the higher echelon. As in (17), upper and lower bounds are established for the optimal system costs, with each set of bounds yielding (s, S)-type ordering policies for each activity. The bounds are established by first computing ordering policies for each lower activity using the standard single-activity formulation. It is shown that the optimal system costs are then bracketed by the sum of the lower activity costs plus a function of system stock alone. For the lower bound, the function is given by the costs obtained by viewing the system as a single installation faced with a demand equal to the sum of the lower installation demands. The upper bound is given by the same cost function, but with period costs (for the higher echelon) being augmented by an implied shortage cost derived from the lower installation computations; these costs are analogous to those previously described but assume a different functional form. As in (17), Hochstaedter suggests the use of levels associated with the upper bound as an approximation to the optimal policy. He also derives an expression for the difference between the upper and lower cost functions which is useful in assessing the accuracy of the approximation.

The extension of the Clark-Scarf series multi-activity model to include combined ordering and disposal policies was accomplished by Fukuda (1961-25). In general, disposal policies determine conditions under which excess stock is removed from inventory at some cost which may be negative indicating a revenue from the disposal. Addressing first the single-activity case, Fukuda assumes that the decision to be made at the beginning of each period is to either order, dispose, or neither. With this assumption and the usual cost structure, he determines optimal ordering-disposal policies in terms of critical levels for each time period. These results extend directly to the series echelon structure wherein critical levels are obtained for each activity and time period. In this case, implied shortage costs for disposal are derived which are analogous to those for ordering in the Clark-Scarf model; such costs apply whenever a higher echelon wants to "sell" some excess stock but cannot because the stock is physically located at some lower level. The implied shortage costs for disposal are passed downward (included in the cost function of the next lower echelon) in contrast to the implied shortage costs for ordering which are passed upward in the structure.

In previous papers, Scarf and Fukuda developed, respectively, Bayesian and maximum likelihood policies for the single-activity problem. Fukuda (1960-24) extended these results, combined with the disposal policies of (25), to the Clark-Scarf series model. For the Bayesian policy, Scarf had shown that the single-activity problem could be reduced to a one-state variable problem if all cost functions are linear and if the demand distributions as well as the prior distribution of the demand parameters are members of the gamma family. Fukuda now shows that this result extends directly to the series multi-activity case; namely, that the solution of an interconnected sequence of one-state variable problems provides optimal Bayes ordering policies for the series structure. Furthermore, the extension holds up in the situation where each activity in the system experiences external demand; thus, Bayes policies are determined for various serial repair and production processes. Finally, Fukuda extends the results to the disposal case, resulting in a Bayesian ordering and disposal policy for the series situation. Analogous results are obtained for the maximum likelihood approach, wherein demand is assumed to be given by a gamma distribution with an unknown parameter and critical levels (ordering and disposal) are obtained for each activity and time period as linear functions of the unknown parameter which are then evaluated by expressions based upon past demand observations.

Veinott (57) used a convexity theorem of Karush (40) to yield a much simpler proof of optimality for the Clark-Scarf model.[1] The theorem was also applied by Iglehart-Morey (1971-36), in the case of two facilities in series, to obtain optimal ordering and shipping policies that take into account the accuracy of demand forecasts; otherwise, the model is the same as in (16). In this model, it is assumed that demand forecasts are made each period which may differ

[1] Veinott (1971-59), using a network approach, also gave a simple solution to a one period problem of a series structure with zero resupply times.

from the true demand. The cost function for the model includes conditional expectations with respect to demand given the forecast. As results, critical levels are obtained that are analogous to those of the Clark-Scarf model except that the levels for the lower activity are expressed as a function of the forecast.

Zacks (1970-64) formulated a Bayesian model of the two-echelon parallel activity structure, assuming that Poisson-distributed demands occur at the lower activities and that the prior distribution of the Poisson parameters is a gamma distribution. He assumes no transshipment among lower activities and no return of stock to the higher activity. In the cost structure of the model, linear holding and shortage costs are assumed, but no ordering or shipping costs are included. With these assumptions, a multi-state variable dynamic programming solution procedure is developed. The optimal Bayesian policies are expressed as functions of on hand and due in amounts at the various activities; the analysis does not provide simple policies in the form of critical levels and rules for their use.

In a later paper (1970-65), Zacks reconsiders the model, this time allowing unrequired stock at lower activities to be returned to the higher facility, linear ordering costs (but no fixed component) at the higher facility, and no shipping costs between echelons. The same general Bayesian approach as in the earlier model is used, and all other assumptions are the same as before. Again, a dynamic programming formulation is presented, from which the form of the optimal policies is derived. The principal result of the analysis is that the optimal ordering policy of the lower activities is obtained by solving an integer convex programming problem with linear constraints. The ordering policy for the higher activity is expressed as the minimizing value of a recursively defined function; ordering amounts depend upon stock on hand plus due in and upon past demand observations.

The case of activities in a series structure, where each activity has a fixed ordering cost in addition to the usual inventory costs was investigated by Williams (1971-61). Using the Clark-Scarf definition of echelon, a multi-state variable dynamic programming model is developed for both the backlog and lost sales case. For the backlogging model, the assignment of shortage costs as a function of the age of the back order is considered, but it is noted that this increases the number of state variables of the model. The special case where a product unit progresses through all activities in the series structure within the same time period is considered in the form of a sub-model. For the sub-model, calculation procedures are presented which are considerably more efficient than those of the more general model.

4. Dynamic Process Analysis.[1]

In previous papers, Veinott developed a technique for analyzing inventory problems, first for the single-product, single-activity case and then for the multi-product, single-activity problem. Veinott's methods represent a departure from the traditional functional equation approach of dynamic programming, being based instead upon a direct analysis of the underlying stochastic process to obtain the form of optimal policies and to investigate their properties. These results were extended by Bessler-Veinott (1966-7) to the multi-activity inventory problem.

In the first section of this paper, the correspondence between the multi-activity and multi-product, single-activity problems is established. In general, when the stock at each facility is viewed as a product, results of the multi-product, single-activity problem have corresponding interpretations for the single-product, multi-facility case. The principal result is that if nonstationary aspects are suitably restricted and the initial stock is small enough the optimal policy in each time period is to order up to a critical stock level at each facility, where the critical levels minimize a one-period cost function which includes the usual inventory costs except for fixed reorder costs.

In the next section, the arrangement of the activities in an arborescence structure is considered. It is assumed that the demands during each period at each activity are satisfied by available stocks at the facility, with excess demands being immediately transmitted to its supplier for possible satisfaction. Excess demands are successively passed up, with backlogging occurring only at the top supplier. With this supply policy, the one-period cost function is established from which the optimal stock levels are derived. In general, the optimal policy is obtained by solving N n-dimensional minimization problems, where N is the number of periods and n is the number of activities.

Having obtained the one-period cost function, several effects of parameter variations are investigated, supported by numeric calculations for a system consisting of three parallel activities supplied by one higher-level activity. Various properties of the shortage cost function and the one-period cost function are also established for use in the subsequent development.

1. This description for the Veinott technique was coined for purposes of this review.

Next, bounds are established for the optimal stock levels. The upper bound for a particular activity is based upon the assumption that there is no stock at any facility supplying or supplied by that facility. The lower bound is then calculated under the assumption that each facility supplied by the facility stocks its upper bound and the supplying facility has a very large stock. An iterative procedure is then given to sharpen these bounds, although it is suggested that, in practice, it may not be worthwhile proceeding beyond the initial values. Finally, an algorithm is presented for computing approximations to the optimal levels based upon the values for the lower bounds; it is noted that the approximations are often optimal for the series activity structure.

In the last two sections of the paper, several relations among facilities having a common supplier are established and simplifications resulting when there are symmetries in the supply structure are discussed. In particular, it is proven that if two facilities having a common direct supplier "look alike" and if certain convexity assumptions are imposed, then it is optimal to stock the same quantities at both facilities.

In the Bessler-Veinott model, it was necessary to place restrictions upon starting stocks in order that optimal policies be given by minimizing one-period cost functions. This restriction was removed by Ignall-Veinott (1969-37) for activity networks having a "nested" property and a specified supply policy; this paper also established the characterization of policies satisfying the one-period minimization criterion as being "myopic". The model presented assumes that at the beginning of a period, stocks are ordered for each activity at a common unit cost. A supply policy is given in terms of a procedure for allocating the starting stocks to satisfy demands at the facilities. This procedure is based upon identification of groups of activities as being suppliers of other groups of activities. The model assumes that stocks at one facility may satisfy shortages at any facility providing that they are replaced from exogenous sources at the beginning of the next period. With these assumptions and the cost structure of the previous model, it is shown that the myopic policy is optimal for any initial stock posture. This result, established first for the stationary case, is then extended to the case where there is a delivery lag and to cases of nonstationary costs and demand distributions.

5. Network Theory.

Although multi-activity inventory systems have long been recognized in terms of a network of activities with flows of material occurring among the nodes, the first application of network theory for the stochastic inventory control problem was by Connors-Zangwill. In principle, this approach is an extension of the network analysis of the deterministic problem.

In the Connors-Zangwill model (1971-18), requirements or availabilities at the nodes are allowed to be discrete random variables with known conditional probability distributions. For a given number of periods and linear objective criterion, the problem is to calculate the network flow that minimizes expected cost. To handle this problem, special kinds of networks, called "replicating networks", are defined which contain two special types of nodes in addition to the normal ones: replicating nodes and collating nodes. A replicating node has a single input arc and several output arcs such that the flow on each output arc is identical with that on the input arc; a collating node has the opposite property. After formulating a problem in terms of a replicating network it is transformed into an equivalent replicating network having only one source and one destination. For the transformed network, an efficient computation algorithm is presented which is analogous to the standard min-cost/max-flow algorithm for deterministic models. It is shown that although this problem can be solved by linear programming under uncertainty, the network approach is much more efficient for large problems. The method and computation procedure are illustrated by a two-period production/inventory problem with backlogging.

IV. ALLOCATION MODELS

The problem of allocating or distributing a given amount of system stock to the various activities in a multi-activity system may be viewed as a restricted type of multi-activity inventory control problem. If such allocations include existing stock postures at the activities, the problem becomes one of determining redistributions in order to satisfy a system cost or performance objective. Such redistribution problems naturally occur when stocks at the various activities get "out-of-balance" or maldistributed because of the random characteristics of demand or previously unknown shifts in demand patterns.

Studies of the allocation problem are characterized by the assumptions that are made concerning the problem to be solved and by the analytic technique that is used. The particular models reviewed below are representative of the various kinds of problem formulations and solution procedures that have been developed.

One of the earliest investigations of the allocation problem was performed by Simpson (1959-52). He assumes that there are a number of activities (warehouses) faced with independent external random demands and that the system as a whole receives a given amount of stock for a particular item which is to last (satisfy demands) for a fixed length of time. Emergency replenishment is allowed whenever the stock at a warehouse is reduced to a previously established trigger level; it is assumed that fixed cost penalty is charged each time this happens and that no distinctions are made as to the source of the replenishment (external or internal). The problem then is to allocate the given quantity to the warehouses to satisfy given performance criteria. Two criteria are considered: system cost minimization and minimization of weighted number of unsatisfied demands.

Under these assumptions (and others that are not explicitly stated), simple allocation rules are obtained for the two performance criteria. For the cost minimization criterion, the rule states that the probability (weighted by emergency resupply penalty) of demand being equal to the quantity allocated minus the emergency trigger level be equalized across all warehouses. For the second criterion, where weights are specified across warehouses for lost sales, the rule states that the probability (times the weighting factor) of demand exceeding the quantity allocated be equalized across the various warehouses.

A redistribution problem involving a one-time decision to relocate stocks among several user locations was investigated by Allen (1958-1). A model is constructed with the objective of minimizing transportation costs plus shortage costs at the end of a defined period of time. It is assumed that transportation costs, for any pair of locations, are proportionate to the amount moved between the pair. With given assumptions about the demand distributions, various properties of the solution to the problem are established which permit the use of a simple iterative computation method. With less general assumptions about unit transportation costs, even simpler methods of solving the redistribution problem are obtained. In particular, the cases of zero costs and costs independent of origin and destination are examined.

Allen continued the study of this problem (1961-2), but this time with a fixed set-up charge for each shipment from one location to another. Also, a normal distribution of demand at each activity is assumed, and only the case where per-unit transportation costs are all zero is considered. With these assumptions, an expression for total system cost of redistribution is obtained which is transformed into a function dependent only upon the total shipments from or to an activity. From this, total system cost is derived as a function of normalized stock positions before and after redistribution and of a 0-1 variable identifying activities that actually ship or receive material. A method for minimizing this function is then derived and a computing procedure is suggested. Further development of the computation procedure is given in (3).

The classical dynamic programming formulation of the inventory control problem is used by Iglehart-Lalchandani (1967-35) in a periodic review allocation model. A system of two activities is considered, each experiencing external random demands that are not necessarily independent. Period costs (expected holding and shortage costs) are assumed to be convex; shipping costs are linear without a fixed component. A critical final assumption is that the sum of the inventory levels at the two installations after delivery of any allocated stock must be less than a given fixed amount. Thus, the problem is one of periodically allocating a fixed total system stock to the two activities. Under these assumptions, it is found that the optimal (least cost) policy is given by critical levels for each activity and time period, and simple rules are given for their use which depend upon the respective stock levels at the two activities. It is stated that an extension of the problem to more than two installations would require additional techniques than those used in this study.

V. PLANNING AND EVALUATION MODELS.

In contrast with inventory control models that are directed towards managing the everyday flow of material in a multi-activity inventory system, a number of models have been developed for use in various kinds of planning or evaluation problems. As examples of such problems, it may be desired to choose particular activities in a multi-activity system which should carry each product in a multi-product inventory in order to minimize overall costs, or evaluations are desired of the relative merits of several different postulated multi-activity inventory control policies, or locations are to be found in an existing multi-activity system at which stocks are to be positioned to meet possible one-time requirements. Models that have been developed for these and other problems of similar nature are reviewed in this section.

1. Inventory Positioning Models.

Given an inventory system consisting of a number of interconnected activities that collectively carry a variety of products, it can well happen that the mix of external demands in terms of where they occur, as well as identifications of external sources can vary from one product to another. Also, even though demands can occur at a given activity, it may not be economical to carry inventories there in view of alternative supply opportunities elsewhere in the system. An important problem in this situation is one of determining, for each product, the particular activities at which inventories should be carried such that total system costs are minimized or some other system performance objective is satisfied. Different aspects of this problem have been investigated in the studies reviewed below.

One of the earliest investigations of this kind of problem was a study by Simpson (1953-51). This work was concerned with a multi-stage manufacturing process where in-process inventories may be carried. The general problem is to determine stages of the process that should carry in-process inventories and amounts to be carried at each such stage in order to achieve a desired level of operating efficiency. To solve this problem, Simpson formulates an expected cost model which assumes an ordering mechanism wherein each unit withdrawn from any stage is immediately reordered from the previous (next higher) stage. Also, it is assumed that for included inventories, stockage objectives are set in advance to provide specified levels of service; thus, assessment of shortage costs are avoided. It is shown that the domain of the cost function under these assumptions is a convex polyhedron and that the minimum is attained at a vertex of the polyhedron; each vertex corresponds to a particular combination of production stages that should carry inventories. It is suggested that the cost function be evaluated at each vertex in order to select the minimum cost solution.

Hanssmann (1959-30) continued work on the problem presented by Simpson but under less restricting assumptions. He permits periodic ordering for each inventory, where ordering decisions are based upon the actual inventory level, quantities due in, and forecasts of future demands (sales). Demand is assumed to be stationary and given by normal distributions. Sales revenues and inventory holding costs are considered, but a functional relationship between average delivery time and expected demands used in lieu of shortage costs. System optimization is therefore defined as the maximization of the difference between sales revenues and inventory carrying costs. To solve this problem, Hannsmann first derives expressions for expected overages and shortages as functions of a target (critical) level for a particular product and activity. These results are used to derive an expression for expected profit (revenue minus inventory cost) as a function of delivery time which is then maximized to find the optimal delivery time from which the optimal ordering policy is found. These results are then related to the multi-activity situation, where delivery times for each activity are dependent upon the existence of stockages at higher activities. An expected profit function is developed containing expected delivery times as variables. Again, maximizing yields optimal delivery times and, hence, optimal ordering policies; if the resulting target level is zero for a given activity, then no inventory is carried there. Finally, the same approach is extended to the multi-product, multi-activity problem. A dynamic programming procedure is used to find delivery times and corresponding positionings of inventories that maximize expected profit.

In another study, Brown-Silverman-Perlman (1971-10) consider a parallel system of activities (aircraft carriers) supported by a forward supply point where all included activities carry all products of interest. This system, in turn, is supported by a higher level system which is abstractly represented in terms of alternative resupply processes and subprocesses (these processes may be viewed as "activities" in the broad sense). It is noted that the length of time required to resupply the lower system is dependent upon which combination of higher level processes are incurred; this length of time, in turn, affects the amounts and costs of inventories required in the lower system. Since each higher level resupply activity has associated costs, the problem is one of selecting a combination of the resupply activities, from among those available, for which the sum of inventory costs and resupply process costs are minimized.

Inventory costs in the lower system are determined by repeated use of a single-activity model by Brown-Corcoran-Lloyd (9); this is an extension of the classical dynamic programming model to include reparable items. Application of the model to the lower activities (all assumed to be the same) yields critical levels from which a demand probability distribution against the higher activity (forward supply point) is derived and approximated by a Poisson distribution. The model is then applied to the higher stock point considered as a single activity. It is noted that this procedure results in overstated inventory costs but the analysis proceeds under these assumptions.

Next, the higher level resupply system is considered in terms of alternative subprocesses, each having an associated response time and level of expenditure (in general, the level of expenditure increases as response time decreases). The total system cost is then given as the sum of costs for selected processes and resulting inventory costs of the lower system which depend upon the response times of the selected processes. The optimization problem becomes a 0-1 integer programming problem which is solved by a branch-and-bound algorithm. The model is applied to each product independently; no account is made of possible economies of scale in the resupply process expenditures.

A study by Pincus (1971-45) deals directly with the inventory positioning problem where all activities in the multi-activity system represent different stockage facilities. In this study, a basic multi-activity structure is assumed, from which a substructure is to be selected for each product in order to minimize expected costs over all products and facilities. For each possible substructure, inventory costs will result for each product and each included activity will have various fixed (capital expenditures) and recurring (operating) costs. Based upon a matrix of such costs for all possible activity substructures and all products, a 0-1 integer programming problem is formulated, the solution yields a substructure for each product such that total inventory and facility costs are minimized. A branch-and-bound algorithm is presented to solve this problem.

As part of the inputs to this model, it is necessary to derive expected inventory costs for a given product in a given multi-activity substructure. Although it is pointed out that any one of the several available multi-activity inventory control models can be used to obtain these costs, a model based upon the Clark-Scarf (16) approach was formulated for purposes of the study. The use of this model is illustrated by numeric examples. Finally, an example consisting of 30 products and ten activity substructures is presented to illustrate the use of the overall design model.

2. Repair Level Decision Models.

Many multi-activity inventory systems consist of stocks of spare parts used in repair and overhaul processes for some end article or articles. Viewing the end item as a hierarchy of component parts, an important problem is one of determining, for each component in the hierarchy, whether or not to repair it when it fails: this is also known as the repair versus discard decision problem. In a multi-activity inventory system, an interrelated problem is one of deciding where to repair a component if it is to be repaired. Solutions to these problems will control the mix of inventories at each activity in the system as well as the mix of other required resources such as repair equipment and personnel. It may be noted that this problem is analogous to the inventory positioning problem but with additional dimensions of complexity.

Although this problem has been investigated in several studies, the one by Williams (1969-62) is perhaps the most general. Here, a model is developed which accepts as inputs descriptions of the hierarchical structure of the end item (s), both in terms of the physical breakdown of included parts and in terms of steps taken to isolate faults that might occur. Also input are factors describing the multi-activity repair and supply system, reliability factors for included parts, and various cost functions relating to the repair and supply operations under stationary assumptions. Based on these inputs, the model uses a dynamic programming solution method to find combinations of repair actions to be taken which minimize overall support costs. As outputs, the model specifies whether or not a component should be repaired when it fails and, if repaired, which facility in the structure should do the repair. These outputs, in turn, determine for each item the particular activities in the system that will experience demand because of the repair decisions and, hence, where inventories of the item should be established. The outputs also control the positioning of maintenance resources in an analogous fashion.

3. Contingency Support Models.

An important problem in military (and some commercial) organizations is the acquisition, maintenance, and distribution of inventories in support of various kinds of postulated contingency operations. The problem has multi-activity aspects of two general kinds: (a) In the normal supply system, what augmentations of inventories are needed for anticipated contingencies and at what echelon level or levels should the contingency stocks be carried? (b) In actual contingency operations, what supply echelon structures should be constructed to best support operations? (This question is of particular importance in the Army where contingency operations often create three- or four-echelon supply systems that change dynamically in terms of structure and location.) The general problem is further characterized by a practical inability (with certain exceptions) to deal with individual products; instead, the analysis for planning purposes must be based upon the consideration of product aggregates or classes of relatively homogeneous commodities.

It has long been recognized that solutions to these problems are critically dependent upon the types of contingency operations considered and upon the types and capabilities of various kinds of available transportation systems. Past research in this area has been primarily upon "connective" models such as transportation models connecting supply with consumption and planning factor models connecting operations with material requirements. Although a number of studies of the multi-activity aspect have been made, no satisfactory integrated analytic investigation has been achieved so far as is known. An attempt by Clark (1963-15) in this direction did not proceed beyond a formulation of a single-activity problem.

4. Evaluation by Servomechanism Theory.

Several studies have used portions of servomechanism theory to analyze the single-activity inventory control problem in cost-free systems. These studies have considered both deterministic and random demand. Early work assumed continuous (over time) inputs and outputs of the inventory system and used Laplace transforms as the analytic tool. Later work recognized the discrete nature of inventory processes and used z-transform methods.

The multi-activity inventory problem using the discrete time (sampled data) servomechanism approach was investigated by Burns (1970-11). Although potentially the theory could be used for inventory control purposes in the same sense as the other techniques described previously, the main thrust of Burn's study was to apply the method to evaluate the consequences of using a particular decision (ordering) rule at the various activities in a cost-free inventory system. The decision rule studied was that the desired level of inventory is expressed as a certain number of weeks' worth of expected demand, the expectation being a first-order exponentially-smoothed average of past demands. For this decision rule, a model is constructed for the case of a series activity structure which is later interpreted in context with an arborescence structure. The model is used to show that the system amplifies minor variations in consumer demand into major disturbances at higher activities. This amplification is traced to two sources, one being a legitimate and unavoidable inventory adjustment and the other being an unwarranted "false order" effect. A new decision rule is then derived which automatically suppresses the false orders. The two decision rules are compared by use of a simulation model of a three-echelon system. It is demonstrated that the revised rule experiences fewer stockouts, maintains lower average inventories, and greatly reduces the amplification effect.

5. Evaluation by Simulation.

All analytic inventory control models suffer from constraining assumptions concerning the environment within which the inventory system operates. Although simulation models can incorporate richer representations of the environment, they cannot solve inventory problems in the same sense as the analytic approaches. They are useful, however, in an evaluative or comparative role: the behavior of analytically derived policies can be studied, or alternative inventory policies can be compared in context with richer representations of the operating environment than were used in the derivation of the policies.

For multi-activity inventory problems, several simulation models (such as in (11) and (14)) have been constructed to provide realizations of a particular policy in essentially the same operating environment as that assumed in the derivation of the policy; the purpose here is to demonstrate the effects and behavior of the analytically derived policy. Other simulation models have been constructed for the explicit purpose of comparing alternative policies; a recent example is given by Haber (1971-27). In this multi-product simulation model, the operation of a particular three-level arborescence activity system (the Polaris submarine support system) is studied for four principal inventory policies. None of the policies tested were multi-activity policies in the sense of being derived from analytic multi-activity inventory theory; instead, they reflected variations of policies used in practice which tend to be aggregates of single-activity policies. The operation of the simulation model yields expected costs stratified by type of cost and type of policy, and various inventory performance statistics for each type of policy. From these results for the particular inventory system studied, judgments were made concerning the behavior and relative effectiveness of the alternative policies for various classes of products.

VI. DIRECTION OF FUTURE RESEARCH

It is probable that research in multi-activity inventory theory has reached a point where highest returns have already been achieved (the easy problems have been solved) and, therefore, marginal returns from further work are likely to diminish. The principal opportunities for further work, both by individual researchers and research teams, probably lie in refinements and extensions of previous results and in the reduction of currently available theory to practice in actual inventory situations. In addition, there may still be a small probability that a new basic theory can be developed which would supercede much of the previous results.

Before considering directions that future research in multi-activity inventory theory might take, some impressions of the current state of the art are given. In particular, an attempt is made to identify weaknesses in the current theory which future research might be able to overcome.

1. State of the Art.

The review of research on the multi-activity inventory problem in previous sections has included studies which might be considered as obsolete in view of later progress. Although these studies were included to provide some kind of historic perspective, it is possible that some of the techniques used may play a role in future research. Furthermore, the earlier work tends to be simpler and therefore may be more viable with respect to practical use than the more sophisticated results of later research.

Looking at recent work on the deterministic inventory control problem, an impression is gained that the problem as formulated has been solved, at least from a theoretical point of view. The models and solution methods developed by Zangwill, Veinott and others appear to encompass most of the important features of the problem. For a general problem of practical size, the amount of computation involved in finding solutions is much too great to be practical, but this is probably due more to the inherent complexity of the problem than to the theory. However, for special cases and assumptions that are not too unrealistic, efficient solutions methods have been developed which make the computational effort more tractable.

In looking at the various techniques that have been applied to the stochastic multi-activity problem, it is clear that the Bessler-Veinott approach is considerably more general than the others in terms of included features of the problem. The most important feature not included is economy of scale resulting from fixed costs, but this factor for actual inventory problems will probably decline in relevancy as management and production automation continues to increase. Another significant limitation is that conditions necessary for the myopic policies of this approach to be optimal are quite restrictive with respect to consideration of dynamic factors. Within these limitations, the main difficulty with the model is a computational one, again largely attributable to complexities resulting if the full power of the model is exercised. Limitations imposed to make the computation possible tend to be over-restrictive relative to practical situations. However, the theory is very useful in analyzing the characteristics of optimal policies for the multi-activity inventory control problem, particularly where stationary or near stationary conditions pertain.

A sufficient amount of experience with the dynamic programming approach to the multi-activity inventory control problem is now available as to confirm its computational practicability (see (10) and (45) for typical citations of computation experience). Unfortunately, this technique suffers from several inadequacies with respect to included features of the problem. The case of parallel activities is still bothersome, even though there is reason to believe good approximations are available. But the main trouble is that, so far, the method has been confined to single-product problems. Extension to even simple multi-product situations have not yet been made.

The dynamic programming approach and, to a lesser degree, the Bessler-Veinott model consider nonstationary aspects which are very important in practical situations, but they both suffer from assumptions of periodic review. Everything else being the same, continuous review policies should result in less inventory costs than periodic review policies. The expected cost and stationary process approaches yield continuous review policies, but they cannot adequately cope with nonstationary factors. Although continuous review models have been developed for multi-product problems, only simple product interdependencies (such as budget or capacity constraints) have been considered in contrast to the more general inter-product relationships encompassed by the Bessler-Veinott method. The main advantage of expected cost and stationary process models is that the computations involved in obtaining solutions are considerably less than for the dynamic technique.

If a survey were to be made of techniques currently used to control actual multi-activity inventory systems, the use of heuristic methods which, at best, are only slightly tempered by results of inventory theory will probably be found to prevail in the majority of the systems. For the remainder, expected cost models are probably used, but mostly in the form of combinations of single-activity models rather than an integrated system model. A claim is made (19) for the Sherbrooke model that it is one of the first analytic multi-activity models to be translated from theory to practice. Looking into the near term future, it is plausible that both the stationary and dynamic programming multi-activity techniques are candidates for practical implementation, but most probably with nonoptimal modifications to cope with realisms not encompassed by the theory. For the more general Bessler-Veinott approach, a wider base of familiarity with the method and further work to reduce the computation requirements is probably indicated prior to any serious implementation consideration.

Considering next the class of allocation models reviewed in Section IV, it appears that the books are more or less closed with respect to this as a separate problem. Instead, this appears to be subsumed by the class of supply policies derived from the more general inventory control techniques. This is particularly evident in the Bessler-Veinott method, where supply policies are explicitly considered in a formal manner. On the other hand, it is easy to envision special allocation or redistribution problems for which the use of simpler methods is attractive.

The stock positioning, repair level, and contingency support problems discussed in Section V all have a common denominator; namely, the network of activities in a multi-activity system is itself a decision variable as well as imbedded inventory control policies. Thus, the three problems cited may be viewed as special cases of a general problem that has an additional dimension of complexity over the multi-product, multi-activity inventory control problem (an appropriate descriptor for the additional dimension might be "multi-network"). It is probably fair to assert that the state of the art with respect to this larger problem is still primitive, as testified by the confinement of progress to date to special cases. Although the Williams model on the repair-level problem recognizes product interdependencies in terms of parts hierarchy relationships, the model assumes stationarity and inventory policies are not included in the form of decision variables. The Pincus model includes inventory policies in the form of suboptimizations, permits only simple interproduct relationships, and does not consider regeneration through repair. As previously noted, analytic work on the general contingency problem, which is actually a dynamic version of the stock positioning problem, is virtually nonexistent.

2. Short-term Research.

In general, short-term opportunities for research on multi-activity inventory problems lie in extensions and refinements of previously developed approaches, and in several aspects of the problem of translating available theoretical results to practice. Of the several approaches to the inventory control problem, the ones that assume nonstationary stochastic conditions appear to be the most interesting.

In the Clark-Scarf approach, further investigations of the parallel activity case (and the general arborescence structure) might prove fruitful; in particular, the work of Zacks provides a good lead on this problem. Additional research here could take the form of (a) finding conditions such that relatively simple policies are optimal, (b) determining sharper bounds on the optimal solution, or (c) developing good approximations to the optimal solutions. Another useful extension of this approach would be to permit simple inter-product dependencies such as budget or capacity constraints; it may also be possible to consider more detailed interrelationships by appropriate interpretations of activities as products in some manner analogous to the Bessler-Veinott development.

The Bessler-Veinott model is of special interest with respect to further research because it enables the inclusion of more factors of the problem than the other methods; its ability to cope with multi-product as well as multi-activity systems is particularly attractive. Even though several methods for alleviating computational difficulties of the model have been developed, including bounding, approximation, and special assumption methods, further research along these lines may be beneficial. In general, these observations apply also to the deterministic models of Zangwill and Veinott.

Since the use of network theory for the multi-activity inventory problem has only recently been opened up by Connors-Zangwill, it is difficult to judge what directions future research along this line might take. It may be that significant new developments will depend upon advances in general network theory. On the other hand, extensions and refinements within the framework already established may be possible.

It has been pointed out that continuous review assumptions have an advantage over periodic review. The general continuous review, dynamic problem is very difficult. However, further research along the lines of step-function approximations to continuous review policies may be useful. The rather heuristic method used in the Clark-Metcalfe model (referred to in Section III-3) may provide a start in this direction. In this model, it is assumed that critical levels are computed periodically but under the assumption that they are implemented in a continuous review mode throughout the period; in effect, this eliminates the implicit inflation of resupply time induced by the periodicity assumption. A study of this and other approximations to the optimal continuous review policies might provide useable results.

As previously indicated, further research on allocation and distribution problems is most likely to be accomplished in context with the more general inventory control problem. For the repair level and other special problems defined in Section V, further work will undoubtedly be done as extensions of the previous studies, but research on the general problem of which the cited problems are special cases is of a more long-term nature. Insofar as evaluation techniques are concerned, simulation is always available for use in a variety of evaluation and planning problems, but it is hard to categorize such applications *per se* as theoretical research. Although some additional work on the use of servomechanism theory may occur for evaluation or inventory control purposes, it will probably be in context with special kinds of products or systems.

Even though it may not be appropriate to view efforts to reduce multi-activity inventory theory to practice as theoretical research, this field should not be overlooked. The adaptation of the theory in the form of more-or-less heuristic models of multi-activity systems that are intuitively appealing for practical application provides important opportunities for further work. Also, the susceptibility of a particular analytic method with respect to its practical use may be considered as some kind of ultimate criterion of its worth.

3. Long-term Research.

Over the long run, the real frontier for research on multi-activity inventory systems lies in the area of integrated logistics, where the surface so far has barely been scratched. In many organizations, the pure inventory problem does not exist and an inventory model, no matter how comprehensive in terms of activity structures, cost functions, and optimization, fall considerably short of satisfying overall logistics objectives. A full appreciation of the interplay among operations, maintenance, transportation, and supply, all being tempered and affected by associated management and data processing systems, tends to relegate pure inventory models to a less relevant status. Even though the full problem is exceedingly complex, there are several avenues worthwhile exploring by analytic techniques.

As a start in this direction, the extension of existing theory to what might be called the multi-network problem (which, by definition, includes multi-product and multi-activity considerations) could be contemplated. This problem is roughly defined as one where parts hierarchy structures and related reliability and end-item usage factors are given, together with a network representation of the potential logistics support system. It is then desired to find a sub-structure of the parts hierarchy for which inventories are to be provided and, for each such item, a subnetwork of the support system at which inventories are to be maintained such that total system costs are minimized. Theoretically, this problem encompasses almost all of the multi-activity inventory problems so far investigated. Also, to properly solve this problem, the notion of "inventory" should be generalized to include such commodities as manpower, maintenance equipment, and other support resources.

In dealing with this problem, it is first necessary to define it in more rigorous terms. Next, an appropriate solution technique must be found. Of the ones now available, the Bessler-Veinott model may be a possible point of departure since it enables the potential consideration of dependencies among items in a parts hierarchy. Unfortunately, however, the problem is characterized by features involving economies of scale which this technique cannot handle; some way around this difficulty would have to be found.

Assuming that a formal model for the problem can be constructed, the next step would be to look for possible decomposition opportunities which would almost certainly be necessary if any use were to be made of the model. Also, if this stage is reached, there would probably be a number of research possibilities with respect to special cases and computation procedures, sensitivity analyses, etc. Even though the full multi-network problem may turn out to be insolvable in any practical sense, a formal recognition of the problem is useful in identifying just what kinds of suboptimizations are being made in current and future multi-activity inventory studies.

Although a solution to the multi-network problem would be a big step toward the full integrated logistics problem, it would still contain rather abstract representations of the real world processes. It is quite possible that advantages of analytic optimization techniques would be overcome by effects of features not included or inadequately represented in the formulation; whether or not this is true would be very difficult to prove. As a result, it might be profitable to resort to a decision-making apparatus which might be called "micro-simulation".

The use of the micro-simulation technique is based upon a recognition of an advanced real time management system as being, in effect, a "model" of the real world system being managed. Since such a management system contains status records in varying degrees of detail, supported by data flows to keep the records current, it is theoretically possible to add components that would simulate the external stresses upon the system as caused by the real world operations. With such additions, it would then be possible to make short-term accelerated-time, simulated "excursions" into possible futures of the system for decision-making purposes. Such micro-simulations might employ evaluation-selection procedures for postulated alternatives, or they might contain adaptive decision-making models.

A very crude example of the latter possibility is given by a model developed by Fisher, et al (1968-23). This model, which is primarily a Monte Carlo simulation model for military logistics applications, contains two innovative features: the use of task networks to describe operational and support environments, and the use of an embedded decision model for resources determinations.

The task network concept involves the establishment, by user-provided data, of networks of interconnected tasks or activities to represent equipment utilization, repair and overhaul functions, and other aspects of the operational and logistics environment. The user also identifies types and amounts of resources needed to accomplish each kind of task; categories of included support resources are maintenance personnel, repair equipment, and spare parts. Other input data includes time distributions for task accomplishment, priority rules, shift policies, end-item utilization schedules, etc. With this data, the model simulates the operation of end-equipment, the incurrence of malfunctions, the accomplishment of servicing tasks, the accomplishment of repair and overhaul functions, the utilization and interaction of resources in the demand process, the changes in resource availability according to shift policies, and other facets of the overall operation. In the simulation, inventories of spare parts are treated identically the same as inventories of other needed resources such as maintenance specialists, shop equipment, etc. Also, the model permits tradeoffs within and among the various types of resources whenever shortages occur; included is task preemption, use of substitutes, task deferment, cannibilization, etc.

The embedded decision model was designed to operate during the course of a particular simulation according to criteria established at the outset. Its general function is to identify particular resource items (bottleneck items) tending to inhibit the achievement of overall end-item utilization objectives and to cause selective injections of additional amounts of these resources according to a cost-effectiveness criterion. Thus, the "reordering" of a resource is a consequence not of before-the-fact demand estimates, but rather of need against end objectives after all reasonable advantages are taken of resource tradeoffs, deferment of needs, etc. The decision model uses an automatic exponential-smoothing forecasting technique to predict degradation in system performance, and a simple marginal analysis procedure for resource augmentations.

Extrapolating the methodology of this model to the micro-simulation procedure, the simulation aspects would be replaced by the real time management system, augmented by components for simulating external stresses. It is clear, however, that this procedure, even though extensively refined and supported by analytic models and micro-simulations for decision-making over longer-term futures of the system, cannot achieve "optimal" policies or resource allocation decisions in the same sense as in purely analytic techniques. Nevertheless, it may represent a viable approach with respect to the practical aspects of the integrated logistics problem.

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