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COUPLE-STRESS SOLUTION TO AN
INFINITE PLATE BOUNDED BY AN ELLIPTICAL HOLE

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by

F. D. Ju
Y. C. Hsu
W. J. Wang

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13. ABSTRACT

Couple-stresses solutions are obtained for an infinite tension elastic plate bounded at the interior by an elliptical hole with the static equilibrating tractions. The nominal tension in the plate is uniform along the major axis. The selection of the Mathieus' functions and the form of weighting functions in the boundary conditions match a particular class of boundary values which reduces upon limiting processes to the three limiting cases. These cases are ones with free stresses on the interior boundary: the couple-stresses solution for the degenerate circle, the couple-stresses solution for the degenerate crack, and the classical solution for the elliptical hole.

Of particular interest is the degenerate crack problem. The couple-stresses solution for the degenerate crack are the same as the classical one of the crack. This is true because couple-stresses are related to curvature and no curvature is induced in this crack problem.

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NOMENCLATURE

A	$= (\cosh 2\xi - \cos 2\eta)$
a, b	= semi-major and semi-minor axes of the ellipse with $\xi = \xi_o$
$2c$	= interfocal distance of ellipse
$ce_2(\eta,-q)$	= Mathieu function of the first kind, of the order of 2, defined by Equation (A1.6)
E	= Young's modulus
e	= eccentricity of ellipse
$Fek_2 = Fek_2(\xi,-q)$	= k-type modified Mathieu function of the second kind, of the order of 2, which is defined by Equation (A1.4)
$(Fek_2)'$	$= \frac{\partial}{\partial \xi}[Fek_2(\xi,-q)]$
$(Fek_2)''$	$= \frac{\partial^2}{\partial \xi^2}[Fek_2(\xi,-q)]$
$\tilde{F}$	$= F/(c/2)^2$
$H, B, C, D, E, \tilde{F}, \tilde{M}$	= superposition constants
h_1, h_2	= scale factors
$I_n(z), K_n(z)$	= modified Bessel functions of the first and second kinds, respectively, where n denotes order
J	= the Jacobian of transformation
k	$= q^{1/2} = \frac{1}{2}\frac{c}{\ell}$
ℓ	= couple-stress characteristic length
$\tilde{M}$	$= M\frac{\pi}{(p_2)'}$
p	= uniaxial uniform load
$(p_2)'$	$= -ce_2(0,q)\ ce_2(\frac{\pi}{2},q)/\pi_o^{(2)}$

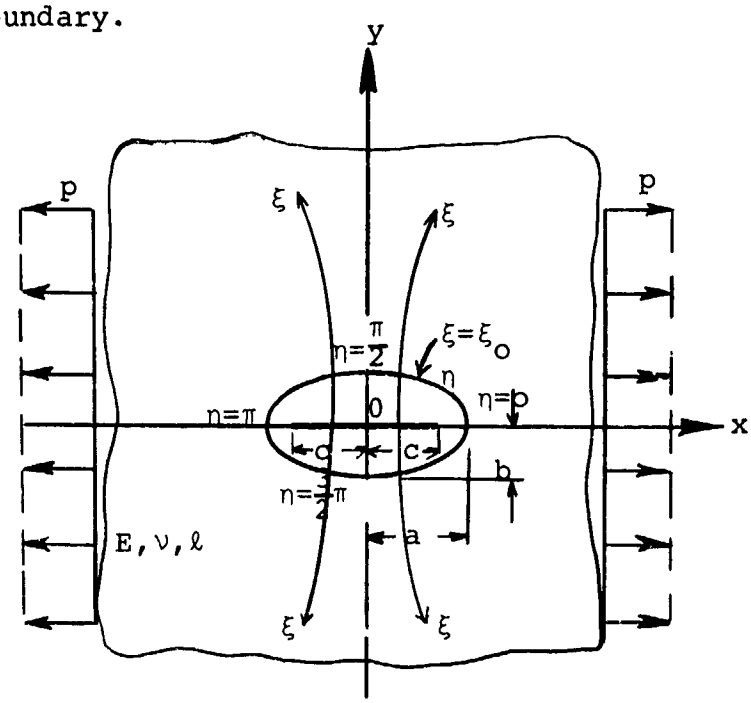
q	$= \left[\frac{1}{2}\left(\frac{c}{\ell}\right)\right]^2 = k^2$. Here q is the real positive number
r, r'	= semi-major and semi-minor axes of any ellipse
r, ϕ	= polar coordinates
$se_2 = se_2(\eta, -q)$	= Mathieu function of the first kind, of the order of 2, which is defined by Equation (A1.5)
$(se_2)'$	$= \frac{\partial}{\partial \eta}[se_2(\eta, -q)$
$(se_2)''$	$= \frac{\partial^2}{\partial \eta^2}[se_2(\eta, -q)$
U	= Airy stress function
x, y	= Cartesian coordinates
α, β	= orthogonal curvilinear coordinates
Γ	= defined by Equation (3.35)
Δ	= defined by Equation (AIII.17)
θ	= the inclination of the curve β = a constant to the x-axis in orthogonal curvilinear coordinates (α, β)
μ	= modulus of rigidity
μ_x, μ_y	= couple-stress components in Cartesian coordinates
μ_α, μ_β	= couple-stress components in orthogonal curvilinear coordinates
μ_ξ, μ_η	= couple-stress components in elliptical coordinates
ν	= Poisson's ratio
ν_1	$= ke^{-\xi}$
ν_2	$= ke^{\xi}$
ξ, η	= elliptical coordinates
$\sigma_{xx}, \sigma_{xy}, \sigma_{yx}, \sigma_{yy}$	= stress components in Cartesian coordinates

$\sigma_{\alpha\alpha}, \sigma_{\alpha\beta}, \sigma_{\beta\alpha}, \sigma_{\beta\beta}$	= stress components in orthogonal curvilinear coordinates
$\sigma_{\xi\xi}, \sigma_{\xi\eta}, \sigma_{\eta\xi}, \sigma_{\eta\eta}$	= stress components in elliptical coordinates
ψ	= couple-stress function
ω_z	= local rigid rotation

1.0 INTRODUCTION

1.1 Objective of Research

The objective of this research is to obtain solutions for an infinite tension plate bounded at the interior by an elliptical hole. The nominal tension in the plate is uniform along the major axis (see Fig. 1). The couple-stress effect is considered. Two limiting cases for the problem are for the ellipticity of 0 and 1 for the interior boundary. That is, for the first case the interior boundary is a circle, for the second case, a crack along the major axis. Of course, the problem of a crack is the major interest of the research. The present investigation includes a study of a class of solutions which satisfies the static equilibrating traction on the interior boundary.



$c \cosh \xi_o = a$

$c \sinh \xi_o = b$

$e = \frac{c}{a}$

Figure 1.

1.2. Review of Pertinent Literature

The origin of couple-stress theory of elasticity is attributed to E. and F. Cosserat, with modern developments by Truesdell, Toupin, Grioli, and Aero and Kuvshinskii. A discussion of the origin and development of the couple-stress theory is given by Mindlin and Tiersten [1].

Recently, specific plane problems in couple-stress elasticity have been studied by many investigators. The following is a list of those studies which are related to the present problem. Mindlin [2] found the couple-stress effects on the stress concentration factors for a circular hole in various two-dimensional fields of stress when the diameter of the hole is comparable in size to the couple-stress constant ℓ. Weitsman [3], [4] generalized the solution to the cases of cylindrical inclusion in fields of cylindrical symmetric and uniaxial tension, respectively. Sternberg [5], [6] found the effect of couple-stresses on the stress concentration around a crack by assuming that the stress singularities at the crack-tips are of the same order as those in the well-known classical solutions. An earlier report by Ju and Hsu [7] (AFOSR-69-1908 TR) contains a comprehensive review of the theory and the basic equations for the plane problems. All pertinent notations and definitions, therefore, will be referred to [7] to avoid repetition.

2.0 TWO-DIMENSIONAL COUPLE-STRESS THEORY

The approach toward the solution of a crack problem may be either by means of the degeneration of an elliptical hole [7] or by the use of a half-plane [5]. The choice of using an elliptical hole has the advantages of (a) no a priori assumption, (b) ready check for the problem of a circular hole [2], (c) ready check for the classical solution [8]. In order that the boundary may be described by a coordinate line, the elliptical coordinate is used. Hence, all equations for the couple-stress theory of elasticity are expressed in such coordinates in this problem.

2.1. Rectangular Cartesian Coordinates

According to Ju and Hsu [7]* or Mindlin [2], the field equations of two-dimensional couple-stress theory in a state of plane strain without body forces or body couples are given as follows:

U and Ψ must satisfy

$$\nabla^4 U = 0 \tag{2.1}$$

and

$$\nabla^2(\Psi - \ell^2\nabla^2\Psi) = 0 \tag{2.2}$$

respectively, where ℓ is the couple-stress characteristic length so that ℓ^2 is the ratio of the Cosserat modulus to the

*Numbers in brackets designate references at end of the report. Equations in Section 2.1 refer to Equations (1.40) through (1.51) in [7].

modulus of rigidity (μ) [7]. U and Ψ must also be related to each other by the Cauchy-Riemann equations

$$\frac{\partial}{\partial x}(\Psi - \ell^2\nabla^2\Psi) = -2(1 - \nu)\ell^2\frac{\partial}{\partial y}(\nabla^2 U)$$

$$\frac{\partial}{\partial y}(\Psi - \ell^2\nabla^2\Psi) = 2(1 - \nu)\ell^2\frac{\partial}{\partial x}(\nabla^2 U) \tag{2.3}$$

Stresses, couple-stresses, displacements, and rigid-body rotation are expressed in terms of the two generating stress functions U and Ψ as follows:

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x \partial y}$$

$$\sigma_{yy} = \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 \Psi}{\partial x \partial y}$$

$$\sigma_{xy} = -\frac{\partial^2 U}{\partial x \partial y} - \frac{\partial^2 \Psi}{\partial y^2}$$

$$\sigma_{yx} = -\frac{\partial^2 U}{\partial x \partial y} + \frac{\partial^2 \Psi}{\partial x^2} \tag{2.4}$$

$$\mu_x = \frac{\partial \Psi}{\partial x}$$

$$\mu_y = \frac{\partial \Psi}{\partial y}$$

2.2. Orthogonal Curvilinear Coordinates

To generalize the results [2], [7] which are derived for rectangular Cartesian and polar coordinates, consider those

for the general two-dimensional orthogonal curvilinear coordinates (α,β). The two generating stress functions $U(\alpha,\beta)$ and $\Psi(\alpha,\beta)$ must satisfy (2.1)* and (2.2), respectively. Here the Laplacian operator, ∇^2, and the biharmonic operators, ∇^4, are expressed in terms of orthogonal curvilinear coordinates α and β.

$$\nabla^2 = h_1 h_2 \left[\frac{\partial}{\partial\alpha}\left(\frac{h_1}{h_2}\frac{\partial}{\partial\alpha}\right) + \frac{\partial}{\partial\beta}\left(\frac{h_2}{h_1}\frac{\partial}{\partial\beta}\right)\right]$$

and (2.5)

$$\nabla^4 = \nabla^2\nabla^2$$

where h_1 and h_2 are scale factors in α and β, respectively. The Cauchy-Riemann equations (2.3) can be transformed to α and β coordinates by use of the chain rule that

$$\frac{\partial}{\partial x} = \frac{\partial\alpha}{\partial x}\frac{\partial}{\partial\alpha} + \frac{\partial\beta}{\partial x}\frac{\partial}{\partial\beta}$$

$$\frac{\partial}{\partial y} = \frac{\partial\alpha}{\partial y}\frac{\partial}{\partial\alpha} + \frac{\partial\beta}{\partial y}\frac{\partial}{\partial\beta} \tag{2.6}$$

The stresses, couple-stresses, displacements, and rigid-body rotation are expressed in terms of U and Ψ as follows:

---

*Numbers in parentheses are references to equations in the text.

$$\sigma_{\alpha\alpha} = h_2^2 \frac{\partial^2 U}{\partial \beta^2} - \frac{h_1^2}{h_2} \frac{\partial h_2}{\partial \alpha} \frac{\partial U}{\partial \alpha} + h_2 \frac{\partial h_2}{\partial \beta} \frac{\partial U}{\partial \beta} - h_1 h_2 \frac{\partial^2 \Psi}{\partial \alpha \partial \beta}$$

$$- h_2 \frac{\partial h_1}{\partial \beta} \frac{\partial \Psi}{\partial \alpha} - h_1 \frac{\partial h_2}{\partial \alpha} \frac{\partial \Psi}{\partial \beta}$$

$$\sigma_{\beta\beta} = h_1^2 \frac{\partial^2 U}{\partial \alpha^2} + h_1 \frac{\partial h_1}{\partial \alpha} \frac{\partial U}{\partial \alpha} - \frac{h_2^2}{h_1} \frac{\partial h_1}{\partial \beta} \frac{\partial U}{\partial \beta} + h_1 h_2 \frac{\partial^2 \Psi}{\partial \alpha \partial \beta}$$

$$+ h_2 \frac{\partial h_1}{\partial \beta} \frac{\partial \Psi}{\partial \alpha} + h_1 \frac{\partial h_2}{\partial \alpha} \frac{\partial \Psi}{\partial \beta}$$

$$\sigma_{\alpha\beta} = -h_1 h_2 \frac{\partial^2 U}{\partial \alpha \partial \beta} - h_2 \frac{\partial h_1}{\partial \beta} \frac{\partial U}{\partial \alpha} - h_1 \frac{\partial h_2}{\partial \alpha} \frac{\partial U}{\partial \beta} \tag{2.7}$$

$$- h_2^2 \frac{\partial^2 \Psi}{\partial \beta^2} + \frac{h_1^2}{h_2} \frac{\partial h_2}{\partial \alpha} \frac{\partial \Psi}{\partial \alpha} - h_2 \frac{\partial h_2}{\partial \beta} \frac{\partial \Psi}{\partial \beta}$$

$$\sigma_{\beta\alpha} = -h_1 h_2 \frac{\partial^2 U}{\partial \alpha \partial \beta} - h_2 \frac{\partial h_1}{\partial \beta} \frac{\partial U}{\partial \alpha} - h_1 \frac{\partial h_2}{\partial \alpha} \frac{\partial U}{\partial \beta}$$

$$+ h_1^2 \frac{\partial^2 \Psi}{\partial \alpha^2} + h_1 \frac{\partial h_1}{\partial \alpha} \frac{\partial \Psi}{\partial \alpha} - \frac{h_2^2}{h_1} \frac{\partial h_1}{\partial \beta} \frac{\partial \Psi}{\partial \beta}$$

$$\mu_\alpha = h_1 \frac{\partial \Psi}{\partial \alpha}$$

$$\mu_\beta = h_2 \frac{\partial \Psi}{\partial \beta}$$

2.3. Elliptical Coordinates

The elliptical coordinates ξ and η (see Fig. 2) are related to the rectangular cartesian coordinates x and y by

$$x = c \cosh\xi \cos\eta$$

$$(0 \leq \xi < \infty,\ 0 \leq \eta < 2\pi) \qquad (2.8)$$

$$y = c \sinh\xi \sin\eta$$

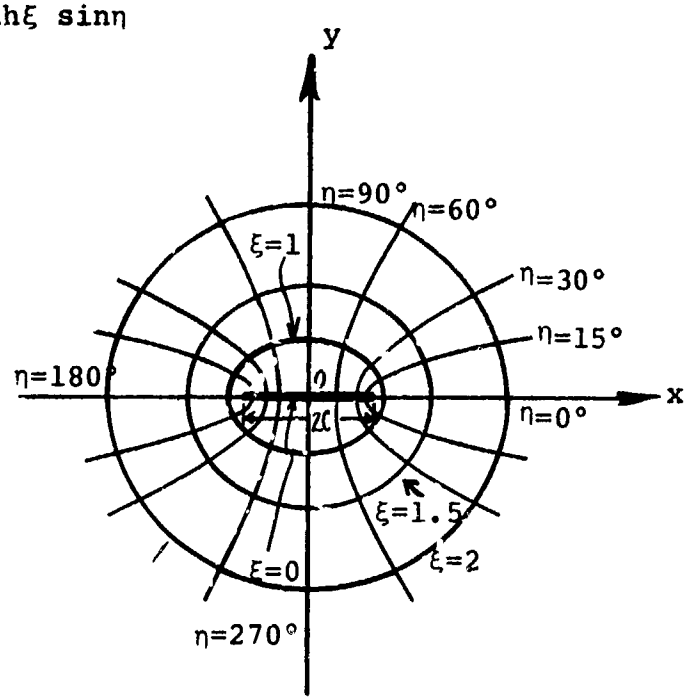


Figure 2.

Here the curves ξ = constant and η = constant form an orthogonal system of confocal ellipses and hyperbolas, with the common foci being the points $(\pm c, 0$. Figure 3 shows two limiting cases when the ellipticity e is 1 and 0, respectively.

The Jacobian of transformation is

$$J = \frac{\partial(x,y)}{\partial(\xi,\eta)} = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = c^2(\cosh^2\xi - \cos^2\eta)$$

At $\xi = 0$, $\eta = 0$ or π, $J = 0$. Hence the only singular points of the transformation are located at the foci $(\pm c, 0)$. The scale factors h_ξ and h_η can be calculated by

$$\frac{1}{h_\xi^2} = \frac{\partial x}{\partial \eta}^2 + \frac{\partial y}{\partial \xi}^2$$

and

$$\frac{1}{h_\eta^2} = \frac{\partial x}{\partial \eta}^2 + \frac{\partial y}{\partial \eta}^2$$

as

$$h_\xi = h_\eta = \frac{\sqrt{2}}{q\sqrt{\cosh 2\xi - \cos 2\eta}} \tag{2.9}$$

This denotes that

$$A = (\cosh 2\xi - \cos 2\eta) \tag{2.10}$$

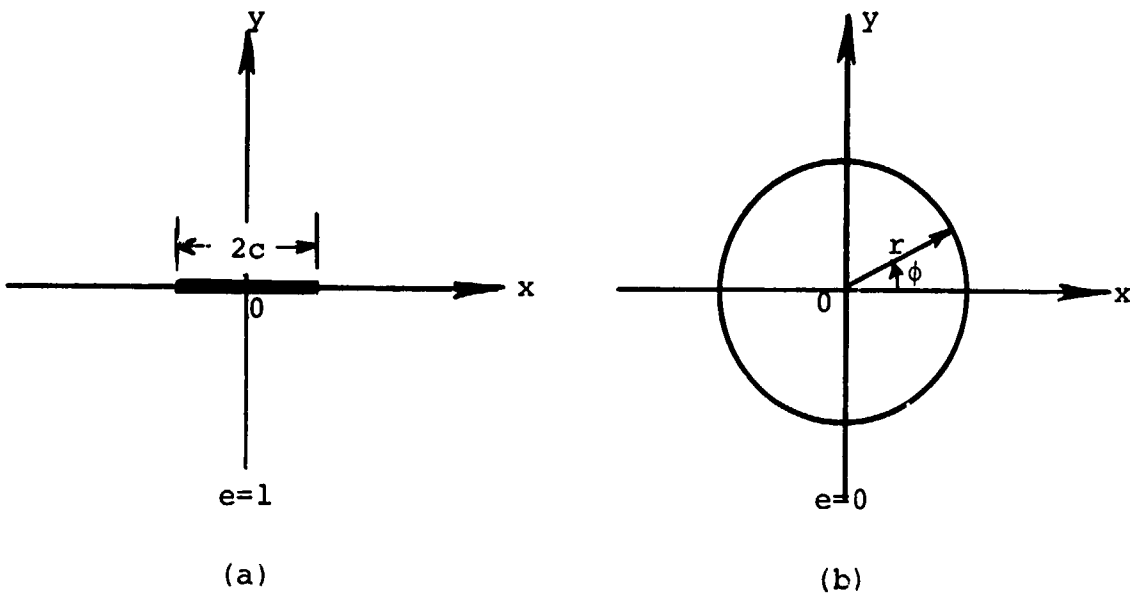


Figure 3.

The following quantities involving h_ξ and h_η are

$$\frac{\partial h_\xi}{\partial \xi} = \frac{\partial h_\eta}{\partial \xi} = -\frac{\sqrt{2}}{c} A^{-\frac{3}{2}} \sinh 2\xi$$

$$\frac{\partial h_\xi}{\partial \eta} = \frac{\partial h_\eta}{\partial \eta} = -\frac{\sqrt{2}}{c} A^{-\frac{3}{2}} \sin 2\eta$$

$$h_\xi^2 = h_\eta^2 = \frac{2}{c^2} A^{-1}$$

$$h_\xi = h_\eta = \frac{\sqrt{2}}{c} A^{-\frac{1}{2}} \tag{2.11}$$

$$h_\xi \frac{\partial h_\xi}{\partial \xi} = h_\xi \frac{\partial h_\eta}{\partial \xi} = -\frac{2}{c^2} A^{-2} \sinh 2\xi$$

$$h_\eta \frac{\partial h_\xi}{\partial \eta} = h_\xi \frac{\partial h_\eta}{\partial \eta} = -\frac{2}{c^2} A^{-2} \sin 2\eta$$

Equation (2.5) becomes, by use of (2.9) and (2.10),

$$\nabla^2 = \frac{2}{c^2 A}\left[\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\right]$$

$$\nabla^4 U = \nabla^2 \nabla^2 U \tag{2.12}$$

The Cauchy-Riemann equations (2.3) can be transformed to elliptical coordinates ξ and η by use of (2.6) with $\alpha = \xi$, $\beta = \eta$

$$\frac{\partial}{\partial \xi}(\Psi - \ell^2 \nabla^2 \Psi) = -2(1 - \nu)\ell^2 \frac{\partial}{\partial \eta}(\nabla^2 U)$$

$$\frac{\partial}{\partial \eta}(\Psi - \ell^2 \nabla^2 \Psi) = 2(1 - \nu)\ell^2 \frac{\partial}{\partial \xi}(\nabla^2 U) \tag{2.13}$$

Similarly, stresses, couple-stresses, displacements, and rigid-body rotation are expressed in terms of elliptical coordinates by use of (2.7) and (2.11).

$$\sigma_{\xi\xi} = \frac{2}{c^2} A^{-1} \frac{\partial^2 U}{\partial \eta^2} + \frac{2}{c^2} A^{-2} \sinh 2\xi \frac{\partial U}{\partial \xi} - \frac{2}{c^2} A^{-2} \sin 2\eta \frac{\partial U}{\partial \eta}$$

$$- \frac{2}{c^2} A^{-1} \frac{\partial^2 \Psi}{\partial \xi \partial \eta} + \frac{2}{c^2} A^{-2} \sin 2\eta \frac{\partial \Psi}{\partial \xi} + \frac{2}{c^2} A^{-2} \sinh 2\xi \frac{\partial \Psi}{\partial \eta} \tag{2.14}$$

$$\sigma_{\eta\eta} = \frac{2}{c^2} A^{-1} \frac{\partial^2 U}{\partial \xi^2} - \frac{2}{c^2} A^{-2} \sinh 2\xi \frac{\partial U}{\partial \xi} + \frac{2}{c^2} A^{-2} \sin 2\eta \frac{\partial U}{\partial \eta}$$

$$+ \frac{2}{c^2} A^{-1} \frac{\partial^2 \Psi}{\partial \xi \partial \eta} - \frac{2}{c^2} A^{-2} \sin 2\eta \frac{\partial \Psi}{\partial \xi} - \frac{2}{c^2} A^{-2} \sinh 2\xi \frac{\partial \Psi}{\partial \eta} \tag{2.15}$$

$$\sigma_{\xi\eta} = - \frac{2}{c^2} A^{-1} \frac{\partial^2 U}{\partial \xi \partial \eta} + \frac{2}{c^2} A^{-2} \sin 2\eta \frac{\partial U}{\partial \xi} + \frac{2}{c^2} A^{-2} \sinh 2\xi \frac{\partial U}{\partial \eta}$$

$$- \frac{2}{c^2} A^{-1} \frac{\partial^2 \Psi}{\partial \eta^2} - \frac{2}{c^2} A^{-2} \sinh 2\xi \frac{\partial \Psi}{\partial \xi} + \frac{2}{c^2} A^{-2} \sin 2\eta \frac{\partial \Psi}{\partial \eta} \tag{2.16}$$

$$\sigma_{\eta\xi} = - \frac{2}{c^2} A^{-1} \frac{\partial^2 U}{\partial \xi \partial \eta} + \frac{2}{c^2} A^{-2} \sin 2\eta \frac{\partial U}{\partial \xi} + \frac{2}{c^2} A^{-2} \sinh 2\xi \frac{\partial U}{\partial \eta}$$

$$+ \frac{2}{c^2} A^{-1} \frac{\partial^2 \Psi}{\partial \xi^2} - \frac{2}{c^2} A^{-2} \sinh 2\xi \frac{\partial \Psi}{\partial \xi} + \frac{2}{c^2} A^{-2} \sin 2\eta \frac{\partial \Psi}{\partial \eta} \tag{2.17}$$

$$\mu_\xi = \frac{(2)^{\frac{1}{2}}}{c} A^{-\frac{1}{2}} \frac{\partial \Psi}{\partial \xi} \tag{2.18}$$

$$\mu_\eta = \frac{(2)^{\frac{1}{2}}}{c} A^{-\frac{1}{2}} \frac{\partial \Psi}{\partial \eta} \tag{2.19}$$

Components of stress and couple-stress in elliptical coordinates are shown in Fig. 4.

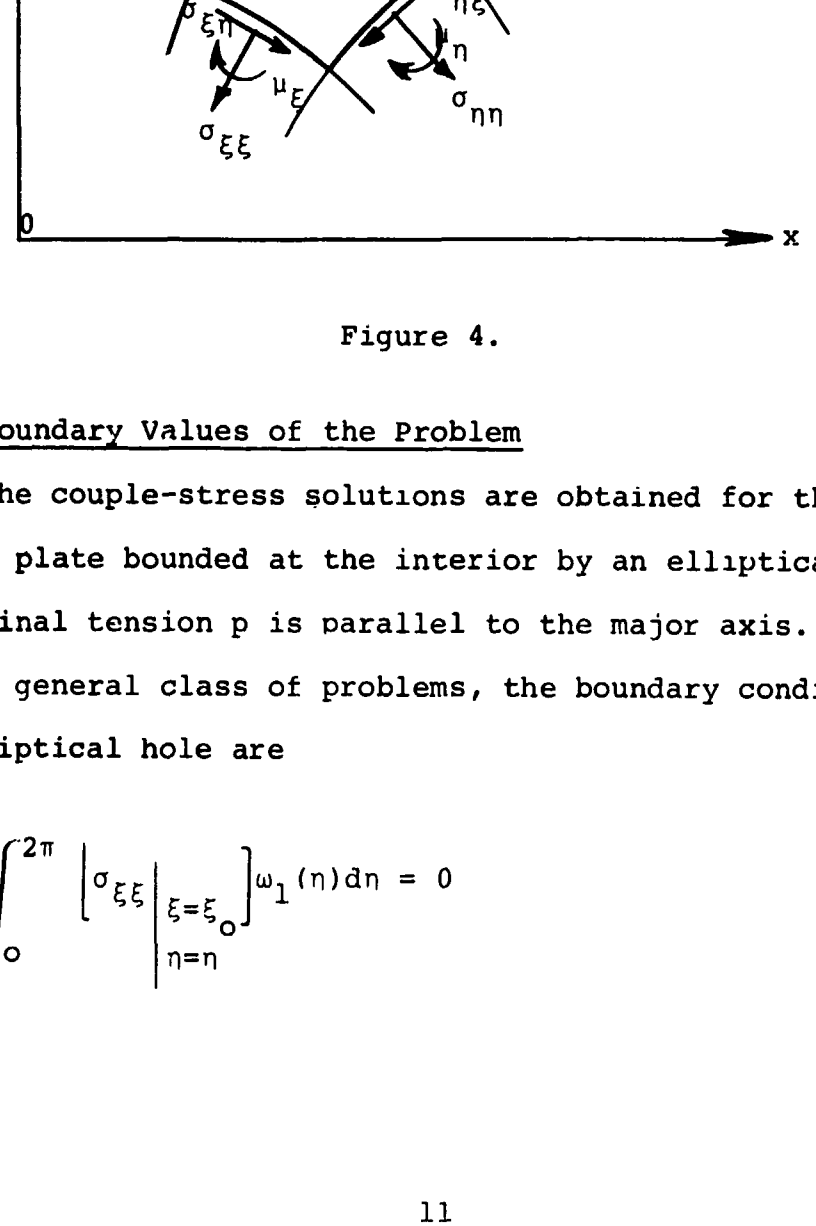


Figure 4.

2.4. Boundary Values of the Problem

The couple-stress solutions are obtained for the uniform tension plate bounded at the interior by an elliptical hole. The nominal tension p is parallel to the major axis. For the present general class of problems, the boundary conditions at the elliptical hole are

$$\int_0^{2\pi} \left[\sigma_{\xi\xi} \Big|_{\substack{\xi=\xi_0 \\ \eta=\eta}} \right] \omega_1(\eta)\, d\eta = 0 \tag{2.20}$$

$$\int_0^{2\pi} \left[\sigma_{\xi\eta} \Big|_{\substack{\xi=\xi_o \\ \eta=\eta}} \right] \omega_2(\eta) d\eta = 0 \tag{2.21}$$

$$\int_0^{2\pi} \left[\mu_{\xi} \Big|_{\substack{\xi=\xi_o \\ \eta=\eta}} \right] \omega_2(\eta) d\eta = 0 \tag{2.22}$$

where* $\omega_1(\eta)$ and $\omega_2(\eta)$ are some weighting functions defined on the interior boundary.

The regularity conditions at infinity are

$$\sigma^{\infty}_{yy} = p, \sigma^{\infty}_{xx} = \sigma^{\infty}_{xy} = \sigma^{\infty}_{yx} = \mu^{\infty}_{x} = \mu^{\infty}_{y} = 0 \tag{2.23}$$

---

*$\omega_1(\eta) = ce_2(\eta,-q)$ and $\omega_2(\eta) = se_2(\eta,-q)$

3.0 SOLUTION OF THE PROBLEM

3.1. Selection of Stress Functions $U(\xi,\eta)$ and $\Psi(\xi,\eta)$

As $U(\xi,\eta)$ is biharmonic in (2.1), with ∇^2 taking the form of (2.12), we choose five solutions of $U(\xi,\eta)$

$$U_1 = e^{2\xi} + \cos 2\eta$$

$$U_2 = e^{-2\xi} + \cos 2\eta$$

$$U_3 = e^{-2\xi} \cos 2\eta \tag{3.1}$$

$$U_4 = \xi$$

$$U_5 = e^{2\xi} \cos 2\eta$$

The selection of these fundamental biharmonic functions is based on the same argument as that used by Filon and Coker [8].

Notice that (2.2) can be rewritten as

$$\nabla^2(\Psi - \ell^2\nabla^2\Psi) = 0 \tag{3.2}$$

and

$$(1 - \ell^2\nabla^2)\nabla^2\Psi = 0 \tag{3.3}$$

From (3.2), we obtain the wave-function solution as the products of two types of Mathieu function (see Appendix I). Here we choose

$$\Psi_2 = (Fek_2)(se_2) \tag{3.4}$$

Equation (3.3) implies that one solution of Ψ is harmonic, Ψ_1. Furthermore, Cauchy-Riemann equations (2.13) conclude that this harmonic Ψ and $2(1-\nu)\ell^2\nabla^2 U$ must be conjugate harmonic functions [2] [7]. As such, Ψ_1 is selected with U in (3.1), as

$$\Psi_1 = \frac{\sin 2\eta}{(\cosh 2\xi - \cos 2\eta)} \tag{3.5}$$

Finally, U and Ψ are formulated by a superposition of stress functions given in (3.1), (3.4), and (3.5).

$$U = HU_1 + BU_2 + CU_3 + DU_4 + EU_5 \tag{3.6}$$

$$\Psi = \tilde{F}\Psi_1 + \tilde{M}\Psi_2 \tag{3.7}$$

3.2. General Expressions of Stresses and Couple-Stresses

Define $\sigma_{\xi\xi_i}$, $\sigma_{\eta\eta_i}$, $\sigma_{\xi\eta_i}$, $\sigma_{\eta\xi_i}$, μ_{ξ_i}, and μ_{η_i} equivalent to U_i for i = 1,2,3,4, and 5 and $\Psi_{(i-5)}$ for i=6 and 7. Thus we have

$$\left.\begin{aligned}
\sigma_{\xi\xi} &= H\sigma_{\xi\xi_1} + B\sigma_{\xi\xi_2} + C\sigma_{\xi\xi_3} + D\sigma_{\xi\xi_4} + E\sigma_{\xi\xi_5} + \tilde{F}\sigma_{\xi\xi_6} + \tilde{M}\sigma_{\xi\xi_7} \\
\sigma_{\eta\eta} &= H\sigma_{\eta\eta_1} + B\sigma_{\eta\eta_2} + C\sigma_{\eta\eta_3} + D\sigma_{\eta\eta_4} + E\sigma_{\eta\eta_5} + \tilde{F}\sigma_{\eta\eta_6} + \tilde{M}\sigma_{\eta\eta_7} \\
\sigma_{\eta\xi} &= H\sigma_{\eta\xi_1} + B\sigma_{\eta\xi_2} + C\sigma_{\eta\xi_3} + D\sigma_{\eta\xi_4} + E\sigma_{\eta\xi_5} + \tilde{F}\sigma_{\eta\xi_6} + \tilde{M}\sigma_{\eta\xi_7} \\
\sigma_{\xi\eta} &= H\sigma_{\xi\eta_1} + B\sigma_{\xi\eta_2} + C\sigma_{\xi\eta_3} + D\sigma_{\xi\eta_4} + E\sigma_{\xi\eta_5} + \tilde{F}\sigma_{\xi\eta_6} + \tilde{M}\sigma_{\xi\eta_7}
\end{aligned}\right\}$$

$$\left.\begin{aligned}
\mu_\xi &= H\mu_{\xi_1} + B\mu_{\xi_2} + C\mu_{\xi_3} + D\mu_{\xi_4} + E\mu_{\xi_5} + \tilde{F}\mu_{\xi_6} + \tilde{M}\mu_{\xi_7} \\
\mu_\eta &= H\mu_{\eta_1} + B\mu_{\eta_2} + C\mu_{\eta_3} + D\mu_{\eta_4} + E\mu_{\eta_5} + \tilde{F}\mu_{\eta_6} + \tilde{M}\mu_{\eta_7}
\end{aligned}\right\} \quad (3.8)$$

Some important quantities are calculated by the use of (2.14) to (2.19), and by the use of (2.12) equivalent to U_i for $i = 1$ to 5 and Ψ_1, Ψ_2. The results are listed as follows (however, some of them, which are independent of couple-stress, have been worked out in Filon and Coker [8]*).

$$\left.\begin{aligned}
\sigma_{\xi\xi_1} &= \frac{1}{c^2} A^{-2}[2\cos 4\eta - 8\cos 2\eta \cosh 2\xi + 4 + 2e^{4\xi}] \\
\sigma_{\eta\eta_1} &= \frac{1}{c^2} A^{-2}[2\cos 4\eta - 8(\cos 2\eta)e^{2\xi} + 4 + 2e^{\xi}] \\
\sigma_{\xi\eta_1} &= \sigma_{\eta\xi_1} = \frac{1}{c^2} A^{-2}[4\sin 2\eta \cosh 2\xi] \\
\mu_{\xi_1} &= \mu_{\eta_1} = 0 \\
\nabla^2 U_1 &= \frac{8}{c^2} A^{-1}[e^{2\xi} - \cos 2\eta] \\
\frac{\partial}{\partial\eta}\nabla^2 U_1 &= -\frac{16}{c^2} A^{-2}[\sin 2\eta \sinh 2\xi] \\
\frac{\partial}{\partial\xi}\nabla^2 U_1 &= \frac{16}{c^2} A^{-2}[1 - \cos 2\eta \cosh 2\xi]
\end{aligned}\right\} \quad (3.9)$$

---

*See Equations (6.2351) to (6.2363) and (6.2421) to (6.2423) on pp. 541 and 543, respectively.

For U_2, we have

$$\left.\begin{aligned}
\sigma_{\xi\xi_2} &= \frac{1}{c^2} A^{-2}[2\cos 4\eta - 8\cos 2\eta \cosh 2\xi + 4 + 2e^{-4\xi}] \\
\sigma_{\eta\eta_2} &= \frac{1}{c^2} A^{-2}[2\cos 4\eta - 8\cos 2\eta e^{-2\xi} + 4 + 2e^{-4\xi}] \\
\sigma_{\xi\eta_2} &= \sigma_{\eta\xi_2} = \frac{1}{c^2} A^{-2}[-4\sin 2\eta \cosh 2\xi] \\
\mu_{\xi_2} &= \mu_{\eta_2} = 0 \\
\nabla^2 U_2 &= \frac{8}{c^2} A^{-1}[e^{-2\xi} - \cos 2\eta] \\
\frac{\partial}{\partial\eta}\nabla^2 U_2 &= \frac{16}{c^2} A^{-2} \sin 2\eta \sinh 2\xi \\
\frac{\partial}{\partial\xi}\nabla^2 U_2 &= -\frac{16}{c^2} A^{-2}[1 - \cos 2\eta \cosh 2\xi]
\end{aligned}\right\} \quad (3.10)$$

For U_3, we have

$$\left.\begin{aligned}
\sigma_{\xi\xi_3} &= \frac{2}{c^2} A^{-2}[\cos 4\eta \cdot e^{-2\xi} - \cos 2\eta(e^{-4\xi} + 3) + 3e^{-2\xi}] \\
\sigma_{\eta\eta_3} &= \frac{2}{c^2} A^{-2}[-\cos 4\eta \cdot e^{-2\xi} - 3e^{-2\xi} + \cos 2\eta(e^{-4\xi} + 3)] \\
\sigma_{\xi\eta_3} &= \sigma_{\eta\xi_3} = \frac{2}{c^2} A^{-2}[\sin 4\eta \cdot e^{-2\xi} - \sin 2\eta(e^{-4\xi} + 3)] \\
\mu_{\xi_3} &= \mu_{\eta_3} = \nabla^2 U_3 = \frac{\partial}{\partial\eta}\nabla^2 U_3 = \frac{\partial}{\partial\xi}\nabla^2 U_3 = 0
\end{aligned}\right\} \quad (3.11)$$

For U_4, we have

$$\left.\begin{aligned}
\sigma_{\xi\xi_4} &= \frac{1}{c^2} A^{-2}[2 \sinh 2\xi] \\
\sigma_{\eta\eta_4} &= \frac{1}{c^2} A^{-2}[-2 \sinh 2\xi] \\
\sigma_{\xi\eta_4} &= \sigma_{\eta\xi_4} = \frac{1}{c^2} A^{-2}[\sin 2\eta] \\
\mu_{\xi_4} &= \mu_{\eta_4} = \nabla^2 U_4 = \frac{\partial}{\partial\eta} \nabla^2 U_4 = \frac{\partial}{\partial\xi} \nabla^2 U_4 = 0
\end{aligned}\right\} \tag{3.12}$$

For U_5, we have

$$\left.\begin{aligned}
\sigma_{\xi\xi_5} &= \frac{1}{c^2} A^{-2}[2 \cos 4\eta \cdot e^{2\xi} - \cos 2\eta(2e^{4\xi} + 6) + 6e^{2\xi}] \\
\sigma_{\eta\eta_5} &= \frac{1}{c^2} A^{-2}[-2 \cos 4\eta \cdot e^{2\xi} + \cos 2\eta(2e^{4\xi} + 6) - 6e^{2\xi}] \\
\sigma_{\xi\eta_5} &= \sigma_{\eta\xi_5} = \frac{1}{c^2} A^{-2}[-2 \sin 4\eta \cdot e^{2\xi} + \sin 2\eta(2e^{4\xi} + 6)] \\
\mu_{\xi_5} &= \mu_{\eta_5} = \nabla^2 U_5 = \frac{\partial}{\partial\eta} \nabla^2 U_5 = \frac{\partial}{\partial\xi} \nabla^2 U_5 = 0
\end{aligned}\right\} \tag{3.13}$$

For Ψ_1, we have

$$\left.\begin{aligned}
\sigma_{\xi\xi_6} &= \frac{12}{c^2} A^{-4} \sinh 2\xi[1 + \sin^2 2\eta - \cos 2\eta \cosh 2\xi] \\
\sigma_{\eta\eta_6} &= \frac{12}{c^2} A^{-4} \sinh 2\xi[1 + \sin^2 2\eta - \cos 2\eta \cosh 2\xi]
\end{aligned}\right|$$

$$\left.\begin{aligned}
\sigma_{\xi\eta_6} &= \sigma_{\eta\xi_6} = \frac{12}{c^2} A^{-4} \sin 2\eta [\sinh^2 2\xi - 1 \\
&\quad + \cos 2\eta \cosh 2\xi] \\
u_{\eta_6} &= -\frac{2\sqrt{2}}{c} A^{-5/2} [1 - \cos 2\eta \cosh 2\xi] \\
u_{\xi_6} &= -\frac{2\sqrt{2}}{c} A^{-5/2} [\sin 2\eta \sinh 2\xi] \\
\nabla^2 \Psi_1 &= 0 \\
\Psi_1 - \ell^2 \nabla^2 \Psi_1 &= A^{-1} \sin 2\eta \\
\frac{\partial}{\partial \eta}(\Psi_1 - \ell^2 \nabla^2 \Psi_1) &= -2A^{-2} [1 - \cos 2\eta \cosh 2\xi] \\
\frac{\partial}{\partial \xi}(\Psi_1 - \ell^2 \nabla^2 \Psi_1) &= -2A^{-2} [\sin 2\eta \sinh 2\xi]
\end{aligned}\right\} \quad (3.14)$$

For Ψ_2, we have

$$\sigma_{\xi\xi_7} = \frac{2}{c^2 A^2}[-A(Fek_2)'(se_2)' + (Fek_2)'(se_2) \sin 2\eta + (Fek_2)(\sinh 2\xi)(se_2)']$$

$$\sigma_{\eta\eta_7} = -\frac{2}{c^2 A^2}[-A(Fek_2)'(se_2)' + (Fek_2)'(se_2) \sin 2\eta + (Fek_2)(\sinh 2\xi)(se_2)']$$

$$
\left.
\begin{aligned}
\sigma_{\xi\eta_7} &= \frac{2}{c^2A^2}[-A(Fek_2)(se_2)'' - (\sinh 2\xi)(Fek_2)'(se_2) \\
&\quad + (Fek_2)(se_2)' \sin 2\eta] \\
\sigma_{\eta\xi_7} &= \frac{2}{c^2A^2}[A(Fek_2)''(se_2) - (\sinh 2\xi)(Fek_2)'(se_2) \\
&\quad + (Fek_2)(se_2)' \sin 2\eta] \\
\mu_{\xi_7} &= \frac{1}{c}\left(\frac{2}{A}\right)^{1/2} (Fek_2)'(se_2) \\
\mu_{\eta_7} &= \frac{1}{c}\left(\frac{2}{A}\right)^{1/2} (Fek_2)(se_2)' \\
\Psi_2 - \ell^2\nabla^2\Psi_2 &= \frac{\partial}{\partial\eta}(\Psi_2 - \ell^2\nabla^2\Psi_2) \\
&= \frac{\partial}{\partial\xi}(\Psi_2 - \ell^2\nabla^2\Psi_2) = 0
\end{aligned}
\right\} (3.15)
$$

Substituting the expressions in (3.9) to (3.15) into (2.14) to (2.19), the results are as follows:

$$
\begin{aligned}
\sigma_{\xi\xi} = \frac{2}{c^2A^2}\{&H[\cos 4\eta - 4\cos 2\eta \cosh 2\xi + 2 + e^{4\xi}] \\
&+ B[\cos 4\eta - 4\cos 2\eta \cdot e^{-2\xi} + 2 + e^{-4\xi}] \\
&+ C[\cos 4\eta \cdot e^{-2\xi} - \cos 2\eta(e^{-4\xi} + 3) + 3e^{-2\xi}] \\
&+ D[\sinh 2\xi] +
\end{aligned}
$$

$$+ E[\cos 4\eta \cdot e^{2\xi} - \cos 2\eta(e^{4\xi} + 3) + 3e^{2\xi}]$$

$$- \tilde{F}[\frac{6}{A^2} \sinh 2\xi(1 + \sin^2 2\eta - \cos 2\eta \cosh 2\xi)]$$

$$- \tilde{M}[-A(Fek_2)'(se_2)' + (Fek_2)'(se_2)\sin 2\eta$$

$$+ (Fek_2)\sinh 2\xi(se_2)']\} \tag{3.16}$$

$$\sigma_{\eta\eta} = \frac{2}{c^2A^2}\{H[\cos 4\eta - 4\cos 2\eta e^{2\xi} + 2 + e^{4\xi}]$$

$$+ B[\cos 4\eta - 4\cos 2\eta e^{-2\xi} + 2 + e^{-4\xi}]$$

$$- C[\cos 4\eta \cdot e^{-2\xi} + 3e^{-2\xi} - \cos 2\eta(e^{-4\xi} + 3]$$

$$- D[\sinh 2\xi]$$

$$- E[\cos 4\eta \cdot e^{2\xi} + 3e^{2\xi} - \cos 2\eta(e^{4\xi} + 3)]$$

$$+ \tilde{F}[\frac{6}{A^2} \sinh 2\xi(1 + \sin^2 2\eta - \cos 2\eta \cosh 2\xi)]$$

$$- \tilde{M}[-A(Fek_2)'(se_2)' + (Fek_2)'(se_2)\sin 2\eta$$

$$+ (Fek_2)\sinh 2\xi(se_2)']\} \tag{3.17}$$

$$\sigma_{\xi\eta} = \frac{2}{c^2A^2}\{2(H - B)[\sin 2\eta \cosh 2\xi]$$

$$+ C[\sin 4\eta \cdot e^{-2\xi} - \sin 2\eta(e^{-4\xi} + 3)]$$

$$+ D[\sin 2\eta]$$

$$- E[\sin 4\eta \cdot e^{2\xi} - \sin 2\eta(e^{4\xi} + 3)]$$

$$+ \tilde{F}\frac{6}{c^2} \sin 2\eta(\sinh^2 2\xi - 1 + \cos 2\eta \cosh 2\xi)]$$

$$+ \tilde{M}[-A(Fek_2)(se_2)'' - \sinh 2\xi(Fek_2)'(se_2)$$

$$+ (Fek_2)(se_2)' \sin 2\eta]\} \tag{3.18}$$

$$\sigma_{\eta\xi} = \sigma_{\xi\eta} + \frac{2}{c^2A^2}\tilde{M}[A(Fek_2)''(se_2) + A(Fek_2)(se_2)''] \tag{3.19}$$

$$\mu_\xi = \frac{\sqrt{2}}{cA^{-5/2}}\{-2F[\sin 2\eta \sinh 2\xi] + \tilde{M}[A^2(Fek_2)'(se_2)]\} \tag{3.20}$$

$$\mu_\eta = \frac{\sqrt{2}}{cA^{-5/2}}\{-2F[1 - \cos 2\eta \cosh 2\xi] + \tilde{M}[A^2(Fek_2)(se_2)']\} \tag{3.21}$$

The seven unknown coefficients H, B, C, D, E, $\tilde{F}$, and $\tilde{M}$ in (3.15) to (3.21) will be determined by using boundary conditions* at $\xi = \xi_o$, Cauchy-Riemann equations, and regularity conditions at $\xi \to \infty$ in Sections 3.3 through 3.5.

3.3. Determination of H, E, and D

Since the plane is subjected to uniform tension, p, parallel to the axis of x at infinity (see Fig. 1), the regularity condition (2.23) is then satisfied at infinity. This can also be expressed in terms of stress function as

*See Equations (2.20) through (2.22).

$$U = \frac{py^2}{2} = \frac{p}{2} c^2 \sinh^2\xi \sin^2\eta = \frac{pc^2}{16}(e^{2\xi} + e^{-2\xi} - 2)(1 - \cos 2\eta)$$

Since at $\xi \to \infty$, both $e^{-2\xi}$ and 2 can be neglected compared with $e^{2\xi}$, U becomes at $\xi \to \infty$

$$U = \frac{pc^2}{16} e^{2\xi} - \frac{pc^2}{16} e^{2\xi} \cos 2\eta \tag{3.22}$$

From (3.1) and (3.4) through (3.7), U becomes at $\xi \to \infty$

$$U = He^{2\xi} + Ee^{2\xi} \cos 2\eta \tag{3.23}$$

Comparing (3.22) with (3.23), we get

$$H = \frac{pc^2}{16} \tag{3.24}$$

$$E = -\frac{pc^2}{16} \tag{3.25}$$

Notice that D is the coefficient of U_4. Since U_4 is independent of couple-stress effects, D is assumed to be in the same form as one obtained in the classical case [8],* namely,

$$D = -\frac{pc^2}{4}(\cosh 2\xi_o - 1) \tag{3.26}$$

3.4. Determination of the Relation Between B, F, and p by Use of Cauchy-Riemann Equations (2.13)

Using the expressions in (3.9) through (3.15) together with (3.6) and (3.7), we have

---

*See Equation (6.2495) on p. 544.

$$\nabla^2 U = H\nabla^2 U_1 + B\nabla^2 U_2 \tag{3.27}$$

$$\Psi - \ell^2\nabla^2\Psi = F(\Psi_1 - \ell^2\nabla^2\Psi_1) = F\Psi_1 \tag{3.28}$$

By direct substitution of (3.27) and (3.28) into (2.13), the Cauchy-Riemann equations read

$$F\frac{\partial\Psi_1}{\partial\xi} = -2(1-\nu)\ell^2[H\frac{\partial}{\partial\eta}(\nabla^2 U_1) + B\frac{\partial}{\partial\eta}(\nabla^2 U_2)] \tag{3.29}$$

$$F\frac{\partial\Psi_1}{\partial\eta} = 2(1-\nu)\ell^2[H\frac{\partial}{\partial\xi}(\nabla^2 U_1) + B\frac{\partial}{\partial\xi}(\nabla^2 U_2)] \tag{3.30}$$

Using the results obtained in (3.9), (3.10), (3.14), and (3.24), either (3.29) or (3.30) leads to the same expression

$$\frac{2F}{c^2} = (1-\nu)\ell^2(B\frac{16}{c^2} - p) \tag{3.31}$$

or

$$B = \frac{\tilde{F}c^2}{16(1-\nu)\ell^2} + \frac{pc^2}{16} \tag{3.32}$$

3.5. Determination of C, $\tilde{F}$, and $\tilde{M}$

The traction conditions on the interior boundary will now be used. On the boundary, $\xi = \xi_o$ and η varies within the limit $(0,2\pi)$.

a. Couple-stress μ_ξ. On the boundary, expression (3.20) becomes

$$\mu_\xi(\xi_o,\eta) = \frac{\sqrt{2}}{cA^{-5/2}}\{-2\tilde{F}\ \sinh 2\xi_o\ \sin 2\eta + M[(Fek_2)']_{\xi_o}\cdot$$

$$\cdot\ [(\tfrac{1}{2} + \cos^2 2\xi_o)(se_2) - (2\cosh 2\xi_o)(se_2)\cdot \tag{3.33}$$

$$\cdot\ \cos 2\eta + \tfrac{1}{2}(se_2)\ \cos 4\eta]\}$$

The boundary condition (2.22) is now applied by multiplying (3.33) on both sides with se_2, and by integrating the resulting expression with respect to η through the range $(0,2\pi)$. The following expression results with the use of equations developed in Appendix II (AII.1) at $n = 2$, (AII.4) and (AII.5) at $n = 0$, $p = 1,2$.

$$\tilde{M} = 2\Gamma\tilde{F} \tag{3.34}$$

where

$$\Gamma = B_2^{(2)}\ \sinh 2\xi_o/\{[(Fek_2)']_{\xi_o}[(\tfrac{1}{2} + \cosh^2 2\xi_o)$$

$$+ (2\cosh 2\xi_o)\Big(\sum_{r=0}^{\infty} B_{2r+2}^{(2)}\ B_{2r+4}^{(2)}\Big) + \tfrac{1}{2}\Big(\sum_{r=0}^{\infty} B_{2r+2}^{(2)}\cdot \tag{3.35}$$

$$\cdot\ B_{2r+6}^{(2)}\Big) - \tfrac{1}{4}(B_2^{(2)})^2]\}$$

b. <u>$\sigma_{\xi\eta}$ on the boundary</u>. After (3.18) is evaluated on the boundary, further reduction is introduced by the use of (3.24), (3.25), (3.26), and (3.32).

$$
\begin{aligned}
[K\sigma_{\xi\eta}]_{\xi=\xi_o} = {} & [\zeta_1(\tfrac{1}{4} + \cosh^2 2\xi_o) - \zeta_2(\cosh 2\xi_o) + \zeta_3]\sin 2\eta \\
& + [-\zeta_1(\cosh 2\xi_o) + \zeta_2(\tfrac{1}{2} + \cosh^2 2\xi_o) + \zeta_4]\sin 4\eta \\
& + [\tfrac{\zeta_1}{4} - \zeta_2 \cosh 2\xi_o]\sin 6\eta + [\tfrac{\zeta_2}{4}]\sin 8\eta \\
& - [\zeta_5(\tfrac{3}{2} + \cosh^2 2\xi_o)\cosh 2\xi_o](se_2)'' \\
& + [\zeta_5(3\cosh^2 2\xi_o + \tfrac{3}{4})](se_2)''\cos 2\eta \\
& - [\zeta_5(\tfrac{3}{2}\cosh 2\xi_o)](se_2)''\cos 4\eta + [\tfrac{\zeta_5}{4}](se_2)''\cos 6\eta \\
& - [\zeta_6(\tfrac{1}{2} + \cosh^2 2\xi_o)](se_2) + [\xi_6(2\cosh 2\xi_o)]\cdot \\
& \cdot (se_2)\cos 2\eta \\
& - [\tfrac{\zeta_6}{2}](se_2)\cos 4\eta + [\xi_7(\tfrac{1}{4} + \cosh^2 2\xi_o)](se_2)'\sin 2\eta \\
& - [\zeta_7(\cosh 2\xi_o)](se_2)'\sin 4\eta + [\tfrac{\zeta_7}{4}](se_2)'\sin 6\eta
\end{aligned}
\tag{3.36}
$$

where K is a constant and

$$
\begin{aligned}
\zeta_1 = {} & -\frac{c}{8(1-\nu)\ell^2}\tilde{F}\cosh 2\xi_o - C(e^{-4\xi_o} + 3) \\
& - \frac{pc^2}{4}(\cosh 2\xi_o - 1) - \frac{pc^2}{16}(e^{4\xi_o} + 3)
\end{aligned}
$$

$$\zeta_2 = Ce^{-2\xi_o} + \frac{pc^2}{16} e^{2\xi_o}$$

$$\zeta_3 = 6\tilde{F}(\sinh^2 2\xi_o - 1)$$

$$\zeta_4 = 3\tilde{F}(\cosh 2\xi_o)$$

$$\zeta_5 = 2\tilde{F}\Gamma(Fek_2)_{\xi=\xi_o}$$

$$\zeta_6 = 2\tilde{F}\Gamma[(Fek_2)']_{\xi=\xi_o} \sinh 2\xi_o$$

$$\zeta_7 = 2\tilde{F}\Gamma(Fek_2)_{\xi=\xi_o}$$

The boundary condition (2.21) is used by multiplying (3.36) on both sides with se_2 and by integrating with respect to n from 0 to 2π. Further simplification is achieved by using (AII.1) with n = 2,4,6,8; (AII.4) and (AII.5) with n = 0, p = 1,2; (AII.6) with n = 0, p = 1,2,3; and (AII.7) with n = 0, p = 1,2,3. All algebraic manipulations are routine but cumbersome, thus omitted. The resulting expression is

$$bC + g\tilde{F} = m \tag{3.37}$$

where

$$b = \Big\{ -[(e^{-4\xi_o} + 3)(\tfrac{1}{4} + \cosh^2 2\xi_o) + e^{-2\xi_o} \cosh 2\xi_o] B_2^{(2)}$$

$$[(e^{-4\xi_o} + 3)(\cosh 2\xi_o) + e^{-2\xi_o}(\tfrac{1}{2} + \cosh^2 2\xi_o)] B_4^{(2)}$$

$$- \left[\frac{1}{4}(e^{-4\xi_o} + 3) + e^{-2\xi_o} \cosh 2\xi_o\right] B_6^{(2)} - \frac{1}{4} e^{-2\xi_o} B_8^{(2)} \Bigg\} \quad (3.33)$$

$$g = \Bigg[\Bigg[-\left(\frac{c^2 \cosh 2\xi_o}{8(1-\nu)\ell^2}\right)\left(\frac{1}{4} + \cosh^2 2\xi_o\right) + 4 \sinh^2 2\xi_o - 6\Bigg] B_2^{(2)}$$

$$- \left[\left(\frac{c^2 \cosh^2 2\xi_o}{8(1-\nu)\ell^2}\right) + 3 \cosh 2\xi_o\right] B_4^{(2)} - \left[\frac{1}{4} \frac{c^2 \cosh 2\xi_o}{8(1-\nu)\ell^2} B_6^{(2)}\right]$$

$$+ \{2\Gamma(Fek_2)_{\xi=\xi_o}\} \Bigg\{\left[\left(\frac{3}{2} + \cosh^2 2\xi_o\right) \cosh 2\xi_o\right] \cdot$$

$$\cdot \left[\sum_{r=0}^{\infty} \left((2r+2) B_{2r+2}^{(2)}\right)^2\right]$$

$$+ \left[\frac{3}{4} + 3 \cosh^2 2\xi_o\right] \left[\sum_{r=0}^{\infty} (2r+3)^2 B_{2r+2}^{(2)} B_{2r+3}^{(2)}\right]$$

$$+ \left[\frac{3}{2} \cosh 2\xi_o\right] \Bigg[2 (B_2^{(2)})^2 +$$

$$+ \sum_{r=0}^{\infty} (2r+5)^2 \cdot B_{2r+2}^{(2)} B_{2r+6}^{(2)}\Bigg]$$

$$+ \frac{1}{4}\left[-9 B_2^{(2)} B_4^{(2)} + \sum_{r=0}^{\infty} (2R+7)^2 B_{2r+2}^{(2)} B_{2r+8}^{(2)}\right]$$

$$- \left[\frac{1}{4} + \cosh^2 2\xi_o\right] \left[\sum_{r=0}^{\infty} B_{2r+2}^{(2)} B_{2r+4}^{(2)}\right]$$

$$- 2[\cosh 2\xi_o] \left[\sum_{r=0}^{\infty} B_{2r+8}^{(2)} B_{2r+6}^{(2)}\right] - \frac{3}{4}\left[\sum_{r=0}^{\infty} B_{2r+2}^{(2)} B_{2r+8}^{(2)}\right]\Bigg\}\Bigg] \quad (3.39)$$

$$m = \frac{pc^2}{16}\{[(e^{4\xi_o} + 4\cosh 2\xi_o - 1)(\frac{1}{4} + \cosh^2 2\xi_o)$$

$$+ e^{2\xi_o}\cosh 2\xi_o]B_2^{(2)}$$

$$+ [(e^{4\xi_o} + 4\cosh 2\xi_o - 1)\cosh 2\xi_o$$

$$+ e^{2\xi_o}(\frac{1}{2} + \cosh^2 2\xi_o)]B_4^{(2)} \tag{3.40}$$

$$+ [\frac{1}{4}(e^{4\xi_o} + 4\cosh 2\xi_o - 1) + e^{2\xi_o}\cosh 2\xi_o]B_6^{(2)}$$

$$+ \frac{1}{4}e^{2\xi_o}B_8^{(2)}\}$$

c. $\underline{\sigma_{\xi\xi}\text{ on the boundary}}$. Equation (3.16) is evaluated first on the boundary. The following expression results with the substitution of Equations (3.24), (3.25), (3.26), and (3.32).

$$[K'\sigma_{\xi\xi}]_{\xi=\xi_o} = [\frac{\gamma_1}{4} + \gamma_2\cosh 2\xi_o + \gamma_3(\frac{1}{2} + \cosh^2 2\xi_o) - \gamma_4]$$

$$+ [-\gamma_1\cosh 2\xi_o - \gamma_2(\cosh^2 2\xi_o + \frac{3}{4})$$

$$- \gamma_3(2\cosh 2\xi_o) + \gamma_6]\cos 2\eta$$

$$+ [\gamma_1(\frac{1}{2} + \cosh^2 2\xi_o) + \gamma_2(\cosh 2\xi_o)$$

$$+ \frac{\gamma_3}{2} + \gamma_5]\cos 4\eta$$

$$+ [-\gamma_1 \cosh 2\xi_o - \frac{\gamma_2}{4}] \cos 6\eta + [\frac{\gamma_1}{4}] \cos 8\eta$$

$$- [\gamma_7(\frac{3}{2} + \cosh^2 2\xi_o) \cosh 2\xi_o$$

$$- \gamma_9(\frac{1}{2} + \cosh^2 2\xi_o)] (se_2)' \tag{3.41}$$

$$+ [\gamma_7(3 \cosh^2 2\xi_o + \frac{3}{4})$$

$$- \gamma_9(2 \cosh 2\xi_o)] \cdot (se_2)' \cos 2\eta$$

$$- [\gamma_7(\frac{3}{2} \cosh 2\xi_o) - \frac{\gamma_9}{2}] (se_2)' \cos 4\eta$$

$$+ [\frac{\gamma_7}{4}] (se_2)' \cos 6\eta$$

$$+ [\gamma_8 (\frac{1}{4} + \cosh^2 2\xi_o)] \cdot (se_2) \sin 2\eta$$

$$- [\gamma_8 \cosh 2\xi_o] (se_2) \sin 4\eta + [\frac{\gamma_8}{4}](se_2) \sin 6\eta = 0$$

where K' is a constant and

$$\gamma_1 = \frac{pc^2}{16}(2 - e^{2\xi_o}) + \frac{\tilde{F}c^2}{16(1 - \nu)\ell^2} + Ce^{-2\xi_o}$$

$$\gamma_2 = \frac{\tilde{F}c^2}{4(1 - \nu)\ell^2} \cosh 2\xi_o + \frac{pc^2}{16}(8 \cosh 2\xi_o - e^{4\xi_o} - 3)$$

$$+ (e^{-4\xi_o} + 3)C$$

$$\gamma_3 = \frac{\tilde{F}c^2}{16(1-\nu)\ell^2}(2 + e^{-4\xi_o}) + \frac{pc^2}{16}(4 + e^{4\xi_o} + e^{-4\xi_o}$$

$$- 3e^{2\xi_o}) + 3Ce^{-2\xi_o} + D \sinh 2\xi_o$$

$$\gamma_4 = 9\tilde{F} \sinh 2\xi_o$$

$$\gamma_5 = 3\tilde{F} \sinh 2\xi_o$$

$$\gamma_6 = 3\tilde{F} \sinh 4\xi_o$$

$$\gamma_7 = 2\Gamma\tilde{F}[(Fek_2)']_{\xi=\xi_o}$$

$$\gamma_8 = 2\Gamma\tilde{F}[(Fek_2)']_{\xi=\xi_o}$$

$$\gamma_9 = 2\Gamma\tilde{F}(Fek_2)_{\xi=\xi_o} \sinh 2\xi_o$$

The boundary condition (2.20) is applied with the multiplication of (3.41) by ce_2, then by integrating the resulting expression with respect to η from 0 to 2π. By using (AII.2) and (AII.3) with η = 2, 4, 6, 8; (AII.8) with n = 0, p = 1,2,3; and (AII.9) with η = 0, p = 1,2,3, the final expression results in

$$\kappa C + s\tilde{F} = t \tag{3.42}$$

where

$$\kappa = \{-[\tfrac{7}{2} e^{-2\xi_o} + (2e^{-4\xi_o} + 6) \cosh 2\xi_o + 6e^{-2\xi_o} \cdot \cosh^2 2\xi_o] A_o^{(2)}$$

$$- [7e^{-2\xi_o} \cosh 2\xi_o$$

$$+ (e^{-4\xi_o} + 3) \cdot (\cosh^2 2\xi_o + \tfrac{3}{2})] A_2^{(2)} \qquad (3.43)$$

$$- [2e^{-2\xi_o} + e^{-2\xi_o} \cosh^2 2\xi_o + (e^{-4\xi_o} + 3) \cdot \cosh 2\xi_o] A_4^{(2)}$$

$$- [e^{-2\xi_o} \cosh 2\xi_o + \tfrac{1}{4}(e^{-4\xi_o} + 3)] A_6^{(2)} - \frac{e^{-2\xi_o}}{4} A_8^{(2)}\}$$

$$s = \{\frac{c^2}{16(1-\nu)\ell^2}\Bigg\{-[\tfrac{1}{2} + 8 \cosh^2 2\xi_o + (2 + e^{-4\xi_o})(1 + 2 \cosh^2 2\xi_o)$$

$$- (18 \sinh 2\xi_o)(\frac{16(1-\nu)\ell^2}{c^2})] A_o^{(2)}$$

$$- [\cosh 2\xi_o + 4(\cosh 2\xi_o)(\tfrac{3}{4} + \cosh^2 2\xi_o)$$

$$+ (2 + e^{-4\xi_o})(2 \cosh 2\xi_o)$$

$$(3 \sinh 4\xi_o)(\frac{16(1-\nu)\ell^2}{c^2})] A_2^{(2)}$$

$$- [(\tfrac{1}{2} + \cosh^2 2\xi_o) + 4 \cosh^2 2\xi_o + (1 + \tfrac{1}{2} e^{-4\xi_o})$$

$$+ (3 \sinh 2\xi_o)(\frac{16(1-\nu)\ell^2}{c^2})] A_4^{(2)}$$

$$- [2 \cosh 2\xi_o] A_6^{(2)} - (\tfrac{1}{4}) A_8^{(2)} \}$$

$$+ 2\Gamma \{ [(\tfrac{1}{2} + \cosh^2 2\xi_o) (\sinh 2\xi_o) (Fek_2)_{\xi=\xi_o}$$

$$- (\tfrac{3}{2} + \cosh^2 2\xi_o) (Fek_2)'_{\xi=\xi_o} \cosh 2\xi_o]$$

$$\cdot \left[\sum_{r=0}^{\infty} (2r + 2) A_{2r+2}^{(2)} B_{2r+2}^{(2)} \right]$$

$$+ [2(\cosh 2\xi_o) (\sinh 2\xi_o) (Fek_2)_{\xi=\xi_o}$$

$$- (3 \cosh^2 2\xi_o + \tfrac{3}{4}) (Fek_2)'_{\xi=\xi_o}] \cdot [A_o^{(2)} B_2^{(2)} +$$

$$+ \frac{1}{2} \sum_{r=0}^{\infty} (2r + 2) A_{2r+4}^{(2)} B_{2r+2}^{(2)}]$$

$$+ [\tfrac{1}{2} \sinh 2\xi_o (Fek_2)_{\xi=\xi_o} - \tfrac{3}{2} (\cosh 2\xi_o) (Fek_2)'_{\xi=\xi_o}]$$

$$\cdot [A_2^{(2)} B_2^{(2)} + 2A_0^{(2)} B_4^{(2)} - \frac{1}{2} \sum_{r=0}^{\infty} (2r + 2) A_{2r+4}^{(2)} B_{2r+2}^{(2)}] \}$$

$$+ 2\Gamma (Fek_2)'_{\xi=\xi_o} \{ - \tfrac{1}{4} [A_4^{(2)} B_2^{(2)} + 2A_2^{(2)} B_4^{(2)}$$

$$+ 3A_0^{(2)} B_6^{(2)} + \frac{1}{2} \sum_{r=0}^{\infty} (2r + 2) A_{2r+8}^{(2)} B_{2r+2}^{(2)}]$$

$$+ \frac{1}{2}\left(\frac{1}{4} + \cosh^2 2\xi_o\right)\left[\sum_{r=0}^{\infty} A_{2r+2}^{(2)} B_{2r+2}^{(2)} - \sum_{r=0}^{\infty} A_{2r}^{(2)} B_{2r+2}^{(2)}\right]$$

$$- \frac{1}{2}(\cosh 2\xi_o)\left[\sum_{r=0}^{\infty} A_{2r+6}^{(2)} B_{2r+2}^{(2)} - \sum_{r=0}^{\infty} A_{2r}^{(2)} B_{2r+6}^{(2)}\right]$$

$$+ \frac{1}{2}\left[\sum_{r=0}^{\infty} A_{2r+8}^{(2)} B_{2r+2}^{(2)} - \sum_{r=0}^{\infty} A_{2r}^{(2)} B_{2r+8}^{(2)}\right]\Bigg\}\Bigg] \tag{3.44}$$

$$t = \frac{pc^2}{16}\Bigg\{-[\frac{1}{2}(e^{2\xi_o} - 2) + 2(3 + e^{4\xi_o} - 8\cosh 2\xi_o)\cosh 2\xi_o$$

$$+ (e^{2\xi_o} + 2e^{-2\xi_o} - 2e^{-4\xi_o} - 4)(1 + 2\cosh^2 2\xi_o)]A_o^{(2)}$$

$$+ [(-3e^{2\xi_o} - 4e^{-2\xi_o} + 4e^{-4\xi_o} + 10)\cosh 2\xi_o$$

$$- (3 + e^{4\xi_o} - 8\cosh 2\xi_o)(\cosh^2 2\xi_o + \frac{3}{4})]A_2^{(2)}$$

$$- [(2\cosh 2\xi_o - e^{-4\xi_o} - 3) + (e^{4\xi_o} + 3)\cosh 2\xi_o$$

$$+ (e^{2\xi_o} - 10)\cosh^2 2\xi_o]A_4^{(2)} - [e^{2\xi_o}\cosh 2\xi_o + \frac{3}{4} + \frac{1}{4}e^{4\xi_o}$$

$$- 4\cosh 2\xi_o]A_6^{(2)} - \frac{1}{4}(e^{2\xi_o} - 2)A_8^{(2)}\Bigg\} \tag{3.45}$$

Equations (3.37) and (3.42) form two linear nonhomogeneous equations with two unknowns, C and $\tilde{F}$. They can readily be solved by using Cramer's rule

$$C = \frac{ms - gt}{bs - g\kappa} \tag{3.46}$$

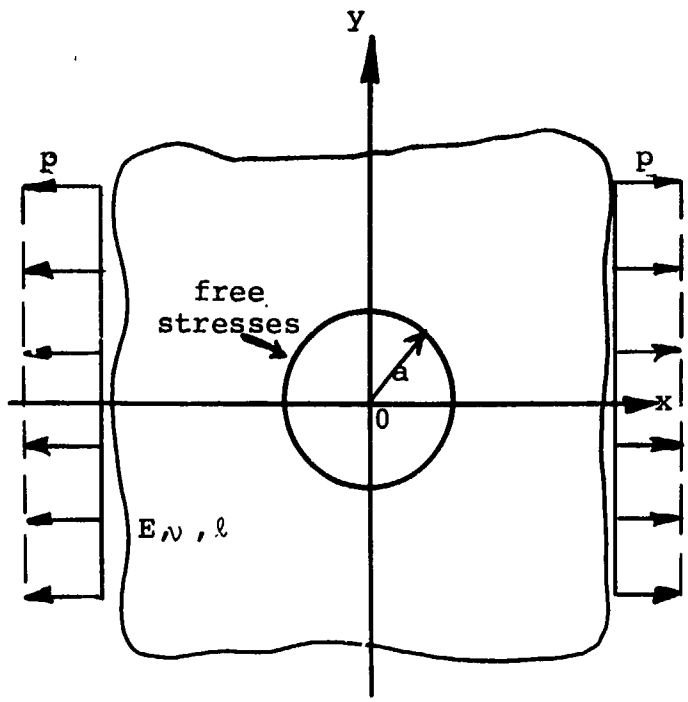
$$F = \frac{bt - m\kappa}{bs - g\kappa} \tag{3.47}$$

provided $(bs - g\kappa) = 0$.

The stresses and couple-stresses of this problem are the general expressions in (3.16) through (3.21), together with the seven determined coefficients in (3.24) through (3.26), (3.32), (3.34), (3.46), and (3.47).

4.0 SPECIAL CASES OF THE SOLUTION

4.1. Infinite Plane with a Circular Hole Under Simple Uniform Tension in the Presence of Couple-Stress (Fig. 5)



$\xi_o \to \alpha, \quad c \to c$

$e \to o, \quad a=b=a$

Figure 5.

Rewrite (3.6) and (3.7) as

$$U = \frac{H}{c^2}(c^2 U_1) + (B)(U_2) + (c^2 C)\frac{U_3}{c^2} + (D)(U_4) + \frac{E}{c^2}(c^2 U_5) \tag{4.1}$$

$$\psi = (F)\frac{\psi_1}{\frac{c^2}{2}} + (M)\frac{\pi}{(p_2)'}\psi_2 \tag{4.2}$$

Solution of this specific problem can be obtained by use of the limiting process described in Appendix III-(a). Equations (4.1) and (4.2) are simplified and reduced, by use of the forms in (AIII.6) and (AIII.7) and the basic theorem on limits

$$U = \frac{p}{4} r^2(1 - \cos\phi) - \frac{pa^2}{2} \ln r + \frac{pa^2}{2(1+\Delta)}\left[-\frac{a^2(1-\Delta)}{2r^2} + 1\right]\cos 2\phi \tag{4.3}$$

$$\psi = \frac{pa\ell}{1+\Delta}\left[\frac{4(1-\nu)a\ell}{r^2} - \frac{\Delta K_2\left(\frac{r}{\ell}\right)}{K_1\left(\frac{a}{\ell}\right)}\right]\sin 2\phi \tag{4.4}$$

which are identical to the forms obtained by [2].*

4.2. Infinite Plane with a Crack under Simple Uniform Tension (Fig. 6)

Solution to this problem can be obtained from the solution of the elliptical hole problem by taking $\xi_o \to 0$, as described in Appendix III-(b). The results in (AIII.19) show that in this crack problem the couple-stress effect vanishes. Furthermore, (AIII.19) shows that the solutions obtained here are identical to the corresponding classical solutions which can also be obtained by proceeding by the same limiting process in [8]. Since couple-stresses are related to curvature and no curvature is induced in this crack problem, it is clear that

*There is a misprint for the expression of E in Equation (38) in [2]. The correct form is $E = (-pa\ell F)/(1 + F)K_1(a/\ell)$.

couple-stress solutions should be the same as classical solutions.

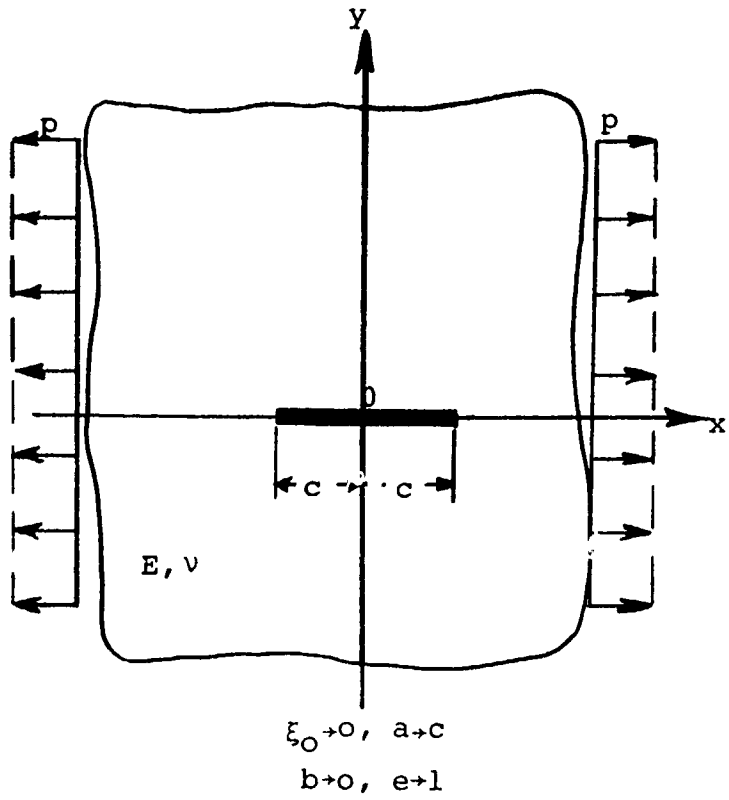


Figure 6.

4.3. Classical Solution with Free Stresses on $\xi = \xi_o$ (Fig. 7)

As $\ell \to 0$, couple-stress solutions of the problem in Fig. 1 obtained in Section 3.0 reduce to classical solutions of the problem in Fig. 7. The results in Equation (AIV.9) are identical to the forms obtained in [8].*

*See Equations (6.2481), (6.2494), and (6.2495) on pp. 543-544 of [8].

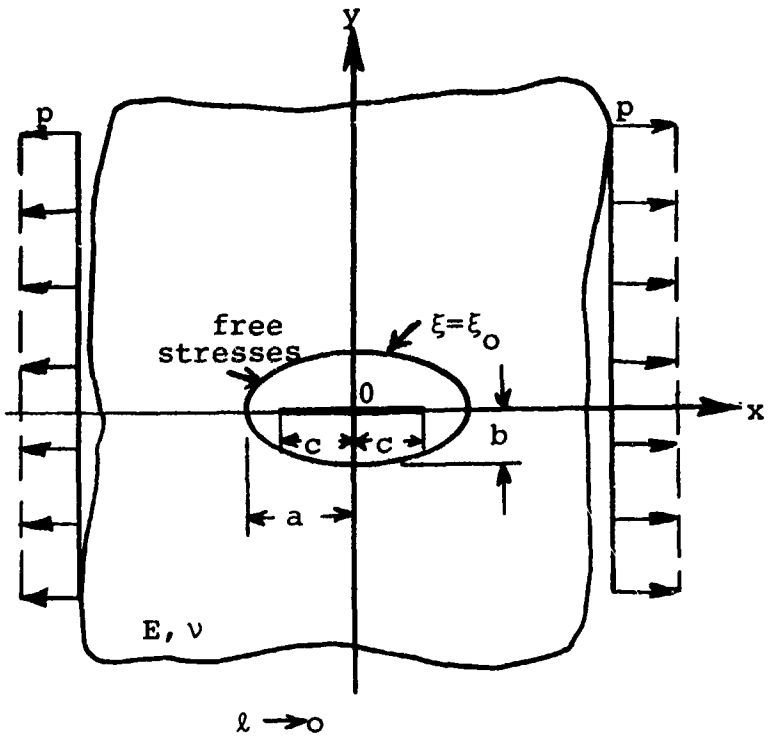


Figure 7.

5.0 DISCUSSION AND CONCLUSION

The solutions to this problem are given either as stress and couple-stress functions in (3.6) and (3.7) or as stresses and couple-stresses in (3.16) through (3.21), together with the seven determined coefficients in (3.24), (3.25), (3.26), (3.32), (3.34), (3.46), and (3.47). All series in (3.32), (3.24), (3.46), and (3.47) are shown to be convergent in Appendix IV. Special cases of the solutions are also obtained by the proper limiting process as discussed in Section 4.0. The three limiting cases are: (4.3) and (4.4) for the degenerate circle in Fig. 5, (AIII.19) for the degenerate crack in Fig. 6, and (AIV.9) for the classical solution in Fig. 7.

The result obtained is a subclass of solutions to the general self-equilibrated boundary-value problems. The selection of the Mathieus' functions and the form of weighting functions in the boundary conditions (2.20), (2.21), and (2.22) match a particular class of boundary-values, which gratifyingly does reduce upon limiting processes to <u>various classical solutions</u> (Figs. 6 and 7, and Fig. 5 for $\ell = 0$).

Here we summarize some important results about the three limiting cases. The solutions for the degenerate circle in (4.3) and (4.4) are shown to be identical to the results in [2]. The solutions for the degenerate crack in (AIII.19) are seen to be the same as those obtained in [8], by proceeding according to the same limiting process. Since couple-stresses are related to curvature and no curvature is induced in this crack problem, it is clear that couple-stress solutions in

(AIII.19) should be the same as the classical solutions in [8]. The classical solutions for the general elliptical hole problem (Fig. 7) in (AIV.9) are obtained by taking $\ell \to 0$ from (3.6) and (3.7) and they are identical to the results in [8].

APPENDIX I

Solution of a Two-dimensional Wave Equation in Elliptical Coordinates

The two-dimensional wave equation

$$\Psi - \ell^2 \nabla^2 \Psi = 0$$

can be expressed in elliptical coordinates ξ and η by using (2.12) as

$$\frac{\partial^2 \Psi}{\partial \xi^2} + \frac{\partial^2 \Psi}{\partial \eta^2} - 2q\ (\cosh 2\xi - \cos 2\eta)\Psi = 0 \qquad \text{(AI.1)}$$

Using the method of separation of variables and the form of solution of (AI.1) as $\Psi(\xi,\eta) = \Psi(\xi)\phi(\eta)$, (AI.1) is reduced to the following two ordinary differential equations known as Mathieu's equations.

$$\frac{d^2 \Psi}{d\xi^2} - (d + 2q \cosh 2\xi)\Psi = 0 \qquad \text{(AI.2)}$$

$$\frac{d^2 \phi}{d\eta^2} + (d + 2q \cos 2\eta)\phi = 0 \qquad \text{(AI.3)}$$

Here d is the separation constant.

From the solutions of (AI.2),* we choose the following one

---

*See Section 8.30 on p. 165 and Section 13.30 [Equation (5) on p. 248] in Reference [9].

$$\mathrm{Fek}_2(\xi,-q) = \frac{(p_2)'}{\pi A_o^{(2)}} \sum_{r=0}^{\infty} A_{2r}^{(2)} \; I_r(\nu_1)K_r(\nu_2) \tag{AI.4}$$

From the solutions of (AI.3),* we select, for the current problem,

$$\mathrm{se}_2(\eta,-q) = \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2)} \sin\,(2r + 2)\eta \tag{AI.5}$$

However, another Mathieu function of (AI.3)

$$\mathrm{ce}_2(\eta,-q) = - \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2)} \cos 2r\eta \tag{AI.6}$$

will be used in the text for evaluation of integrals.

Derivatives of $\mathrm{se}_2(\eta,-q)$, $\mathrm{Fek}_2(\xi,-q)$

Since the series expressed in (AI.4) and (AI.5) are uniformly convergent** for real ξ and η, respectively, they may be differentiated term by term.

$$\frac{\partial}{\partial\xi}[\mathrm{Fek}_2(\xi,-q)] = \frac{(p_2)'k}{\pi A_o^{(2)}} \sum_{r=0}^{\infty} A_{2r}^{(2)} \left[e^{\xi} I_r(\nu_1)\frac{\partial K_r(\nu_2)}{\partial \nu_2} - e^{-\xi} \frac{\partial I_r(\nu_1)}{\partial \nu_1} K_r(\nu_2)\right] \tag{AI.7}$$

---

*See Section 2.18 on p. 21 in Reference [9].

**See Sections 3.21 and 3.22 (on pp. 37-38) and Section 13.60 on p. 257 in Reference [9].

$$\frac{\partial}{\partial\eta}[se_2(\eta,-q)] = \sum_{r=0}^{\infty} (-1)^r (2r + 2) B_{2r+2}^{(2)} \cos(2r + 2)\eta \qquad \text{(AI.8)}$$

$$\frac{\partial^2}{\partial\eta^2}[se_2(\eta,-q)] = \sum_{r=0}^{\infty} (-1)^{r+1} (2r + 2)^2 B_{2r+2}^{(2)} \sin(2r + 2)\eta \qquad \text{(AI.9)}$$

APPENDIX II

Integrals Quoted from Reference [10]*

$$\int_0^{2\pi} se_2 \sin 2n\eta d\eta = (-1)^{n+1}\pi B_n^{(2)} \qquad (n \geq 1) \qquad \text{(AII.1)}$$

$$\int_0^{2\pi} ce_2 d\eta = -2\pi A_o^{(2)} \qquad \text{(AII.2)}$$

$$\int_0^{2\pi} ce_2 \cos 2n\eta d\eta = (-1)^{n+1}\pi A_n^{(2)} \qquad (n \geq 1) \qquad \text{(AII.3)}$$

$$\int_0^{2\pi} (se_2)^2 d\eta = \pi \qquad \text{(AII.4)}$$

More Integrals Established

All the integrands in (AII.5) through (AII.9) can be shown to be convergent absolutely and uniformly for all real η by using Reference [11],**,† hence term-by-term integration can be applied to all of them (Reference [11]).‡ By direct integration,¶ the following results are obtained.

---

*See Equations (20.5.3) and (20.5.4) on p. 732.

**See Theorem on p. 146.

†See "Weierstrass's test" on p. 345.

‡See Theorem 3 on p. 341.

¶See Section 5 on p. 384 in Reference [9].

$$\frac{1}{\pi}\int_0^{2\pi} [se_{2n+2}(\eta,-q)]^2 \cos 2p\eta d\eta =$$

$$= \begin{cases} \sum_{r=0}^{\frac{(p-3)}{2}} B_{2r+2}^{(2n+2)} B_{2p-2r-2}^{(2n+2)} - \\ - \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} B_{2p+2r+2}^{(2n+2)} \qquad (p \text{ is odd}, \geq 1) \\ - \frac{1}{2}\left[B_p^{(2n+2)}\right]^2 - \sum_{r=0}^{\frac{(p-4)}{2}} B_{2r+2}^{(2n+2)} B_{2p-2r-2}^{(2n+2)} + \\ + \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} B_{2p+2r+2}^{(2n+2)} \qquad (p \text{ is even}, \geq 2) \end{cases} \quad \text{(AII.5)}$$

$$\frac{1}{\pi}\int_0^{2\pi} [se_{2n+2}(\eta,-q)][se'_{2n+2}(\eta,-q)] \sin 2p\eta d\eta =$$

$$= (-1)^p\, p \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} B_{2p+2r+2}^{(2n+2)} \qquad (p \geq 1) \quad \text{(AII.6)}$$

$$\frac{1}{\pi}\int_0^{2\pi} se_{2n+2}(\eta,-q)\, se''_{2n+2}(\eta,-q) =$$

$$= - \sum_{r=0}^{\infty} (2r+2) B_{2r+2}^{(2n+2)\,2}$$

$$\frac{1}{\pi}\int_0^{2\pi} se_{2n+2}(\eta,-q)\, se''_{2n+2}(\eta,-q) \cos 2p\eta d\eta = \quad \text{(AII.7)}$$

$$= \begin{cases} -\sum_{r=0}^{\frac{1}{2}(p-3)} (2p-2r-3)^2 B_{2r+2}^{(2n+2)} B_{2p-2r-2}^{(2n+2)} \\ \quad + \sum_{r=0}^{\infty} (2p+2r+1)^2 B_{2r+2}^{(2n+2)} B_{2p+2r+2}^{(2n+2)} \quad (p \text{ is odd}, \geq 1) \\ -\frac{1}{2} p B_p^{(2n+2)\,2} + \sum_{r=0}^{\frac{(p-4)}{2}} (2p-2r-3)^2 B_{2r+2}^{(2n+2)} B_{2p-2r-2}^{(2n+2)} \\ \quad - \sum_{r=0}^{\infty} (2p+2r+1) B_{2r+2}^{(2n+2)} B_{2p+2r+2}^{(2n+2)} \quad (p \text{ is even}, \geq 2) \end{cases} \tag{AII.7}$$

$$\left.\begin{aligned} &\frac{2}{\pi}\int_0^{2\pi} ce_{2n+2}(\eta,-q)\,se_{2n+2}(\eta,-q)\,\sin 2p\eta\,d\eta = \\ &= \sum_{r=0}^{\infty} A_{2r+2p}^{(2n+2)} B_{2r+2}^{(2n+2)} - \sum_{r=0}^{\infty} A_{2r+2}^{(2n+2)} B_{2r+2p}^{(2n+2)} \quad (p \geq 1) \end{aligned}\right\} \tag{AII.8}$$

$$\frac{1}{\pi}\int_0^{2\pi} ce_{2n+2}(\eta,-q)\,se'_{2n+2}(\eta,-q)\,d\eta =$$

$$= \sum_{r=0}^{\infty} (2r+2) A_{2r+2}^{(2n+2)} B_{2r+2}^{(2n+2)}$$

$$\frac{2}{\pi}\int_0^{2\pi} ce_{2n+2}(\eta,-q)\,se'_{2n+2}(\eta,-q)\,\cos 2p\eta\,d\eta =$$

$$= \begin{cases} \sum_{r=0}^{\frac{1}{2}(p-1)} (-1)^p (2r+2) A_{2p-2r-2}^{(2n+2)} B_{2r+2}^{(2n+2)} + \\ \quad + \sum_{r=0}^{\infty} (-1)^p (2r+2) A_{2p+2r+2}^{(2n+2)} B_{2r+2}^{(2n+2)} \quad (p \text{ is odd}, \geq 1) \end{cases} \tag{AII.9}$$

$$\left\{ \begin{array}{l} \displaystyle\sum_{r=0}^{\frac{1}{2}(p-2)} (-1)^p \, (2r + 2) \, A_{2p-2r-2}^{(2n+2)} \, B_{2r+2}^{(2n+2)} \; + \\[2ex] \displaystyle + \sum_{r=0}^{\infty} (-1)^p \, (2r + 2) \, A_{2p+2r+2}^{(2n+2)} \, B_{2r+2}^{(2n+2)} \quad (p \text{ is even}, \geq 0) \end{array} \right.$$

APPENDIX III

Let r and r' represent the semi-major axis and the semi-minor axis, respectively, for any confocal ellipse. By using (2.8) r,r' can be expressed in terms of c and ξ

$$r = x = \pm c \cosh\xi, \text{ at } \eta = 0,\pi,\ y = 0$$
$$r' = y = \pm c \sinh\xi, \text{ at } \eta = \frac{\pi}{2}, \frac{3}{2}\pi,\ x = 0 \tag{AIII.1}$$

By definition

$$c = re \tag{AIII.2}$$

From (AIII.1) and (AIII.2), we have

$$\cosh\xi = e^{-1} \tag{AIII.3}$$

<u>Some Useful Limiting Forms When the Ellipse Tends to a Circle with $e \to 0$</u>

For fixed r, as $e \to 0$, the confocal ellipse of the semi-major axis r, tends to a circle with r as radius, and the confocal hyperbolas become radii of the circle, with $\eta = \phi$ (see Figure 3(b)). By use of (AIII.2) and (AIII.3), it is clear that $e \to 0$ corresponds to $c \to 0$ and $\xi \to \infty$. Since $q^{1/2} = k = \frac{1}{2}\frac{c}{\xi}$. [illegible] or $k \to 0$ as $c \to 0$. By means of these equivalent limiting processes, we obtain the following*

$$\lim_{e\to 0}\left(\frac{1}{2}\, ce^{\xi}\right)^{\pm n} = r^{\pm n} \qquad (n = 1,2,3) \tag{AIII.4}$$

---

*See Appendix I on p. 367-370 in Reference [9].

$$\lim_{e\to 0}\left[\frac{\pi}{(p_2)'}\,Fek_2\right] = K_2\left(\frac{r}{\ell}\right) \tag{AIII.5}$$

$$\lim_{e\to 0}\left[\frac{\pi}{(p_2)'}\,(Fek_2)'\right] = r\,\frac{\partial}{\partial r}\left[K_2\left(\frac{r}{\ell}\right)\right] \tag{AIII.6}$$

$$\lim_{e\to 0}(se_2) = \sin 2\phi \tag{AIII.7}$$

$$\lim_{e\to 0}(ce_2) = \cos 2\phi \tag{AIII.8}$$

For the specific ellipse of the semi-major axis a with $\xi = \xi_o$, (AIII.7) and (AIII.8) still hold while (AIII.4) through (AIII.6) take the following forms.

$$\lim_{e\to 0}\left[\left(\frac{1}{2}\,ce^{\xi_o}\right)^{\pm n}\right] = a^{\pm n} \qquad (n = 1,2,3,\ldots) \tag{AIII.9}$$

$$\lim_{e\to 0}\left[\frac{\pi}{(p_2)'}\,(Fek_2)_{\xi=\xi_o}\right] = K_2\left(\frac{a}{\ell}\right) \tag{AIII.10}$$

$$\lim_{e\to 0}\left[\frac{\pi}{(p_2)'}\,(Fek_2)'_{\xi=\xi_o}\right] = \left\{r\,\frac{\partial}{\partial r}\left[K_2\left(\frac{r}{\ell}\right)\right]\right\}_{r=a} \tag{AIII.11}$$

<u>Some Useful Limiting Forms When the Ellipse Tends to a Crack with $e \to 1$ or $\xi_o \to 0$.</u>

The ellipse with the semi-major axis, $a = c\cosh\xi_o$, and the semi-minor axis, $b = c\sinh\xi_o$, tends to the inter-focal line of length 2c, when $e \to 1$, while $a \to c$, $b \to 0$ (Figure 3(b)). By use of (AIII.3) with $\xi = \xi_o$, $e \to 1$ is seen to be equivalent to $\xi_o \to 0$. The following useful limiting quantities are obtained by direct computation.

$$\lim_{\xi_o \to 0} e^{\pm n \xi_o} = 1 \qquad (n = 1,2,3,\ldots) \qquad \text{(AIII.12)}$$

$$\left.\begin{aligned} &\lim_{\xi_o \to 0}[\sinh 2\xi_o] = 0 \\ &\lim_{\xi_o \to 0}[\cosh 2\xi_o] = 1 \end{aligned}\right\} \qquad \text{(AIII.13)}$$

$$\lim_{\xi_o \to 0}\left[(Fek_2)\Big|_{\xi=\xi_o}\right] = Fek_2(0,-q) \qquad \text{(AIII.14)}$$

$$\lim_{\xi_o \to 0}\left[(Fek_2)'\Big|_{\xi=\xi_o}\right] = Fek_2'(0,-q) = -\frac{p_2' ce_2(o,q)}{\pi A_o^{(2)}} \qquad \text{(AIII.15)}$$

<u>Limiting Values of Solutions as e → 0</u>

By means of the forms in (AIII.4) through (AIII.11), the limiting values of the modified forms of U_i ($i = 1,2,\ldots,5$) in (3.1) and the modified forms of Ψ_1, Ψ_2 in (3.4) and (3.5) are obtained as follows

$$\left.\begin{aligned} &\lim_{e \to 0}[c^2 U_1] = 4r^2 \\ &\lim_{e \to 0}[U_2] = \cos 2\phi \\ &\lim_{e \to 0}\left[\frac{U_3}{c^2}\right] = \frac{1}{4r^2}\cos\phi \\ &\lim_{e \to 0}[U_4] = \ln r \\ &\lim_{e \to 0}[c^2 U_5] = 4r^2 \cos 2\phi \end{aligned}\right\} \qquad \text{(AIII.16)}$$

$$\left.\begin{aligned}
&\lim_{e\to 0}\left[\frac{2V_1}{c^2}\right] = \frac{1}{r^2}\sin 2\phi \\
&\lim_{e\to 0}\left[\frac{\pi}{(p_2)'}V_2\right] = K_2\left(\frac{r}{\ell}\right)\sin 2\phi
\end{aligned}\right)$$

By use of the forms in (AIII.4) through (AIII.11), the limiting values of D in (3.26), B in (3.32), c^2C in (3.46),* F in (3.47),**,† Γ/c^2 in (3.35), and M in (3.34) are obtained as follows.

$$\left.\begin{aligned}
&\lim_{e\to 0}[D] = -\frac{pa^2}{2} \\
&\lim_{e\to 0}[d^2C] = -\frac{pa^4(1-\Delta)}{(1+\Delta)} \\
&\lim_{e\to 0}[F] = \frac{4(1-\nu)a^2\ell^2 p}{(1+\Delta)} \\
&\lim_{e\to 0}[B] = \frac{pa^2}{2(1+\Delta)} \\
&\lim_{e\to 0}\left[\frac{\Gamma}{c^2}\right] = -\frac{\pi}{(p_2)'}\,\frac{\Delta}{16a\ell(1-\nu)K_1\frac{a}{\ell}}
\end{aligned}\right\} \qquad \text{(AIII.17)}$$

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*C^2D is obtained by multiplying (3.46) by c^2.

**F is obtained by multiplying (3.47) by $c^2/2$.

†b, g, M, n, s, and t in (3.46) are expressed in (3.38), (3.39), (3.40), (3.43), (3.44), and (3.45), respectively.

‡Γ/c^2 is obtained by dividing (3.35) by c^2.

¶M is obtained by multiplying (3.34) by $(p_2)'/\pi$.

$$\lim_{e\to 0}[M] = -\frac{pa\ell\Delta}{(1+\nu)K_1\frac{a}{\ell}}$$

where

$$\Delta = \frac{8(1-\nu)}{4 + \frac{a}{\ell} + 2\frac{a}{\ell}\frac{K_o\frac{a}{\ell}}{K_1\frac{a}{\ell}}}$$

Since, from (3.24) and (3.25), we have $H/c^2 = -E/c^2 = p/16$ which is a constant, the following limiting processes are obvious.

$$\left.\begin{aligned} \lim_{e\to 0}\frac{H}{c^2} &= \frac{p}{16} \\ \lim_{e\to 0}\frac{E}{c^2} &= -\frac{p}{16} \end{aligned}\right\} \qquad \text{(AIII.18)}$$

Limiting Values of Solutions as $\xi_o \to 0$

By means of the forms in (AIII.12) through (AIII.15), the limiting values of H in (3.24), B in (3.32), C in (3.46), D in (3.26), E in (3.25), $\tilde{F}$ in (3.47), and $\tilde{M}$ in (3.34) can be obtained as follows.

$$\left.\begin{aligned} \lim_{\xi_o\to 0} H &= \lim_{\xi_o\to 0} B = \frac{pc^2}{16} \\ \lim_{\xi_o\to 0} C &= \lim_{\xi_o\to 0} E = -\frac{pc^2}{16} \end{aligned}\right\} \qquad \text{(AIII.19)}$$

$$\left.\begin{aligned} &\lim_{\xi_o \to 0} D = 0 \\ &\lim_{\xi_o \to 0} F = \lim_{\xi_o \to 0} M = 0 \end{aligned}\right\}$$

APPENDIX IV

Some Useful Limiting Forms as $\ell \to 0$

Since $q^{1/2} = \frac{1}{2}\frac{c}{\ell} = k$, for fixed c, $\ell \to 0$ implies that $q \to \infty$ or $k \to \infty$. The following limiting quantities are obtained by using [9].*

$$\lim_{\ell \to 0} \frac{A_{2r}^{(2)}}{A_0^{(2)}} = 2(-1)^r \qquad (r = 1,2,3,\ldots) \qquad \text{(AIV.1)}$$

$$\lim_{\ell \to 0} \frac{B_{2r+2}^{(2)}}{B_2^{(2)}} = (-1)^r (r + 1) \qquad (r = 1,2,3,\ldots) \qquad \text{(AIV.2)}$$

$$\lim_{\ell \to 0} A_{2r}^{(2)} = 0 \qquad (r = 0,1,2,3,\ldots) \qquad \text{(AIV.3)}$$

$$\lim_{\ell \to 0} B_{2r}^{(2)} = 0 \qquad (r = 1,2,3,\ldots) \qquad \text{(AIV.4)}$$

$$\lim_{\ell \to 0} \frac{B_2^{(2)}}{A_o^{(2)}} = \lim_{\ell \to 0} \frac{A_o^{(2)}}{B_2^{(2)}} = 1 \qquad \text{(AIV.5)}$$

For large arguments, z, we have**

$$I_n(z) \sim \frac{e^z}{\sqrt{2\pi z}} \sim I_n'(z) \qquad \text{(AIV.6)}$$

$$K_n(z) \sim \frac{e^{-z}}{\sqrt{2x/\pi}} \sim -K_n'(z) \qquad \text{(AIV.7)}$$

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*See Section 3.34 on p. 47.

**See Equations (9.7.1) to (9.7.4) on pp. 377-378 in [9].

Substituting (AIII.6) and (AIII.7) into (AI.4) and (AI.7) for the large value of q (small value of ℓ), and taking the limit of the resulting form with $\ell \to 0$, we obtain

$$\lim_{\ell \to 0} \frac{[Fek_2]_{\xi=\xi_o}}{[Fek_2]'_{\xi=\xi_o}} = \frac{\sinh \xi_o}{2} \tag{AIV.8}$$

All the series forms in expression g in (3.39) and s in (3.44) can be shown to be absolutely and uniformly convergent.* As such, by use of [11],** (AIV.3) and (AIV.4), all these series tend to be zero as $\ell \to 0$. Based upon this argument and (AIV.8), it is seen that in $g/B_2^{(2)}$ and $s/A_o^{(2)}$, terms containing ℓ as denominators dominate for the small value of ℓ. Hence, other terms in $g/B_2^{(2)}$ and $s/A_o^{(2)}$ can be neglected as compared with terms having ℓ as denominators, for $\ell \to 0$.

Limiting Values of Solutions Like $\ell \to 0$

By use of (AIV.1) through (AIV.8), the limiting value of H in (3.24), B in (3.32), C in (3.46), D in (3.26), E in (3.25), $\tilde{F}$ in (3.47), and $\tilde{M}$ in (3.34) can be obtained as follows.

$$\lim_{\ell \to 0} [H] = \frac{1}{16} pc^2$$

$$\lim_{\ell \to 0} [B] = \frac{1}{16} pc^2 \left(2e^{2\xi_o} - 1\right)$$

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*See Section 3.22 on p. 38 in [9].

**See Theorem 2 on pp. 339-340.

REFERENCES

1. Mindlin, R. D., and Tiersten, H. F., "Effects of Couple-Stresses in Linear Elasticity," Archives for Rational Mechanics and Analysis, 11, 1962, pp. 415-448.

2. Mindlin, R. D., "Influence of Couple-Stresses on Stress Concentration," Exp. Mechanics, Vol. 3, Jan. 1963, pp. 1-7.

3. Weitsman, Y., "Couple-Stress Effects on Stress Concentration Around a Cylindrical Inclusion in a Field of Uniaxial Tension," J. App. Mech., June 1965, pp. 424-428.

4. Weitsman, Y., "Strain Gradient Effects Around Cylindrical Inclusions and Cavities in a Field of Cylindrically Symmetric Tension," J. App. Mech., March 1966, pp. 57-67.

5. Muki, R., and Sternberg, E., "The Effects of Couple-Stresses on the Stress Concentration Around a Crack," Int. J. Solids Struct., Vol. 3, 1967, pp. 69-95.

6. Sternberg, E., Couple-Stresses and Singular Stress Concentrations in Elastic Solids, ONR Report NR-064-431, No. 12, Sept., 1967.

7. Ju, F. D., and Hsu, Y. C., Review of Couple-Stress Theory of Elasticity and Elastic Theory of Fracture Mechanics, AFOSR 69-1903 TR, June 1969, University of New Mexico Technical Report ME-40(69)AFOSR-D74.

8. Coker, E. G., and Filon, L. N. G., A Treatise on Photo-Elasticity, Cambridge University Press, 1931.

9. McLachlan, N. W., Theory and Application of Mathieu Functions, Dover Publications, Inc., New York, 1964.

10. Abramowitz, M., and Stegun, I. A., eds., Handbook of Mathematical Functions, National Bureau of Standards, Applied Mathematics Series-55, U.S. Government Printing Office, Washington, D.C., 1964, 3rd Printing, March 1965.

11. Knopp, K., Theory and Application of Infinite Series, Blackie and Sons Limited, London and Glasgow, 1957.