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A BOUNDARY LAYER METHOD FOR OPTIMAL CONTROL OF SINGULARLY PERTURBED SYSTEMS

by

Robert Reynolds Wilde

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13. ABSTRACT

A method is developed for approximating the solution of an optimally controlled singularly perturbed system . The method is applicable to both fixed and free end-point problems where in the latter problem a terminal cost is added to the performance index. Although the optimal solution is generally difficult to obtain using existing numerical algorithms, this method avoids such difficulties. The approximate solution is obtained by properly combining the solutions of three systems: a "reduced" $2n_1$-dimensional system, a "left layer" time invariant initial value n_2-dimensional system, and a "right layer" time invariant initial value n_2-dimensional system. The layer solutions can be interpreted as the results of two boundary layer regulators: one acting in forward time from the initial point and the other acting in reverse time from the end point. Example problems are worked which illustrate the method developed.

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A BOUNDARY LAYER METHOD FOR OPTIMAL CONTROL OF SINGULARLY PERTURBED SYSTEMS

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A method is developed for approximating the solution of an optimally controlled singularly perturbed system

$$\dot{x}_1 = A_{11}(t,\mu)x_1 + A_{12}(t,\mu)x_2 + B_1(t,\mu)u, \quad x_1(t_o) = x_1^o$$

$$\mu\dot{x}_2 = A_{21}(t,\mu)x_1 + A_{22}(t,\mu)x_2 + B_2(t,\mu)u, \quad x_2(t_o) = x_2^o$$

with respect to the performance index

$$\int_{t_o}^{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} Q_{11}(t,\mu) & Q_{12}(t,\mu) \\ Q_{12}'(t,\mu) & Q_{22}(t,\mu) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u'R(t,\mu)u\right)dt$$

such that the approximate solution converges to the optimal solution as $\mu \to 0$ uniformly on the entire interval $[t_o,T]$. Here x_1, x_2, and u are n_1-, n_2-, and m-dimensional vectors respectively, and μ is a small positive scalar parameter. The method is applicable to both fixed and free end-point problems where in the latter problem a terminal cost is added to the performance index. Although the optimal solution is generally difficult to obtain using existing numerical algorithms, this method avoids such difficulties. The approximate solution is obtained by properly combining the solutions of three systems: a "reduced" $2n_1$-dimensional system, a

"left layer" time invariant initial value n_2-dimensional system, and a "right layer" time invariant initial value n_2-dimensional system. The layer solutions can be interpreted as the results of two boundary layer regulators: one acting in forward time from the initial point and the other acting in reverse time from the end point. Example problems are worked which illustrate the method developed.

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1. INTRODUCTION

1.1 Problem Description

This thesis treats the problem of finding an approximate solution to a linear class of optimally controlled singularly perturbed systems

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,x_2,u,t,\mu) \\ \mu\dot{x}_2 &= f_2(x_1,x_2,u,t,\mu) \end{aligned} \tag{1.1}$$

where x_1, x_2, and u are n_1-, n_2-, and m-dimensional vectors respectively and μ is a small positive parameter. System (1.1) is called singularly perturbed since its dimension is reduced from $n_1 + n_2$ to n_1 if the scalar parameter μ is set equal to zero. Physically this parameter may represent a small time constant, mass, moment of inertia and other possibly negligible parameters.† As is customary in engineering design, such parameters would be initially neglected enabling the designer to solve an n_1-th dimensional problem instead of the original $(n_1 + n_2)$-th dimensional problem. An additional advantage of such a reduced-dimensional design is the avoidance of fast transients present in the high-dimensional problem. However, the best a reduced-dimensional design can do in general is to approximate the optimal solution on an open subinterval of the operation interval $[t_o,T]$. For even when μ is very small, large discrepancies between the optimal and the reduced solution may occur at

†For example, if τ is a small time constant and m is a small mass, then one can write $\tau = \alpha_1\mu$ and $m = \alpha_2\mu$ where α_1 and α_2 are appropriately chosen coefficients.

the ends of the interval because some of the boundary conditions were disregarded in the reduced order design. The end intervals in which these discrepancies rapidly diminish are called boundary layers due to analogy with fluids. A boundary layer method is developed in this thesis to approximate the optimal solution over the entire interval $[t_o,T]$. The method is straight forward and avoids having to find the optimal solution which is often difficult to find using existing numerical methods. The difficulty is a result of both widely varying decay transients and widely varying growth transients associated with the solution of a two point boundary value (TPBV) problem.

The boundary layer method developed is directly applicable to two types of optimal control problems for the system

$$\begin{aligned} \dot{x}_1 &= A_{11}(t,\mu)x_1 + A_{12}(t,\mu)x_2 \\ \mu\dot{x}_2 &= A_{21}(t,\mu)x_1 + A_{22}(t,\mu)x_2 \end{aligned} \tag{1.2}$$

with quadratic performance indices: regulator problems and trajectory optimization problems.

The approach taken to the regulator problem in this thesis is through a stabilizability analysis of singularly perturbed systems. The main result of this analysis is that a high-dimensional system is stabilizable if two lower-dimensional systems are stabilizable: a reduced system and a boundary layer system. The optimal regulator problem is then seen as the selection of the best of the stabilizing controls. The existence and the singular perturbation properties of the optimal regulator are shown by a direct method not involving usual optimality conditions. This result

represents further extension of related work in [41,55].† However, the methodology used here is different from either of these. In this thesis, optimality of the regulator design follows immediately from properties on the reduced solution.

The major part of this thesis is devoted to trajectory optimization. Both fixed end-point and terminal cost problems are treated by the same method. Boundary layer correction terms are explicitly obtained for each end of the interval, thus revealing the two-time scale properties of singularly perturbed optimal trajectories. From a control designer's point of view, a useful interpretation of the fast transients in the boundary layers is that they can be viewed as the results of two boundary layer regulators: one acting in forward time from the initial point and the other acting in reverse time from the end point. To summarize, the approach taken is to decompose a trajectory optimization problem into a "slow" trajectory optimization problem for a reduced system and into two regulator problems for boundary layer systems. The solutions of these problems are obtained in separate time scales, and when properly combined, they form an approximation to the exact solution which is valid uniformly on the entire interval $[t_o,T]$.

It should be pointed out that the singularly perturbed optimization problem with fixed end-points has not previously been considered from a control point of view, although techniques for the solution of such problems are reported in [24,26]. The problem with terminal cost is treated in [55] where the approach was through use of a positive definite solution of a

---

†The references [21,40,41] are for publication in the near future.

singularly perturbed Riccati system. The approximation obtained by this approach was not valid in the terminal boundary layer. This difficulty does not occur in the method developed in this thesis where both positive and negative definite Riccati solutions are employed to form a dichotomy transformation. The dichotomy transformation is used to separate the TPBV problem associated with the trajectory optimization problem into two independent singularly perturbed initial value problems. This result not only simplifies calculations but permits the treatment of a singularly perturbed TPBV problem by the more common treatments used for initial value problems. It is hoped that the avoidance of TPBV theory [13,15,16,18] will help control engineers to understand and apply singular perturbation methods in system design. In the same spirit, most of the conditions used in lemmas and theorems of the thesis are given in terms of notions familiar in control theory such as controllability, observability, stabilizability, etc. It is also shown, in the case of terminal cost, that the use of a feedback control whose optimal Riccati gains are approximated by reduced and boundary layer terms will result in an approximate solution to the optimal problem which is valid uniformly on the closed interval $[t_o,T]$. An alternate approach to using Riccati equations for the same class of problems is the use of singularly perturbed TPBV theory. Such an approach is developed in a yet unpublished work [40].

In the method of this thesis, the main tool is a transformation involving two solutions of a singularly perturbed matrix Riccati system. In general, these solutions are not continuous at the ends of the interval due to the presence of the boundary layer terms. However, the transformation

needed here is to be twice continuously differentiable on the whole interval. This difficulty can be avoided if the boundary conditions of the singularly perturbed Riccati system are at a designer's disposal. In this thesis, the end conditions for the Riccati transformation variables are free and are selected to guarantee the continuity properties. By using the end value of the reduced solution as the end condition for the singularly perturbed system, the zero order boundary layer is eliminated as can be seen from [21,39,53]. If by construction this boundary condition is an appropriate function of μ, boundary layers can be eliminated to any desired order.

1.2 Singular Perturbation Results in Control Theory

The first major analysis [28,45,46] of singularly perturbed optimal control problems dealt with finding an approximate solution to the optimal control problem on the open interval (t_o,T). Thus, only "outer" expansions were considered. The analysis avoided a direct study of the singularly perturbed TPBV equations by using two different approaches, each of which resulted in the need to analyze only singularly perturbed initial value problems. The first approach was to assume that the control and its derivative were continuous in t and μ for $t \in [t_o,T]$ and $\mu \in [0,\mu^*]$. The approach suffered from not being able to define a reasonable class of problems for which this assumption is valid, yet did provide a correct outer expansion. The second approach was for a linear-quadratic free end-point problem where the feedback control was expressed in terms of Riccati gains. The approach was successful but was applied under the following three restrictions: the system is time invariant, the fast variable x_2 is not in the performance index, and $\bar{A}_{22}$ is negative definite.

The Riccati approach was extensively developed in [29,55]. Time varying systems were permitted, the fast variable x_2 could appear in the performance index, and the requirement that $\bar{A}_{22}$ be negative definite was relaxed by requiring only boundary layer controllability and boundary layer observability--notions introduced there. Both "inner" and "outer" expansion terms were treated making it possible to approximate the Riccati solution on the closed interval $[t_o,T]$. The singularly perturbed initial value problem resulting upon insertion of the feedback control containing Riccati gains with boundary layer jumps at $t = T$ into the system equations was successfully analyzed but only for a subinterval of $[t_o,T)$. The problem of an end-point jump was avoided when various sub-optimal feedback designs were proposed and analyzed by not permitting boundary layer jumps to occur in the gain matrices. In this thesis, the results of [55] are extended by showing that the use of two Riccati systems can avoid the difficulty of having to analyze a system with boundary layer jumps at both ends. Thus an approximation to the optimal solution is given which is valid on the closed interval $[t_o,T]$. Furthermore, the two Riccati approach permits an analysis of a trajectory optimization problem; a problem which could not be treated by the approach in [55].

Singularly perturbed optimal control applications in flight problems are reported in [24,25,26]. In [26], a heuristic approach is given for construction of an approximate solution to a non-linear optimal control problem which accounts for boundary layer jumps at both ends of a fixed time interval. The approach is in agreement with the expansion in [50].

A complete expansion for the state variables, control and performance index is given in [40] for the same linear-quadratic optimization

problem considered in [55]. Each term of the state and control variables contained a left and right inner term in addition to an outer term. The validity of the asymptotic correctness of the method was based on [51]. The expansion would be similar to that of [50] for such a problem. Thus the approximation was shown to be valid for the closed interval $[t_o,T]$. The hypotheses were not control oriented, and the treatment given assumed that the eigenvalues of

$$\begin{bmatrix} A_{22} & -S_{22} \\ -Q_{22} & -A_{22}' \end{bmatrix}$$

have multiplicity one, an assumption not made in this thesis. In [41], the Riccati method was used to solve a time invariant regulator problem for a scalar system. A formal expansion of the algebraic Riccati gains was made about the origin and shown to be convergent there and without boundary layers. This paper assumed conditions on the high-dimensional system in contrast to that used in this thesis where assumptions on only the low-dimensional auxiliary systems are made. Since no boundary layers appear in the system equations when the optimal feedback control is inserted into them, a standard singularly perturbed initial value problem results to which expansions are well known and valid for $t \in [t_o,T)$.

1.3 Stability Problems

There are two types of results for the high-dimensional infinite time initial value problem. The first result guarantees that the solution

converges uniformly in t to the reduced solution on any interval of the form $[t',\infty)$ where $t_o < t'$. The second result guarantees that a solution of the high-dimensional problem is asymptotically (or conditionally asymptotically) stable. The first result is used to approximate the solution of the high-dimensional problem by that of the low-dimensional reduced problem. A crucial hypothesis often assumed in proving similar approximation results on a finite time interval is that the real part of all the eigenvalues of the matrix $\frac{\partial f_2}{\partial x_2}$ evaluated along the reduced trajectory is less than a fixed negative number.† An additional hypothesis is generally assumed for the infinite time problem. This hypothesis is the uniform asymptotic (or conditional asymptotic) stability in the small of the reduced solution. Since both of the hypotheses stated are for linearized systems, the results obtained based on these hypotheses are only valid for initial conditions starting near the reduced solution. Such hypotheses were made in [5,27] with the exception that in [27] the eigenvalue-criteria of $\frac{\partial f_2}{\partial x_2}$ was replaced by Krasovskii's condition [14]. Both of these hypotheses were relaxed in [18,20]. Different techniques were used to establish these results: asymptotic [21], Lyapunov [18,20,27], and successive approximation [5]. Conditionally stable systems were treated only in [5,21]. The approximation and stability results of [27] were extended in [18] and [20] respectively to treat a much wider class of problems, and [18] permitted f_1 and f_2 to depend on μ which was not permitted in [20,27]. Both of the conditionally stable works [5,21] permitted f_1 and f_2 to depend on μ.

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†A matrix in which the real part of all its eigenvalues is less than a fixed negative number is called stable.

1.4 Two Point Boundary Value Problems

The crucial hypothesis used in all the singularly perturbed TPBV theorems surveyed here is that the absolute value of the real part of all the eigenvalues of the matrix $\frac{\partial f_2}{\partial x_2}$ evaluated along the reduced trajectory is greater than a fixed positive number. The basic method for analysis of such problems was first made evident in [31] for an initial value singularly perturbed system. The method consisted of finding an initial stable manifold such that solutions starting on this manifold would rapidly converge and then remain close to the reduced solution. For finite time problems, it was also shown there that solutions starting slightly off this initial stable manifold could also be made to remain close to the reduced solution by making μ sufficiently small. The singularly perturbed TPBV theory is based upon recognizing that a terminal stable manifold exists similar to that of the initial stable manifold. Solutions starting on the terminal manifold also rapidly converge and then remain close to the reduced solution, but in reverse time. Thus the manifolds exist as a consequence of the crucial hypothesis stated.

For the boundary conditions to be on such manifolds at each end of the time interval, it is necessary that there exists a particular association of eigenvalues with the state variables. Assuming the eigenvalues are continuous, knowledge that the eigenvalues at any time $\tilde{t}$ are associated with the correct states implies the correct association with the states for all time. To guarantee the proper association, an additional hypothesis is needed. For this purpose a transformation hypothesis was given in [15,16] and still another type was given in [13].

A linear system was analyzed in [15,16] for the most general combination of boundary values at $t = t_o$ and $t = T$. The hypotheses given were very difficult to check and the results valid only on an open subinterval of $[t_o,T]$. Non-linear systems were analyzed in [13,21,50]. The problem treated in [13] assumed the system was composed of two slow systems y_1, y_2 and two fast systems ω_1, ω_2 satisfying boundary conditions on y_1 and ω_1 at $t = t_o$ and y_2 and ω_2 at $t = T$. The treatment adapted the approach in [31] and proved the closeness of the reduced solution to the high-dimensional solution providing the boundary condition on ω_1 at $t = t_o$ and on ω_2 at $t = T$ were close to the reduced solution. The most general theorem available thus far is stated in [21]. If y is the fast system and ω the slow system, the boundary conditions at $t = t_o$ were given there by

$$\mathcal{A}(y(t_o),\omega(t_o),\mu) = 0, \qquad \mathcal{B}(y(T),\omega(T),\mu) = 0$$

where $\mathcal{A}$ and $\mathcal{B}$ are smooth functions of their arguments. The theorem guarantees the closeness behavior providing the boundary value at $t = t_o$ was on the intersection of $\mathcal{A}$ and an initial stable manifold and at $t = T$ was on the intersection of $\mathcal{B}$ and a terminal stable manifold. In [50], a method is given for finding an asymptotic expansion for a system in which the fast variable boundary conditions are given on the variables either at $t = t_o$ or at $t = T$. The slow variable boundary conditions were arbitrary.

1.5 Chapter Preview

Chapter 2 develops a methodology for proving the existence of an optimal solution to a regulator problem which should be generally applicable

to broadening the class of linear systems with quadratic performance indices known to have an optimal solution. It is shown that the existence of stabilizing controls for two low-dimensional systems not only implies the existence of a stabilizing control for the high-dimensional system but also that an optimal solution exists to the regulator problem.

Chapter 3 is a preparatory chapter for the remaining chapters where a dichotomy transformation is introduced and a comparison is made between a non-singularly perturbed system and a singularly perturbed system. Conditions are imposed so that both systems behave similarly to point out the interchanging roles of the operation interval in the first case and μ in the second.

Chapter 4 contains the main results of the thesis where a method is given to approximate the optimal solution of a fixed end-point problem whose exact solution would in general be difficult to solve. An application of the method is given for an example problem in Chapter 5.

The example problem of Chapter 5 graphically illustrates the important points: two-time scale property, boundary layers at both ends of the time interval, closeness of the approximate solution to the actual optimal, stiffness, and how the interval in which the reduced solution may be a good approximation can be extended by decreasing μ.

Chapter 6 applies the method to a free end-point problem and also includes an example problem. Furthermore, it is shown that the optimal solution is approximated uniformly on the interval $[t_o,T]$ when an approximate feedback structure is used consisting of Riccati gains approximated by their reduced and boundary layer terms.

2. REGULATOR PROBLEM

2.1 Problem Statement and Approach

This chapter considers the design of a feedback control to regulate the singularly perturbed system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,x_2,u,t,\mu) \\ \mu\dot{x}_2 &= f_2(x_1,x_2,u,t,\mu) \end{aligned} \tag{2.1}$$

for which

$$f_1(0,0,0,t,\mu) = 0, \qquad f_2(0,0,0,t,\mu) = 0 \tag{2.2}$$

for all $t \geq t_o$, $\mu \epsilon (0,\mu^*]$. Here x_1, x_2, and u are n_1-, n_2-, and m-dimensional vectors respectively, and μ is a small positive scalar parameter. The zero solution of (2.1) might represent any nominal trajectory of a system having the form (2.1) but which has been translated into the origin in a new coordinate system. In this translated coordinate system, (2.1) describes the motion about the original from a perturbed initial condition. Thus one could consider the problem of regulation for the translated system as that of finding the control to make a desired trajectory asymptotically stable. Restricting the class of non-linear problems to those in which the behavior of the linearized system determines the behavior of the non-linear system such as [27,52], only the linearized system of (2.1) needs to be analyzed. A stabilizing control is defined here as one which makes the zero solution of (2.1) asymptotically stable. If such a control is applied to the system, the system is then said to be stabilized. The first

result of this chapter is to show in section 2.2 that the existence of stabilizing controls for two lower-dimensional systems guarantees the existence of a stabilizing control for the higher-dimensional system (2.1).

This theorem is applied in section 2.3 to a time invariant system to justify the regulator design proposed in [55]. In general many stabilizing controls can be found for a stabilizable system. Of interest is the selection of one of these which can easily be found, implemented, and which yields a performance cost close to the optimal cost. A performance index will be given for the same system which was shown to possess a stabilizing control. The second result of this chapter is to show that the existence of stabilizing controls for the two low-dimensional systems not only implies the existence of a stabilizing control for the high-dimensional system but also that an optimal solution exists to this regulator problem. Lastly it will be shown that the proposed design is a good approximation to the optimal design.

2.2 Stabilizing Controls

Since the existence of stabilizing controls follows from a stability theorem, this theorem will be given first. Few stability results exist for singularly perturbed systems, and the two main results, of which one is given by Klimushev and Krasovskii [27] and the other by Hoppensteadt [20], are not well known. This is evidenced by the recent articles [10,11,47] for linear time-invariant systems of form (2.1) applied to networks with small and large parasitics. The stability theorems of [10,11,47] are encompassed by the earlier work done by Klimushev and Krasovskii. Hoppensteadt's

hypotheses are numerous and not well suited from an engineering standpoint even though he treats a more general problem than that covered in [20]. Neither of these stability works considers the case when f_1 and f_2 depend on μ. Such a result appears in [5] whose proof is based on successive approximations. A stability theorem for a linear system whose matrices depend on μ will be given which is an extension of the theorem in [27] whose proof is based on Lyapanov functions.

The following theorem deals with the uniform asymptotic stability of the linear system

$$\begin{aligned} \dot{x}_1 &= A_{11}(t,\mu)x_1 + A_{12}(t,\mu)x_2 \\ \mu\dot{x}_2 &= A_{21}(t,\mu)x_1 + A_{22}(t,\mu)x_2 \end{aligned} \tag{2.3}$$

The stability property of (2.3) for μ sufficiently small is deduced from stability properties of two auxiliary systems: the n_2-dimensional system

$$\dot{q} = A_{22}(\theta,0)q \tag{2.4}$$

where $\theta \geq t_o$ is a fixed parameter, and the n_1-dimensional system

$$\dot{p} = [A_{11}(t,0) - A_{12}(t,0)A_{22}^{-1}(t,0)A_{21}(t,0)]p. \tag{2.5}$$

<u>Theorem 2.2.1</u> If

(i) all the matrices $A_{ij}(t,\mu)$ in (2.3) and their derivatives with respect to t and μ are bounded and continuous for all $t \geq t_o$, $\mu \epsilon [0,\mu^*]$,

(ii) the real parts of all the eigenvalues of $A_{22}(\theta,0)$ are smaller than a fixed negative number for all $\theta \geq t_o$,

(iii) system (2.5) is uniformly asymptotically stable,

then there exists a $\mu^* > 0$ such that system (2.3) is uniformly asymptotically stable for all $\mu \epsilon (0,\mu^*]$.

<u>Proof</u>: Define δx and δy using

$$\begin{aligned} \hat{x}_1 &= \tilde{x}_1 + \delta x \\ \hat{x}_2 &= \tilde{x}_1 + \delta y - \bar{A}_{22}^{-1}\bar{A}_{21}\delta x \end{aligned} \tag{2.6}$$

where $(\tilde{x}_1,\tilde{x}_2)$ and $(\hat{x}_1,\hat{x}_2)$ are solutions of (2.3) corresponding to two different initial conditions. For brevity, arguments of functions are dropped when no confusion results, and a bar is used to indicate that $\mu = 0$. Thus A_{11} denotes $A_{11}(t,\mu)$ and $\bar{A}_{11}$ denotes $A_{11}(t,0)$. Upon substitution of (2.6) into (2.3),

$$\begin{aligned} \delta\dot{x} &= (\bar{R} + \Delta A_{11} - \Delta A_{12}\bar{S})\delta x + A_{12}\delta y \\ \mu\delta\dot{y} &= (\frac{\Delta A_{21}}{\mu} - \frac{\Delta A_{22}}{\mu}\bar{S} + \dot{\bar{S}} + \bar{S}\bar{R} + \bar{S}\,\Delta A_{11} - \bar{S}\,\Delta A_{12}\bar{S})\delta x \\ &\quad + (\frac{\bar{A}_{22}}{\mu} + \frac{\Delta A_{22}}{\mu} + \bar{S}A_{12})\delta y \end{aligned} \tag{2.7}$$

where $\bar{R} = \bar{A}_{11} - \bar{A}_{12}\bar{A}_{22}^{-1}\bar{A}_{21}$, $\bar{S} = \bar{A}_{22}^{-1}\bar{A}_{21}$, and $\Delta A_{ij} = A_{ij}(t,\mu) - \bar{A}_{ij}$. Clearly, when (2.7) is uniformly asymptotically stable for $\mu > 0$ so is (2.3). Let $M(\theta,0)$ be the unique positive definite solution of

$$A_{22}'(\theta,0)M + MA_{22}(\theta,0) = - I_{n_2} \tag{2.8}$$

for all $\theta \geq t_o$. Here and in (2.11) I_k denotes a k x k identity matrix. From (ii) it follows that $q'M(\theta,0)q$ is a Lyapunov function for (2.4). Let the function $p'\bar{N}p$, whose derivative for (2.5) is $-p'p$, be a Lyapunov function guaranteeing (iii). This function exists by a well known Lyapunov theorem, such as Theorem 3 of [23],

It is now shown that, for a sufficiently small positive μ, the function

$$w = \delta x'\bar{N}\delta x + \delta y'\bar{M}\delta y \tag{2.9}$$

is a Lyapunov function for (2.7) satisfying the requirements for uniform asymptotic stability such as the conditions of Theorem 1 of [23]. By definition of $\bar{M} = M(t,0)$ and $\bar{N}$ there exist continuous nondecreasing functions α and β of the norm $\|\delta x,\delta y\|$ such that $\alpha(0) = 0$, $\beta(0) = 0$ and

$$0 < \alpha(\|\delta x,\delta y\|) \leq w \leq \beta(\|\delta x,\delta y\|) \tag{2.10}$$

holds for all $t \geq t_o$ and all $\delta x \neq 0$, $\delta y \neq 0$. The derivative of w for (2.7) is

$$\dot{w} = \begin{bmatrix} \delta x \\ \delta y \end{bmatrix}' \begin{bmatrix} -I_{n_1} + L_{11} & L_{12} \\ L_{12}' & -\frac{1}{\mu} I_{n_2} + L_{22} \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} \tag{2.11}$$

where

$$L_{11} = \bar{N}(\Delta A_{11} - \Delta A_{12}\bar{S}) + (\Delta A_{11} - \Delta A_{12}\bar{S})'\bar{N}$$

$$L_{12} = \bar{N}A_{12} + \left[\frac{1}{\mu}\Delta A_{21} - \frac{1}{\mu}\Delta A_{22}\bar{S} + \dot{\bar{S}} + \bar{S}(\bar{R} + \Delta A_{11} - \Delta A_{12}\bar{S})\right]'\bar{M}$$

$$L_{22} = \dot{\bar{M}} + (\frac{\Delta A_{22}}{\mu} + \bar{S}A_{12})'\bar{M} + \bar{M}(\frac{\Delta A_{22}}{\mu} + \bar{S}A_{12}).$$

The $\frac{\Delta A_{ij}}{\mu}$ are bounded functions for $t \geq t_o$ and for all $\mu \epsilon [0,\mu^*]$ by (i). After substitution of θ by t in (2.8) and differentiation with respect to t, it follows that

$$\dot{\bar{M}} = \int_{t_o}^{\infty} e^{\bar{A}_{22}'\sigma} (\dot{\bar{A}}_{22}'\bar{M} + \bar{M}\dot{\bar{A}}_{22}) e^{\bar{A}_{22}\sigma} d\sigma \tag{2.12}$$

Hence from (2.12) and hypothesis (i), the L_{ij} are bounded for all $t \geq t_o$ and for all $\mu \epsilon [0,\mu^*]$; moreover $L_{11} \to 0$ as $\mu \to 0$. Thus L_{11} is dominated by $- I_{n_1}$ for sufficiently small μ. Also L_{22} is dominated by $- \frac{1}{\mu} I_{n_2}$ for μ sufficiently small. Inspection of leading principal minors of the symmetric matrix in (2.11) shows that there exists a positive μ^* such that for all $\mu \epsilon (0,\mu^*]$, all $t \geq t_o$ and all $\delta x \neq 0$, $\delta y \neq 0$

$$\dot{w} \leq -\gamma(\|\delta x, \delta y\|) < 0 \tag{2.13}$$

where γ is a nondecreasing function and $\gamma(0) = 0$. Properties (2.10) and (2.13) of w and $\dot{w}$ prove that (2.7) is a uniformly asymptotically stable system for $\mu \epsilon (0,\mu^*]$.

The purpose for including the δx term in transformation (2.6) was to avoid the appearance of $\frac{1}{\mu}A_{ij}$ terms in the off-diagonal terms of the

coefficient matrix in (2.11). The smoothness assumptions made in (i) could have been relaxed as is evident from the requirements that the coefficient matrix in (2.11) be positive definite. It is further noted that if one were to search for the least upper bound of μ which makes the i-th leading principal minor positive but which is not to exceed that found for the (i-1)-th minor, then by repeating this process through $i = n_1 + n_2$ a μ^* is obtainable.

The stabilization theorem will now be given for the system

$$\begin{aligned} \dot{x}_1 &= A_{11}(t,\mu)x_1 + A_{12}(t,\mu)x_2 + B_1(t,\mu)u \\ \mu\dot{x}_2 &= A_{21}(t,\mu)x_1 + A_{22}(t,\mu)x_2 + B_2(t,\mu)u \end{aligned} \tag{2.14}$$

The theorem is based on the existence of two controls: $u_2 = D_2(\theta,0)q$ to make the system

$$\frac{dq}{dt} = A_{22}(\theta,0)q + B_2(\theta,0)u_2 \tag{2.15}$$

asymptotically stable for all $\theta \geq t_o$ and $u_1 = \bar{D}p$ to make the system

$$\begin{aligned} \frac{dp}{dt} = &\{\bar{A}_{11}-(\bar{A}_{12}+\bar{B}_1\bar{D}_2)(\bar{A}_{22}+\bar{B}_2\bar{D}_2)^{-1}\bar{A}_{21}\}p \\ &+ \{\bar{B}_1-(\bar{A}_{12}+\bar{B}_1\bar{D}_2)(\bar{A}_{22}+\bar{B}_2\bar{D}_2)^{-1}\bar{B}_2\}u_1 \end{aligned} \tag{2.16}$$

uniformly asymptotically stable. It will be shown that when u_1 and u_2 do exist, then the control

$$u = D_1(t,\mu)x_1 + D_2(t,\mu)x_2 \tag{2.17}$$

stabilizes (2.14). To formulate the corollary it is convenient to first express u in (2.14) in terms of D_1 and D_2. Hence

$$\begin{aligned} x_1 &= C_{11}(t,\mu)x_1 + C_{12}(t,\mu)x_2 \\ \mu\dot{x}_2 &= C_{21}(t,\mu)x_1 + C_{22}(t,\mu)x_2 \end{aligned} \tag{2.18}$$

where

$$\begin{aligned} C_{11} &= A_{11}+B_1D_1, \qquad C_{12} = A_{12}+B_1D_2 \\ C_{21} &= A_{21}+B_2D_1, \qquad C_{22} = A_{22}+B_2D_2 \end{aligned} \tag{2.19}$$

<u>Corollary 2.2.2</u> If there exist D_1 and D_2 such that the C_{ij} satisfy the hypotheses of Theorem 2.2.1 where the C_{ij} replace the A_{ij} matrices, then there exists a $\mu^* > 0$ such that the control (2.17) makes system (2.14) uniformly asymptotically stable for all $\mu\epsilon(0,\mu^*]$.

The proof is obvious in view of Theorem 2.2.1.

2.3 A Regulator Design

As an application of the stability theorem 2.2.1, it is now investigated whether the control proposed in [55], for a time invariant system of the form (2.14), is a stabilizing control for all $\mu\epsilon(0,\mu^*]$. The proposed control was to not only stabilize the system but yield a low cost for the performance index

$$J = \frac{1}{2}\int_{t_o}^{\infty}\left\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u'Ru\right\} dt \tag{2.20}$$

Since optimality is meaningless unless it can be shown that the system can be stabilized, the latter is analyzed first. The control proposed in [55] is given by

$$u = -R^{-1}\begin{bmatrix} B_1' & \frac{B_2'}{\mu} \end{bmatrix}\begin{bmatrix} \bar{K}_{11} & \mu\bar{K}_{12} \\ \mu\bar{K}_{12}' & \mu\bar{K}_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \tag{2.21}$$

With (2.21), system (2.14) becomes

$$\begin{bmatrix} \dot{x}_1 \\ \mu\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11}-S_{11}\bar{K}_{11}-S_{12}\bar{K}_{12}' & A_{12}-\mu S_{11}\bar{K}_{12}-S_{12}\bar{K}_{22} \\ A_{21}-S_{12}'\bar{K}_{11}-S_{22}\bar{K}_{12}' & A_{22}-\mu S_{12}'\bar{K}_{12}-S_{22}\bar{K}_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.22}$$

where $S_{11} = B_1R^{-1}B_1'$, $S_{12} = B_1R^{-1}B_2'$, $S_{22} = B_2R^{-1}B_2'$. From the stability theorem it is evident that (2.21) is a stabilizing control if the following two matrices are stable:

$$\bar{A}_{22}-\bar{S}_{22}\bar{K}_{22} \tag{2.23}$$

$$(\bar{A}_{11}-\bar{S}_{11}\bar{K}_{11}-\bar{S}_{12}\bar{K}_{12}')-(\bar{A}_{12}-\bar{S}_{12}\bar{K}_{22})(\bar{A}_{22}-\bar{S}_{22}\bar{K}_{22})^{-1}(\bar{A}_{21}-\bar{S}_{12}'\bar{K}_{11}-\bar{S}_{22}\bar{K}_{12}') \tag{2.24}$$

This is indeed the case for the proposed control as a result of the hypotheses given in [55] and of the selection of the $\bar{K}_{ij}$'s as the unique root for $\mu = 0$ of the algebraic Riccati system

$$0 = -K_{11}(A_{11}-S_{12}K_{12}') - (A_{11}-S_{12}K_{12}')'K_{11}$$

$$- K_{12}A_{21} - A_{21}'K_{12}' + K_{11}S_{11}K_{11} + K_{12}S_{22}K_{12}' - Q_{11}$$

$$0 = -K_{12}(A_{22}-S_{22}K_{22}) - K_{11}A_{12} + K_{11}S_{12}K_{22} - A_{21}'K_{22} - Q_{12} \tag{2.25}$$

$$- \mu(A_{11}'K_{12} - K_{11}S_{11}K_{12} - K_{12}S_{12}'K_{12})$$

$$0 = -K_{22}A_{22} - A_{22}K_{22} + K_{22}S_{22}K_{22} - Q_{22}$$

$$- \mu[K_{12}'(A_{12}-S_{12}K_{22}) + (A_{12}-S_{12}K_{22})'K_{12} - \mu K_{12}'S_{11}K_{12}]$$

having the property that $\bar{K}_{11}$ and $\bar{K}_{22}$ are symmetrical positive definite matrices. Thus $\bar{K}_{22}$ is the symmetrical positive definite root of

$$K_{22}\bar{A}_{22} + \bar{A}_{22}'K_{22} - K_{22}\bar{S}_{22}K_{22} + \bar{Q}_{22} = 0, \tag{2.26}$$

$\bar{K}_{11}$ is the symmetrical positive definite root of

$$K\hat{A} + \hat{A}'K - K\hat{B}R^{-1}\hat{B}'K + \hat{Q} = 0, \tag{2.27}$$

and

$$\bar{K}_{12} = \bar{K}_{11}\bar{E}_1 - \bar{E}_2 \tag{2.28}$$

where

$$\hat{A} = \bar{A}_{11} + \bar{E}_1\bar{A}_{21} + \bar{S}_{12}\bar{E}_2 + \bar{E}_1\bar{S}_{11}\bar{E}_2, \qquad \hat{B} = \bar{B}_1 + \bar{E}_1\bar{B}_2$$

$$\hat{Q} = -\bar{E}_2'\bar{A}_{21} - \bar{A}_{21}'\bar{E}_2 - \bar{E}_2'\bar{S}_{22}\bar{E}_2 + \bar{Q}_{11}$$

$$\bar{E}_1 = (\bar{S}_{12}\bar{K}_{22} - \bar{A}_{12})(\bar{A}_{22} - \bar{S}_{22}\bar{K}_{22})^{-1}$$

$$\bar{E}_2 = (\bar{A}_{21}\bar{K}_{22} + \bar{Q}_{12})(\bar{A}_{22} - \bar{S}_{22}\bar{K}_{22})^{-1}$$

The hypotheses which have been assumed in [55] are

(i) The coefficient matrices of the time invariant plant (2.14), Q, R and their derivatives are bounded and continuous functions of μ for all $\mu \epsilon [0,\mu^*]$,

(ii) For all $\mu \epsilon [0,\mu^*]$, the symmetrical matrix Q is positive semi-definite and R is positive definite,

(iii) The pair $(\bar{A}_{22},\bar{B}_2)$ is controllable and the pair $(\bar{A}_{22},\bar{C}_2)$ is observable where $C_2'C_2 = Q_{22}$,

(iv) The pair $(\hat{A},\hat{B})$ is controllable and the pair $(\hat{A},\hat{C})$ is observable where $\hat{C}'\hat{C} = \hat{Q}$.

Hypothesis (iii) guarantees the (2.23) is a stable matrix and hypothesis (iv) guarantees that (2.24) is a stable matrix.

2.4 Optimal Regulator Design

Now that it has been established that the system can be stabilized, it will be shown that an optimal solution exists for the problem. This is done by two lemmas. The first verifies the existence of a unique symmetrical positive definite root of (2.25) in the neighborhood of $\mu = 0$. The second justifies the optimality of this root. It will then be shown that the proposed design is near optimal for sufficiently small μ.

Lemma 2.4.1 There exists a μ^* such that for all $\mu \epsilon (0,\mu^*]$,

$$\begin{bmatrix} K_{11} & \mu K_{12} \\ \mu K_{12}' & \mu K_{22} \end{bmatrix} \tag{2.29}$$

is the unique symmetrical positive definite root of (2.25) and that

$$\begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\mu} & \frac{A_{22}}{\mu} \end{bmatrix} - \begin{bmatrix} S_{11} & \frac{S_{12}}{\mu} \\ \frac{S'_{12}}{\mu} & \frac{S_{22}}{\mu^2} \end{bmatrix} \begin{bmatrix} K_{11} & \mu K_{12} \\ \mu K'_{12} & \mu K_{22} \end{bmatrix} \tag{2.30}$$

is a stable matrix.

Proof: It follows from an application [48] of the implicit function theorem that for μ small there exist unique positive definite K_{11} and K_{22} satisfying (2.25) and that $K_{ij} = \bar{K}_{ij} + O(\mu)$.
Inspection of the leading principal minors of (2.29) shows that there exists a μ^* such that for all $\mu \epsilon (0,\mu^*]$, this matrix (2.29) is positive definite. That (2.30) is a stable matrix now follows upon application of Theorem 2.2.1 and the fact that $K_{ij} = \bar{K}_{ij} + O(\mu)$.

Lemma 2.4.2 If there exists a unique symmetrical positive semi-definite root P_∞ to the algebraic Riccati equation

$$A'K + KA - KSK + Q = 0 \tag{2.31}$$

and if the control $u = -R^{-1}B'P_\infty x$ makes the time invariant system

$$\dot{x} = Ax + Bu \tag{2.32}$$

asymptotically stable, then this control minimizes the performance index

$$J = \int_{t_o}^{\infty} (x'Qx + u'Ru)dt \ . \tag{2.33}$$

The minimum value of the performance index where R is positive definite is given by $x'(t_o)P_\infty x(t_o)$.

<u>Proof</u>: Let $\gamma = u + R^{-1}B'P_\infty x$. Then (2.32) and (2.33) become

$$\dot{x} = (A - SP_\infty)x + B\gamma \tag{2.34}$$

$$J = \int_{t_o}^{\infty} [x'Qx + (\gamma - R^{-1}B'P_\infty x)'R(\gamma - R^{-1}B'P_\infty x)]dt \tag{2.35}$$

$$= \int_{t_o}^{\infty} [\gamma'R\gamma + x'(Q + P_\infty SP_\infty)x - 2\gamma'B'P_\infty x]dt$$

and recognizing the fact that P_∞ is a solution of (2.27), it follows that

$$J = \int_{t_o}^{\infty} (\gamma'R\gamma - 2[(A-SP_\infty)x + B\gamma]' P_\infty x) \ dt$$

$$= \int_{t_o}^{\infty} \gamma'R\gamma dt - 2 \int_{t_o}^{\infty} \dot{x}'P_\infty x dt$$

$$= \int_{t_o}^{\infty} \gamma'R\gamma dt + x'(t_o)P_\infty x(t_o) \tag{2.36}$$

Thus the minimizing control is given for $\gamma = 0$ since R is positive definite, and P_∞ is positive semi-definite. This proof is based on [2].

<u>Theorem 2.4.3</u> For all $\mu \epsilon (0,\mu^*]$, there exists a unique positive definite root of (2.25) such that the feedback control

$$u = -R^{-1} \begin{bmatrix} B_1' & \frac{B_2'}{\mu} \end{bmatrix} \begin{bmatrix} K_{11} & \mu K_{12} \\ \mu K_{12}' & \mu K_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} . \tag{2.37}$$

minimizes (2.33).

<u>Proof</u>: This immediately follows from Lemma 2.4.1 and Lemma 2.4.2 defining

$$A = \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\mu} & \frac{A_{22}}{\mu} \end{bmatrix} , \qquad B = \begin{bmatrix} B_1 \\ \frac{B_2}{\mu} \end{bmatrix}$$

$$S = \begin{bmatrix} S_{11} & \frac{S_{12}}{\mu} \\ \frac{S_{12}'}{\mu} & \frac{S_{22}}{\mu^2} \end{bmatrix} , \qquad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}$$

for use of Lemma 2.4.2.

The performance index is defined for $\mu \epsilon [0,\mu^*]$ and hence $J \to \bar{J}$ as $\mu \to 0$ and is represented by $J = \bar{J} + 0(\mu)$.[‡] Thus the proposed design gives a performance cost close to that resulting from the optimal control. An alternate control having the same general properties as (2.21) is given by

$$u = -R^{-1}[B_1' \; B_2'] \begin{bmatrix} \bar{K}_{11} & 0 \\ \bar{K}_{12}' & \bar{K}_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.38}$$

[‡] $q(t) = 0(\mu)$ if the norm $|q(t)|$ of the vector or matrix q satisfies the inequality $|q| \leq \alpha\mu$ for some positive scalar constant α and for $\mu \leq \mu^*$.

3. A DICHOTOMY IN LINEAR CONTROL THEORY

3.1 Introduction and Statement of Problem

This chapter introduces a dichotomy transformation which serves as a tool in solving TPBV singular perturbation problems. Also, a comparison is made between a singularly perturbed problem and a nonsingularly perturbed problem in which similarities are pointed out.

A 2n-dimensional system

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A(t) & -S(t) \\ -Q(t) & -A'(t) \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{3.1}$$

is said to possess an "exponential dichotomy" if there exist positive constants α, β, γ and ξ such that for all $t \geq t_o$

$$|x(t)| + |\lambda(t)| \leq \alpha e^{-\gamma(t-t_o)}, \quad \text{for} \begin{bmatrix} x(t_o) \\ \lambda(t_o) \end{bmatrix} \in Y \tag{3.2}$$

$$|x(t)| + |\lambda(t)| \geq \beta e^{\xi(t-t_o)}, \quad \text{for} \begin{bmatrix} x(t_o) \\ \lambda(t_o) \end{bmatrix} \notin Y \tag{3.3}$$

where Y is a linear subspace of R^{2n} and $|x(t)|$ and $|\lambda(t)|$ are norms of the n-dimensional vectors $x(t)$ and $\lambda(t)$.

This chapter shows how, under suitable conditions, "dichotomy transformations" are constructed which diagonalize (3.1) into two n-dimensional systems, one exponentially stable in forward time and the other exponentially stable in reverse time. In this way $x(t)$ and $\lambda(t)$ can be found by solving differential equations in their stable direction.

This development is purposely introduced to stress similarities between (3.1) and the singularly perturbed system

$$\mu\begin{bmatrix}\dot{x}\\ \dot{\lambda}\end{bmatrix} = \begin{bmatrix}A(t) & -S(t)\\ -Q(t) & -A'(t)\end{bmatrix}\begin{bmatrix}x\\ \lambda\end{bmatrix} \tag{3.4}$$

A typical application is the optimization of

$$\dot{x} = A(t)x + B(t)u \tag{3.5}$$

with respect to

$$J = \frac{1}{2}\int_{t_o}^{T}[x'Q(t)x + u'R(t)u]dt \tag{3.6}$$

with x fixed at both ends. If in this problem the system (3.1), with $S(t) = B(t)R(t)B'(t)$, possesses an exponential dichotomy, and if the interval $[t_o,T]$ is large, then the corresponding TPBV problem can be approximately solved by solving two independent initial value problems.

3.2 Negative Definite Riccati Matrix

Conditions for existence and uniqueness of the symmetrical positive definite solution P(t) of

$$\dot{K} = -KA(t) - A'(t)K + KS(t)K - Q(t), \tag{3.7}$$

subject to a symmetric positive semi-definite end condition π,

$$K = \pi \qquad \text{at } t = T, \tag{3.8}$$

are known as well as the conditions for existence and uniqueness of the symmetrical positive definite root P_∞ of the time invariant system

$$- KA - A'K + KSK - Q = 0 . \tag{3.9}$$

In the construction of dichotomy transformations in this chapter, symmetrical negative definite solutions $N(t)$ and N_∞ of (3.7) and (3.9) will also be used. The transformations follow after first presenting two lemmas which depend upon the hypothesis

<u>H 3.2.1</u> Let for all $t \epsilon [t_o, T]$ the matrices $A(t)$, $B(t)$, $Q(t)$ and $R(t)$ be continuously differentiable functions of t, $Q(t)$ be symmetric positive semi-definite and $R(t)$ be symmetric positive definite.

<u>Lemma 3.2.2</u> Let H 3.2.1 be satisfied. Then for all $t \epsilon [t_o, T]$ there exists a unique symmetric negative definite solution $N(t)$ of (3.7) subject to an initial condition

$$K = -\Gamma \qquad \text{at } t = t_o \tag{3.10}$$

where Γ is a symmetric positive semi-definite matrix.

<u>Proof</u>: Consider the minimization of

$$\hat{J} = \frac{1}{2} \hat{x}'\Gamma\hat{x}_T + \frac{1}{2} \int_{t_o}^{T} [\hat{x}'Q(t_o+T-\tau)\hat{x} + \hat{u}'R(t_o+T-\tau)\hat{u}]d\tau \tag{3.11}$$

subject to

$$\frac{d\hat{x}}{d\tau} = -A(t_o+T-\tau)\hat{x} - B(t_o+T-\tau)\hat{u} \tag{3.12}$$

with $\hat{x}$ given at $\tau = t_o$ and free at $\tau = T$. The corresponding Riccati equation

$$\frac{d\hat{K}}{d\tau} = KA(t_o+T-\tau) + A'(t_o+T-\tau)\hat{K} + \hat{K}S(t_o+T-\tau)\hat{K} - Q(t_o+T-\tau) \tag{3.13}$$

has a unique positive definite solution $\hat{K}(\tau)$ satisfying the end condition

$$\hat{K} = \Gamma \quad \text{at } \tau = T \tag{3.14}$$

The substitution of $\tau = t_o+T-t$ in (3.13) and (3.14) shows that $N \equiv -\hat{K}(t_o+T-t)$ uniquely satisfies (3.7) and (3.10).

<u>Lemma 3.2.3</u> Let H 3.2.1 be satisfied where A, B, R and $Q = C'C$ are constant matrices, $[A,B]$ is a controllable pair and $[A,C]$ is an observable pair. Then the algebriac equation (3.9) has a unique symmetrical negative definite root N_∞, and $-(A-SN_\infty)$ is a stable matrix.

<u>Proof</u>: In (3.11). disregard the terminal cost term and let $T \to \infty$. Then (3.11) and (3.12) constitute a well defined infinite time regulator problem. Thus

$$\hat{K}A + A'\hat{K} + \hat{K}S\hat{K} - Q = 0 \tag{3.15}$$

has the unique symmetrical positive definite root $\hat{P}_\infty$ and $-[A+S\hat{P}_\infty]$ is a

stable matrix. The substitution of $-N_\infty$ for $\hat{P}_\infty$ into (3.15) then shows that N_∞ is the unique symmetrical negative definite root of (3.9). The uniqueness is a direct consequence of the non-singular transformation $\hat{P}_\infty = -N_\infty$ and the uniqueness of P_∞ of (3.15).

3.3 Dichotomy Transformations

Note that if $z = W(t)\xi$ transforms $\dot{z} = L(t)z$ into $\dot{\xi} = D(t)\xi$, then

$$\dot{W} = L(t)W - WD(t) \ . \tag{3.16}$$

We will consider $L(t)$ as the 2n by 2n coefficient matrix in (3.1) and construct a nonsingular transformation $W(t)$ which will make the resulting matrix $D(t)$ block-diagonal.

Lemma 3.3.1 Under the conditions of Lemma 3.2.2, the transformation

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} I & I \\ P(t) & N(t) \end{bmatrix} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} \tag{3.17}$$

is nonsingular for all $t \in [t_o, T]$ and transforms (3.1) into

$$\begin{bmatrix} \dot{y} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A(t)-S(t)P(t) & 0 \\ 0 & A(t)-S(t)N(t) \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} \tag{3.18}$$

Proof: Using (3.1), (3.17) and (3.18) to form (3.16) and noting that $P(t)$ and $N(t)$ satisfy (3.7), (3.16) is satisfied as an identity.

Transformation (3.17) is nonsingular since both $P(t)$ and $P(t) - N(t)$ are nonsingular.

Theorem 3.3.2 Let the assumptions of Lemma 3.2.3 be satisfied and use $\pi = P_\infty$ and $-\Gamma = N_\infty$ in (3.8) and (3.10). Then the transformation (3.17) becomes

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} I & I \\ P_\infty & N_\infty \end{bmatrix} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} \tag{3.19}$$

and the subspace Y in (3.2) is defined by $\eta = 0$.

Proof: Note that $\dot{W} = 0$ and (3.16) is satisfied. Since $A - SP_\infty$ is stable and $-(A - SN_\infty)$ is stable by Lemma 3.2.3, inequality (3.2) holds only when $\eta(t_o) = 0$.

Theorem 3.3.3 Consider (3.5) and (3.6) and, using $T = \infty$ and $Q(t) = C'(t)C(t)$, where $C(t)x(t)$ is the output of (3.5), define an output regulator problem as in [22]. When this problem satisfies the stability theorem by Kalman [22, Theorem 6.10], then the subspace Y generating the dichotomy (3.2), (3.3) is defined by $\eta = 0$ in (3.17).

Proof: From [22], it is known that

$$\frac{dy}{dt} = [A(t) - S(t)P(t)]y \tag{3.20}$$

is uniformly asymptotically stable. From the definitions of uniform complete controllability and observability [22] applied to (3.11) and

(3.12), and using the same change of time variables as in Lemma 3.2.2, it can also be shown that

$$\frac{d\eta}{dt} = [A(t) - S(t)N(t)]\eta \ , \tag{3.21}$$

is uniformly asymptotically stable in reverse time.

Thus (3.17) and (3.19) are transformations which make transparent the dichotomy properties of (3.1) guaranteed to exist by the assumptions in Lemma 3.2.3 for time-invariant systems and in Theorem 3.3.3 for time-varying systems. P(t) and N(t) are related by the expression

$$N(t) = P(t) - \frac{1}{2} H^{-1}(t) \tag{3.22}$$

where H(t) is the unique symmetrical positive definite solution of

$$\frac{dH}{dt} = [A(t)-S(t)P(t)]H + H[A(t)-S(t)P(t)]' + \frac{1}{2} S(t) \tag{3.23}$$

with $H = \frac{1}{2}[P(t_o) - N(t_o)]^{-1}$ at $t = t_o$. This follows upon recognition that N(t), P(t), and H(t) are unique solutions of their respective differential equations and that H(t) defined in (3.22) satisfies (3.23) as an identity. Another transformation which could have been used to show the dichotomy properties is given by

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} I & H(t) \\ P(t) & N(t)H(t) \end{bmatrix} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} \tag{3.24}$$

which transforms (3.1) into

$$\begin{bmatrix} \dot{y} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} A(t)-S(t)P(t) & 0 \\ 0 & -[A(t)-S(t)P(t)]' \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix}. \tag{3.25}$$

For a time invariant problem a similar transformation was used in [33].

3.4 An Application

Consider the minimization of (3.6) subject to (3.5) and with x fixed at both ends,

$$x(t_o) = x^o, \qquad x(T) = x^T. \tag{3.26}$$

To solve the boundary value problem (3.1) and (3.26) using the transformation (3.19), y^o and η^T are determined from

$$\begin{bmatrix} x^o \\ x^T \end{bmatrix} = \begin{bmatrix} I & \Psi(t_o,T) \\ \Phi(T,t_o) & I \end{bmatrix} \begin{bmatrix} y^o \\ \eta^T \end{bmatrix}, \tag{3.27}$$

where $\Phi(t,t_o)$ and $\Psi(t,T)$ are the fundamental matrices of (3.20) and (3.21) respectively. However, if (3.1) possesses the dichotomy (3.2), (3.3) and if the interval $[t_o,T]$ is sufficiently large, then $|\Psi(t_o,T)|<<1$, $|\Phi(T,t_o)|<<1$ and

$$y^o \approx x^o, \qquad \eta^T \approx x^T. \tag{3.28}$$

Using these boundary conditions, the approximate $y(t)$ and $\eta(t)$ are obtained from the independent initial value problems (3.20) and (3.21). The approximate solution of the boundary value problem (3.1) and (3.26) is then found using

$$\begin{aligned} x(t) &= y(t) + \eta(t) \\ \lambda(t) &= P(t)y(t) + N(t)\eta(t) \end{aligned} \quad . \tag{3.29}$$

This procedure is particularly convenient when A, B, Q and R are constant matrices, since then $P(t) = P_\infty$ and $N(t) = N_\infty$ can be obtained by an algebraic method.

3.5 Example

Consider finding an approximate solution of system (3.1) satisfying (3.26) where $A = 0.5$, $B = 1$, $Q = 2$, and $R = 1$. Then (3.1) becomes

$$\begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} 0.5 & -1 \\ -2 & -0.5 \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix}; \qquad \begin{aligned} x(0) &= x^o \\ x(T) &= x^T \end{aligned} \tag{3.30}$$

For this example, $P_\infty = 2$ is the positive and $N_\infty = -1$ is the negative root of

$$K^2 - K - 2 = 0 \ . \tag{3.31}$$

The transformation

$$\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} y(t) \\ \eta(t) \end{bmatrix} \tag{3.32}$$

transforms (3.30) into

$$\begin{bmatrix} \dot{y} \\ \dot{\eta} \end{bmatrix} = \begin{bmatrix} -1.5 & 0 \\ 0 & 1.5 \end{bmatrix} \begin{bmatrix} y \\ \eta \end{bmatrix} \tag{3.33}$$

and (3.27) gives

$$\begin{aligned} x^o &= y^o + \eta^T e^{-1.5T} \\ x^T &= y^o e^{-1.5T} + \eta^T \quad . \end{aligned} \tag{3.34}$$

If T is large, $y^o \gg \eta^T e^{-1.5T}$ and $\eta^T \gg y^o e^{-1.5T}$. Hence

$$\begin{aligned} x(t) &\approx x^o e^{-1.5t} + x^T e^{1.5(t-T)} \\ \lambda(t) &\approx 2x^o e^{-1.5t} - x^T e^{1.5(t-T)} \end{aligned} \tag{3.35}$$

Note that both y(t) and $\eta(t)$ are obtained by solving (3.20) and (3.21) in their stable directions.

3.6 Discussion

An interpretation of (3.20) is that it is the solution of a regulator problem whose performance index has a penalty term at t = T and whose system is subject to an initial boundary condition at $t = t_o$. Similarly, an

interpretation of (3.21) is that it is the solution of a regulator problem whose performance index has a penalty term at $t = t_o$ and whose system is subject to a final boundary condition at $t = T$. The stability of these regulated systems is assured by the assumptions given in Lemma 3.2.3 and Theorem 3.3.3. For a sufficiently large interval, a simplifying approximation is possible and the TPBV problem is approximately solved by solving only initial value regulated systems and adding the resulting solutions. As either the time constant associated with the exponential bound on the regulator solutions is decreased or the time interval increased, the TPBV solution appears as the summation of two decaying transients; one at each end of the time interval. For a very small time constant, the transient behavior at the initial time could be approximated by the solution of (3.20) whose coefficient matrix is assumed constant (the functional value at $t = t_o$). Similarly, the transient behavior at the final time could be approximated by the solution of (3.21) whose coefficient matrix is assumed constant (the functional value at $t = T$). The differential equations (3.20) and (3.21) are called "stiff" when the transients occur very rapidly.

3.7 A Preliminary Singular Perturbation Problem

It will now be shown that the solution of the singularly perturbed time invariant system

$$\mu \begin{bmatrix} \dot{x} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} A & -S \\ -Q & -A' \end{bmatrix} \begin{bmatrix} x \\ \lambda \end{bmatrix} \tag{3.36}$$

satisfying the boundary conditions

$$x(t_o) = x^o, \qquad x(T) = x^T \tag{3.37}$$

behaves similar to the solution of the TPBV problem (3.1) satisfying (3.37) under suitable stability conditions and for μ sufficiently small. Specifically, the previous requirement that the time interval $[t_o,T]$ be large relative to the transient intervals at the initial and final times will be shown equivalent to the condition that μ be sufficiently small for the specified interval $[t_o,T]$ of the problem. Furthermore, it will be shown that an approximate solution of (3.36) and (3.37) is given by the summation of solutions of two time-invariant regulated systems.

If the hypotheses of Lemma 3.2.3 are satisfied, then there exists a unique symmetrical positive definite constant solution P_∞ and a unique symmetrical negative definite constant solution N_∞ of the Riccati equation

$$\mu\dot{K} + KA + A'K - KSK + Q = 0 \tag{3.38}$$

for (3.36). Also, $[A - SP_\infty]$ and $-[A - SN_\infty]$ are both stable matrices. Thus (3.36) is transformed by

$$\begin{bmatrix} x \\ \lambda \end{bmatrix} = \begin{bmatrix} I & I \\ P_\infty & N_\infty \end{bmatrix} \begin{bmatrix} y \\ \ell \end{bmatrix} \tag{3.39}$$

into the two decoupled singularly perturbed systems

$$\mu\dot{y} = (A-SP_\infty)y \tag{3.40}$$

$$\mu\dot{\eta} = (A-SN_\infty)\eta \ . \tag{3.41}$$

Assuming that an initial condition y^o is given for (3.40) and a final condition η^T is given for (3.41), the solutions for y and η are given by

$$y = e^{(A-SP_\infty)(t-t_o)/\mu} y^o \tag{3.42}$$

$$\eta = e^{(A-SN_\infty)(t-t_o)/\mu} \eta^T \tag{3.43}$$

The boundary conditions x^o, x^T are related to those of y^o, η^T by

$$\begin{bmatrix} x^o \\ x^T \end{bmatrix} = \begin{bmatrix} I & e^{(A-SN_\infty)(t_o-T)/\mu} \\ e^{(A-SP_\infty)(T-t_o)/\mu} & I \end{bmatrix} \begin{bmatrix} y^o \\ \eta^T \end{bmatrix} \tag{3.44}$$

Clearly for whatever t_o and T ($t_o \le T$) have been given, it is always possible to find a $\mu^* > 0$ such that for all $\mu \in [0,\mu^*]$

$$\left| e^{(A-SP_\infty)(T-t_o)/\mu} \right| \ll 1, \qquad \left| e^{(A-SN_\infty)(t_o-T)/\mu} \right| \ll 1$$

since $[A-SP_\infty]$ and $-[A-SN_\infty]$ are stable matrices. Thus

$$y^o \approx x^o \ , \qquad \eta^T \approx x^T \tag{3.45}$$

for $\mu \leq \mu^*$. Hence the approximate $x(t,\mu)$ solution is given by

$$x(t,\mu) = e^{(A-SP_\infty)(t-t_o)/\mu} x^o + e^{(A-SN_\infty)(t-T)/\mu} x^T \tag{3.46}$$

This simplified problem points out the interchanging roles of T and μ and represents a special case of a more general problem rigorously analyzed in Chapter 4.

4. FIXED END-POINT PROBLEM: THEORY

4.1 Introduction and Problem Statement

The goal of this chapter is to develop an approximation of the optimal solution of a trajectory optimization problem over the entire operation interval $[t_o,T]$. An approximate solution is sought instead of the actual solution since the latter is often quite difficult to find using existing methods. The complication is a result of both widely varying decay transients and widely varying growth transients associated with the solution of a TPBV problem. The design objectives for the approximate design solution is:

(i) solution of a lower-dimensional problem than the original problem when the approximation is to be valid only on an open interval of $[t_o,T]$;

(ii) accounting for boundary layer phenomena by finding correction terms which, when added to the reduced solution, make the approximation valid over the whole interval $[t_o,T]$;

(iii) determination of correction terms separately in a stretched time scale thus avoiding "stiff" problems.

This approximation design is developed for the problem of optimally controlling the $(n_1 + n_2)$-dimensional system

$$\begin{aligned} \dot{x}_1 &= A_{11}(t,\mu)x_1 + A_{12}(t,\mu)x_2 + B_1(t,\mu)u \\ \mu\dot{x}_2 &= A_{21}(t,\mu)x_1 + A_{22}(t,\mu)x_2 + B_2(t,\mu)u \end{aligned} \tag{4.1}$$

with respect to the performance index

$$J = \frac{1}{2}\int_{t_o}^{T}\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}' & Q_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u'Ru\right)dt \tag{4.2}$$

where the boundary conditions for (4.1) are

$$\begin{aligned} x_1 &= x_1^{o} \quad \text{and} \quad x_2 = x_2^{o} \quad \text{at } t = t_o \\ x_1 &= x_1^{T} \quad \text{and} \quad x_2 = x_2^{T} \quad \text{at } t = T \end{aligned} \tag{4.3}$$

The following hypotheses are made about the matrices appearing in (4.1) and (4.2).

<u>H 4.1.1</u> For all $t \in [t_o,T]$, $\mu \in [0,\mu^*]$ the symmetrical matrix $Q = Q(t,\mu)$ is positive semi-definite, and $R = R(t,\mu)$ is positive definite.

<u>H 4.1.2</u> For all $t \in [t_o,T]$, $\mu \in [0,\mu^*]$ the elements of the matrices in (4.1) and (4.2) are three times continuously differentiable functions of their arguments.

From the optimality conditions for (4.1), (4.2), it follows that

$$u = -R^{-1}(B_1'\lambda_1 + B_2'\lambda_2) \tag{4.4}$$

and hence

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\lambda}_1 \\ \mu\dot{x}_2 \\ \mu\dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & -S_{11} & A_{12} & -S_{12} \\ -Q_{11} & -A_{11}' & -Q_{12} & -A_{21}' \\ A_{21} & -S_{12}' & A_{22} & -S_{22} \\ -Q_{12}' & -A_{12}' & -Q_{22} & -A_{22}' \end{bmatrix} \begin{bmatrix} x_1 \\ \lambda_1 \\ x_2 \\ \lambda_2 \end{bmatrix} \tag{4.5}$$

Using

$$z_1 = \begin{bmatrix} x_1 \\ \lambda_1 \end{bmatrix}, \qquad z_2 = \begin{bmatrix} x_2 \\ \lambda_2 \end{bmatrix}$$

the system (4.5) is rewritten in the compact form

$$\begin{bmatrix} \dot{z}_1 \\ \mu\dot{z}_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \tag{4.6}$$

where the definition of D_{ij} is evident from comparison of (4.6) with (4.5). When μ is set equal to zero, the $2(n_1 + n_2)$-dimensional system (4.6) reduces to a $2n_1$-dimensional system, and if the indicated inverse exists, the reduced system becomes

$$\dot{\bar{z}}_1 = (\bar{D}_{11} - \bar{D}_{12}\bar{D}_{22}^{-1}\bar{D}_{21})\bar{z}_1 \ . \tag{4.7}$$

Variable $\bar{z}_2$ is not present in the differential system (4.7) but is algebraically related to $\bar{z}_1$ by

$$\bar{z}_2 = -\bar{D}_{22}^{-1}\bar{D}_{21}\bar{z}_1 \ . \tag{4.8}$$

Thus in general $\bar{x}_2$ will not satisfy the x_2 boundary conditions in (4.3). A solution $\bar{z}_1$ satisfying (4.7) and the x_1 boundary conditions of (4.3) with the corresponding $\bar{z}_2$ obtained from (4.8) is called a reduced solution.

The approximation is based on the requirement that an auxiliary time invariant system depending on a fixed parameter η

$$\frac{d\hat{x}_2}{dt} = A_{22}(\eta,0)\hat{x}_2 + B_2(\eta,0)\hat{u} \tag{4.9}$$

be stabilizable for each η in the interval $[t_o,T]$. The full meaning of stabilizability of system (4.9) will be clear later when this system appears in so called "layer regulators" and makes it possible to use algebraic Riccati equations for calculation of the correction terms. Controllability and observability hypothesis H 4.1.3 used here is more restrictive than the stabilizability requirement but simplifies the forthcoming derivations. This is the crucial hypothesis of Theorem 4.2.1.

<u>H 4.1.3</u> For all $t \in [t_o,T]$,

$$\text{rank}\left[\bar{B}_2,\bar{A}_{22}\bar{B}_2,\bar{A}_{22}^2\bar{B}_2,\ldots,\bar{A}_{22}^{n_2-1}\bar{B}_2\right] = n_2 \tag{4.10}$$

$$\text{rank}\left[\bar{C}_2',\bar{A}_{22}'\bar{C}_2',(\bar{A}_{22}')^2\bar{C}_2',\ldots,(\bar{A}_{22}')^{n_2-1}\bar{C}_2'\right] = n_2 \tag{4.11}$$

It is seen that hypothesis H 4.1.3 guarantees the existence of the inverse in (4.7) and (4.8). The last hypothesis needed is

<u>H 4.1.4</u> There exists a unique (bounded) reduced solution satisfying the x_1 boundary conditions in (4.3).

4.2 Main Theorem

The theorem of this section establishes the existence of an approximate solution for the trajectory optimization problem (4.1)-(4.3) which accomplishes all of the design objectives (i)-(iii) set forth in section 4.1. This permits numerically complicated problems to be approximately solved in a simplified manner.

Theorem 4.2.1 Let H 4.1.1, H 4.1.2, H 4.1.3, and H 4.1.4 be satisfied for problem (4.1), (4.2), and (4.3). Then there exists a positive constant μ^* such that for $\mu \leq \mu^*$ and $t \in [t_o, T]$

$$
\begin{aligned}
x_1(t,\mu) &= \bar{x}_1(t) + O(\mu) \\
x_2(t,\mu) &= \bar{x}_2(t) + \mathcal{L}_2(\tau) + \mathcal{R}_2(\sigma) + O(\mu) \\
\lambda_1(t,\mu) &= \bar{\lambda}_1(t) + O(\mu) \\
\lambda_2(t,\mu) &= \bar{\lambda}_2(t) + \bar{P}_{22}(t_o)\mathcal{L}_2(\tau) + \bar{N}_{22}(T)\mathcal{R}_2(\sigma) + O(\mu) \ .
\end{aligned}
\tag{4.12}
$$

Here $\mathcal{L}_2(\tau)$ is the solution of the "left" layer system

$$
\frac{d\mathcal{L}_2}{d\tau} = \left[\bar{A}_{22}(t_o) - \bar{S}_{22}(t_o)\bar{P}_{22}(t_o)\right] \mathcal{L}_2 \tag{4.13}
$$

subject to initial condition

$$
\mathcal{L}_2 = x_2^{\,o} - \bar{x}_2(t_o) \quad \text{at } \tau = 0
$$

where $\tau = (t-t_o)/\mu$, $\mathcal{R}_2(\sigma)$ is the solution of the "right" layer system

$$\frac{d\mathcal{R}_2}{d\sigma} = \left[\bar{A}_{22}(T) - \bar{S}_{22}(T)\bar{N}_{22}(T)\right] \mathcal{R}_2 \tag{4.14}$$

subject to terminal condition

$$\mathcal{R}_2 = x_2^T - \bar{x}_2(T) \quad \text{at } \sigma = 0$$

where $\sigma = (t-T)/\mu$, and $\bar{P}_{22}(t_o)$ and $\bar{N}_{22}(T)$ are the symmetrical positive and negative definite algebraic solutions of

$$\bar{A}_{22}'K_{22} + K_{22}\bar{A}_{22} - K_{22}\bar{S}_{22}K_{22} + \bar{Q}_{22} = 0 \tag{4.15}$$

evaluated at t_o and T respectively.

Discussion: This theorem can be used for different types of approximations of the optimal solution. For an approximation on the interval $[t^1,t^2]$ where $t_o < t^1 \leq t^2 < T$, one neglects the $\mathcal{L}_2,\mathcal{R}_2$ terms in (4.12) and approximates the high-dimensional solution by the low-dimensional reduced solution found from (4.7), (4.8). This approximation is within $O(\mu)$ for μ sufficiently small. For the approximation to be valid on the interval $[t_o,t^2]$, the left layer correction term is added to the reduced solution; for the approximation to be valid on the interval $[t^1,T]$, the right layer correction term is added to the reduced solution. When both left and right layer correction terms are added, the approximation is valid on the entire interval $[t_o,T]$.

It should be noted that the theorem has the desired lower-dimensional and time scale separation properties. The correction terms are evaluated independently and each is the solution of a time invariant initial value problem. For computation of $\mathcal{L}_2$, only the symmetrical positive definite root $\bar{P}_{22}$ at $t = t_o$ needs to be found from the algebraic Riccati equation (4.15). The symmetrical negative root $\bar{N}_{22}$ of (4.15) at $t = T$ is needed for $\mathcal{R}_2$.

Finally, a similar theorem could be formulated for an asymptotic expansion as common in singular perturbation theory. This was not done here in order to provide a clearer proof of the theorem.

The proof of this theorem will be given after first proving a set of lemmas. The first two lemmas are used to establish certain properties of two different solutions of a singularly perturbed Riccati system. One solution is symmetrical positive semi-definite and the other is symmetrical negative semi-definite. The third lemma uses these solutions in defining a transformation which enables one to solve two initial value problems in place of a TPBV problem.

4.3 Properties of Riccati Matrices

Let the co-state variables λ_1 and λ_2 be related to the state variables x_1 and x_2 by the Riccati transformation

$$\begin{bmatrix} \lambda_1 \\ \mu\lambda_2 \end{bmatrix} = \begin{bmatrix} K_{11} & \mu K_{12} \\ \mu K_{12}' & \mu K_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.16}$$

The Riccati equations resulting after elimination of λ_1 and λ_2 from (4.5) using (4.16) are given by

$$\dot{K}_{11} = -K_{11}(A_{11}-S_{12}K_{12}') - (A_{11}-S_{12}K_{12}')'K_{11}$$
$$- K_{12}A_{21} - A_{21}'K_{12}' + K_{11}S_{11}K_{11} + K_{12}S_{22}K_{12}' - Q_{11} \tag{4.17a}$$

$$\mu\dot{K}_{12} = -K_{12}(A_{22}-S_{22}K_{22}) - K_{11}A_{12} + K_{11}S_{12}K_{22} - A_{21}'K_{22} - Q_{12}$$
$$- \mu(A_{11}'K_{12}-K_{11}S_{11}K_{12}-K_{12}S_{12}'K_{12}) \tag{4.17b}$$

$$\mu\dot{K}_{22} = -K_{22}A_{22} - A_{22}'K_{22} + K_{22}S_{22}K_{22} - Q_{22}$$
$$- \mu\left[K_{12}'(A_{12}-S_{12}K_{22}) + (A_{12}-S_{12}K_{22})'K_{12} - \mu K_{12}'S_{11}K_{12}\right] \tag{4.17c}$$

The corresponding reduced Riccati system obtained by setting $\mu = 0$ in (4.17) is

$$\dot{K}_{11} = -K_{11}(\bar{A}_{11}-\bar{S}_{12}K_{12}') - (\bar{A}_{11}-\bar{S}_{12}K_{12}')K_{11}$$
$$- K_{12}\bar{A}_{21} - \bar{A}_{21}'K_{12}' + K_{11}\bar{S}_{11}K_{11} + K_{12}\bar{S}_{22}K_{12}' - \bar{Q}_{11} \tag{4.18a}$$

$$0 = -K_{12}(\bar{A}_{22}-\bar{S}_{22}K_{22}) - K_{11}\bar{A}_{12} + K_{11}\bar{S}_{12}K_{22} - \bar{A}_{21}'K_{22} - \bar{Q}_{12} \tag{4.18b}$$

$$0 = -K_{22}\bar{A}_{22} - \bar{A}_{22}'K_{22} + K_{22}\bar{S}_{22}K_{22} - \bar{Q}_{22} \tag{4.18c}$$

and is to satisfy the same K_{11} boundary condition as imposed on K_{11} for the solution of (4.17). In general, it cannot satisfy the end conditions for K_{12} and K_{22}. Since the system (4.18) has an algebraic part, it can have many solutions satisfying the symmetrical positive semi-definite boundary condition on K_{11} at $t = T$. Only one of these solutions denoted by $\bar{K}_{ij}$ meets the previous hypotheses H 4.1.1 - H 4.1.3 and the requirement that $\bar{K}_{11}$, $\bar{K}_{22}$ by symmetrical positive semi-definite matrices for all $t \in [t_o,T]$. When $\bar{K}_{12}$ and $\bar{K}_{22}$ at $t = T$ are equal to the respective boundary conditions imposed on K_{12} and K_{22} in

(4.17), then the solution of (4.18) is designated by $\bar{P}_{ij}$ and the solution of (4.17) satisfying these boundary conditions is designated by $P_{ij}(t,\mu)$. Likewise, of the many solutions of (4.18) satisfying the symmetrical negative semi-definite boundary condition on K_{11} at $t = t_o$, there is only one $\bar{N}_{ij}$ satisfying the conditions imposed. These conditions are: the previous hypotheses H 4.1.1 - H 4.1.3 must be met, $\bar{N}_{11}$ and $\bar{N}_{22}$ are to be symmetrical negative semi-definite for all $t \in [t_o,T]$, and the values of $\bar{N}_{12}$ and $\bar{N}_{22}$ at $t = t_o$ are to be the same as the boundary conditions imposed on these variables in (4.17). The solution of (4.17) satisfying these boundary conditions is designated by $N_{ij}(t,\mu)$.

The basis of the proof of the main theorem is the existence of the Riccati solutions P_{ij}, N_{ij}. The singularly perturbed Riccati solution of (4.17) is known to generally have a boundary layer jump at $t = T$ when the boundary conditions are given at $t = T$. Thus the Riccati solution would rarely be continuous for all $t \in [t_o,T]$, $\mu \in [0,\mu^*]$ which is a crucial requirement in the approach used to prove our Main Theorem. But by properly selecting the boundary conditions for (4.17), the boundary layer jumps have been eliminated and the continuity of $P_{ij}(t,\mu)$ and $N_{ij}(t,\mu)$ can be guaranteed.

<u>Lemma 4.3.1</u> Let H 4.1.1, H 4.1.2, and H 4.1.3 be satisfied and let the boundary conditions for (4.17) be given by

$$\begin{bmatrix} K_{11} & \mu K_{12} \\ \mu K_{12}' & \mu K_{22} \end{bmatrix} = \begin{bmatrix} \pi_{11}(\mu) & \mu\bar{P}_{12} \\ \mu\bar{P}_{12}' & \mu\bar{P}_{22} \end{bmatrix} \quad \text{at } t = T \tag{4.19}$$

where $\pi_{11}(\mu)$ is an arbitrary symmetrical positive semi-definite matrix which is a three times continuously differentiable function of μ for all $\mu \in [0,\mu^*]$. Then there exists a $\mu^* > 0$ such that for all $\mu \leq \mu^*$ the unique solution $P_{ij}(t,\mu)$ of (4.17) satisfying (4.19) can be asymptotically approximated on the entire interval $[t_o,T]$ by

$$P_{ij}(t,\mu) = \bar{P}_{ij} + O(\mu) \qquad (i,j = 1,2) \tag{4.20}$$

where $\bar{P}_{11}$ and $\bar{P}_{22}$ are symmetrical positive definite matrices, except possibly at $t = T$ where $\bar{P}_{11}$ can be positive semi-definite. Furthermore $[\bar{A}_{22}-\bar{S}_{22}\bar{P}_{22}]$ is a stable matrix.

The proof is based on showing that the hypotheses of the singularly perturbed initial value Lemma A.4 of the appendix are satisfied. In addition to certain smoothness assumptions, these hypotheses require first, that the reduced solution exists, and second, the stability of the boundary layer equation. This proof in part uses facts established in [55] but differs from the proof given there in two ways. First the problem considered here requires only that the initial conditions of the Riccati problem lie within a neighborhood of the reduced Riccati solution at $t = T$. Thus global properties, which can be proven, are not needed here. Second, a proof of boundary layer stability is given where Kronecker products were not needed and the continuity of P_{ij} at $\mu = 0$ is established.

<u>Proof of Lemma 4.3.1</u>: The existence of the reduced solution will first be shown and then boundary layer stability; both of which are hypotheses of

the appendix. A unique algebraic symmetrical positive definite root $\bar{P}_{22}$ of (4.18c) is guaranteed to exist for all $t \in [t_o,T]$ by H 4.1.1 and H 4.1.3. This can easily be seen at a fixed time by interpreting (4.18c) as the Riccati equation resulting from a time invariant optimal regulator problem. The continuity of $\bar{P}_{22}(t)$ for $t \in [t_o,T]$ then follows from the implicit function theorem and H 4.1.2. This interpretation also makes evident that $[\bar{A}_{22}-\bar{S}_{22}\bar{P}_{22}]$ is a stable matrix for each $t \in [t_o,T]$. From this and (4.18b) it follows that $\bar{P}_{12}$ is uniquely expressible in terms of $\bar{P}_{11}$ and $\bar{P}_{22}$. The existence of the unique symmetrical positive definite matrix $\bar{P}_{11}$ (semi-definite at $t = T$ if $\bar{\pi}_{11}$ is semi-definite there) was established in [55] by showing it was the solution of a Riccati system similar to (3.6). Thus the reduced solution $\bar{P}_{ij}$ exists. The stability of the boundary layer equation

$$\begin{aligned}
\frac{d\hat{K}_{12}}{d\tau} &= -\hat{K}_{12}(\bar{A}_{22}-\bar{S}_{22}\hat{K}_{22}) - \hat{K}_{11}\bar{A}_{12} + \hat{K}_{11}\bar{S}_{12}\hat{K}_{22} - \bar{A}_{21}'\hat{K}_{22} - \bar{Q}_{12} \\
\frac{d\hat{K}_{22}}{d\tau} &= -\hat{K}_{22}\bar{A}_{22} - \bar{A}_{22}'\hat{K}_{22} + \hat{K}_{22}\bar{S}_{22}\hat{K}_{22} - \bar{Q}_{22}
\end{aligned} \tag{4.21}$$

will now be shown about the reduced solution $\bar{K}_{ij} = \bar{P}_{ij}$ for each $t \in [t_o,T]$. The linearized system of (4.21) about the reduced solution where $\hat{K}_{11} = \bar{P}_{11}$, $\hat{K}_{12} = \bar{P}_{12} + \delta K_{12}$, and $\hat{K}_{22} = \bar{P}_{22} + \delta K_{22}$ becomes

$$\frac{d\delta K_{12}}{d\tau} = -\delta K_{12}(\bar{A}_{22}-\bar{S}_{22}\bar{P}_{22}) - (\bar{A}_{21}'-\bar{P}_{11}\bar{S}_{12}-\bar{P}_{12}\bar{S}_{22})\delta K_{22} \tag{4.22a}$$

$$\frac{d\delta K_{22}}{d\tau} = -\delta K_{22}(\bar{A}_{22}-\bar{S}_{22}\bar{P}_{22}) - (\bar{A}_{22}-\bar{S}_{22}\bar{P}_{22})\delta K_{22} \tag{4.22b}$$

whose coefficient matrices are functions of the fixed parameter t. Since

$[\bar{A}_{22}-\bar{S}_{22}\bar{P}_{22}]$ is a stable matrix, it immediately follows that the solution of (4.22) is asymptotically stable. Upon substitution of this solution in (4.22a) and again noting the presence of the stable matrix $[\bar{A}_{22}-\bar{S}_{22}\bar{P}_{22}]$, it then follows that the solution of (4.22a) is asymptotically stable. Since the smoothness assumptions are met, it remains only to note that the boundary layer correction terms for P_{12} and P_{22} at $t = T$ are eliminated as a direct consequence of the special way in which the boundary conditions were chosen in the hypothesis of this lemma. Thus satisfaction of Lemma A.4 guarantees the existence of the μ^* of this lemma.

<u>Lemma 4.3.2</u> Let H 4.1.1, H 4.1.2, and H 4.1.3 be satisfied and let the boundary conditions for (4.17) be given by

$$\begin{bmatrix} K_{11} & \mu K_{12} \\ \mu K_{12}' & \mu K_{22} \end{bmatrix} = \begin{bmatrix} -\Gamma_{11}(\mu) & \mu\bar{N}_{12} \\ \mu\bar{N}_{12}' & \mu\bar{N}_{22} \end{bmatrix} \quad \text{at } t = t_o \tag{4.23}$$

where $-\Gamma_{11}(\mu)$ is an arbitrary symmetrical negative semi-definite matrix which is three times continuously differentiable function of μ for all $\mu \in [0,\mu^*]$. Then there exists a $\mu^* > 0$ such that for all $\mu \leq \mu^*$, the unique solution $N_{ij}(t,\mu)$ of (4.17) satisfying (4.19) can be asymptotically approximated in the entire interval $[t_o,T]$ by

$$N_{ij}(t,\mu) = \bar{N}_{ij} + O(\mu) \qquad (i,j = 1,2) \tag{4.24}$$

where $\bar{N}_{11}$ and $\bar{N}_{22}$ are symmetrical negative definite matrices, except

possibly at $t = T$ where $\bar{N}_{11}$ can be negative semi-definite. Furthermore, $-[\bar{A}_{22}-\bar{S}_{22}\bar{N}_{22}]$ is a stable matrix.

<u>Proof</u>: Consider the Riccati system

$$\frac{d\hat{K}_{11}}{d\gamma} = -\hat{K}_{11}(-A_{11}-S_{12}\hat{K}_{12}') - (-A_{11}-S_{12}\hat{K}_{12}')'\hat{K}_{11} + \hat{K}_{12}A_{21} + A_{21}'\hat{K}_{12}' + \hat{K}_{11}S_{11}\hat{K}_{11} + \hat{K}_{12}S_{22}\hat{K}_{12}' - Q_{11} \tag{4.25a}$$

$$\mu\frac{d\hat{K}_{12}}{d\gamma} = -\mu(-A_{11}'\hat{K}_{12}-\hat{K}_{11}S_{11}\hat{K}_{12}-\hat{K}_{12}S_{12}'\hat{K}_{12}) - \hat{K}_{12}(-A_{22}-S_{22}\hat{K}_{22}) + \hat{K}_{11}A_{12} + \hat{K}_{11}S_{12}\hat{K}_{22} + A_{21}'\hat{K}_{22} - Q_{12} \tag{4.25b}$$

$$\mu\frac{d\hat{K}_{22}}{d\gamma} = -\mu[\hat{K}_{12}'(-A_{12}-S_{12}\hat{K}_{22}) + (-A_{12}-S_{12}\hat{K}_{22})'\hat{K}_{12}-\mu\hat{K}_{12}'S_{11}\hat{K}_{12}] + \hat{K}_{22}A_{22} + A_{22}'\hat{K}_{22} + \hat{K}_{22}S_{22}\hat{K}_{22} - Q_{22} \tag{4.25c}$$

whose coefficients are functions of $t_o + T - \gamma$ and whose boundary conditions are given by

$$\begin{bmatrix} \hat{K}_{11} & \mu\hat{K}_{12} \\ \mu\hat{K}_{12}' & \mu\hat{K}_{22} \end{bmatrix} = \begin{bmatrix} \Gamma_{11}(\mu) & -\mu\bar{N}_{12}(\mu) \\ -\mu\bar{N}_{12}'(\mu) & -\mu\bar{N}_{22}(\mu) \end{bmatrix} \quad \text{at } t = T \tag{4.26}$$

Noting that (4.25) and (4.26) satisfy Lemma 4.3.1 and making the substitution $\gamma = t_o + T - t$ in (4.25) and (4.26), it is clear $N_{ij} = -\hat{K}_{ij}(t_o + T-t)$ uniquely satisfies (4.17) and (4.26).

4.4 Dichotomy Transformation

Let the following transformation be defined

$$\begin{bmatrix} x_1 \\ \lambda_1 \\ x_2 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} I_{n_1} & I_{n_1} & 0 & 0 \\ P_{11} & N_{11} & \mu P_{12} & \mu N_{12} \\ 0 & 0 & I_{n_2} & I_{n_2} \\ P_{12}' & N_{12}' & P_{22} & N_{22} \end{bmatrix} \begin{bmatrix} \ell_1 \\ r_1 \\ \ell_2 \\ r_2 \end{bmatrix} \tag{4.27}$$

or equivalently

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{4.28}$$

where

$$q_1 = \begin{bmatrix} \ell_1 \\ r_1 \end{bmatrix}, \qquad q_2 = \begin{bmatrix} \ell_2 \\ r_2 \end{bmatrix} \tag{4.29}$$

and the definition of W_{ij} is evident upon comparison of (4.28) with (4.27). The non-singularity of transformation (4.27) for all $t \in [t_o,T]$ and $\mu \in [0,\mu^*]$ can be seen from the determinant of the coefficient matrix of (4.28)

$$\det\begin{vmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{vmatrix} = \det\begin{vmatrix} I_{n_1} & I_{n_1} \\ P_{11} & N_{11} \end{vmatrix} \cdot$$

$$\det\left\{\begin{bmatrix} I_{n_2} & I_{n_2} \\ P_{22} & N_{22} \end{bmatrix} - \mu\begin{bmatrix} 0 & 0 \\ P_{12}' & N_{12}' \end{bmatrix}\begin{bmatrix} I_{n_1} & I_{n_1} \\ P_{11} & N_{11} \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ P_{12} & N_{12} \end{bmatrix}\right\}$$

$$= \det\begin{vmatrix} I_{n_1} & I_{n_1} \\ P_{11} & N_{11} \end{vmatrix} \cdot \det\begin{vmatrix} I_{n_2} & I_{n_2} \\ P_{22} & N_{22} \end{vmatrix} + O(\mu) \tag{4.30}$$

$\neq 0$ since the bracketed terms of (4.30) are non-zero.

The following lemma shows that transformation (4.28) transforms (4.6) into two decoupled systems; one containing ℓ_1 and ℓ_2 and the other containing r_1 and r_2. The transformation also results in the ℓ boundary layer system being stable in forward time and that of the r boundary layer system being stable in reverse time. Thus transformation (4.28) dichotomizes the x_2, λ_2 boundary layer system as the transformation of Chapter 3 did for the original system.

<u>Lemma 4.4.1</u> Let the conditions of Lemmas 4.3.1 and 4.3.2 be satisfied. Then the non-singular transformation (4.27) transforms system (4.5) into

$$\begin{bmatrix} \dot{\ell}_1 \\ \mu\dot{\ell}_2 \end{bmatrix} = \begin{bmatrix} A_{11}-S_{11}P_{11}-S_{12}P_{12}' & A_{12}-\mu S_{11}P_{12}-S_{12}P_{22} \\ A_{21}-S_{12}'P_{11}-S_{22}P_{12}' & A_{22}-\mu S_{12}'P_{12}-S_{22}P_{22} \end{bmatrix}\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \tag{4.31}$$

$$\begin{bmatrix} \dot{r}_1 \\ \mu\dot{r}_2 \end{bmatrix} = \begin{bmatrix} A_{11}-S_{11}N_{11}-S_{12}N_{12}' & A_{12}-\mu S_{11}N_{12}-S_{12}N_{22} \\ A_{21}-S_{12}'N_{11}-S_{22}N_{12}' & A_{22}-\mu S_{12}'N_{12}-S_{22}N_{22} \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{4.32}$$

or in more compact notation,

$$\begin{bmatrix} \dot{\ell}_1 \\ \mu\dot{\ell}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \tag{4.33}$$

$$\begin{bmatrix} \dot{r}_1 \\ \mu\dot{r}_2 \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \tag{4.34}$$

where the definition of F_{ij} is evident from comparison of (4.33) with (4.31) and that of G_{ij} from comparison of (4.34) with (4.32).

<u>Proof</u>: Upon elimination of z_1, z_2 from (4.6) using (4.28) and comparing this result with what it must be to satisfy (4.31) and (4.32), it suffices to show that

$$\begin{bmatrix} \dot{W}_{11} & \dot{W}_{12} \\ \mu\dot{W}_{21} & \mu\dot{W}_{22} \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix} - \begin{bmatrix} W_{11} & W_{12}/\mu \\ \mu W_{21} & W_{22} \end{bmatrix} \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} \tag{4.35}$$

where

$$E_{11} = \begin{bmatrix} F_{11} & 0 \\ 0 & G_{11} \end{bmatrix}, \qquad E_{12} = \begin{bmatrix} F_{12} & 0 \\ 0 & G_{12} \end{bmatrix}$$

$$E_{21} = \begin{bmatrix} F_{21} & 0 \\ 0 & G_{21} \end{bmatrix}, \qquad E_{22} = \begin{bmatrix} F_{22} & 0 \\ 0 & G_{22} \end{bmatrix} \tag{4.36}$$

It can be readily shown that (4.35) is satisfied as an identity.

4.5 Proof of Main Theorem

The systems (4.31) and (4.32) are convenient for a straight forward proof of the theorem. It will first be assumed that initial conditions $\ell_1^o(\mu)$, $\ell_2^o(\mu)$ are given at $t = t_o$ for system (4.31) and final conditions $r_1^T(\mu)$, $r_2^T(\mu)$ are given at $t = T$ for system (4.32). These boundary conditions are assumed to be twice continuously differentiable, Thus two independent singularly perturbed initial value problems exist, and Theorem A.5 can be applied providing its assumptions are satisfied since P_{ij} and N_{ij} are twice continuous differentiable functions for all $t \in [t_o,T]$ and for all $\mu \in [0,\mu^*]$ in view of Lemmas 4.3.1 and 4.3.2. Also the boundary layer parts of 4.3.1 and 4.3.2 are stable in their respective time directions since $[\bar{A}_{22}-\bar{S}_{22}\bar{P}_{22}]$ and $-[\bar{A}_{22}-\bar{S}_{22}\bar{N}_{22}]$ are stable matrices by Lemmas 4.3.1 and 4.3.2 respectively. Thus when ℓ_1^o, ℓ_2^o are given, Theorem A.5 permits one to write for (4.33)

$$\ell_1 = \bar{\ell}_1 + O(\mu), \qquad \ell_2 = \bar{\ell}_2 + \mathcal{L}_2(\tau) + O(\mu) \tag{4.37}$$

where the reduced solution $\bar{\ell}_1$ satisfies

$$\dot{\bar{\ell}}_1 = [\bar{F}_{11} - \bar{F}_{12}\bar{F}_{22}^{-1}\bar{F}_{21}]\bar{\ell}_1 \tag{4.38}$$

and the initial condition

$$\bar{\ell}_1 = \bar{\ell}_1^o \text{ at } t = t_o, \tag{4.39}$$

and where $\bar{\ell}_2$ is algebraically related to $\bar{\ell}_1$ by

$$\bar{\ell}_2 = -\bar{F}_{22}^{-1}\bar{F}_{21}\bar{\ell}_1 \ . \tag{4.40}$$

Also, the left layer solution $\mathcal{L}_2$ satisfies

$$\frac{d\mathcal{L}_2}{d\tau} = [\bar{A}_{22}(t_o) - \bar{S}_{22}(t_o)\bar{P}_{22}(t_o)]\,\mathcal{L}_2 \tag{4.41}$$

and the initial condition

$$\mathcal{L}_2 = \bar{\ell}_2^{\,o} - \bar{\ell}_2(t_o) \quad \text{at } \tau = 0\ . \tag{4.42}$$

Similarly, Theorem A.5 permits one to write for (4.34)

$$r_1 = \bar{r}_1 + O(\mu), \qquad r_2 = \bar{r}_2 + \mathcal{R}_2(\sigma) + O(\mu) \tag{4.43}$$

where the reduced solution $\bar{r}_1$ satisfies

$$\dot{\bar{r}}_1 = [\bar{G}_{11} - \bar{G}_{12}\bar{G}_{22}^{-1}\bar{G}_{21}]\,\bar{r}_1 \tag{4.44}$$

and the end condition

$$\bar{r}_1 = \bar{r}_1^{\,T} \qquad \text{at } t = T, \tag{4.45}$$

and where $\bar{r}_2$ is algebraically related to $\bar{r}_1$ by

$$\bar{r}_2 = -\bar{G}_{22}^{-1}\bar{G}_{21}\bar{r}_1\ . \tag{4.46}$$

Also, the right layer solution $\mathcal{R}_2$ satisfies

$$\frac{d\mathcal{R}_2}{d\sigma} = [\bar{A}_{22}(T) - \bar{S}_{22}(T)\bar{N}_{22}(T)]\,\mathcal{R}_2 \tag{4.47}$$

and the end condition

$$\mathcal{R}_2 = \bar{r}_2^{\,T} - \bar{r}_2(T) \qquad \text{at } \sigma = 0 \,. \tag{4.48}$$

Thus far it has been assumed that the boundary conditions $\ell_1^{\,o}$, $\ell_2^{\,o}$, $r_1^{\,T}$, and $r_2^{\,T}$ have been given. What now must be shown is how to find these boundary conditions $\bar{\ell}_1^{\,o}$, $\bar{\ell}_2^{\,o}$, $\bar{r}_1^{\,T}$, and $\bar{r}_2^{\,T}$ in terms of the boundary conditions $x_1^{\,o}$, $x_2^{\,o}$, $x_1^{\,T}$, and $x_2^{\,T}$ specified in (4.3) to satisfy the theorem. Using the transformation in (4.27)

$$\begin{aligned} x_1(t,\mu) &= \bar{\ell}_1(t) + [\ell_1(t,\mu) - \bar{\ell}_1(t)] + \bar{r}_1(t) + [r_1(t,\mu) - \bar{r}_1(t)] \\ x_2(t,\mu) &= \bar{\ell}_2(t) + \mathcal{L}_2(\tau) + [\ell_2(t,\mu) - \bar{\ell}_2(t) - \mathcal{L}_2(\tau)] \\ &\quad + \bar{r}_2(t) + \mathcal{R}_2(\sigma) + [r_2(t,\mu) - \bar{r}_2(t) - \mathcal{R}_2(\sigma)] \end{aligned} \tag{4.49}$$

is satisfied as an identity for all $t \in [t_o, T]$ and for all $\mu \in [0, \mu^*]$. By selecting boundary conditions in accordance with (4.39), (4.42), (4.45), and (4.48) as required by Theorem A.5, (4.49) becomes

$$\begin{aligned} x_1(t,\mu) &= [\bar{\ell}_1(t) + \bar{r}_1(t)] + O(\mu) \\ x_2(t,\mu) &= [\bar{\ell}_2(t) + \bar{r}_2(t)] + \mathcal{L}_2(\tau) + \mathcal{R}_2(\sigma) + O(\mu) \end{aligned} \tag{4.50}$$

By H 4.1.4, there exists a unique reduced solution $\bar{x}_1$, $\bar{\lambda}_1$ of system (4.7) satisfying the $x_1^{\,o}$, $x_1^{\,T}$ boundary conditions and it is uniquely related to $\bar{x}_2$, $\bar{\lambda}_2$ in (4.8) by H 4.1.3. Recall the non-singular $W(t,\mu)$ transformation

in (4.28) is defined for $\mu = 0$. Thus solving the reduced $\bar{x}_i$, $\bar{\lambda}_i$ system (4.7), (4.8) is equivalent to solving the reduced $\bar{\ell}_i, \bar{r}_i$ systems (4.38), (4.44) providing the boundary condition requirements are met which are readily seen from (4.28) for $\mu = 0$ to be

$$\bar{\ell}_1(t_o) + \bar{r}_1(t_o) = x_1^{\,o}, \qquad \bar{\ell}_1(T) + \bar{r}_1(T) = x_1^{\,T} \tag{4.51}$$

Thus the existence of $\bar{\ell}_1^{\,o}$, and $\bar{r}_1^{\,T}$ is assured by letting $\bar{\ell}_1^{\,o} = \bar{\ell}_1(t_o)$ and $\bar{r}_1^{\,T} = \bar{r}_1(T)$ and $\bar{x}_1$ and $\bar{x}_2$ can be expressed by

$$\bar{x}_1(t) = \bar{\ell}_1(t) + \bar{r}_1(t), \qquad \bar{x}_2(t) = \bar{\ell}_2(t) + \bar{r}_2(t) \tag{4.52}$$

Since $\mathcal{R}_2(\sigma) \to 0$ as $\sigma \to -\infty$, this term is negligible when evaluating (4.50) at $t = t_o$ for small μ. Thus for $\mu = 0$, (4.50) gives

$$\mathcal{L}_2(0) = x_2^{\,o} - \bar{x}_2(t_o) = [x_2^{\,o} - \bar{r}_2(t_o)] - \bar{\ell}_2(t_o) \tag{4.53}$$

Hence $\bar{\ell}_2^{\,o}$ exists and is equal to $x_2^{\,o} - \bar{r}_2(t_o)$ upon comparison with (4.42). Similarly, since $\mathcal{L}_2(\tau) \to 0$ as $\tau \to \infty$, this term is negligible when evaluating (4.50) at $t = T$ for small μ. Thus for $\mu = 0$, (4.50) gives

$$\mathcal{R}_2(0) = x_2^{\,T} - \bar{x}_2(T) = [x_2^{\,T} - \bar{\ell}_2(T)] - \bar{r}_2(T) \tag{4.54}$$

Hence $\bar{r}_2^{\,T}$ exists since $\bar{\ell}_2(T)$ exists and is equal to $x_2^{\,T} - \bar{\ell}_2(T)$ upon comparison with (4.48). Thus it has been established that

$$x_1(t,\mu) = \bar{x}_1(t) + 0(\mu)$$
$$x_2(t,\mu) = \bar{x}_2(t) + \mathcal{L}_2(\tau) + \mathcal{R}_2(\sigma) + 0(\mu) \tag{4.55}$$

The same argument used to establish (4.52) holds for proving

$$\bar{\lambda}_1(t) = \bar{P}_{11}\bar{\ell}_1 + \bar{N}_{11}\bar{r}_1$$
$$\bar{\lambda}_2(t) = \bar{P}_{12}'\bar{\ell}_1 + \bar{N}_{12}'\bar{r}_1 + \bar{P}_{22}\bar{\ell}_2 + \bar{N}_{22}\bar{r}_2 \tag{4.56}$$

Using transformation (4.28)

$$\begin{aligned}
\lambda_1(t,\mu) &= [\bar{P}_{11}+0(\mu)][\bar{\ell}_1+0(\mu)] + [\bar{N}_{11}+0(\mu)][\bar{r}_1+0(\mu)] \\
&\quad + \mu[\bar{P}_{12}+0(\mu)][\bar{\ell}_2+\mathcal{L}_2(\tau) + 0(\mu)] \\
&\quad + \mu[\bar{N}_{12}+0(\mu)][\bar{r}_2+\mathcal{R}_2(\sigma) + 0(\mu)] \\
&= [\bar{P}_{11}\bar{\ell}_1 + \bar{N}_{11}\bar{r}_1] + 0(\mu) \\
&= \bar{\lambda}_1 + 0(\mu)
\end{aligned} \tag{4.57}$$

$$\begin{aligned}
\lambda_2(t,\mu) &= [\bar{P}_{12}'+0(\mu)][\bar{\ell}_1+0(\mu)] + [\bar{N}_{12}'+0(\mu)][\bar{r}_1+0(\mu)] \\
&\quad + [\bar{P}_{22}+0(\mu)][\bar{\ell}_2+\mathcal{L}_2(\tau) + 0(\mu)] \\
&\quad + [\bar{N}_{22}+0(\mu)][\bar{r}_2+\mathcal{R}_2(\sigma) + 0(\mu)] \\
&= [\bar{P}_{12}'\bar{\ell}_1+\bar{N}_{12}'\bar{r}_1+\bar{P}_{22}\bar{\ell}_2+\bar{N}_{22}\bar{r}_2] + \bar{P}_{22}\mathcal{L}_2(\tau)+\bar{N}_{22}\mathcal{R}_2(\sigma) + 0(\mu) \\
&= \bar{\lambda}_2(t) + \bar{P}_{22}(t)\mathcal{L}_2(\tau) + \bar{N}_{22}(t)\mathcal{R}_2(\sigma) + 0(\mu) \\
&= \bar{\lambda}_2(t) + \bar{P}_{22}(t_o)\mathcal{L}_2(\tau) + \bar{N}_{22}(T)\mathcal{R}_2(\sigma) + 0(\mu)
\end{aligned} \tag{4.58}$$

The last step is justified by the continuity hypothesis H 4.1.2.

4.6 Control and Performance Index Approximation

Two corollaries will be proven in this section. The first establishes the existence of an approximate control solution for the optimal control problem (4.1) - (4.3), and the second establishes the existence of an approximate performance index for this problem.

Corollary 4.6.1 Let the hypotheses of Theorem 4.2.1 be satisfied. Then there exists a positive constant $\mu^* > 0$ such that for $\mu \le \mu^*$ and $t \in [t_o, T]$,

$$u(t,\mu) = \bar{u}(t) + u_L(\tau) + u_R(\sigma) + O(\mu) \tag{4.59}$$

where

$$\begin{aligned} \bar{u}(t) &= -\bar{R}^{-1}(t)[\bar{B}_1'(t)\bar{\lambda}_1(t) + \bar{B}_2'(t)\bar{\lambda}_2(t)] \\ u_L(\tau) &= -\bar{R}^{-1}(t_o)\bar{B}_2'(t_o)\bar{P}_{22}(t_o)\mathcal{L}_2(\tau) \\ u_R(\sigma) &= -\bar{R}^{-1}(T)\bar{B}_2'(T)\bar{N}_{22}(T)\mathcal{R}_2(\sigma) \end{aligned} \tag{4.60}$$

Proof: Recall from (4.4) that the control u is written as

$$u = -R^{-1}(B_1'\lambda_1 + B_2'\lambda_2) \tag{4.61}$$

Using the expansion for λ_1 and λ_2 given in Theorem 4.2.1

$$\begin{aligned} u = -R^{-1}(&B_1'[\bar{\lambda}_1 + O(\mu)] \\ &+ B_2'[\bar{\lambda}_2 + \bar{P}_{22}(t_o)\mathcal{L}_2(\tau) + \bar{N}_{22}(T)\mathcal{R}_2(\sigma) + O(\mu)]) \end{aligned} \tag{4.62}$$

But from H 4.1.2, (4.62) can be written as

$$u = -\bar{R}^{-1}(\bar{B}_1'\bar{\lambda}_1+\bar{B}_2'\bar{\lambda}_2) - \bar{R}^{-1}\bar{B}'_2\bar{P}_{22}(t_o)\mathcal{L}_2(\tau)$$
$$-\bar{R}^{-1}\bar{B}_2'\bar{N}_{22}(T)\mathcal{R}_2(\sigma) + 0(\mu) \tag{4.63}$$

or

$$u = \bar{u} - \bar{R}^{-1}(t_o)\bar{B}_2'(t_o)\bar{P}_{22}(t_o)\mathcal{L}_2(\tau)$$
$$- \bar{R}^{-1}(T)\bar{B}_2'(T)\bar{N}_{22}(T)\mathcal{R}_2(\sigma) + 0(\mu) \tag{4.64}$$

Note from the corollary that the approximate control solution is composed of three control vectors: $\bar{u}(t)$, $\mathcal{U}_L(\tau)$, and $\mathcal{U}_R(\sigma)$. The $\mathcal{U}_L(\tau)$ control is the stabilizing control for the boundary layer system

$$\frac{d\mathcal{L}_2}{d\tau} = \bar{A}_{22}(t_o)\mathcal{L}_2 + \bar{B}_2(t_o)u \tag{4.65}$$

and is responsible for steering the system

$$\begin{bmatrix} \dot{x}_1 \\ \mu\dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \tag{4.66}$$

from its initial state rapidly to its reduced state while optimizing the performance index J. The reduced $\bar{u}(t)$ control retains the system solution near its reduced solution while minimizing J until close to t = T. The $\mathcal{U}_R(\sigma)$ control results in the boundary layer system

$$\frac{d\mathcal{R}_2}{d\tau} = \bar{A}_{22}(T)\mathcal{R}_2 + \bar{B}_2(T)u \tag{4.67}$$

being completely instable (no state stable) and is responsible for steering (4.66) from its reduced state near $t = T$ rapidly to its final state while minimizing J.

<u>Corollary 4.6.2</u> Let the hypotheses of Theorem 4.2.1 be satisfied. Then there exists a positive constant $\mu^* > 0$ such that for $\mu \le \mu^*$ and $t \in [t_o, T]$,

$$J(\mu) = \bar{J} + O(\mu) \tag{4.68}$$

where

$$\bar{J} = \frac{1}{2}\int_{t_o}^{T}\left(\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix}' \begin{bmatrix}\bar{Q}_{11} & \bar{Q}_{12}\\ \bar{Q}_{12}' & \bar{Q}_{22}\end{bmatrix}\begin{bmatrix}\bar{x}_1\\ \bar{x}_2\end{bmatrix} + \bar{u}'\bar{R}\bar{u}\right)dt$$

<u>Proof</u>: From (4.2),

$$J = \frac{1}{2}\int_{t_o}^{T}(x_1'Q_{11}x_1 + 2x_1'Q_{12}x_2 + x_2'Q_{22}x_2 + u'Ru)dt \tag{4.69}$$

Using the asymptotic expressions for x_1, x_2, and u and H 4.1.2,

$$\begin{aligned} J = \bar{J} &+ \int_{t_o}^{T}[a_1'(t)\mathcal{L}_2(\tau) + a_2'(t)\mathcal{R}_2(\sigma)]dt \\ &+ \int_{t_o}^{T}[\mathcal{L}_2'(\tau)A_1(t)\mathcal{L}_2(\tau) + \mathcal{R}_2'(\sigma)A_2(t)\mathcal{R}_2(\sigma)]dt \\ &+ \int_{t_o}^{T}\mathcal{L}_2'(\tau)A_3(t)\mathcal{R}_2(\sigma)\,dt + O(\mu) \end{aligned} \tag{4.70}$$

whose a_i vectors ($i = 1,2$) and A_i matrix coefficients ($i = 1,3$) are continuous. Recognizing that the norms of $\mathcal{L}_2(\tau)$ and $\mathcal{R}_2(\sigma)$ are bounded by

$$|\mathcal{L}_2(\tau)| \le Ke^{-\alpha(t-t_0)/\mu}, \qquad |\mathcal{R}_2(\sigma)| \le Ke^{\alpha(t-T)/\mu} \tag{4.71}$$

for some positive constants K and α, it is readily seen that the norm of the integral terms of (4.70) are bounded by $O(\mu)$. Thus (4.70) can be written as (4.68).

4.7 Discussion and Interpretation

Theorem 4.2.1 gives an approximate solution for (4.3), (4.5) and proves it is asymptotically valid uniformly for $t \in [t_0,T]$. The essence of the proof was to transform the singularly perturbed TPBV problem by a non-singular transformation into two singularly perturbed initial value problems which would satisfy the hypotheses of theorems for such systems. To comply with a continuity hypothesis of Theorem A.5, the boundary conditions for two singularly perturbed initial-value Riccati systems, whose Riccati gains composed the transformation, were appropriately selected. The non-singularity of the transformation was a consequence of its determinant being asymptotically represented by the product of terms involving the determinant of the difference of symmetrical positive definite Riccati matrices and symmetrical negative definite Riccati matrices. Upon approximating the solutions of the transformed singularly perturbed systems involving ℓ and r variables respectively by their zero-order asymptotic expansions†, and by appropriately selecting the

† See Appendix for definition.

ℓ and r boundary conditions, the summation of these terms yielded the x_1 and x_2 asymptotic approximations given in the theorem. Here use was made of the relation between $\bar{x}_1$, $\bar{x}_2$ and the reduced $\bar{\ell}$ and $\bar{r}$ terms expressed by the transformation.

The left boundary variable $\mathcal{L}_2$, satisfying (4.13) and the associated boundary condition, is the τ term of the zero-order asymptotic approximation of ℓ_2. It can be seen as the solution of the following optimal regulator problem

$$\min_u \int_0^\infty [\hat{y}'\hat{y} + \hat{u}'\bar{R}(t_o)\hat{u}]d\tau \tag{4.72}$$

subject to the constraint

$$\begin{aligned} \frac{d\mathcal{L}_2}{d\tau} &= \bar{A}_{22}(t_o)\mathcal{L}_2 + \bar{B}_2(t_o)\hat{u} \ , \\ \mathcal{L}_2 &= x_2^{\,o} - \bar{x}_2(t_o) \quad \text{at } \tau = 0 \end{aligned} \tag{4.73}$$

and observed through

$$\hat{y} = \bar{C}_2(t_o)\mathcal{L}_2 \ . \tag{4.74}$$

The solution exists and the coefficient matrix $[\bar{A}_{22}(t_o)-\bar{S}_{22}(t_o)\bar{P}_{22}(t_o)]$ is a stable matrix; thus $\mathcal{L}_2 \to 0$ as $\tau \to \infty$. Similarly $\mathcal{R}_2(\sigma)$ is seen to satisfy the optimal regulator problem

$$\min_u \int_{-\infty}^0 [\tilde{y}'\tilde{y}+\tilde{u}'\bar{R}(T)\tilde{u}]d\sigma \tag{4.75}$$

subject to constraint

$$\frac{d\mathcal{R}_2}{d\sigma} = \bar{A}_{22}(T)\mathcal{R}_2 + \bar{B}_2(T)\tilde{u},$$
$$\mathcal{R}_2 = x_2^T - \bar{x}(T) \quad \text{at } \sigma = 0 \tag{4.76}$$

and observed through

$$\tilde{y} = \bar{C}_2(T)\mathcal{R}_2 \quad . \tag{4.77}$$

Similar to $\mathcal{L}_2$, $-[\bar{A}_{22}(T)-\bar{S}_{22}(T)\bar{N}_{22}(T)]$ is a stable matrix, and $\mathcal{R}_2(\sigma) \to 0$ and $\sigma \to -\infty$. Thus the solution of x_2 for example is approximated by the sum of: the reduced solution $\bar{x}_2$, the solution of a left layer regulator problem $\mathcal{L}_2(\tau)$, and the solution of a right layer regulator problem $\mathcal{R}_2(\sigma)$. Corollary 4.6.1 shows that the optimal control u is also the sum of three different controls--each of which is computed independently of the others. The theorem permits an engineer to approximate the solution on any open interval by using only the reduced solution and can add either a left, right, or both left and right correction terms depending on his application

Since $\mathcal{L}_2(\tau) \to 0$ as $\tau \to \infty$ and $\mathcal{R}_2(\sigma) \to 0$ as $\sigma \to -\infty$, it can readily be seen when expressing τ and σ as functions of time that a fast transient occurs at $t = t_o$ from $\mathcal{L}_2$ decaying with increasing time and that a fast transient occurs at $t = T$ from $\mathcal{R}_2$ decaying with decreasing time. The time constants for the decay are directly proportional to μ. Therefore the presence of $\mathcal{L}_2$, $\mathcal{R}_2$ in the state and control variables demonstrates the two-time scale separation properties.

It is further noted that even though the approximate performance index may be close to the optimal one, this in no way infers that the approximating solutions for the state and control vectors are close to the actual in the boundary layer. This is true since the integral value from a boundary layer term is negligible for μ small enough.

The existence of a solution to (4.5) satisfying boundary conditions (4.3) could have been shown using fundamental matrices in a manner analogous to (3.44). This method would show that ℓ_1^o, ℓ_2^o, r_1^T, and r_2^T are uniquely determined from x_1^o, x_2^o, x_1^T, and x_2^T. A useful reference for proving the existence of the non-singularly perturbed linear TPBV problems of the type considered is found in [3].

5. FIXED END-POINT PROBLEM: EXAMPLES

5.1 Design Example

The example problem worked here illustrates the important points stressed in Chapter 4: two-time scale property, boundary layers at both ends of the time interval, closeness of the approximate solution to the actual optimal, stiffness, and how the interval in which the reduced solution may be a good approximation can be extended by decreasing μ. Graphs are given which clearly show these. The graphs not only compare the approximating and actual solutions which shows the closeness of the approximation, but also compare the actual and reduced solutions which shows the need to include boundary layer terms. The selection of a time-invariant problem resulted in being able to actually show how the system eigenvalues approach those of the reduced and boundary layer systems as μ approaches zero.

The optimal control problem is to minimize with respect to the control u the performance index

$$J = \int_0^T (2x_1^2 + x_2^2 + u^2)dt \tag{5.1}$$

for the singularly perturbed system

$$\begin{aligned} \dot{x}_1 &= \frac{3}{2} x_2 \\ \mu\dot{x}_2 &= -\frac{3}{2} x_1 + \frac{1}{2} x_2 - u \end{aligned} \tag{5.2}$$

whose boundary constraints are given by

$$x_1 = x_1^{o} \text{ at } t = 0 \text{ and } x_1 = x_1^{T} \text{ at } t = T \tag{5.3a}$$

$$x_2 = x_2^{o} \text{ at } t = 0 \text{ and } x_2 = x_2^{T} \text{ at } t = T \tag{5.3b}$$

From (4.6), it is seen that the necessary conditions for an extremal are given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{\lambda}_1 \\ \mu\dot{x}_2 \\ \mu\dot{\lambda}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{3}{2} & 0 \\ -4 & 0 & 0 & \frac{3}{2} \\ -\frac{3}{2} & 0 & \frac{1}{2} & -1 \\ 0 & -\frac{3}{2} & -2 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ \lambda_1 \\ x_2 \\ \lambda_2 \end{bmatrix} \tag{5.4}$$

subject to (5.3). From (4.7) and (4.8), the reduced $\bar{x}_1$, $\bar{\lambda}_1$ solution is to satisfy system

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{\lambda}}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -4 & 0 \end{bmatrix} - \begin{bmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -1 \\ -2 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{3}{2} & 0 \\ 0 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{\lambda}_1 \end{bmatrix} \tag{5.5}$$

and boundary condition (5.3a). Variables $\bar{x}_2$, $\bar{\lambda}_2$ are related to $\bar{x}_1$, $\bar{\lambda}_1$ by

$$\begin{bmatrix} \bar{x}_2 \\ \bar{\lambda}_2 \end{bmatrix} = - \begin{bmatrix} \frac{1}{2} & -1 \\ -2 & -\frac{1}{2} \end{bmatrix}^{-1} \begin{bmatrix} -\frac{3}{2} & 0 \\ 0 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{\lambda}_1 \end{bmatrix} \tag{5.6}$$

where the inverse can easily be seen to exist.

It will first be shown that the hypotheses of Theorem 4.2.1 are satisfied. Upon comparison of (4.6) and (5.4) it is seen that

$$Q = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}, \; R = 1 \; , \; A_{22} = 1/2, \; B_2 = -1, \text{ and } C_2 = \sqrt{2}$$

Thus H 4.1.1 is satisfied since both Q and R are positive definite. The continuity assumptions of H 4.1.2 are certainly satisfied since this is a time-invariant problem. The rank of $[\bar{B}_2 \,\vdots\, \bar{A}_{22}\bar{B}_{22}]$ and $[\bar{C}_2' \,\vdots\, \bar{A}_{22}'\bar{C}_2']$ is one guaranteeing that H 4.1.3 is met. To satisfy H 4.1.4 it must be shown there exists a unique reduced solution. Rewriting (5.5) and (5.6),

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{\lambda}}_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -1 \\ -6 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{\lambda}_1 \end{bmatrix} \tag{5.7}$$

$$\begin{bmatrix} \bar{x}_2 \\ \bar{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & -\frac{2}{3} \\ -\frac{4}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{\lambda}_1 \end{bmatrix} \tag{5.8}$$

Explicitly solving (5.7), (5.8), and (5.3a), one obtains

$$\begin{aligned} \bar{x}_1 &= \alpha_1(\alpha_2 e^{-\frac{5}{2}t} + \alpha_3 e^{\frac{5}{2}t}) \\ \bar{\lambda}_1 &= \alpha_1(3\alpha_2 e^{-\frac{5}{2}t} - 2\,\alpha_3 e^{\frac{5}{2}t}) \\ \bar{x}_2 &= \frac{5}{3}\,\alpha_1(-\alpha_2 e^{-\frac{5}{2}t} + \alpha_3 e^{\frac{5}{2}t}) \\ \bar{\lambda}_2 &= -\frac{1}{3}\,\alpha_1(7\,\alpha_2 e^{-\frac{5}{2}t} + 2\,\alpha_3 e^{\frac{5}{2}t}) \end{aligned} \tag{5.9}$$

where

$$\alpha_1 = 1/(e^{\frac{5}{2}T} - e^{-\frac{5}{2}T}), \qquad \alpha_2 = e^{\frac{5}{2}T} x_1^{o} - x_1^{T}$$
$$\alpha_3 = - e^{-\frac{5}{2}T} x_1^{o} + x_1^{T} \tag{5.10}$$

Thus all the hypotheses are satisfied. The boundary layer correction terms will now be found. The Riccati equation (4.16) to which the unique algebraic symmetrical positive and negative definite roots are to be found is

$$\bar{K}_{22} - \bar{K}_{22}^{2} + 2 = 0 \tag{5.11}$$

The positive root $\bar{P}_{22}(0)$ and negative root $\bar{N}_{22}(T)$ are

$$\bar{P}_{22}(0) = 2 \quad \text{and} \quad \bar{N}_{22}(T) = - 1 \tag{5.12}$$

Using (4.14) and (4.15), the boundary layer correction terms $\mathcal{L}_2$ and $\mathcal{R}_2$ are given by

$$\frac{d\mathcal{L}_2}{d\tau} = - \frac{3}{2} \mathcal{L}_2, \; \mathcal{L}_2 = x_2^{o} - \bar{x}_2(0) \quad \text{at } \tau = 0 \tag{5.13}$$

$$\frac{d\mathcal{R}_2}{d\sigma} = \frac{3}{2} \mathcal{R}_2, \; \mathcal{R}_2 = x_2^{T} - \bar{x}_2(T) \quad \text{at } \sigma = 0 \tag{5.14}$$

Thus

$$\mathcal{L}_2(\tau) = [x_2^{\,o} - \bar{x}_2(0)]\, e^{-\frac{3}{2}\tau} \quad , \qquad \tau = t/\mu \tag{5.15a}$$

$$\mathcal{R}_2(\sigma) = [x_2^{\,T} - \bar{x}_2(T)]\, e^{\frac{3}{2}\sigma} \quad , \qquad \sigma = (t-T)/\mu \tag{5.15b}$$

where from (5.9),

$$\begin{aligned} \bar{x}_2(0) &= \frac{5}{3}\alpha_1(-\alpha_2 + \alpha_3) \\ \bar{x}_2(T) &= \frac{5}{3}\alpha_1(-\alpha_2 e^{-\frac{5}{2}T} + \alpha_3\, e^{\frac{5}{2}T}) \end{aligned} \tag{5.16}$$

Using (5.9), (5.12), and (5.15)

$$\begin{aligned} x_1(t,\mu) &= \bar{x}_1 + 0(\mu) \\ x_2(t,\mu) &= \bar{x}_2 + \mathcal{L}_2(\tau) + \mathcal{R}_2(\sigma) + 0(\mu) \\ \lambda_1(t,\mu) &= \bar{\lambda}_1 + 0(\mu) \\ \lambda_2(t,\mu) &= \bar{\lambda}_2 + \bar{P}_{22}(0)\mathcal{L}_2(\tau) + \bar{N}_{22}(T)\mathcal{R}_2(\sigma) + 0(\mu) \end{aligned} \tag{5.17}$$

Also, from (4.54), $\bar{u} = \bar{\lambda}_2$ for the problem and

$$u(t,\mu) = \bar{u} + 2\mathcal{L}_2(\tau) - \mathcal{R}_2(\sigma) + 0(\mu) \tag{5.18}$$

A comparison of the actual solution with the reduced and zero-order approximation will be done for the specific case when

$$x_1^{\,o} = 4.0, \quad x_2^{\,o} = 3.0, \quad x_1^{\,T} = 0.5, \quad \text{and } x_2^{\,T} = -1.3 \tag{5.19}$$

Figures 5.1 and 5.2 show this comparison for $\mu = 0.1$. Although the approximating solutions for x_1 and x_2 are reasonably close to their actual solutions, the difference is expectable since the value of μ is only a fifth of the value of the smallest coefficient in the system description. The transients in the boundary layers shown in Figure 5.2 are evident but are not exceptionally steep. Figures 5.3 and 5.4 show the same comparison for $\mu = 0.01$. Here the difference is almost negligible on the scales shown and the transients shown in Figure 5.4 are much steeper than in Figure 5.2.

5.2 Eigenvalue Discussion

In the simple problem it is possible to explicitly show the dependence of the eigenvalues on μ. It can be easily shown that the eigenvalues γ of (5.4) are determined from the expression

$$16\mu^2\gamma^4 + 36(2\mu - 1)\gamma^2 + 9(25) = 0 \tag{5.20}$$

and given by

$$\gamma = \pm\frac{3}{2\mu}\sqrt{\frac{1}{2}\left(1 - 2\mu \pm \sqrt{1 - 4\mu - \frac{64}{9}\mu^2}\right)} \tag{5.21}$$

An analysis of these eigenvalues will now be done as is common in the design of stable feedback systems. In the latter case, one is often concerned with how large a feedback constant must be to stabilize a system. Root locus techniques can be employed to accomplish this. Meerov [37] applied this technique to analyze the stability of a multiloop feedback structure, each loop containing a feedback gain expressible as a constant

coefficient times the variable gain coefficient K. By letting $\mu = 1/K$, he developed theorems guaranteeing the existence of a positive constant μ^* such that for $\mu \leq \mu^*$ all the eigenvalues of the system considered were stable. The hypotheses of his theorems were based on showing the stability of simpler auxiliary systems--one such hypothesis was to show the stability of a reduced equation formed by setting $\mu = 0$ in the characteristic equation. Thus his theory is similar to that employed in singular perturbation theory. By writing (5.20) in the form

$$\frac{8\gamma^2(2\mu\gamma^2+9)}{9(-4\gamma^2+25)} = -\frac{1}{\mu} \tag{5.22}$$

it is easy to see that the eigenvalues for $\mu = 0$ correspond to those of the poles of (5.22). This is termed the reduced solution and is most easily obtained by setting $\mu = 0$ in (5.20). The other values can be found by consideration of

$$\frac{9(8P^2+25\mu)}{4(4P^2-9)P^2} = -\frac{1}{\mu} \tag{5.23}$$

where $P = \gamma/\mu$ in (5.20). These eigenvalues, those not observed by looking at the poles of (5.22), for $\mu = 0$ correspond to the poles of (5.23). The poles of (5.22) are given by $\gamma = \pm\frac{5}{2}$ and are identical to the eigenvalues of the reduced system (5.7). The poles of (5.23) are given by $P = \pm\frac{3}{2}$ and are identical to the roots of the auxiliary system (5.13) and (5.14). In (5.24), these roots $\pm\gamma_1$, $\pm\gamma_2$ are shown for different values of μ. γ_1 is scaled by μ.

μ	$\mu \cdot \gamma_1$	γ_2
.188	.83	4.45
.1	1.31	2.86
.01	1.48	2.52
↓	↓	↓
0.0	1.50	2.50

(5.24)

From (5.24) it is noted that the eigenvalues for $\mu = 0$ are very close to those for $\mu = .01$ but not so close for $\mu = .1$ and certainly not for $\mu = .188$ where double roots occur. This is in agreement with the graphs shown in Figures 5.1 - 5.4.

5.3 Numerical Aspect

Should the problem be very stiff (μ much less than .01), an explicit solution to this problem becomes very difficult and a special technique would have to be determined for solving this problem. A straight forward way to solve this time-invariant problem is to first find the eigenvalues and then determine the c_i coefficients of the exponential terms describing the x_1, x_2, λ_1, and λ_2 variables to match the boundary conditions. The trouble with this method is in matching the boundary conditions. For the example problem, let

$$x_1 = c_1 e^{\gamma_1 t} + c_2 e^{\gamma_2 t} + c_3 e^{-\gamma_1 t} + c_4 e^{-\gamma_2 t}.$$

Then one must find the solution of

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{2}{3}\gamma_1 & \frac{2}{3}\gamma_2 & -\frac{2}{3}\gamma_1 & -\frac{2}{3}\gamma_2 \\ e^{\gamma_1 T} & e^{\gamma_2 T} & e^{-\gamma_1 T} & e^{-\gamma_2 T} \\ \frac{2}{3}\gamma_1 e^{\gamma_1 T} & \frac{2}{3}\gamma_2 e^{\gamma_2 T} & -\frac{2}{3}\gamma_1 e^{-\gamma_1 T} & -\frac{2}{3}\gamma_2 e^{-\gamma_2 T} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} x_1^{\,o} \\ x_2^{\,o} \\ x_1^{\,T} \\ x_2^{\,T} \end{bmatrix} \tag{5.25}$$

for $\mu = 0.01$, (5.25) becomes

$$\begin{bmatrix} .100E+01 & .100E+01 & .100E+01 & .100E+01 \\ .990E+02 & .168E+01 & -.990E+02 & -.168E+01 \\ .302E+65 & .125E+02 & .321E-64 & .800E-01 \\ .299E+67 & .210E+02 & -.328E-62 & -.135E+00 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 4.0 \\ 3.0 \\ 0.5 \\ -1.3 \end{bmatrix} \tag{5.26}$$

whose solution vector c' is given by

$$-.354E-66 \quad .147E-01 \quad -.995E-01 \quad .408E+01 \tag{5.27}$$

It can easily be seen that overflow or underflow will occur for μ much less than 0.01 from (5.26), (5.27) respectively. Thus for a very stiff system, it may not be practical to solve for the actual solution. Yet the approximation described in this thesis is simple to find and will be very close to the actual solution for μ sufficiently small.

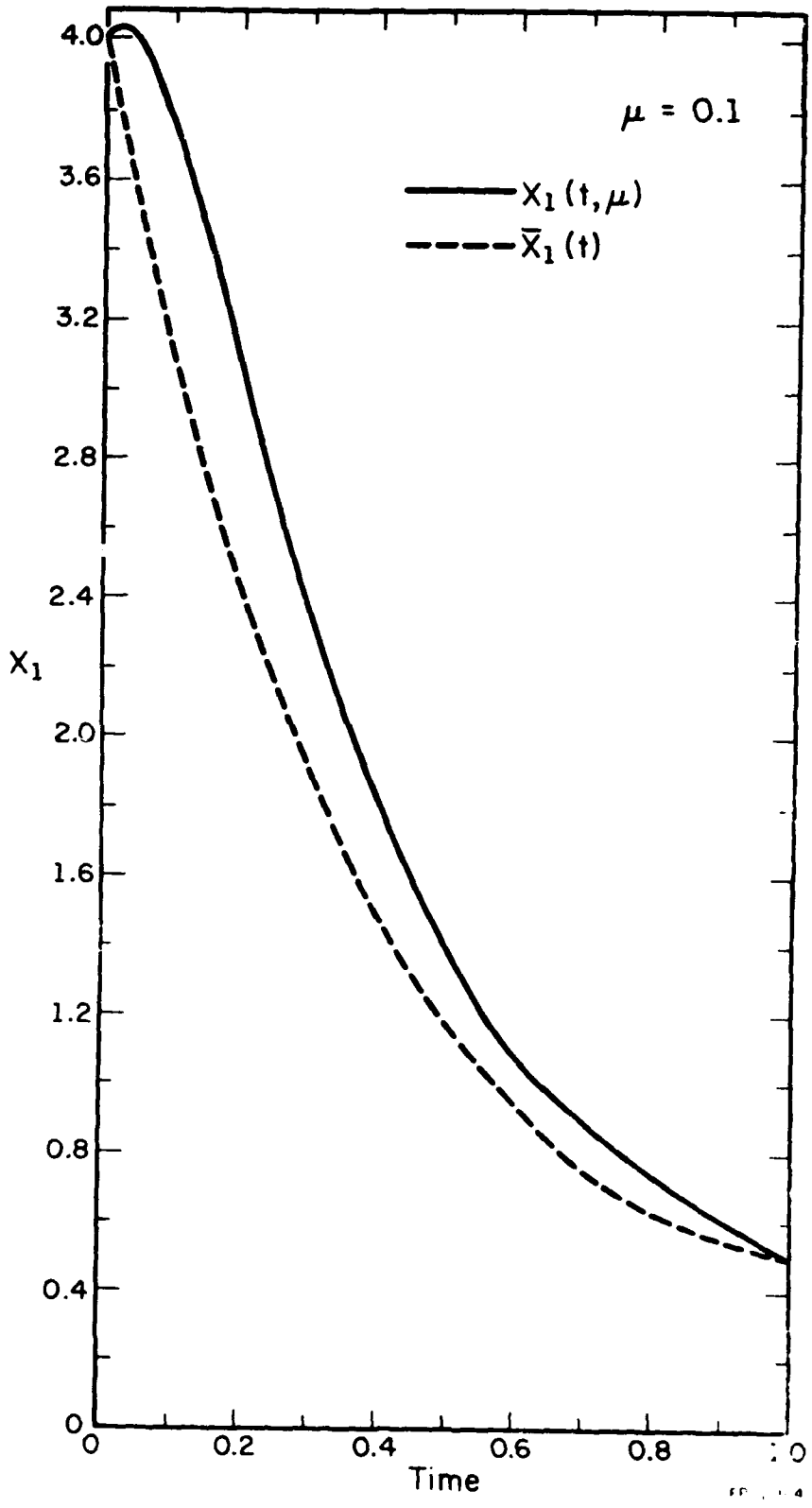

Fig. 5.1. Fixed End Point Problem

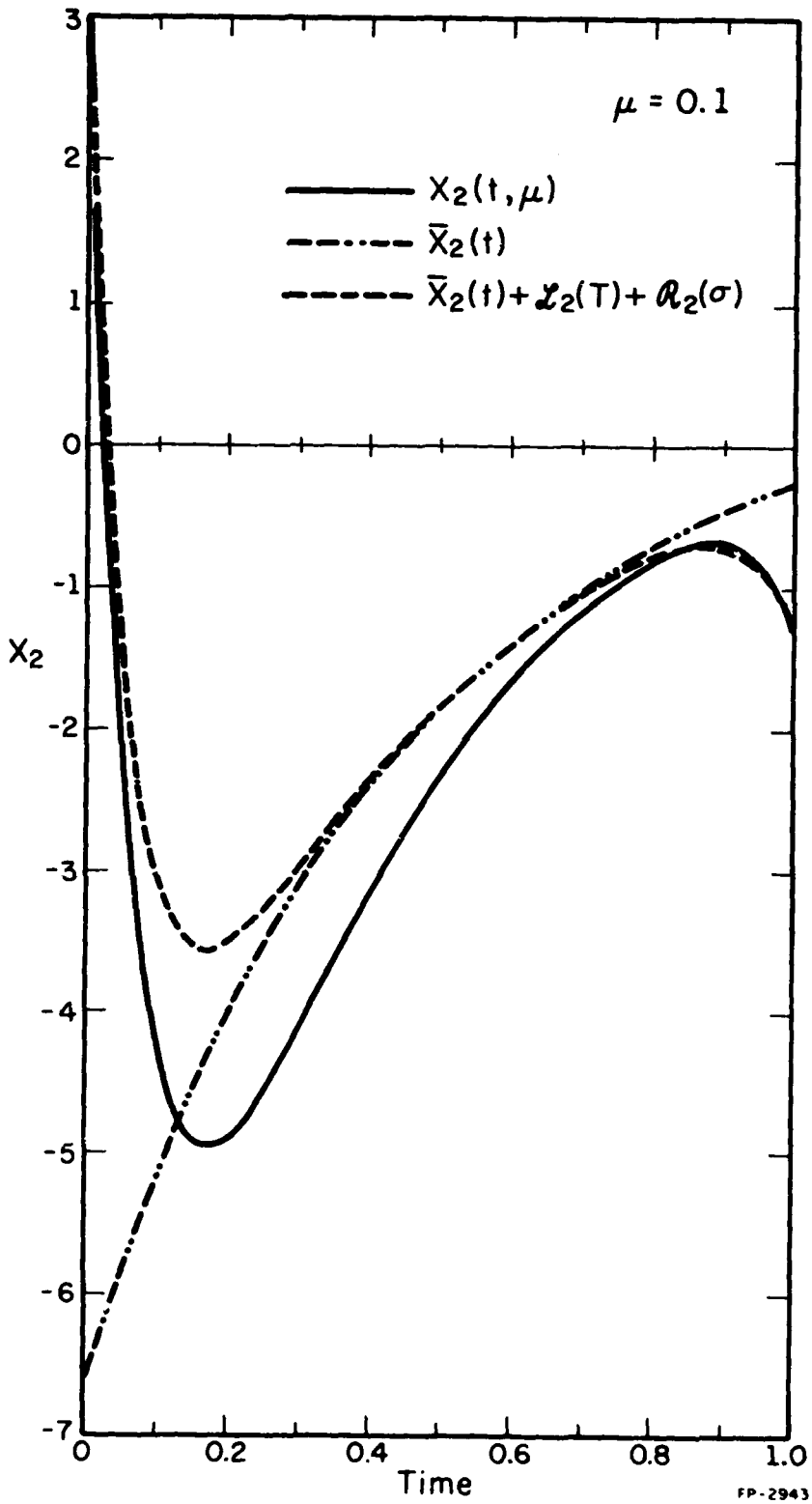

Fig. 5.2. Fixed End Point Problem.

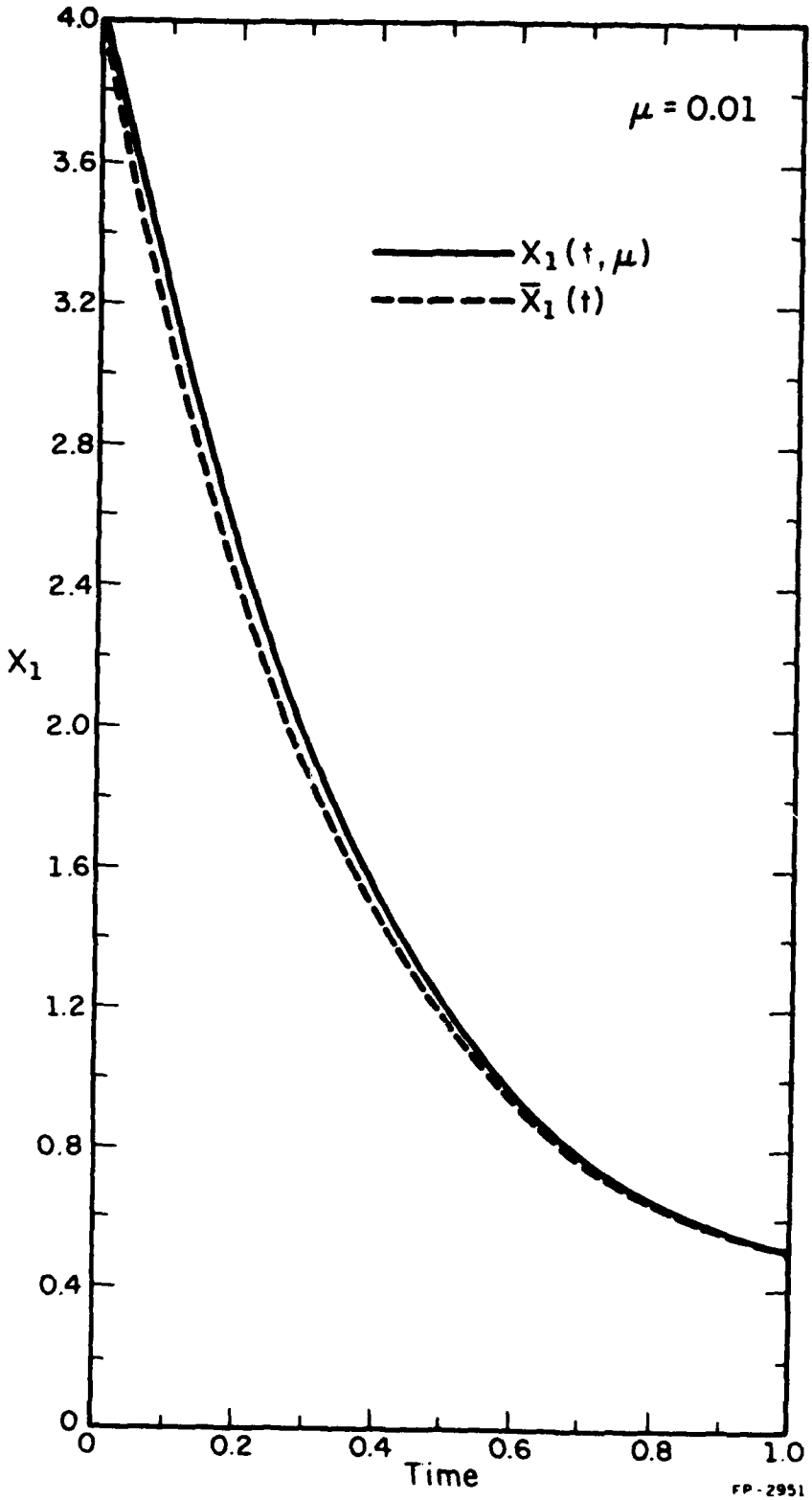

Fig. 5.3. Fixed End Point Problem.

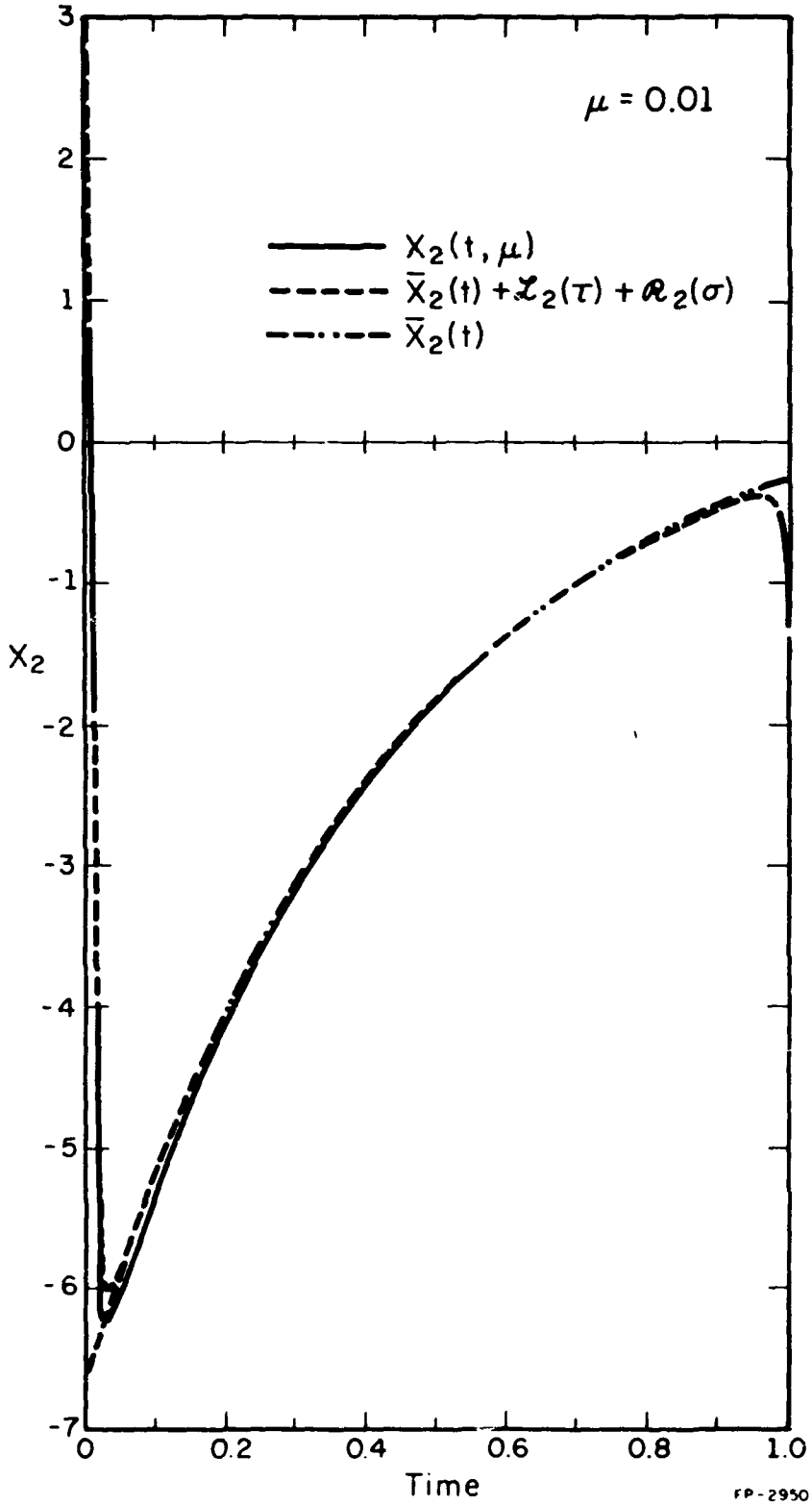

Fig. 5.4. Fixed End Point Problem.

6. TERMINAL COST PROBLEMS AND EXAMPLES

6.1 Introduction and Statement of Problem

This chapter analyzes the optimal open and closed loop control of the same singularly perturbed linear system (4.1) as in Chapters 4 and 5 but with free end point and with a terminal cost in the performance index

$$J = \frac{1}{2} \; x'\pi x\Big|_T + \frac{1}{2} \int_{t_o}^{T} (x'Qx + u'Ru)dt \tag{6.1}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \pi = \begin{bmatrix} \pi_{11}(\mu) & \mu\pi_{12}(\mu) \\ \mu\pi_{12}'(\mu) & \mu\pi_{22}(\mu) \end{bmatrix} \tag{6.2}$$

where π is symmetrical positive semi-definite and its π_{ij} matrix elements are three times continuously differentiable functions of μ. The two-time scale design procedure presented here for obtaining an expression asymptotic to the exact solution is similar to that presented in Chapter 4. After the asymptotic correctness for the expression has been shown, it will be shown that a similar expression results if the singularly perturbed Riccati gains are substituted by their zero-order terms in finding the optimal solution of the system The closeness of the approximate open and closed loop solutions to the optimal one will be shown graphically for an example problem. The results of this chapter are immediately applicable to the linear tracking problem.

The boundary conditions for the necessary optimality equations (4 5) are given by

$$x_1 = x_1^o \quad \text{and} \quad x_2 = x_2^o \qquad \text{at } t = t_o$$

$$\begin{bmatrix} \lambda_1 \\ \mu\lambda_2 \end{bmatrix} = \begin{bmatrix} \pi_{11} & \mu\pi_{12} \\ \mu\pi_{12}' & \mu\pi_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{at } t = T \tag{6.3}$$

and those for the reduced system (4.7) are given by

$$\bar{x}_1 = x_1^o \qquad \text{at } t = t_o$$

$$\bar{\lambda}_1 = \bar{\pi}_{11}\bar{x}_1 \qquad \text{at } t = T \ . \tag{6.4}$$

As pointed out in the proof of Lemma 4.3.1, under the hypotheses of the lemma, $\bar{\lambda}_1$ is related to $\bar{x}_1$ for all $t \in [t_o,T]$ by

$$\bar{\lambda}_1 = \bar{P}_{11}\bar{x}_1 \tag{6.5}$$

where $\bar{P}_{11}$ is the unique symmetrical Riccati gain satisfying its associated Riccati equation and boundary condition determined from 6.4. Thus the reduced solution $\bar{x}_1$, $\bar{\lambda}_1$ of this problem is known to exist and H 4.1.4 is not needed.

6 2 Open Loop Solution

Theorem 6.2.1 Let H 4.1.1, H 4.1.2, and H 4.1.3 be satisfied for problem (4.1), (6.1) and (6.2). Then there exists a positive constant μ^* such that for all $\mu \leq \mu^*$ and for all $t \in [t_o,T]$

$$
\begin{aligned}
x_1(t,\mu) &= \bar{x}_1(t) + O(\mu) \\
x_2(t,\mu) &= \bar{x}_2(t) + \mathcal{L}_2(\tau) + \mathcal{R}_2(\sigma) + O(\mu) \\
\lambda_1(t,\mu) &= \bar{\lambda}_1(t) + O(\mu) \\
\lambda_2(t,\mu) &= \bar{\lambda}_2(t) + \bar{P}_{22}(t_o)\mathcal{L}_2(\tau) + \bar{N}_{22}(T)\mathcal{R}_2(\sigma) + O(\mu) \quad .
\end{aligned}
\tag{6.6}
$$

Here $\mathcal{L}_2(\tau)$ is the solution of the "left" layer system

$$\frac{d\mathcal{L}_2}{d\tau} = [\bar{A}_{22}(t_o) - \bar{S}_{22}(t_o)\bar{P}_{22}(t_o)]\mathcal{L}_2 \tag{6.7}$$

subject to initial condition

$$\mathcal{L}_2 = x_2^{\,o} - \bar{x}_2(t_o) \quad \text{at } \tau = 0 \ , \tag{6.8}$$

$\mathcal{R}_2(\sigma)$ is the solution of the "right" layer system

$$\frac{d\mathcal{R}_2}{d\sigma} = [\bar{A}_{22}(T) - \bar{S}_{22}(T)\bar{N}_{22}(T)]\mathcal{R}_2 \tag{6.9}$$

subject to terminal condition

$$\mathcal{R}_2 = -[\bar{N}_{22}(T)-\bar{\pi}_{22}]^{-1}[\bar{\lambda}_2(T)-\bar{\pi}_{12}'(T)\bar{x}_1(T)-\bar{\pi}_{22}(T)\bar{x}_2(T)] \quad \text{at } \sigma = 0 \tag{6.10}$$

and $\bar{P}_{22}(t_o)$ and $\bar{N}_{22}(T)$ are the symmetrical positive and negative algebraic solutions of

$$\bar{A}_{22}'K_{22} + K_{22}\bar{A}_{22} - K_{22}\bar{S}_{22}K_{22} + \bar{Q}_{22} = 0 \tag{6.11}$$

evaluated at t_o and T respectively.

Remark: Only the differences in the proof compared with that of Theorem 4.2.1 will be emphasized. The essential difference lies in the determination of the boundary condition.

Proof: The proof consists in showing that the boundary conditions $\bar{\ell}_1^{\,o}$, $\bar{r}_1^{\,T}$, $\bar{\ell}_2^{\,o}$, $\bar{r}_2^{\,T}$ for (4.42) and (4.48) can be found in terms of the boundary conditions (6.3) to satisfy the theorem. From transformation (4.27) and boundary conditions (6.3), it is readily seen that the reduced $\bar{\ell}$, $\bar{r}$ solutions satisfy

$$\bar{\ell}_1(t_o) + \bar{r}_1(t_o) = x_1^{\,o} \tag{6.12a}$$

$$\bar{r}_1(T) = -[\bar{N}_{11}(T)-\bar{\pi}_{11}(T)]^{-1}[\bar{P}_{11}(T)-\bar{\pi}_{11}(T)]\bar{\ell}_1(T) \tag{6.12b}$$

Since the reduced $\bar{x}_1$, $\bar{x}_2$, $\bar{\lambda}_1$, $\bar{\lambda}_2$ solution exists and is related to the $\bar{\ell}$ and $\bar{r}$ solution by a non-singular transformation, the $\bar{\ell}$ and $\bar{r}$ solution exists satisfying (6.12). Hence there exists an $\bar{\ell}_1^{\,o}$ and $\bar{r}_1^{\,T}$ and,

$$\begin{aligned} \bar{x}_1 &= \bar{\ell}_1 + \bar{r}_1, \quad & \lambda_1 &= \bar{P}_{11}\bar{\ell}_1 + \bar{N}_{11}\bar{r}_1 \\ \bar{x}_2 &= \bar{\ell}_2 + \bar{r}_2, \quad & \bar{\lambda}_2 &= \bar{P}_{12}'\bar{\ell}_1 + \bar{N}_{12}'\bar{r}_1 + \bar{P}_{22}\bar{\ell}_2 + \bar{N}_{22}\bar{r}_2 \quad . \end{aligned} \tag{6.13}$$

The existence of $\bar{\ell}_2^{\,o}$ follows from an argument identical to that in Theorem 4.2.1. The last step of the proof is to show the existence of $\bar{r}_2^{\,T}$. Using

(4.27) and (6.3),

$$\lambda_2 = P_{12}'\ell_1 + N_{12}'r_1 + P_{22}\ell_2 + N_{22}r_2 = \pi_{12}'(\ell_1 + r_1) + \pi_{22}(\ell_2 + r_2) \text{ at } t = T \tag{6.14}$$

Grouping like terms and using the approximating ℓ and r expressions (4.37) and (4.43) results in

$$\begin{aligned}(\bar{P}_{12}' - \bar{\pi}_{12}')(\bar{\ell}_1 + O(\mu)) &+ (\bar{N}_{12}' - \bar{\pi}_{12}')(\bar{r}_1 + O(\mu)) \\ &+ (\bar{P}_{22} - \bar{\pi}_{22})(\bar{\ell}_2 + \mathcal{L}_2(\tau) + O(\mu)) \\ &+ (\bar{N}_{22} - \bar{\pi}_{22})(\bar{r}_2 + \mathcal{R}_2(\sigma) + O(\mu))\Big|_{t = T} = 0\end{aligned} \tag{6.15}$$

Since $\mathcal{L}_2(\tau) \to 0$ as $\tau \to \infty$, the $\mathcal{L}_2(\tau)$ term is negligible when evaluating (6.15) at $t = T$ for sufficiently small μ. Thus for $\mu = 0$, (6.15) becomes upon use of (6.13)

$$\mathcal{R}_2(0) = -[\bar{N}_{22}(T) - \bar{\pi}_{22}]^{-1}[\bar{\lambda}_2(T) - \bar{\pi}_{12}'(T)\bar{x}_1(T) - \bar{\pi}_{22}(T)\bar{x}_2(T)] \tag{6.16}$$

but from (4.48), $\mathcal{R}_2(0) = \bar{r}_2^{\,T} - \bar{r}_2(T)$ or

$$\bar{r}_2^{\,T} = \mathcal{R}_2(0) + \bar{r}_2(T) \tag{6.17}$$

Here $\mathcal{R}_2(0)$ given by (6.16) exists since $\bar{\lambda}_2$, $\bar{x}_1$, and $\bar{x}_2$ exist and is related to them by the non-singular matrix $[\bar{N}_{22}(T) - \bar{\pi}_{22}(T)]$. Likewise, $\bar{r}_2(T)$ exists as previously stated. Hence $\bar{r}_2^{\,T}$ exists and a μ^* exists satisfying the theorem.

Corollaries identical to 4.6.1 - 4.6.2 hold providing the reduced solution and boundary layer terms are evaluated in accordance with the Theorem and the performance index is approximated by

$$J = \bar{J} + O(\mu) \tag{6.18}$$

where

$$\bar{J} = \frac{1}{2}\, \bar{x}_1' \bar{\pi}_{11} \bar{x}_1 \Big|_{t=T} + \frac{1}{2} \int_{t_o}^{T} \left(\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}' \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} \\ \bar{Q}_{12}' & \bar{Q}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \bar{u}'\bar{R}\bar{u} \right) dt \tag{6.19}$$

6.3 Discussion

The following observation is made since it will be used in the subsequent section. Suppose π_{11} is symmetrical positive definite. Then by choosing $\bar{P}_{11} = \bar{\pi}_{11}$ at $t = T$, $\bar{r}_1(T)$ in (6.12b) is zero, $\bar{r}_1^{\,T} = 0$. From (4.44) and (4.46), this is seen to imply $\bar{r}_1(t) = 0$ and $\bar{r}_2(t) = 0$ for all $t \in [t_o, T]$ which in turn implies $\bar{\ell}_1^{\,o} = x_1^{\,o}$ from (6.12a). Thus from the non-singularity of (4.27), it follows that

$$\begin{aligned} \bar{x}_1(t) &= \bar{\ell}_1(t), \qquad \bar{x}_2(t) = \bar{\ell}_2(t) \\ \bar{\lambda}_1(t) &= \bar{P}_{11}(t)\bar{\ell}_1(t), \quad \bar{\lambda}_2(t) = \bar{P}_{12}'(t)\bar{\ell}_1(t) + \bar{P}_{22}(t)\bar{\ell}_2(t) \end{aligned} \tag{6.20}$$

where $\bar{\ell}_1$ and $\bar{\ell}_2$ are evaluated from (4.38) and (4.40) with $\bar{\ell}_1^{\,o} = x_1^{\,o}$. From (4.53) it is seen that $\mathcal{L}_2(0) = x_2^{\,o} - \bar{\ell}_2(t_o)$ and from (6.16) that the expression for $\mathcal{R}_2(0)$ there remains unchanged and is given by

$$\mathcal{R}_2(0) = -[\bar{N}_{22}(T)-\bar{\pi}_{22}]^{-1}[\bar{\lambda}_2(T)-\bar{\pi}_{12}'(T)\bar{x}_1(T)-\bar{\pi}_{22}(T)\bar{x}_2(T)] \tag{6.21}$$

Thus the role of the r-system is to produce the boundary layer jump at $t = T$ if one exists. For the special case when there is no slow system for the terminal cost problem (6.1) and (6.2), it can be seen from (4.8) that $\bar{x}_2 = 0$ and $\bar{\lambda}_2 = 0$ since $\bar{x}_1 = 0$ and $\bar{\lambda}_1 = 0$. From this fact it is seen that $\mathcal{R}_2(0) = 0$ and no zero-order boundary jump occurs at $t = T$.

6.4 Closed Loop Solution

Theorem 6.4.1 Let the hypotheses of H 4.1.1, H 4.1.2, and H 4.1.3 be satisfied, let π_{11} be symmetrical positive definite, and let $K_{ij}(t,\mu)$ be the solution of the Riccati system (4.17) satisfying the π_{ij} boundary conditions from (6.3). Then the substitution of the zero-order Riccati gains for $K_{ij}(t,\mu)$ given by $\bar{K}_{ij}(t) + \mathcal{K}_{ij}(\sigma)$ in the determination of the state and co-state variables yields an approximate solution which is asymptotic to the correct solution $x_1(t,\mu)$, $x_2(t,\mu)$, $\lambda_1(t,\mu)$, and $\lambda_2(t,\mu)$ and given by the difference of the actual solution and $O(\mu)$,

$$\begin{aligned}
x_1(t,\mu) &= \bar{x}_1(t) + O(\mu) \\
x_2(t,\mu) &= \bar{x}_2(t) + \mathcal{L}_2(\tau) + \mathcal{R}_2(\sigma) + O(\mu) \\
\lambda_1(t,\mu) &= \bar{\lambda}_1(t) + O(\mu) \\
\lambda_2(t,\mu) &= \bar{\lambda}_2(t) + \bar{P}_{22}(t_o)\mathcal{L}_2(\tau) + \bar{N}_{22}(T)\mathcal{R}_2(\sigma) + O(\mu)
\end{aligned} \tag{6.22}$$

Variables $\bar{x}_1(t)$, $\bar{x}_2(t)$, $\bar{\lambda}_1(t)$ and $\bar{\lambda}_2(t)$ are related to $\bar{\ell}$ and $\bar{r}$ parameters as in (6.20). Variable $\mathcal{L}_2(\tau)$ is the solution of

$$\frac{d\mathcal{L}_2}{d\tau} = [\bar{A}_{22}(t_o) - \bar{S}_{22}(t_o)\bar{P}_{22}(t_o)]\mathcal{L}_2 \tag{6.23}$$

subject to initial condition

$$\mathcal{L}_2 = x_2^{\,o} - \bar{x}_2(t_o) \quad \text{at } \tau = 0$$

and $\mathcal{R}_2(\sigma)$ is given by

$$\mathcal{R}_2(\sigma) = -[\bar{K}_{22}(T)+\mathcal{K}_{22}(\sigma)-\bar{N}_{22}(T)]^{-1}[\mathcal{K}_{12}(\sigma)\bar{x}_1(T)+\mathcal{K}_{22}(\sigma)\bar{x}_2(T)] \tag{6.24}$$

<u>Proof</u>: The proof is based on first finding an asymptotic expression for the state and co-state variables using the equivalent asymptotic expressions for the $K_{ij}(t,\mu)$ solutions and then recognizing that these same expressions result when using only the zero-order terms of $K_{ij}(t,\mu)$.

Let the actual Riccati gains be given by $K_{ij}(t,\mu) = \bar{K}_{ij}(t)+\mathcal{K}_{ij}(\sigma)+O(\mu)$ and the auxiliary "positive" and "negative" Riccati gains by $P_{ij}(t,\mu) = \bar{P}_{ij}(t)+O(\mu)$ and $N_{ij}(t,\mu) = \bar{N}_{ij}(t)+O(\mu)$ respectively where $i,j = 1,2$. Here $\mathcal{K}_{11}(\sigma) = 0$ since no zero-order boundary layer occurs in this variable and $\bar{P}_{ij}(t)$ are chosen equal to $\bar{K}_{ij}(t)$ for $i,j = 1,2$. Now λ_1 and λ_2 are expressible in terms of x_1 and x_2 by

$$\begin{bmatrix} \lambda_1 \\ \mu\lambda_2 \end{bmatrix} = \begin{bmatrix} K_{11} & \mu K_{12} \\ \mu K_{12}' & \mu K_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{for all } t \in [t_o,T] \tag{6.25}$$

and in terms of ℓ_1, ℓ_2, r_1, and r_2 using transformation (4.27) as

$$\lambda_1 = P_{11}\ell_1 + N_{11}r_1 + \mu P_{12}\ell_2 + \mu N_{12}r_2$$
$$\lambda_2 = P_{12}'\ell_1 + N_{12}'r_1 + P_{22}\ell_2 + N_{22}r_2 \tag{6.26}$$

Thus it follows after elimination of λ_1, λ_2 using (6.25), (6.26) and (4.27) and then grouping

$$(K_{11}-P_{11})\ell_1 + (K_{11}-N_{11})r_1 + \mu(K_{12}-P_{12})\ell_2 + \mu(K_{12}-N_{12})r_2 = 0$$
$$(K_{12}'-P_{12}')\ell_1 + (K_{12}'-N_{12}')r_1 + (K_{22}-P_{22})\ell_2 + (K_{22}-N_{22})r_2 = 0 \tag{6.27}$$

Rewriting (6.27) using the expanded forms for K_{ij}, P_{ij}, and N_{ij}

$$[O(\mu)]\ell_1+[\bar{K}_{11}-\bar{N}_{11}+O(\mu)]r_1+\mu[\mathcal{K}_{12}+O(\mu)]\ell_2+\mu[\bar{K}_{12}+\mathcal{K}_{12}-\bar{N}_{12}+O(\mu)]r_2 = 0$$
$$[\mathcal{K}_{12}'+O(\mu)]\ell_1+[\bar{K}_{12}'+\mathcal{K}_{12}'-\bar{N}_{12}'+O(\mu)]r_1 \tag{6.28}$$
$$+ [\mathcal{K}_{22}+O(\mu)]\ell_2+[\bar{K}_{22}+\mathcal{K}_{22}-\bar{N}_{22}+O(\mu)]r_2 = 0$$

Solving (6.28) for r_1, r_2

$$\begin{bmatrix} r_1 \\ r_2 \end{bmatrix} = -\left(\begin{bmatrix} 0 & 0 \\ (\bar{K}_{22}+\mathcal{K}_{22}-\bar{N}_{22})^{-1}\mathcal{K}_{12}' & (\bar{K}_{22}+\mathcal{K}_{22}+\bar{N}_{22})^{-1}\mathcal{K}_{22} \end{bmatrix} + O(\mu)\right)\begin{bmatrix} \ell_1 \\ \ell_2 \end{bmatrix} \tag{6.29}$$

Therefore

$$r_1 = O(\mu)\ell_1+O(\mu)\ell_2$$
$$r_2 = -(\bar{K}_{22}+\mathcal{K}_{22}-\bar{N}_{22})^{-1}(\mathcal{K}_{12}'\ell_1+\mathcal{K}_{22}\ell_2)+O(\mu)\ell_1+O(\mu)\ell_2 \tag{6.30}$$

Thus if $\bar{\ell}_1^{\,o}$ and $\bar{\ell}_2^{\,o}$ can be found in terms of the specified boundary conditions $x_1^{\,o}$ and $x_2^{\,o}$, then ℓ_1 and ℓ_2 can be approximated by

$$\ell_1(t,\mu) = \bar{\ell}_1(t)+0(\mu), \qquad \ell_2(t,\mu) = \bar{\ell}_2(t)+\mathcal{L}_2(\tau)+0(\mu) \tag{6.31}$$

where $\mathcal{L}_2(\tau)$ is the solution of (6.23) satisfying the boundary condition $\mathcal{L}_2 = \bar{\ell}_2^{\,o} - \bar{\ell}_2(t_o) = \bar{\ell}_2^{\,o} - \bar{x}_2(t_o)$ at $\tau = 0$. Once $\ell_1(t,\mu)$ and $\ell_2(t,\mu)$ are found, $r_1(t,\mu)$ and $r_2(t,\mu)$ are known from (6.29). The $x_1^{\,o}$ boundary conditions, related through the transformation (4.27), is

$$\begin{aligned} x_1^{\,o} &= \ell_1^{\,o} + r_1^{\,o} \\ &= \ell_1^{\,o} + 0(\mu)\ell_1^{\,o} + 0(\mu)\ell_2^{\,o} \qquad \text{by (6.30)} \end{aligned} \tag{6.32}$$

thus $\bar{\ell}_1^{\,o} = x_1^{\,o}$ (6.32) evaluated at $\mu = 0$. Similarly for x_2

$$\begin{aligned} x_2^{\,o} &= \ell_2^{\,o} + r_2^{\,o} \\ &= \ell_2^{\,o} \text{ for } \mu = 0 \text{ since } r_2^{\,o} \text{ is zero from (6.30)} \end{aligned} \tag{6.33}$$

upon recognizing the $\mathcal{K}_{12}$ and $\mathcal{K}_{22}$ variables are negligible at $t = t_o$. Therefore,

$$\bar{\ell}_2^{\,o} = x_2^{\,o} \text{ and from before, } \bar{\ell}_1^{\,o} = x_1^{\,o} \quad . \tag{6.34}$$

Hence the existence of $\bar{\ell}_1^{\,o}$ and $\bar{\ell}_2^{\,o}$ in terms of $x_1^{\,o}$ and $x_2^{\,o}$ has been established. From the transformation

$$x_1(t,\mu) = \ell_1(t,\mu) + r_1(t,\mu)$$

$$= \bar{\ell}_1(t)+0(\mu) \text{ using (6.30)}$$

$$= \bar{x}_1(t)+0(\mu) \text{ by (6.20)} \tag{6.35}$$

$$x_2(t,\mu) = \ell_2(t,\mu) + r_2(t,\mu)$$

$$= \bar{\ell}_2+\mathcal{L}_2-(\bar{K}_{22}+\mathcal{K}_{22}-\bar{N}_{22})^{-1}(\mathcal{K}_{22}\bar{\ell}_2+\mathcal{K}_{12}\bar{\ell}_1)$$

$$-(\bar{K}_{22}+\mathcal{K}_{22}-\bar{N}_{22})^{-1}\mathcal{K}_{22}\mathcal{L}_2+0(\mu) \text{ using (6.30)}$$

$$= \bar{\ell}_2+\mathcal{L}_2-(\bar{K}_{22}+\mathcal{K}_{22}-\bar{N}_{22})^{-1}(\mathcal{K}_{22}\bar{\ell}_2+\mathcal{K}_{12}\bar{\ell}_1)+0(\mu) \tag{6.36}$$

The latter step is justified after observing that the product of a σ and τ function are negligible for small enough μ. Using H 4.1.2 and replacing $\bar{\ell}_2$ by $\bar{x}_2$, $x_2(t,\mu)$ can be written as

$$x_2(t,\mu) = \bar{x}_2(t) + \mathcal{L}_2(\tau) + \mathcal{R}_2(\sigma) + 0(\mu) \tag{6.37}$$

where

$$\mathcal{R}_2(\sigma) = -[\bar{K}_{22}(T)+\mathcal{K}_{22}(\sigma)-\bar{N}_{22}(t)]^{-1}[\mathcal{K}_{12}(\sigma)\bar{x}_1(T)+\mathcal{K}_{22}(\sigma)\bar{x}_2(T)] \tag{6.38}$$

Knowing (6.35) and (6.37), from (6.25), (6.35) and (6.37)

$$\lambda_1(t,\mu) = K_{11}(t,\mu)x_1(t,\mu)+\mu K_{12}(t,\mu)x_2(t,\mu)$$

$$= [\bar{P}_{11}+O(\mu)][\bar{\ell}_1+O(\mu)] + O(\mu)$$

$$= \bar{P}_{11}\bar{\ell}_1 + O(\mu) = \bar{\lambda}_1(t) + O(\mu) \text{ from (6.20)} \tag{6.39a}$$

$$\lambda_2(t,\mu) = K_{12}'(t,\mu)x_1(t,\mu) + K_{22}(t,\mu)x_2(t,\mu)$$

$$= [K_{12}'+\mathcal{K}_{12}+O(\mu)][\bar{\ell}_1+O(\mu)]$$

$$+ [\bar{K}_{22}+\mathcal{K}_{22}+O(\mu)][\bar{\ell}_2+\mathcal{L}_2+\mathcal{R}_2+O(\mu)]$$

$$= [K_{12}'\ell_1+\bar{K}_{22}\bar{\ell}_2]+[\mathcal{K}_{12}\ell_1+\mathcal{K}_{22}\bar{\ell}_2]+[\bar{K}_{22}+\mathcal{K}_{22}][\mathcal{L}_2+\mathcal{R}_2] + O(\mu)$$

$$= \bar{\lambda}_2-[\bar{K}_{22}+\mathcal{K}_{22}-\bar{N}_{22}]\mathcal{R}_2+[K_{22}+\mathcal{K}_{22}][\mathcal{L}_2+\mathcal{R}_2] + O(\mu)$$

$$= \bar{\lambda}_2+[\bar{K}_{22}+\mathcal{K}_{22}]\mathcal{L}_2+\bar{N}_{22}\mathcal{R}_2 + O(\mu)$$

$$= \bar{\lambda}_2+\bar{K}_{22}(t_o)\mathcal{L}_2(\tau)+\bar{N}_{22}(T)\mathcal{R}_2(\sigma) + O(\mu) \tag{6.39b}$$

Since the same asymptotic expressions result when the zero-order terms of $K_{ij}(t,\mu)$ are used, the theorem is proved.

If λ_1 and λ_2 had been eliminated from (4.5) using (6.25), the resulting x_1 and x_2 equations would generally be discontinuous at $t = T$ for $\mu = 0$. By using the ℓ and r systems it was possible to treat well behaved functions. The ℓ system contained no boundary jumps and the r variables which did were determined algebraically once the ℓ parameters were found.

6.5 Open Loop Design Example

The optimal control problem is to minimize with respect to the control u the performance index

$$J = \frac{1}{2} x_1(T)^2 + \int_{t_o}^{T} (2x_1^2 + x_2^2 + u^2)dt \tag{6.40}$$

for the singularly perturbed system

$$\begin{aligned} \dot{x}_1 &= \frac{3}{2} x_2 \\ \mu\dot{x}_2 &= -\frac{3}{2} x_1 + \frac{1}{2} x_2 - u \end{aligned} \tag{6.41}$$

whose boundary constraint is given by

$$x_1 = x_1^o \quad \text{and} \quad x_2 = x_2^o \quad \text{at } t = t_o \tag{6.42}$$

The optimality conditions for this problem are identical to those given for the problem in Chapter 5 except for the boundary constraints which are now (6.42) and

$$\lambda_1(T) = x_1(T), \quad \lambda_2(T) = 0 \tag{6.43}$$

The solution $\bar{x}_1$, $\bar{\lambda}_1$ is to satisfy the reduced system (5.7) and the x_1 boundary condition of (6.42) and the λ_1 boundary condition of (6.43). The variables $\bar{x}_2$, $\bar{\lambda}_2$ are related to $\bar{x}_1$, $\bar{\lambda}_1$ by (5.8). Since the hypotheses of

Theorem 6.2.1 are identical to those of Theorem 4.2.1, the hypotheses are met as previously established in Chapter 5. The reduced solution is thus given by

$$
\begin{aligned}
\bar{x}_1 &= \alpha_4 \left[e^{-\frac{5}{2}(t-T)} + \frac{2}{3} e^{\frac{5}{2}(t-T)} \right] x_1^{o} \\
\bar{\lambda}_1 &= \alpha_4 \left[3e^{-\frac{5}{2}(t-T)} - \frac{4}{3} e^{\frac{5}{2}(t-T)} \right] x_1^{o} \\
\bar{x}_2 &= \frac{5}{3} \alpha_4 \left[-e^{-\frac{5}{2}(t-T)} + \frac{2}{3} e^{\frac{5}{2}(t-T)} \right] x_1^{o} \\
\bar{\lambda}_2 &= -\frac{1}{3} \alpha_4 \left[7e^{-\frac{5}{2}(t-T)} + \frac{4}{3} e^{\frac{5}{2}(t-T)} \right] x_1^{o}
\end{aligned}
\tag{6.44}
$$

where $\alpha_4 = e^{-\frac{5}{2} T/(1 + \frac{2}{3} e^{-5T})}$. The boundary layer correction terms are found next. Recall $\bar{P}_{22}(0) = 2$ and $\bar{N}_{22}(T) = -1$ from (5.12). Thus these correction terms are given as the solution of

$$
\begin{aligned}
\frac{d\mathcal{L}_2}{d\tau} &= -\frac{3}{2} \mathcal{L}_2, \quad \mathcal{L}_2 = x_2^{o} - \bar{x}_2(0) \text{ at } \tau = 0 \\
\frac{d\mathcal{R}_2}{d\sigma} &= \frac{3}{2} \mathcal{R}_2, \quad \mathcal{R}_2 = \bar{\lambda}_2(T) \text{ at } \sigma = 0
\end{aligned}
\tag{6.45}
$$

as seen from (6.7) - (6.10) where from (6.44),

$$
\begin{aligned}
\bar{x}_2(0) &= \frac{5}{3} \alpha_4 (-e^{\frac{5}{2} T} + \frac{2}{3} e^{-\frac{5}{2} T}) \\
\bar{\lambda}_2(T) &= -\frac{25}{9} \alpha_4 x_1^{o} .
\end{aligned}
\tag{6.46}
$$

Thus from (6.45) and (6.46),

$$\mathcal{L}_2(\tau) = [x_2^{\,o} - x_2(0)]\, e^{-\frac{3}{2}\tau}$$
$$\mathcal{R}_2(\sigma) = (-\frac{25}{9}\alpha_4 x_1^{\,o})\, e^{\frac{3}{2}\sigma} \tag{6.47}$$

Hence all the terms composing the zero-order approximate solution of the variables are known--i.e.,

$$\begin{aligned} x_1(t,\mu) &= \bar{x}_1(t) + 0(\mu) \\ x_2(t,\mu) &= \bar{x}_2(t) + \mathcal{L}_2(\tau) + \mathcal{R}_2(\sigma) + 0(\mu) \\ \lambda_1(t,\mu) &= \bar{\lambda}_1(t) + 0(\mu) \\ \lambda_2(t,\mu) &= \bar{\lambda}_2(t) + \bar{P}_{22}(0)\mathcal{L}_2(\tau) + \bar{N}_{22}(T)\mathcal{R}_2(\sigma) + 0(\mu) \end{aligned} \tag{6.48}$$

Also, from (4.60), $\bar{u} = \bar{\lambda}_2$ where $\bar{\lambda}_2$ is given in (6.44) and

$$u = \bar{u} + 2\mathcal{L}_2(\tau) - \mathcal{R}_2(\sigma) + 0(\mu) \tag{6.49}$$

To compare actual solutions with zero-order solutions, the following boundary conditions were selected.

$$x_1^{\,o} = 4.0 \quad \text{and} \quad x_2^{\,o} = 3.0 \tag{6.50}$$

Figures (6.1) - (6.4) show this comparison for $\mu = 0.1$. The x_1 and x_2 plots,

Figure (6.1) and Figure (6.2), are similar to their plots shown in Figures (5.1) and (5.2) respectively due to the special selection of boundary conditions.

6.6 Closed Loop Design Example

From (4.17) and (4.18) for the free end-point problem just discussed, the singularly perturbed Riccati system is given by

$$\begin{aligned} \dot{K}_{11} &= 3K_{12} + K_{12}^{2} - 4 \\ \mu\dot{K}_{12} &= -\frac{3}{2}K_{11} - \frac{1}{2}K_{12} + \frac{3}{2}K_{22} + K_{12}K_{22} \\ \mu\dot{K}_{22} &= -3\mu K_{12} - K_{22} + K_{22}^{2} - 2 \end{aligned} \tag{6.51}$$

subject to boundary conditions

$$K_{11} = 1, \quad K_{12} = 0, \quad \text{and } K_{22} = 0 \quad \text{at } t = T \tag{6.52}$$

and the reduced solution is to satisfy

$$\begin{aligned} \dot{\bar{K}}_{11} &= 3\bar{K}_{12} + \bar{K}_{22}^{2} - 4 \\ 0 &= -\frac{3}{2}\bar{K}_{11} - \frac{1}{2}\bar{K}_{12} + \frac{3}{2}\bar{K}_{22} + \bar{K}_{12}\bar{K}_{22} \\ 0 &= -\bar{K}_{22} + \bar{K}_{22}^{2} - 2 \end{aligned} \tag{6.53}$$

and the boundary condition

$$\bar{K}_{11} = 1 \quad \text{at } t = T. \tag{6.54}$$

The reduced solution is given by

$$\bar{K}_{11} = \frac{9 - 4\,e^{5(t-T)}}{3 + 2\,e^{5(t-T)}}, \quad \bar{K}_{12} = \frac{3 - 8\,e^{5(t-T)}}{3 + 2\,e^{5(t-T)}}, \quad \bar{K}_{22} = 2. \tag{6.55}$$

The Riccati correction terms are to satisfy

$$\frac{d\mathcal{K}_{12}}{d\sigma} = \frac{3}{2}\,\mathcal{K}_{12} + \mathcal{K}_{22}(\frac{1}{2} + \mathcal{K}_{12}) \tag{6.56}$$

$$\frac{d\mathcal{K}_{22}}{d\sigma} = \mathcal{K}_{22}(\mathcal{K}_{22} + 3) \tag{6.57}$$

and the boundary condition

$$\mathcal{K}_{12} = \bar{\pi}_{12}(T) - \bar{K}_{12}(T), \quad \mathcal{K}_{22} = \bar{\pi}_{22}(T) - \bar{K}_{22}(T) \quad \text{at } \sigma = 0. \tag{6.58}$$

But from (6.55), $\bar{K}_{12} = -1$ and $\bar{K}_{22} = 2$ at $t = T$. Thus $\mathcal{K}_{12} = 1$ and $\mathcal{K}_{22} = -2$ at $t = T$. Hence

$$\begin{aligned} \mathcal{K}_{12}(\sigma) &= \frac{e^{\frac{3}{2}\sigma}}{1+2\,e^{3\sigma}}\,(5 - 2\,e^{\frac{3}{2}\sigma}) \\ \mathcal{K}_{22}(\sigma) &= -\frac{6\,e^{3\sigma}}{1+2\,e^{3\sigma}} \end{aligned} \tag{6.59}$$

The $K_{ij}(t,\mu)$ solutions of (6.51) are thus approximated by

$$K_{11}(t,\mu) = \bar{K}_{11} + O(\mu)$$

$$K_{12}(t,\mu) = \bar{K}_{12} + \mathcal{K}_{12}(\sigma) + O(\mu) \qquad (6.60)$$

$$K_{22}(t,\mu) = \bar{K}_{22} + \mathcal{K}_{22}(\sigma) + O(\mu)$$

Figures (6.5) - (6.7) compare the actual and approximate solutions for K_{11}, K_{12}, and K_{22} for $\mu = 0.1$. Figure (6.8) does the same for $\mu = 0.01$. The two solutions are very close when $\mu = .01$ and the transient occurring at $t = T$ is becoming quite steep. Figures (6.1), (6.2) show that the x_1, x_2 approximate solutions found using the Riccati approximations are very close to the correct values. Figures (6.3), (6.4) show the same information for λ_1, λ_2 computed using (6.25) with the corresponding approximate expressions for the Riccati gains and state parameters. The fact that the closeness is better than in the open loop case is not surprising since the correction terms were computed from non-linear systems in the latter case. Thus this feedback example supports Theorem 6.4.1.

To emphasize the fact that μ should be small for the theorems to hold, recall from the eigenvalue analysis that $\mu = 1$ should be completely unreasonable. To graphically show this, the plot of x_2 is shown in Figure (6.9). Closeness does not apply!

Remark: Recall the expression given for $\mathcal{R}_2(\sigma)$ in Theorem 6.4.1

$$\mathcal{R}_2(\sigma) = -[\bar{K}_{22}(T)+\mathcal{K}_{22}(\sigma)-\bar{N}_{22}(T)]^{-1}[\mathcal{K}_{12}(\sigma)\bar{x}_1(T)+\mathcal{K}_{22}(\sigma)\bar{x}_2(T)] \cdot \qquad (6.61)$$

It will be shown, for the problem considered, to be equivalent to the $R_2(\sigma)$ term used in Theorem 6.2.1 given by (6.47).

$$R_2(\sigma) = -\left[2 - 6\,\frac{e^{3\sigma}}{1+2e^{3\sigma}} + 1\right]^{-1} \cdot$$

$$\left[\frac{e^{\frac{3}{2}\sigma}}{1+2e^{3\sigma}}(5 - 2e^{\frac{3}{2}\sigma}) \cdot \frac{5}{3}\,\alpha_4 x_1^{o} - 6\,\frac{e^{3\sigma}}{1+2e^{3\sigma}} \cdot -\frac{5}{9}\,\alpha_4 x_1^{o}\right]$$

$$= (-\frac{25}{9}\,\alpha_4 x_1^{o})e^{\frac{3}{2}\sigma} \tag{6.62}$$

Thus the equivalency has been shown.

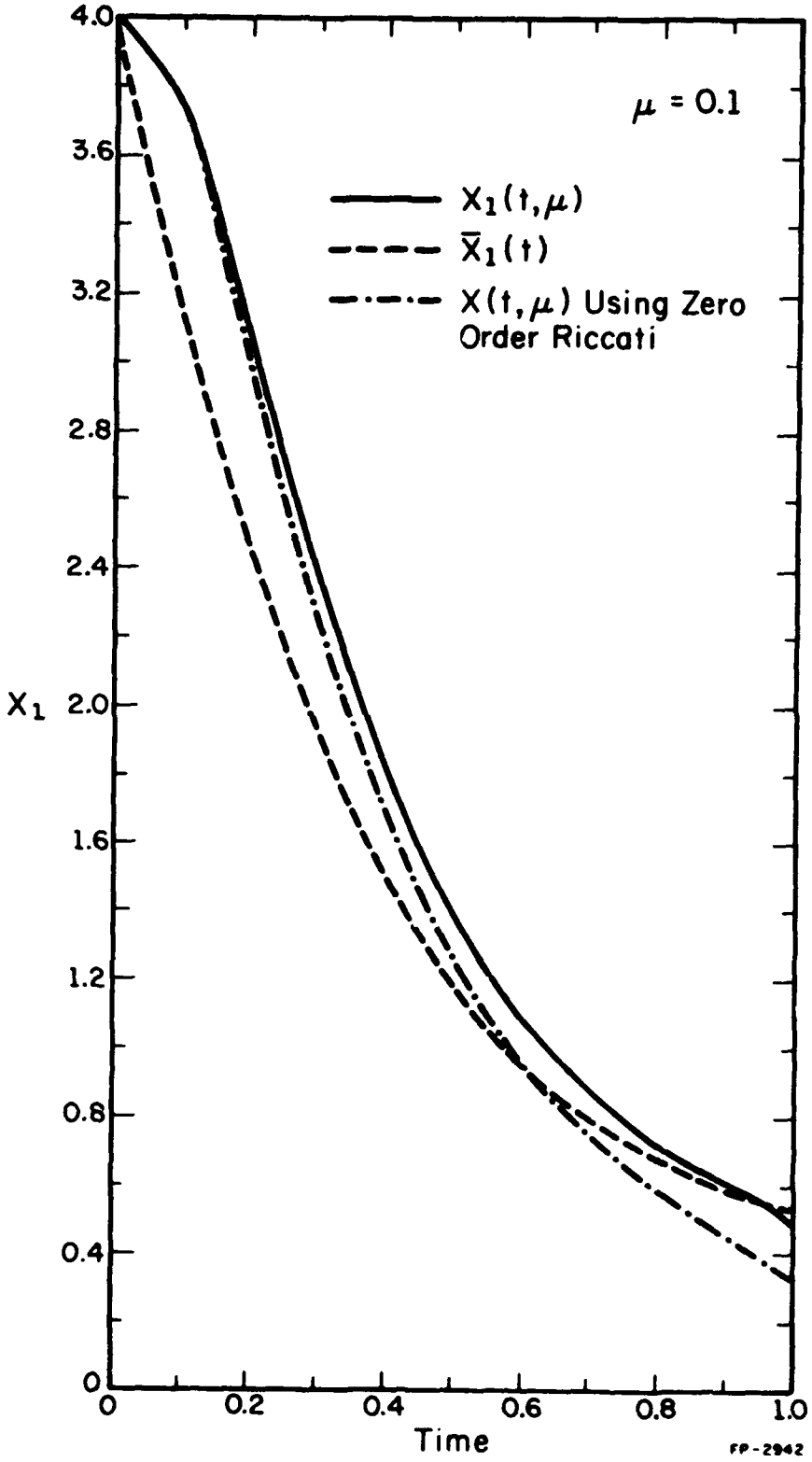

Fig. 6.1. Free End Point Problem.

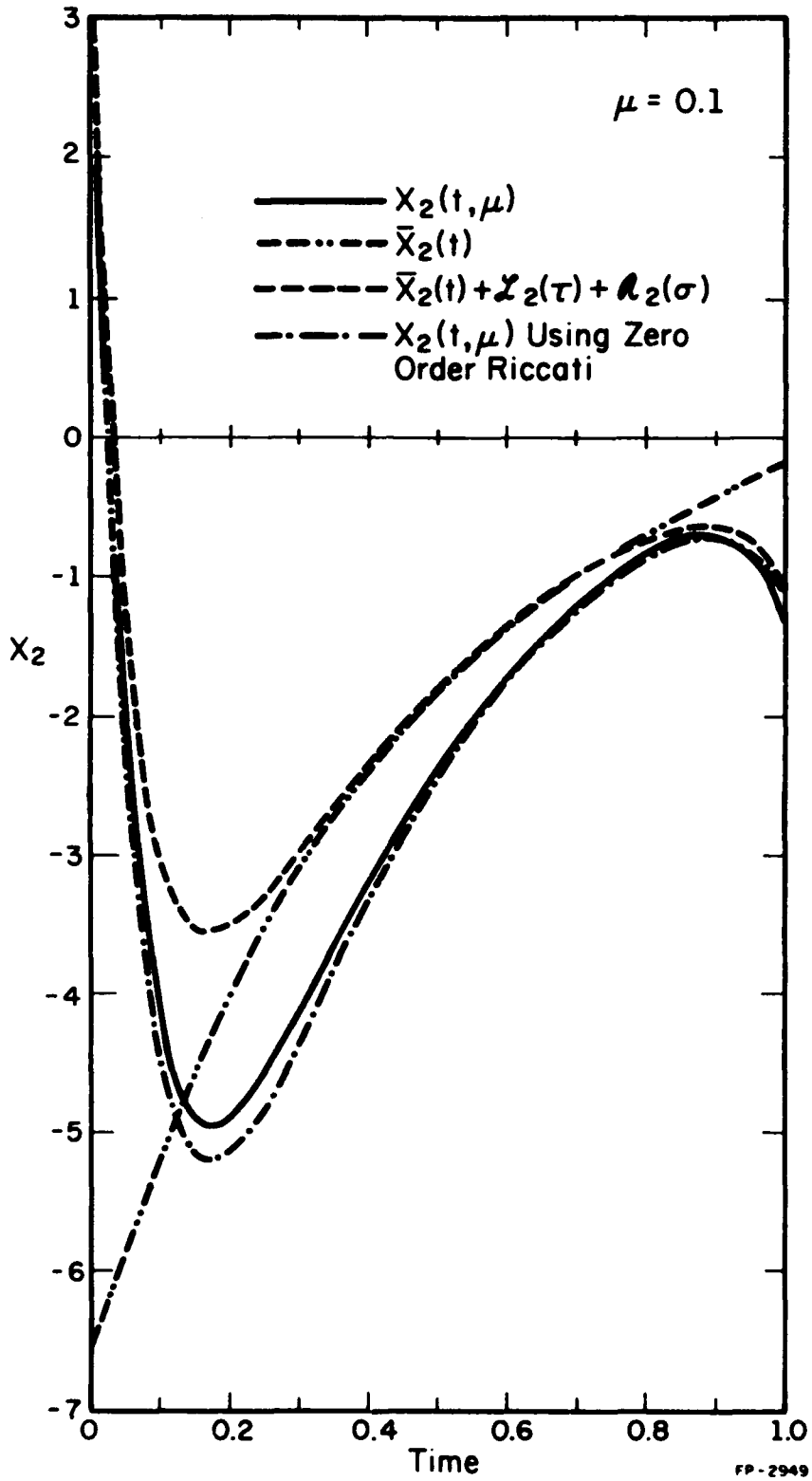

Fig. 6.2. Free End Point Problem.

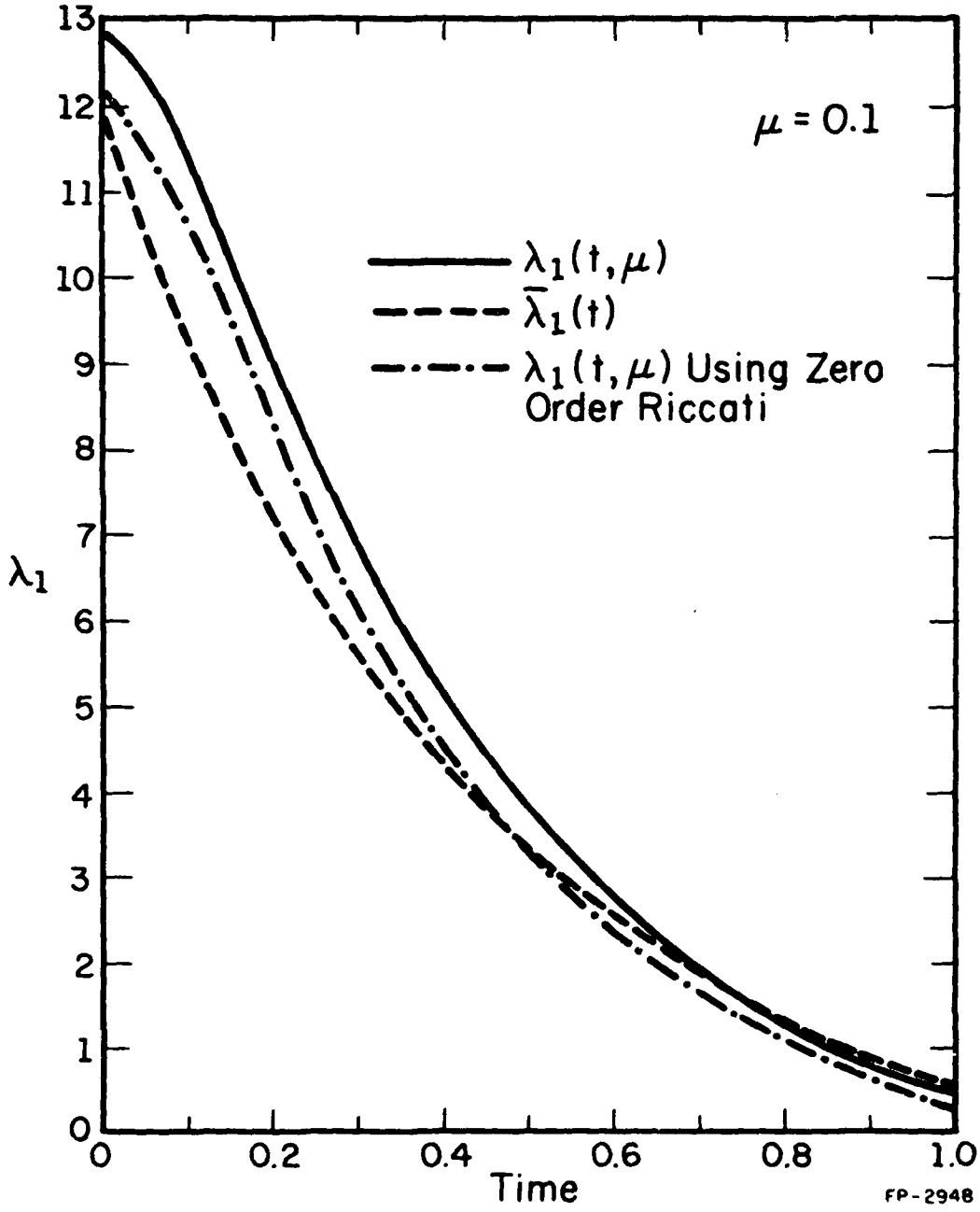

Fig. 6.3. Free End Point Problem.

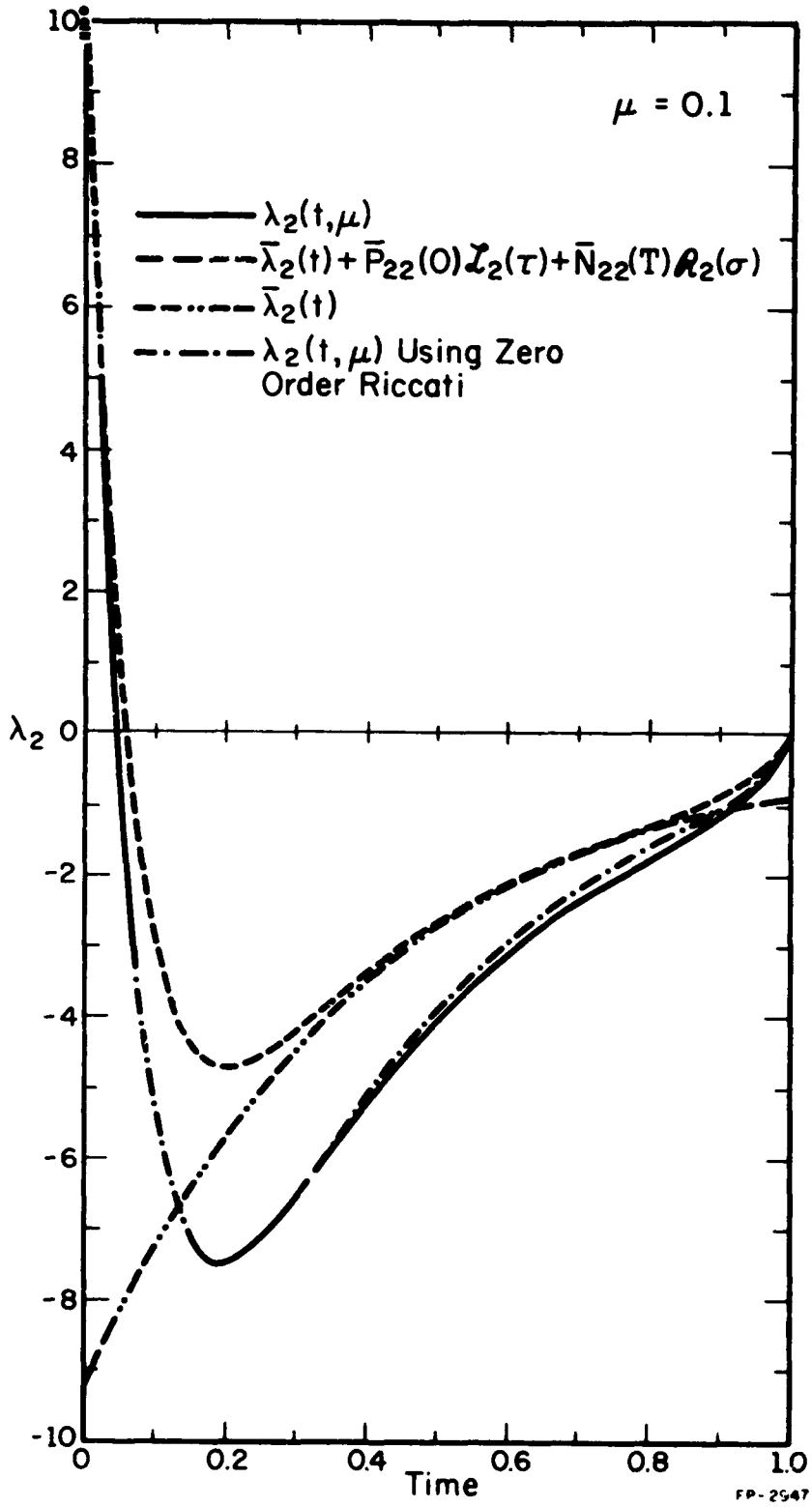

Fig. 6.4. Free End Point Problem.

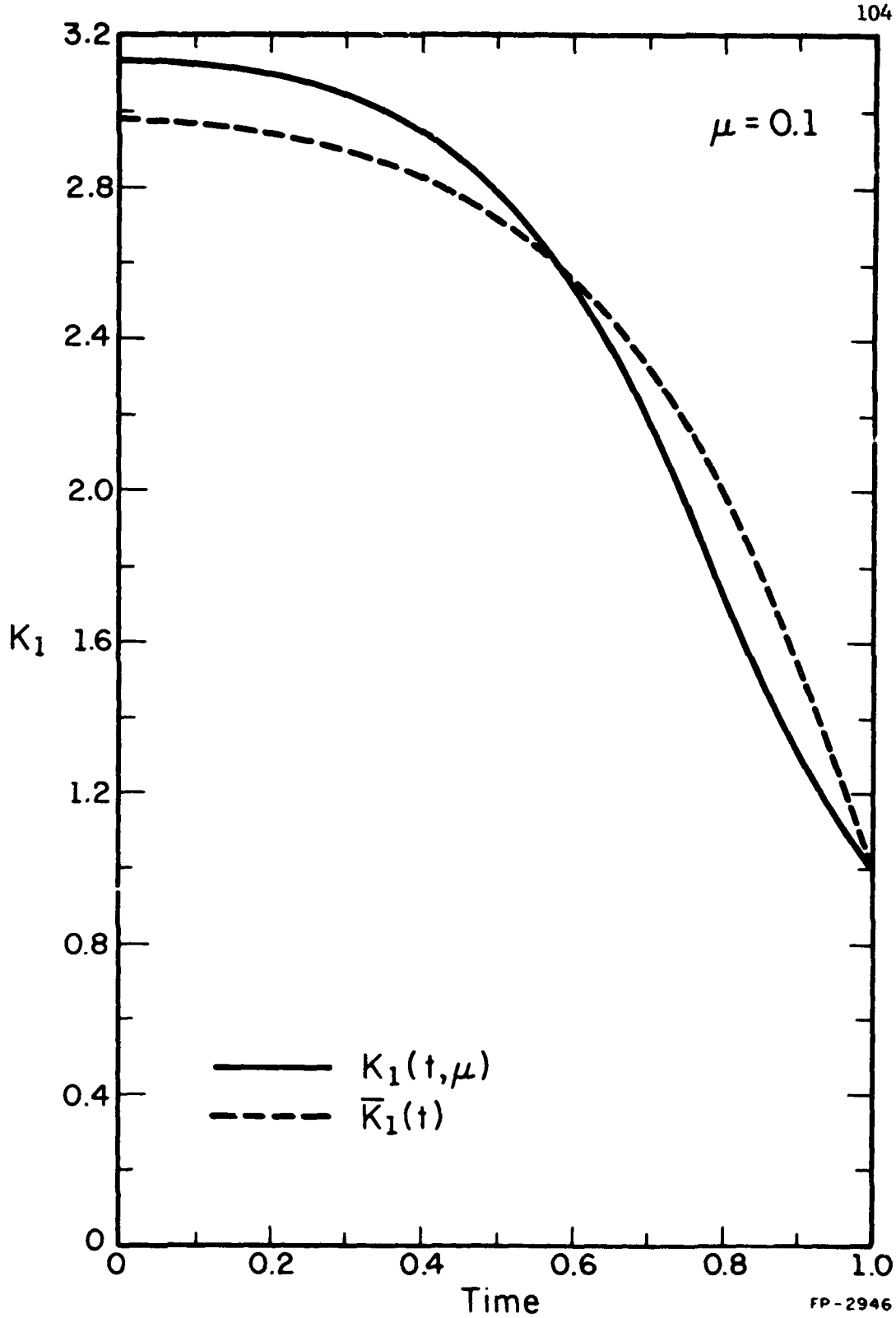

Fig. 6.5. Free End Point Problem.

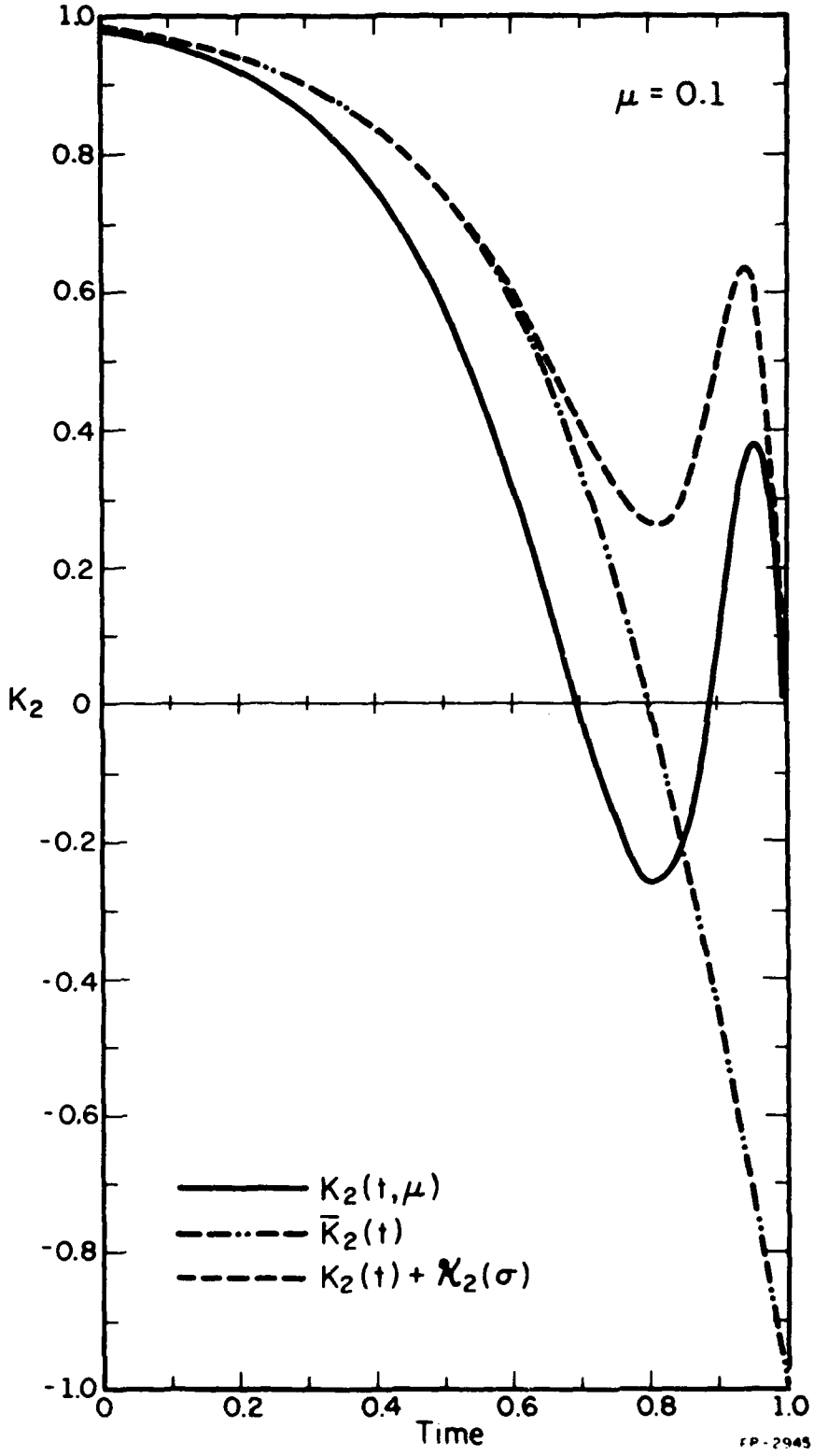

Fig. 6.6. Free End Point Problem.

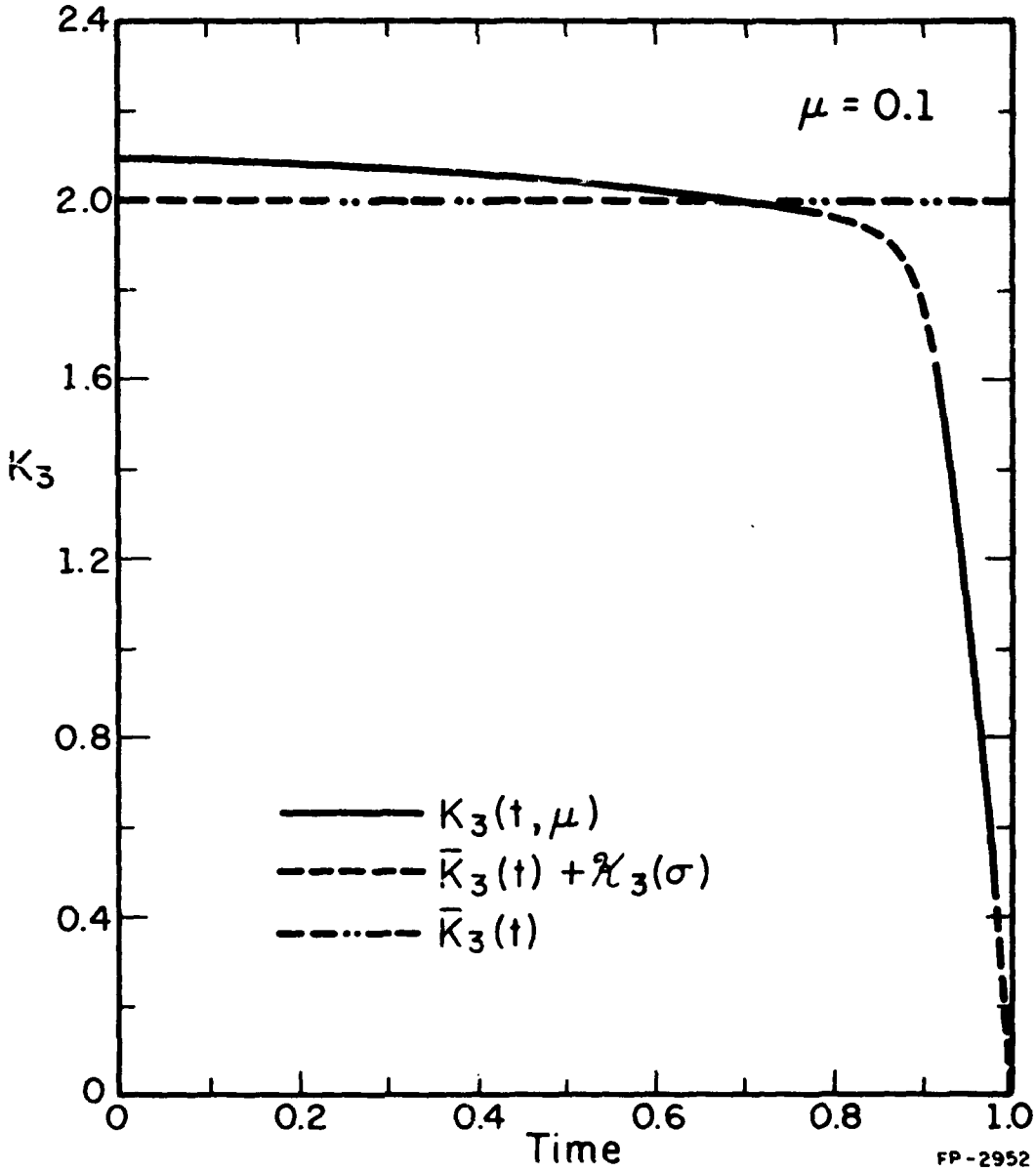

Fig. 6.7. Free End Point Problem.

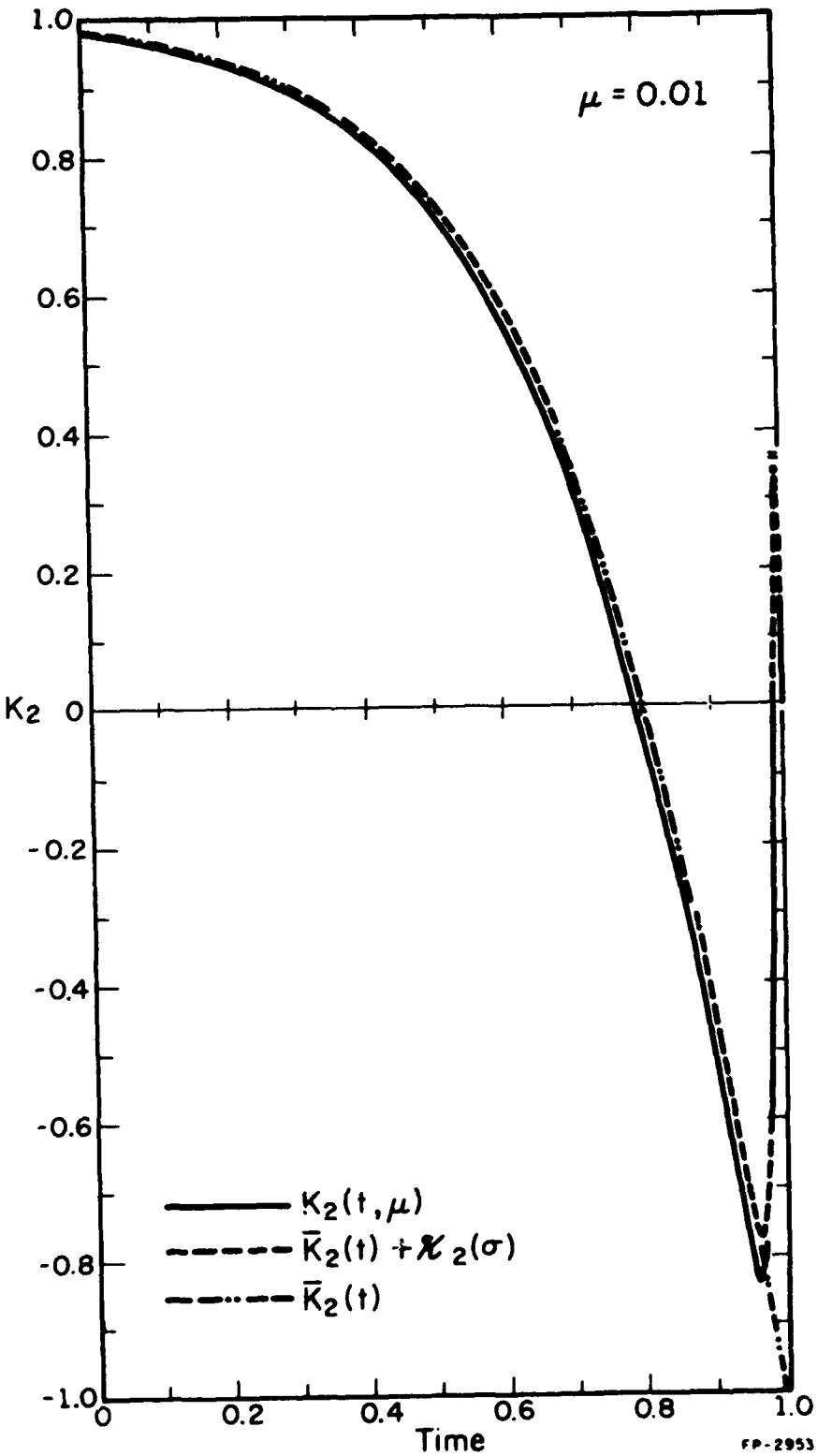

Fig. 6.8. Free End Point Problem.

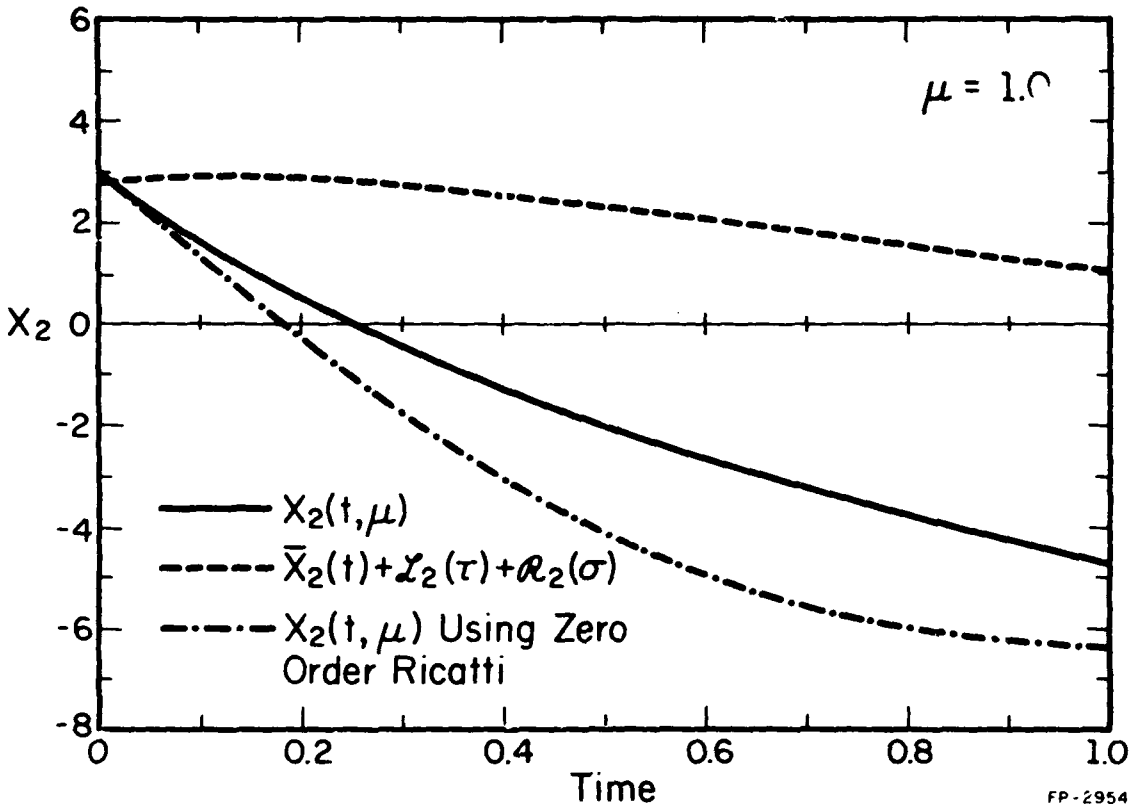

Fig. 6.9. Free End Point Problem.

APPENDIX A: INITIAL VALUE PROBLEM

This appendix contains an initial value singular perturbation lemma and theorem for the non-linear system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,x_2,t,\mu), \qquad x_1 = \rho(\mu) \text{ at } t = t_o \\ \mu\dot{x}_2 &= f_2(x_1,x_2,t,\mu), \qquad x_2 = \eta(\mu) \text{ at } t = t_o \end{aligned} \tag{A.1}$$

where x_1, x_2 are n_1-, n_2-dimensional vectors and μ is a small positive scalar parameter. The lemma establishes that the boundary layer terms of the solution of (A.1) will be identically zero for some η. The theorem establishes that the solution of (A.1) can be approximated uniformly on the entire interval $[t_o,T]$. Let the reduced solution $\bar{x}_1$, $\bar{x}_2$ satisfy the system

$$\begin{aligned} \dot{\bar{x}}_1 &= f_1(\bar{x}_1,\bar{x}_2,t,0), \qquad \bar{x}_1 = \rho(0) \text{ at } t = t_o \\ 0 &= f_2(\bar{x}_1,\bar{x}_2,t,0) \end{aligned} \tag{A.2}$$

formed from (A.1) by setting μ equal to zero. The following hypotheses are assumed.

<u>H A.1</u> System (A.2) has a continuous solution x_1, x_2 for all $t \in [t_o,T]$.

<u>H A.2</u> The functions f_1, f_2 have continuous derivatives to order $R + 2$ with respect to (x_1,x_2,t,μ) in some neighborhood of $(\bar{x}_1,\bar{x}_2,t,0)$, $t \in [t_o,T]$, $\mu \in [0,\mu^*]$ for some $\mu^* > 0$. Also ρ, η have continuous derivatives to order $R + 2$ with respect to μ for $\mu \in [0,\mu^*]$.

<u>H A.3</u> The real parts of the eigenvalues of $\frac{\partial f_2}{\partial x_2}(\bar{x}_1,\bar{x}_2,t,0)$ are less than a fixed negative number for all $t \in [t_o,T]$.

<u>Lemma A.4</u> Let H A.1 - H A 3 be satisfied. Then there exists a μ^*, $\alpha > 0$ such that when $|\rho(\mu)-\bar{x}(t_o)| < \alpha$ and $\mu \in [0,\mu^*]$, a solution of

$$\dot{x}_1 = f_1(x_1,x_2,t,\mu), \quad x_1 = \rho(\mu) \tag{A.3}$$

$$\mu\dot{x}_2 = f_2(x_1,x_2,t,\mu)$$

exists which satisfies

$$\begin{aligned} x_1(t,\mu) &= \sum_{r=0}^{R} x_{1_r}(t)\mu^r + O(\mu^{R+1}) \\ x_2(t,\mu) &= \sum_{r=0}^{R} x_{2_r}(t)\mu^r + O(\mu^{R+1}) \end{aligned} \tag{A.4}$$

for all $t \in [t_o,T]$, $\mu \in (0,\mu^*]$.

<u>Theorem A.5</u> Let H A.1 - H A.3 be satisfied where R = 0 Then there exists a $\mu^* > 0$ such that for all $\mu \leq \mu^*$, all the solutions $x_1(t,\mu)$, $x_2(t,\mu)$ of (A.1) starting in some neighborhood of the reduced solution at $t = t_o$ exist on the interval $[t_o,I]$ and satisfy

$$\begin{aligned} x_1(t,\mu) &= \bar{x}_1(t) + O(\mu) \\ x_2(t,\mu) &= \bar{x}_2(t) + \Lambda(\tau) + O(\mu) \end{aligned} \tag{A.5}$$

where $\Lambda(\tau)$ is the solution of

$$\frac{d\Lambda}{d\tau} = f_2(\rho(0), \Lambda+\bar{x}_2,t_o,0), \quad \Lambda = \eta(0) - \bar{x}_2(t_o) \tag{A 6}$$

and $\Lambda \to 0$ as $\tau \to \infty$

The x_1 and x_2 solutions of (A.3) and their R + 1 derivatives with respect to μ are continuous and equal to the reduced solution at $\mu = 0$. From (A.6) is is evident that the x_2 initial condition implied for problem (A.3) should be $\bar{x}_2(t)$. This lemma and theorem follow from a much more general lemma and theorem given in [21]. The terms $\bar{x}_1(t)$ and $\bar{x}_2(t) + \Lambda(\tau)$ are called the "zero-order" approximation of $x_1(t,\mu)$ and $x_2(t,\mu)$ in (A.5).

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Robert Reynolds Wilde was born September 24, 1939 in St. Louis, Missouri. He received his B.S. degree with honors in 1962 and his M.S. degree in 1963 in electrical engineering from the University of Illinois. As an undergraduate, he received scholarships from International Telephone and Telegraph and was selected the outstanding electrical engineering student of the year at the University in February 1962. Also in 1962 he was commissioned an officer in the United States Air Force and was designated a distinguished military graduate.

During the summers of 1961 and 1962, he worked at Sangamo Electric Company of Springfield, Illinois as an engineer trainee in both the sonar and network filter design departments. Subsequently he served first as a teaching assistant and then as an instructor in the Department of Electrical Engineering of the University of Illinois.

From November 1965 to August 1969, he was assigned to the 6595th Aerospace Test Wing at Vandenberg Air Force Base, California. In the Instrumentation and Electronics Section there, he was responsible for providing equipment for support of missile operations in the areas of telemetry processing processing, microwave, timing, and television as well as associated software and later served as section head. He was selected and has pursued his Ph.D. studies through the Air Force Institute of Technology.

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Wilde, R. R. and P. V. Kokotović, "Stability of Singularly Perturbed Systems and Networks with Parasitics," IEEE Trans. Automatic Control, to appear.

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