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FAST ENVELOPE ALGORITHMS USING LINEAR COMBINATIONS OF QUADRATURE SAMPLES

James K. Beard

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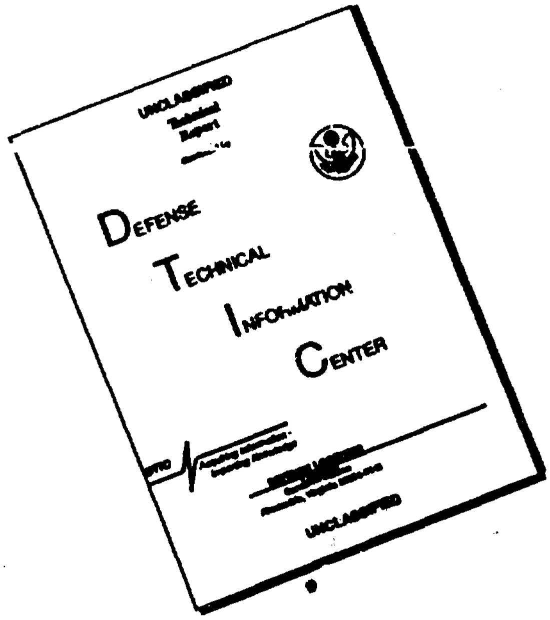

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1.0 Introduction

In many digital signal processing systems, it is necessary to convert an array of quadrature samples to an envelope array, a process equivalent to rectifying in analog systems. For each quadrature sample, the quantity to be computed is

$$r = \sqrt{x_s^2 + y_s^2} \quad , \tag{1-1}$$

where x_s and y_s are the in phase and quadrature components of the signal.

In practical systems, x_s and y_s are carried as integers and the output must also be an integer. Also, it is desirable to avoid overflow problems associated with squaring, and performing a square root is often prohibitive due to computation time-data rate tradeoffs. A simple estimate of r which avoids squaring and square roots, but which does not cause errors that would affect the performance of the system, is therefore required.

This report describes this goal by using linear combinations of

$$x = \max(|x_s|),|y_s|), \;\; y = \min(|x_s|,|y_s|) \tag{1-2}$$

of the form

$$\hat{r} = ax + by \tag{1-3}$$

to compute an approximation or estimate of r as defined by Eq. (1-1). Considerable attention is devoted to the implementation by simple hardware or software oriented algorithms.

The evaluation of approximations of the type in Eq. (1-3) offers many conceptual difficulties. In order to minimize the complexity of the concepts and mathematics required to analyze the approximations, they are presented in terms of a new set of variables. For example, in the (x_s, y_s) or (x,y) plane, r as given by Eq. (1-1) is no single point or curve but is a set of circles centered at the origin. In the $(x_s/r, y_s/r)$ or $(x/r, y/r)$ plane, r becomes a single circle of radius one and centered at the origin, called the unit circle. However, in this plane, linear combinations of x and y, such as $\hat{r}$ as given by Eq. (1-3), are circles passing <u>through</u> the origin. However, the complex variable w, where

$$w = \frac{x + jy}{\hat{r}} = (x/\hat{r}, y/\hat{r}) \qquad (1\text{-}4)$$

and $\hat{r}$ is a linear combination of x and y as given by Eq. (1-3), offers the following advantages: (a) in the w plane, where curves are plotted as Re{w} versus Im{w}, $\hat{r}$ as given by Eq. (1-3) is a straight line, and (b)r as given by Eq. (1-1) is the unit circle (a circle of radius one centered at the origin) as in the $(x/r, y/r)$ plane. In the w plane, it becomes obvious that the optimal estimates of r are polygons approximating the unit circle, and other concepts are equally simplified. The most important of these concepts are (a) the use of (x,y) as given by Eq. (1-2) allows consideration of only half the first quadrant rather than the entire unit circle, (b) the use of n straight lines in this region results in an 8n sided polygon approximating the unit circle in the w plane, and (c) the use of $\hat{r} = x$ (i.e., b=0 in (Eq. 1-3)) for small y results in an 8n-4 sided polygon in the w plane.

The notations used throughout this report are as follows.

Quantity	Definition
x_s	In phase sample.
y_s	Quadrature sample.
x	Largest of absolute values of x_s, y_s.
y	Smallest of absolute values of x_s, y_s.
r	True value of envelope.
$\hat{r}$	Estimate of envelope.
e	Peak absolute error of estimate.
s	rms error of estimate.
b	Multiplicative bias of estimate.

2.0 Basis of Solutions

2.1 Assumptions

The hardware capabilities used in the algorithms presented are (a) absolute value, (b) shifting, (c) magnitude comparison between two positive integers, and (d) addition-subtraction. Multiplication, in the form of shift-adds, is used, but general purpose multiplication hardware is not required.

The properties of the input assumed in the development of the algorithms are as follows. If a variable θ is defined by

$$\theta_s = \tan^{-1}(y_s/x_s) + \pi h(x_s, y_s) \quad , \tag{2-1}$$

the function $h(x_s, y_s)$ being defined by

$$h(x_s, y_s) \begin{cases} = 0 & , \quad x_s \geq 0 \\ = +1 & , \quad x_s < 0 \quad , \quad y_s \geq 0 \\ = -1 & , \quad x_s < 0 \quad , \quad y_s \leq 0 \quad , \end{cases} \tag{2-2}$$

the probability density function of θ is

$$p(\theta_s) \begin{cases} = \dfrac{1}{2\pi} & , \quad -\pi < \theta \leq \pi \\ = 0 & , \quad \text{otherwise} \quad ; \end{cases} \tag{2-3}$$

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i.e., the signal phase θ_s is uniformly distributed over $[-\pi, \pi]$. If the absolute value of both samples is taken and x is arbitrarily taken to be the largest, then

$$p(\theta) \begin{cases} = \frac{4}{\pi} & , \quad 0 \leq \theta \leq \frac{\pi}{4} \\ = 0 & , \quad \text{otherwise} \quad ; \end{cases} \tag{2-4}$$

i.e., the modified signal phase θ is uniformly distributed over $[0, \pi/4]$.

2.2 Coordinate Systems

The most obvious coordinate system for consideration of the problem at hand, namely, the x_s, y_s plane, has the fundamental drawback that in this system r is a family of circles rather than a single locus. Therefore, the x_s, y_s plane coordinate system is not used in this report.

The y/x,r/x plane constrains the envelope to the locus

$$r/x = \sqrt{1+(y/x)^2} \quad , \quad 0 \leq y/x \leq 1 \quad ,$$

but the weakness of this system is that y/x has the probability density

$$p(y/x) = \frac{4/\pi}{1+(y/x)^2} \quad , \quad 0 \leq y/x \leq 1 \quad ,$$

a form which complicates evaluation of b and s. Also, this coordinate system is related to optimal estimates by complicated relationships.

The $x/r, y/r$ plane constrains the envelope value to the unit circle. Estimates of the form $\hat{r}=ax+by$ in normalized form,

$$\hat{r}/r = a\cos\theta + b\sin\theta \quad , \quad \cos\theta = x/r \quad , \quad \sin\theta = y/r \quad , \tag{2-5}$$

are circles passing through the origin with diameter $\left(a^2+b^2\right)^{1/2}$ and centered at $(a/2, b/2)$, as shown in Fig. 1. The angle of maximum $\hat{r}$, ψ, is

$$\psi = \tan^{-1}(b/a), \ \hat{r} \text{ at maximum} \quad . \tag{2-6}$$

The θ of Eq. (2-5) is the same as that of Eq. (2-4). This is the coordinate system in which the solution will be presented.

Another coordinate system of importance is the w plane, where w is the complex variable

$$w = r/\hat{r}\,(\cos\theta + j\sin\theta) = u + jv = \frac{x}{ax+by} + j\,\frac{y}{ax+by} \quad . \tag{2-7}$$

The variable w is the complex conjugate of the reciprocal of $\hat{r}/r$ expressed as a complex variable.

$$\frac{1}{w} = \frac{\hat{r}}{x+jy} = \frac{\hat{r}}{r}\exp(-j\theta) \quad .$$

The true envelope is again the unit circle, but Eq. (2-5) in the w plane is a straight line,

$$au + bv = 1 \quad , \tag{2-8}$$

with slope $-a/b$ (or, $\tan(\psi + \pi/2)$, with closest approach to the origin $\left(a^2+b^2\right)^{-1/2}$. The line corresponding to the circle in Fig. 1 is shown in Fig. 2. The angle ψ is the angle of minimum $|w|$.

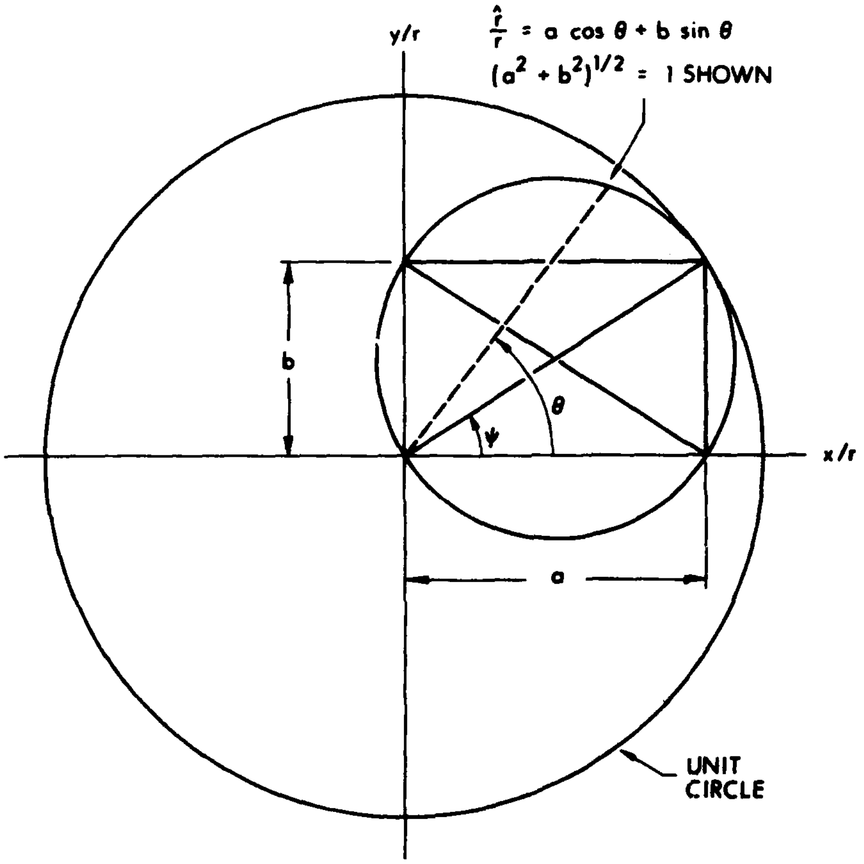

FIGURE 1
x/r, y/r COORDINATE SYSTEM

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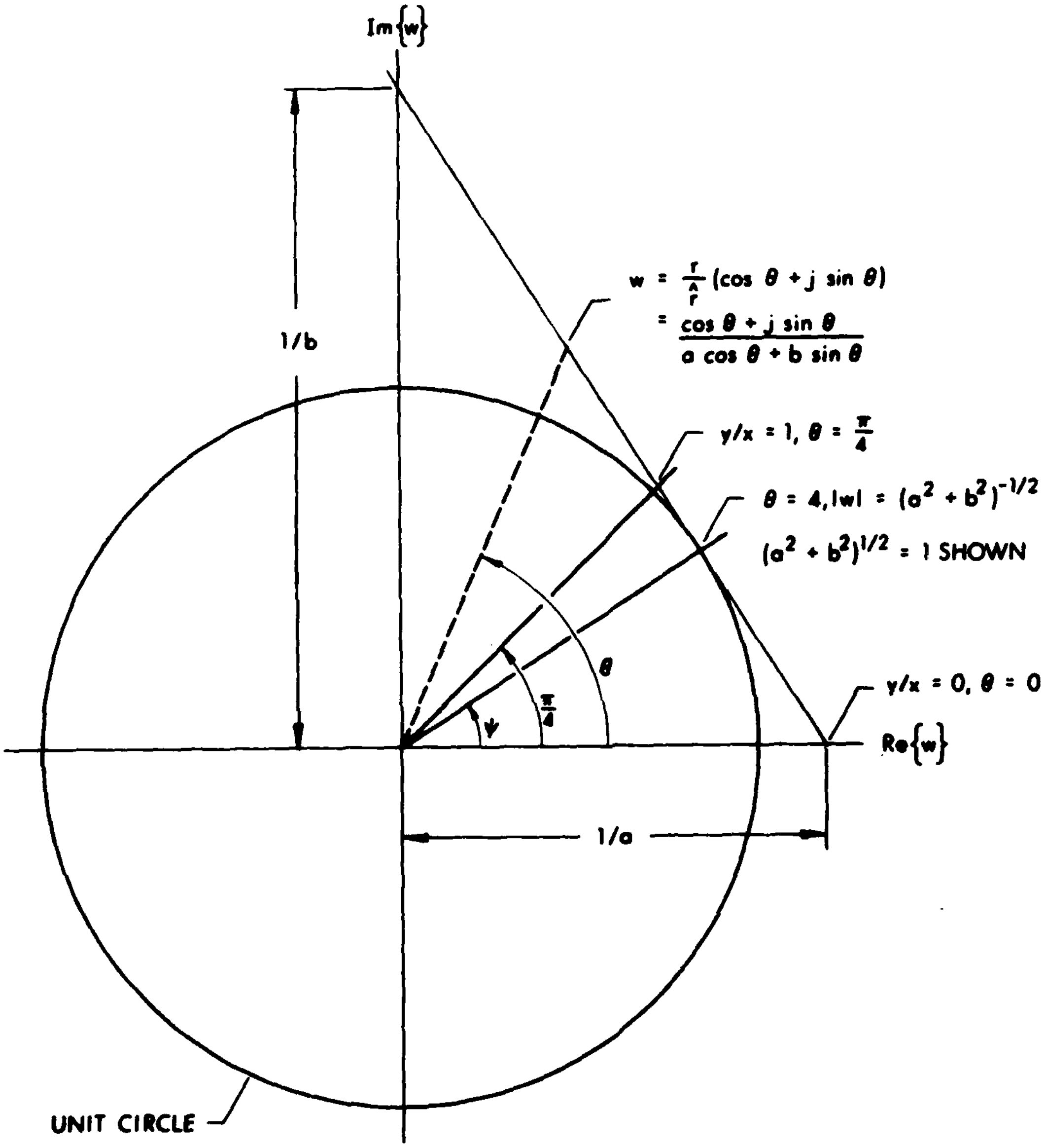

FIGURE 2
THE w PLANE

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3.0 Optimal Estimates

Optimal estimates, in the context of this report, are taken to mean the use of n straight lines in the w plane (i.e., simple linear combinations of x_s and y_s) to approximate the unit circle with minimum rms error s_n. The multiplicative bias b_n is irrelevant to signal processing considerations. In the w plane,

$$w = u + jv \quad , \quad v/u = y/x = \tan\theta \quad ,$$

and θ is uniformly distributed, so an optimal fit for a fixed number of lines will be a regular polygon, independent of the loss or error criterion. The use of the absolute values x and y, $x \geq y \geq 0$, instead of x_s and y_s restricts θ to the range 0 to $\pi/4$, and n linear combinations of x and y are equivalent, in general, to 8n linear combinations of x_s and y_s. If the line at $\theta=0$ in the w plane is vertical ($\hat{r}=cx$), or if the line at $\theta=\pi/4$ in the w plane has a slope of $-1(\hat{r}=c(x+y))$, the lines in the w plane will be parallel to tangents to the unit circle at those points and the polygon will have 8n-4 or 8n-8 sides. Thus, the $\hat{r}=c(x+y)$ case is to be avoided; however, the simplicity of $\hat{r}=cx$ near $\theta=0$ makes this case worthy of consideration. Accordingly, only the 8n and 8n-4 cases are of practical importance. These two cases are shown in Figs. 3 and 4 for n=2.

The parameters e, s, and b will be computed below for the 8n and 8n-4 sided polygon cases. The subscript p will be used in the 8n-4 case equations to prevent confusion.

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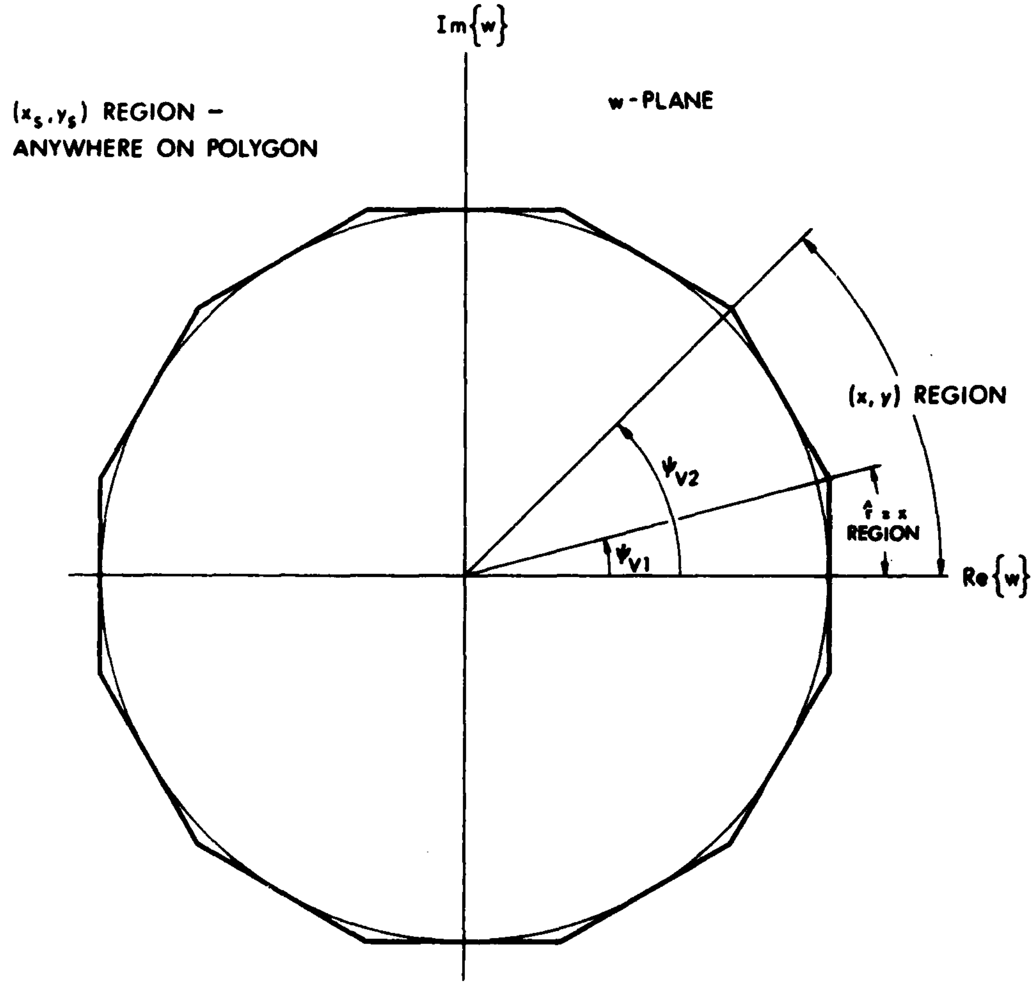

FIGURE 3
8p-4 CASE, p = 2, IN THE w PLANE

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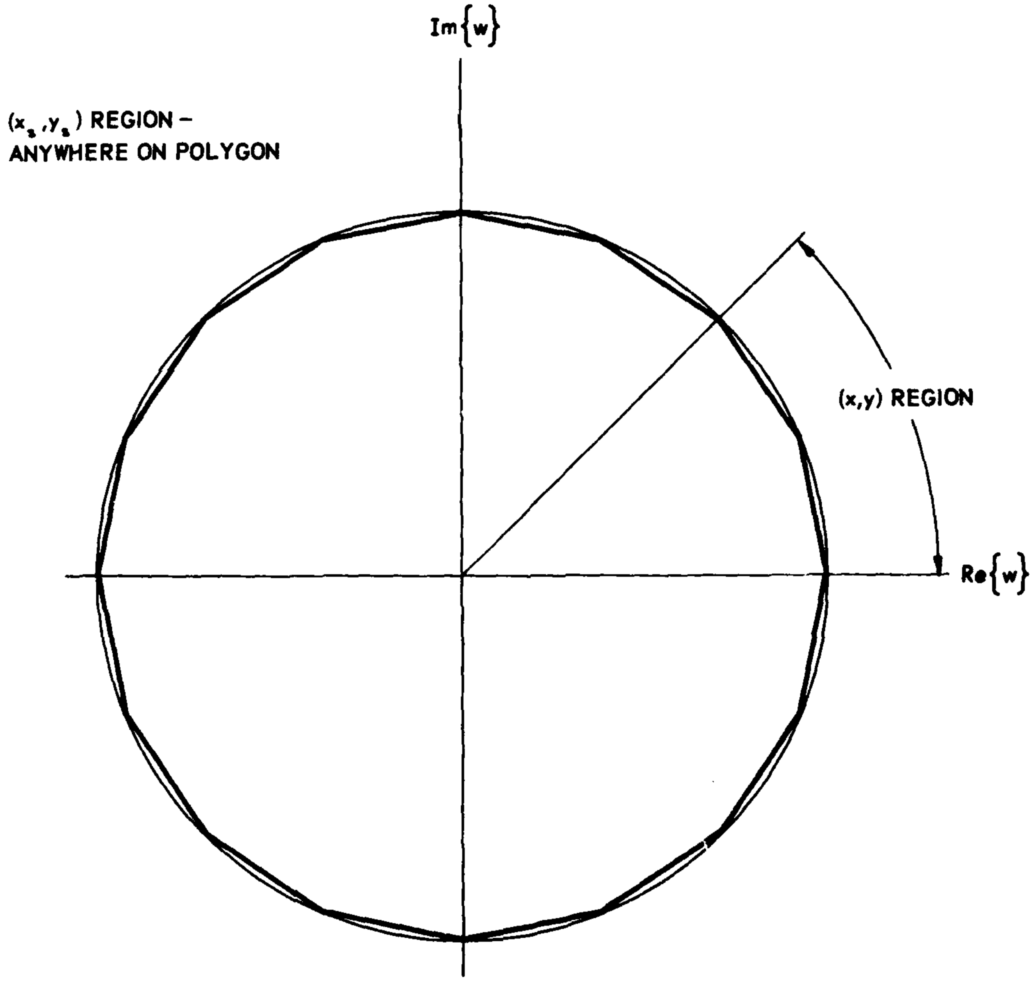

FIGURE 4
8n CASE, n = 2, IN THE w PLANE

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The sides of the regular polygons are defined by the angles of minimum $|w|$,

$$\psi_m = \frac{(m-1/2)\pi}{4n} \quad , \quad 1 \leq m \leq n \; (8n \text{ case}) \quad , \tag{3-1}$$

$$\psi_q = \frac{(q-1)\pi}{4p-2} \quad , \quad 1 \leq q \leq p \; (8p\text{-}4 \text{ case}) \quad , \tag{3-2}$$

the vertices,

$$\psi_{vm} = \frac{m\pi}{4n} \quad , \quad 0 \leq m \leq n \; (8n \text{ case}) \quad , \tag{3-3}$$

$$\psi_{vq} = \frac{(q-1/2)\pi}{4p-2} \quad , \quad 1 \leq q \leq p \; (8p\text{-}4 \text{ case}) \quad , \tag{3-4}$$

and the fact that the vertices are on the unit circle in the 8n case,

$$a_m^2 + b_m^2 = \sec^2\left(\frac{\pi}{8n}\right) \quad , \tag{3-5}$$

and the sides are tangent to the unit circle in the 8p-4 case,

$$a_q^2 + b_q^2 = 1 \quad . \tag{3-6}$$

Using Eqs. (3-5) and (3-6) with $b/a = \tan\psi$, where ψ is given by Eq. (3-1) and (3-2), yields

$$\begin{aligned} \hat{r}_m &= \left[x \cos(\psi_m) + y \sin(\psi_m)\right] \sec\left(\frac{\pi}{8n}\right) \\ &= r \cos(\theta-\psi_m) \sec\left(\frac{\pi}{8n}\right) \quad , \end{aligned} \tag{3-7}$$

and

$$\hat{r}_q = x\cos(\psi_q) + y\sin(\psi_q)$$
$$= r\cos(\theta-\psi_q) \quad . \tag{3-8}$$

For the 8n case, the peak error is at the ψ_m,

$$e_n = \frac{\hat{r}(\psi_m)}{r} - 1 = \sec\left(\frac{\pi}{8n}\right) - 1 \quad , \tag{3-9}$$

while for the 8p-4 case, the peak error is at the vertices,

$$e_p = 1 - \frac{\hat{r}(\psi_{vq})}{r} = 1 - \cos\left(\frac{\pi}{8p-4}\right) \quad . \tag{3-10}$$

The multiplicative bias is found by integrating $\hat{r}/r$ with respect to θ over one side of the polygon and dividing by the angle between the vertices, since θ is uniformly distributed over the sector corresponding to each side of the w plane polygon.

Thus,

$$b_n = \frac{8n}{\pi}\int_0^{\pi/(8n)} \cos\left(\theta - \frac{\pi}{8n}\right)\sec\left(\frac{\pi}{8n}\right) d\theta = \frac{\tan\left(\frac{\pi}{8n}\right)}{\frac{\pi}{8n}} \quad , \tag{3-11}$$

and

$$b_p = \frac{8p-4}{\pi}\int_0^{\pi/(8p-4)} \cos\theta d\theta = \frac{\sin\left(\frac{\pi}{8p-4}\right)}{\frac{\pi}{8p-4}} \quad , \tag{3-12}$$

using the obvious expedient of integrating over half a side.

The rms error is found by averaging $(\hat{r}/br)^2-1$ in the same way. Thus,

$$s_n^2 = \frac{8n}{\pi}\int_0^{\pi/(8n)} \frac{\cos^2\left(\theta - \frac{\pi}{8n}\right)}{\left[\sin\left(\frac{\pi}{8n}\right)\Big/\left(\frac{\pi}{8n}\right)\right]^2}\, d\theta - 1$$

$$= \frac{1}{2}\,\frac{1 + \dfrac{\sin\left(\frac{\pi}{4n}\right)}{\left(\frac{\pi}{4n}\right)}}{\left[\sin\left(\frac{\pi}{8n}\right)\Big/\left(\frac{\pi}{8n}\right)\right]^2} - 1 \quad , \tag{3-13}$$

and

$$s_p^2 = \frac{8p-4}{\pi}\int_0^{\pi/(8p-4)} \frac{\cos^2(\theta)}{\left[\sin\left(\frac{\pi}{8p-4}\right)\Big/\left(\frac{\pi}{8p-4}\right)\right]^2}\, d\theta - 1$$

$$= \frac{1}{2}\,\frac{1 + \dfrac{\sin\left(\frac{\pi}{4p-2}\right)}{\frac{\pi}{4p-2}}}{\left[\sin\left(\frac{\pi}{8p-4}\right)\Big/\left(\frac{\pi}{8p-4}\right)\right]^2} - 1 \quad . \tag{3-14}$$

The multiplicative bias b is unimportant in most system applications, but $-10 \log_{10}(s^2)$ is the upper limit for signal-to-noise out of the system. For quadrature sampled, very narrowband noise, $20 \log_{10}(e)$ is the relative amplitude of the peak-to-peak "flutter", or amplitude modulation, generated by the envelope algorithm.

For a 4k sided polygon, the asymptotic forms for e, b, and s for large k are, from Eq. (3-9) through Eq. (3-14),

$$\begin{aligned}
e &\sim \frac{1}{2}\left(\frac{\pi}{4k}\right)^2 \\
b_n &\sim 1 + \frac{1}{3}\left(\frac{\pi}{8n}\right)^2 \quad (k=2n) \\
b_p &\sim 1 - \frac{1}{6}\left(\frac{\pi}{8p-4}\right)^2 \quad (k=2p-1) \\
s &\sim \frac{1}{3\sqrt{5}}\left(\frac{\pi}{4k}\right)^2 \quad .
\end{aligned} \tag{3-15}$$

Also, for large k, the peak error of $\hat{r}$ is described asymptotically by

$$1 - \frac{1}{3}\, e \leq \frac{\hat{r}}{br} \leq 1 + \frac{2}{3}\, e \quad . \tag{3-16}$$

4.0 Implementation

The simplest way to implement Eq. (3-7) or Eq. (3-8) is to compute

$$\hat{r}_1 = x+y\tan\left(\frac{\pi}{8n}\right) \quad (8n \text{ case}) \tag{4-1}$$

or

$$\hat{r}_1 = x \quad (8p-4 \text{ case}) \tag{4-2}$$

first. Then, for $1 \leq m \leq n-1$ or $1 \leq q \leq p-1$, the quantity

$$\hat{r}_{m+1} - \hat{r}_m = 2\tan\left(\frac{\pi}{8n}\right)\left(-x\sin\left(\frac{m\pi}{4n}\right) + y\cos\left(\frac{m\pi}{4n}\right)\right) \tag{4-3}$$

or

$$\hat{r}_{q+1} - \hat{r}_q = 2\sin\left(\frac{\pi}{8p-4}\right)\left(-x\sin\left(\frac{\left(q-\frac{1}{2}\right)\pi}{4p-2}\right) + y\cos\left(\frac{\left(q-\frac{1}{2}\right)\pi}{4p-2}\right)\right) \tag{4-4}$$

can be used to form $\hat{r}_2$, $\hat{r}_3$, ..., as follows. Compute

$$\Delta_m = y - x\tan\left(\frac{m\pi}{4n}\right) \tag{4-5}$$

or

$$\Delta_q = y - x\tan\left(\frac{\left(q-\frac{1}{2}\right)\pi}{4p-2}\right) \quad . \tag{4-6}$$

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If $\Delta \leqq 0$, the computation is completed; otherwise, use

$$\hat{r}_{m+1} - \hat{r}_m = 2 \tan\left(\frac{\pi}{8n}\right) \cos\left(\frac{m\pi}{4n}\right) \Delta_m \tag{4-7}$$

or

$$\hat{r}_{q+1} - \hat{r}_q = 2 \sin\left(\frac{\pi}{8p-4}\right) \cos\left(\frac{\left(q - \frac{1}{2}\right)\pi}{4p-2}\right) \Delta_q \tag{4-8}$$

to update $\hat{r}_m$ or $\hat{r}_q$. If m=n or q=p, the computation is finished; otherwise, repeat the computation of Δ and subsequent sign check.

The quantities

$$c_1(m) = \tan\left(\frac{m\pi}{4n}\right) \quad , \quad 1 \leqq m \leqq n-1 \quad , \tag{4-9}$$

$$c_2(m) = 2 \tan\left(\frac{\pi}{8m}\right) \cos\left(\frac{m\pi}{4n}\right) \quad , \quad 1 \leqq m \leqq n-1 \quad , \tag{4-10}$$

$$c_1(q) = \tan\left(\frac{\left(q - \frac{1}{2}\right)\pi}{4p-2}\right) \quad , \quad 1 \leqq q \leqq p-1 \quad , \tag{4-11}$$

and

$$c_2(q) = 2 \sin\left(\frac{\pi}{8p-4}\right) \cos\left(\frac{\left(q - \frac{1}{2}\right)\pi}{4p-2}\right) \quad , \quad 1 \leqq q \leqq p-1 \quad , \tag{4-12}$$

along with $\tan(\pi/8n)$, e_n, e_p, b_n, b_p, s_n, and s_p, are given in the appendix, for all n and p from 1 to 12, in both decimal and octal.

The multiplications in Eqs. (4-1), (4-5), (4-6), (4-7), and (4-8) can be accomplished within the purposes of the algorithm by using only the top L bits of $\tan(m\pi/4n)$, c_1 and c_2, where L is the number of zero leading bits in b or S, depending on whether narrowband flutter or signal-to-noise is the limiting factor. In the CDC 3200, if (A)=-y and (Q)=+x, $-\Delta$ can be computed faster by several shift-add (SHQ, AQA) instructions than by a single multiply-add (MUA, ADA) execution if the number of nonzero bits among the most significant L in the multiplier is less than six. An AZJ,GE instruction can then be used to retrieve $\hat{r}$ and exit if $\Delta \leq 0$. The p=2 (regular 12 sided polygon) case is programmed as an example in the appendix; entering with x_s and y_s in (A) and (Q), the execution times are 17 μsec minimum, 24 μsec average, and 34 μsec maximum, including call and return with the result in (A). This routine is on disc file SPAM.

5.0 Appendix

5.1 Example

This is a listing of a COMPASS routine for the 12 sided polygon envelope estimate shown in Fig. 3. The peak error is +0, -0.034_{10}, the multiplicative bias is 0.989, and the rms relative error is 0.01_{10} so that narrowband flutter is 0.15 dB peak-to-peak and the maximum signal-to-noise is 40 dB. The execution times are 17 μsec minimum, 24 μsec average, and 34 μsec maximum, including call and return. Entry is accomplished by a RTJ XY.SQRT instruction with the fixed point quadrature samples in (A) and (Q), and return is to P+1 with the envelope estimate in (A).

LISTING OF XY.SQRT

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```
                                              ENTRY    XY.SQRT
00000  20000024 20 0 P00024 0                 LDA      RHAT             RETRIEVE ENV
00001  01077777 01 0  77777 0      XY.SQRT    UJP      **               QUAD SAMPLES IN (A) AND (Q)
00002  05400000 05 1  00000 0                 ASG,S    0                SET (A) = IABS((A))
00003  16477777 16 1  77777 0                 XOA,S    -0
00004  05500000 05 1  00000 1                 QSG,S    0                SET (Q) = IABS((Q))
00005  16577777 16 1  77777 1                 XOQ,S    -0
00006  03700010 03 1 P00010 3                 AQJ,LT   *+2              PUT SMALLEST IN (A)
00007  13000030 13 0  00030 0                 SHAQ     24
00010  41000024 41 0 P00024 0                 STQ      RHAT             RHAT = X
00011  16477777 16 1  77777 0                 XOA,S    -0               (A) = -Y, (Q) = +X
                                *** AN AQF HERE WOULD SAVE (AQ) FOR P .GT. 2
00012  12477775 12 1  77775 0                 SHQ      -2               (Q) = .2H*X
00013  53040000 53 0  40000 0                 AQA                       (A) = -(Y - .2B*X) = -DEL
00014  03200000 03 0 P00000 2                 AZJ,GE   XY.SQRT-1        EXIT IF DEL .LE. 0
00015  12477773 12 1  77773 0                 SHQ      -4               (Q) = .01H*X
00016  53040000 53 0  40000 0                 AQA                       (A) = -DEL
00017  16477777 16 1  77777 0                 XOA,S    -0               (A) = +DEL = Y - .21B*X
00020  13077746 13 0  77746 0                 SHAQ     -25
00021  20000024 20 0 P00024 0                 LDA      RHAT             (A) = X, (Q) = .4B*DEL
00022  53040000 53 0  40000 0                 AQA                       (A) = FINAL RHAT
00023  01400001 01 1 P00001 0                 UJP,I    XY.SQRT          EXIT
00024                              RHAT       BSS      1
                                              END

                                            NUMBER OF LINES WITH DIAGNOSTICS        0
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5.2 Tables

Following are tables of constants to be used in choosing and implementing an envelope estimate based on a 4k sided polygon. For any k, the peak relative error is +0,-EP for odd k or +EP,-0 for even k, the rms relative error is 20 $\log_{10}$SP, and the multiplicative bias is BP. If L is the number of leading zero bits in BP or SP, implementation is as follows.

Find x = absolute value of largest sample, y = absolute value of smallest sample. Define an initial $\hat{r}$ by

$$\hat{r} = x \quad , \quad k \text{ odd} \quad ; \quad \hat{r} = x + C_o y \quad , \quad k \text{ even} \quad . \tag{5-1}$$

Use shift-adds for the multiplication, using only the nonzero bits in the L bits of C_o following the binary point. Then, compute

$$-\Delta = -y + x*C1(1) \quad , \tag{5-2}$$

using shift-adds with the nonzero bits among the L most significant in C1(1) for the multiplication. If $-\Delta \geq 0$, exit with $\hat{r}$ in (A). Otherwise, update $\hat{r}$ using

$$\hat{r} = \hat{r} + \Delta*C2(1) \quad , \tag{5-3}$$

using shift-adds with the nonzero bits among the L most significant in C2(1) for the multiplication, as before. If no C1's or C2's remain, exit with $\hat{r}$ in (A). Otherwise, repeat the operations and checks beginning with Eq. (5-2) using C1(2) and C2(2), C1(3) and C2(3), etc., until either $-\Delta \geq 0$ or there are no more C1's or C2's.

TABLES OF PARAMETERS

4 SIDED POLYGON, ERROR LIMITS +0.-EP

	EP	RP	SP	CO
DECIMAL	2.9289321881E-01	9.0031631615E-01	9.7720810291E-02	0
OCTAL	.2257542	.7147544	.0620207	0

C1 ARRAY, DECIMAL

C1 ARRAY, OCTAL

.

C2 ARRAY, DECIMAL

C2 ARRAY, OCTAL

.

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8 SIDED POLYGON, ERROR LIMITS +EN,-0

	EN	RN	SP	CO
DECIMAL	8.2392200273E-02	1.0547861751E 00	2.3331699511E-02	4.1421356236E-01
OCTAL	.0521365	1.0340317	.0137442	.3240475

C1 ARRAY, DECIMAL

C1 ARRAY, OCTAL
.

C2 ARRAY, DECIMAL

C2 ARRAY, OCTAL
.

12 SIDED POLYGON. ERROR LIMITS +0.-EP

	EP	RP	SP	C0
DECIMAL	3.4074173716E-02	9.8861592945E-01	1.0284343176E-02	0
OCTAL	.0213443	.7721276	.0052100	0

C1 ARRAY. DECIMAL
2.6794919243E-01

C1 ARRAY. OCTAL
.2111412 .

C2 ARRAY. DECIMAL
5.0000000000E-01

C2 ARRAY. OCTAL
.4000000 .

16 SIDED POLYGON, ERROR LIMITS +EN,-0

	EN	BN	SP	CO
DECIMAL	1.9591158227E-02	1.0130523683E 00	5.7683470148E-03	1.9891236738E-01
OCTAL	.0120176	1.0065355	.0027501	.1456575

C1 ARRAY, DECIMAL
4.1421356236E-01

C1 ARRAY, OCTAL
.3240475 .

C2 ARRAY, DECIMAL
3.6754212998E-01

C2 ARRAY, OCTAL
.2741350 .

```
20 SIDED POLYGON, ERROR LIMITS +0.-EP

                     EP                  BP                  SP                  C0
DECIMAL      1.2311659433E-02    9.9589273524E-01    3.6868452096E-03                    0
OCTAL                .0062333            .7757132            .0017064                    0

C1 ARRAY, DECIMAL
   1.5838444032E-01    5.0952544949E-01

C1 ARRAY, OCTAL
 .1210574  .4047010  .


C2 ARRAY, DECIMAL
   3.0901699437E-01    2.7876825791E-01

C2 ARRAY, OCTAL
 .2361570  .2165653  .
```

24 SIDED POLYGON, ERROR LIMITS +EN,-0

	EN	RN	SP	CO
DECIMAL	8.6289605824E-03	1.0057509965E 00	2.5584745611E-03	1.3165249759E-01
OCTAL	.0043260	1.0027435	.0012366	.1033177

C1 ARRAY, DECIMAL
2.6794919243E-01 5.7735026918E-01

C1 ARRAY, OCTAL
.2111412 .4474647 .

C2 ARRAY, DECIMAL
2.5433309507E-01 2.2802881476E-01

C2 ARRAY, OCTAL
.2021577 .1646003 .

28 SIDED POLYGON, ERROR LIMITS +0,-EP

	EP	BP	SP	C0
DECIMAL	6.2877901073E-03	9.9790319004E-01	1.8788768504E-03	0
OCTAL	.0031602	.7767323	.0007544	0

C1 ARRAY, DECIMAL
1.1267293990E-01   3.4991513395E-01   6.2834164537E-01

C1 ARRAY, OCTAL
.0715404 .2631201 .5015540 .

C2 ARRAY, DECIMAL
2.2252093396E-01   2.1136280516E-01   1.8960606274E-01

C2 ARRAY, OCTAL
.1617344 .1541574 .1410501 .

```
32 SIDED POLYGON, ERROR LIMITS +FN,-0

                        FN                 AN                 SP                 CO
DECIMAL       4.8385723494E-03   1.0032251966E 00   1.4381155725E-04   9.8491403356E-02
OCTAL                 .0023643          1.0015154           .0005711            .0623327

C1 ARRAY, DECIMAL
   1.9891236738E-01    4.1421356236E-01    6.6817863792E-01

C1 ARRAY, OCTAL
 .1456575  .3240475  .5260670  .

C2 ARRAY, DECIMAL
   1.9319783731E-01    1.8198838338E-01    1.6378521793E-01

C2 ARRAY, OCTAL
 .1427255  .1351331  .1236672  .
```

```
36 SIDED POLYGON, ERROR LIMITS +0,-EP

                        EP                  RP                  SP                  C0
DECIMAL      3.8053018799E-03    9.9873124394E-01    1.1360627031E-03
OCTAL                .0017454            .7772633            .0004516                  0
                                                                                       0
C1 ARRAY, DECIMAL
   8.7488663525E-02    2.6794919243E-01    4.6630765815E-01    7.0020753820E-01

C1 ARRAY, OCTAL
 .0546265  .2111412  .3565776  .5464032  .

C2 ARRAY, DECIMAL
   1.7364817767E-01    1.6837196565E-01    1.5797985667E-01    1.4278760968E-01

C2 ARRAY, OCTAL
 .1307207  .1261516  .1207054  .1110667  .
```

40 SIDED POLYGON, ERROR LIMITS +EN,-0

	EN	RN	SP	CO
DECIMAL	3.0921984289E-03	1.0020612537E 00	9.2009359321E-04	7.8701706624E-02
OCTAL	.0014525	1.0010343	.0003612	.0502271

C1 ARRAY, DECIMAL

1.5838444032E-01   3.2491969623E-01   5.0952544949E-01   7.2654252799E-01

C1 ARRAY, OCTAL

.1210574 .2462676 .4047010 .5637726 .

C2 ARRAY, DECIMAL

1.5546551643E-01   1.4969954224E-01   1.4024746849E-01   1.2734203662E-01

C2 ARRAY, OCTAL

.1174623 .1145127 .1076350 .1011460 .

44 SIDED POLYGON. ERROR LIMITS +0.-EP

	EP	BP	SP	CO
DECIMAL	2.5478853786E-03	9.9915056049E-01	7.6032149523E-04	0
OCTAL	.0012337	.7774413	.0003073	0

C1 ARRAY. DECIMAL

7.1521411558E-02 2.1753668456E-01 3.7298071632E-01 5.4604131117E-01 7.4859062328E-01

C1 ARRAY. OCTAL

.0444747 .1573017 .2767565 .4274454 .5772164 .

C2 ARRAY. DECIMAL

1.4231483827E-01 1.3941771856E-01 1.3368245616E-01 1.2522580446E-01 1.1421991649E-01

C2 ARRAY. OCTAL

.1106730 .1073034 .1043440 .1000732 .0723661 .

48 SIDED POLYGON, ERROR LIMITS +EN,-0

	EN	AN	SP	CO
DECIMAL	2.1456707619E-03	1.0014303451E 00	6.3882228408E-04	6.5543462813E-02
OCTAL	.0010624	1.0005670	.0002474	.0414357

C1 ARRAY, DECIMAL

1.3165249758E-01	2.6794919243E-01	4.1421356236E-01	5.7735026918E-01	7.6732698795E-01

C1 ARRAY, OCTAL

.1033177	.2111412	.3240475	.4474647	.6106761 .

C2 ARRAY, DECIMAL

1.2996545880E-01	1.2662024695E-01	1.2110852757E-01	1.1352460770E-01	1.0399825031E-01

C2 ARRAY, OCTAL

.1024255	.1006506	.0760037	.0720776	.0651764 .

```
52 SIDED POLYGON, ERROR LIMITS +0,-EP

                     EP                 BP                 SP                  C0
DECIMAL    1.8244457606E-03   9.9939177739E-01   5.4428658831E-04                   0
OCTAL              .0007362           .7775404           .0002165                   0

C1 ARRAY, DECIMAL
   6.0488856060E-02   1.8325680127E-01   3.1161260248E-01   4.5006346613E-01   6.0452107439E-01
   7.8344996792E-01

C1 ARRAY, OCTAL
 .0367606  .1356475  .2374273  .3463353  .4654075  .6211006  .

C2 ARRAY, DECIMAL
   1.2053668025E-01   1.1877898403E-01   1.1528922275E-01   1.1011828500E-01   1.0334157468E-01
   9.5057911507E-02

C2 ARRAY, OCTAL
 .0755560  .0746412  .0730163  .0703027  .0647223  .0605267  .
```

56 SIDED POLYGON. ERROR LIMITS +EN.-0

	EN	RN	SP	CO
DECIMAL	1.5756637731E-03	1.0010503874E 00	4.6928529175E-04	5.6158725410E-02
OCTAL	.0006350	1.0004233	.0001730	.0346016

C1 ARRAY. DECIMAL
1.1267293990E-01 2.2824347438E-01 3.4991513395E-01 4.8157461880E-01 6.2834164537E-01
7.9747338891E-01

C1 ARRAY. OCTAL
.0715404 .1646705 .2631201 .3664417 .5015540 .6302347 .

C2 ARRAY. DECIMAL
1.1161136138E-01 1.0950155431E-01 1.0601470167E-01 1.0119465265E-01 9.5102022146E-02
8.7813428536E-02

C2 ARRAY. OCTAL
.0711122 .0700411 .0662171 .0636375 .0605423 .0547536 .

60 SIDED POLYGON, ERROR LIMITS +0,-EP

	EP	HP	SP	CO
DECIMAL	1.3704652083E-03	9.9954313647E-01	4.0883162586E-04	0
OCTAL	.0005472	.7776102	.0001531	0

C1 ARRAY, DECIMAL

5.2407779283E-02	1.5838444032E-01	2.6794919244E-01	3.8386403504E-01	5.0952544949E-01
6.4940759321E-01	8.0978403317E-01			

C1 ARRAY, OCTAL

.0326523 .1210574 .2111412 .3044235 .4047010 .5143762 .6364700 .

C2 ARRAY, DECIMAL

1.0452846327E-01	1.0338322755E-01	1.0110530356E-01	9.7719648702E-02	9.3263356923E-02
8.7785252291E-02	8.1345354067E-02			

C2 ARRAY, OCTAL

.0654114 .0647352 .0636101 .0620205 .0576003 .0547443 .0515142 .

64 SIDED POLYGON, ERROR LIMITS +EN.-0

	EN	RN	SP	CO
DECIMAL	1.2059964356E-03	1.0008039653E 00	3.5931127439E-04	4.9126849764E-02
OCTAL	.0004741	1.0003226	.0001362	.0311162

C1 ARRAY, DECIMAL

9.8491403356E-02 1.9891236738E-01 3.0334668361E-01 4.1421356236E-01 5.3451113594E-01
6.6817863792E-01 8.2067879082E-01

C1 ARRAY, OCTAL

.0623327 .1456575 .2332404 .3240475 .4215257 .5260670 .6441400 .

C2 ARRAY, DECIMAL

9.7780581121E-02 9.6365782251E-02 9.4022928225E-02 9.0774581998E-02 8.6652026926E-02
8.1694965463E-02 7.5951136824E-02

C2 ARRAY, OCTAL

.0620405 .0612556 .0601074 .0563640 .0542732 .0516477 .0467061 .

68 SIDED POLYGON, ERROR LIMITS +0.-EP

	EP	BP	SP	C0
DECIMAL	1.0670251504E-03	9.9964429962E-01	3.1815581454E-04	0
OCTAL	.0004276	.7776426	.0001233	0

C1 ARRAY, DECIMAL

4.6232790198E-02	1.3949404272E-01	2.3519786280E-01	3.3516621934E-01	4.4154319217E-01
5.5699634508E-01	6.8501593256E-01	8.3038944041E-01		

C1 ARRAY, OCTAL

.0275275 .1073274 .1703276 .2534657 .3420437 .4351352 .5365647 .6511215 .

C2 ARRAY, DECIMAL

9.2268359466E-02	9.1481158349E-02	8.9913472254E-02	8.7578676117E-02	8.4496689590E-02
8.0693807104E-02	7.6202473501E-02	7.1061007265E-02		

C2 ARRAY, OCTAL

.0571735 .0566552 .0560222 .0546562 .0532062 .0512413 .0470100 .0443042 .

```
72 SIDED POLYGON, ERROR LIMITS +EN,-0

                    EN                      BN                      SP                      CO
DECIMAL     9.5268513539E-04     1.0006351033E 00     2.8393240457E-04     4.3660942906E-02
OCTAL                .0003716             1.0002464             .0001123             .0262654

C1 ARRAY, DECIMAL
   8.7488663525E-02    1.7632698071E-01    2.6794919243E-01    3.6397023427E-01    4.6630765815E-01
   5.7735026918E-01    7.0020753820E-01    8.3909963118E-01

C1 ARRAY, OCTAL
 .0546265  .1322170  .2111412  .2722645  .3565776  .4474647  .5464032  .6554747  .

C2 ARRAY, DECIMAL
   8.6989599678E-02    8.5995270151E-02    8.4346464707E-02    8.2055731735E-02    7.9140505092E-02
   7.5622971425E-02    7.1529901274E-02    6.6892445389E-02

C2 ARRAY, OCTAL
 .0544236  .0540171  .0531367  .0520063  .0504122  .0465601  .0444771  .0421774  .
```

```
76 SIDED POLYGON, ERROR LIMITS +0.-EP

                     EP                  RP                  SP                    C0
DECIMAL      8.5424157439E-04    9.9971523655E-01    2.5481492449E-04                  0
OCTAL                .0003377             .7776653             .0001026                 0

C1 ARRAY, DECIMAL
    4.1360305942E-02    1.2464987001E-01    2.0967795045E-01    2.9771289897E-01    3.9020121084E-01
    4.8887021096E-01    5.9587066271E-01    7.1398627665E-01    8.4695679648E-01

C1 ARRAY, OCTAL
 .0251323  .0776442  .1532657  .2303335  .3076207  .3722323  .4610537  .5554372  .6615105  .


C2 ARRAY, DECIMAL
    8.2579345473E-02    8.2015244807E-02    8.0890896856E-02    7.9213982064E-02    7.6995955448E-02
    7.4251968386E-02    7.1000765085E-02    6.7264554565E-02    6.3068858936E-02

C2 ARRAY, OCTAL
 .0522175  .0517736  .0513251  .0504354  .0473300  .0460106  .0442643  .0423410  .0402251  .
```

80 SIDED POLYGON, ERROR LIMITS +EN,-0

	EN	RN	SP	CO
DECIMAL	7.7155858161E-04	1.0005143592E 00	2.2996012684E-04	3.9290107007E-02
OCTAL	.0003122	1.0002067	.0000742	.0240735

C1 ARRAY, DECIMAL

7.8701706824E-02	1.5838444032E-01	2.4007875908E-01	3.2491969623E-01	4.1421356236E-01
5.0952544949E-01	6.1280078812E-01	7.2654252799E-01	8.5408068546E-01	

C1 ARRAY, OCTAL

.0502271 .1210574 .1727272 .2462676 .3240475 .4047010 .4716020 .5637726 .6652241 .

C2 ARRAY, DECIMAL

7.8337977444E-02	7.7612761182E-02	7.6409036446E-02	7.4734224591E-02	7.2599651391E-02
7.0015483360E-02	6.7000646595E-02	6.3572728561E-02	5.9752863514E-02	

C2 ARRAY, OCTAL

.0500677 .0475716 .0470761 .0462071 .0451272 .0436621 .0422337 .0404312 .0364477 .

84 SIDED POLYGON, ERROR LIMITS +0,-EP

	EP	BP	SP	CO
DECIMAL	6.9929516758E-04	9.9976689070E-01	2.0852127637E-06	0
OCTAL	.0002573	.7777027	.0000665	0

C1 ARRAY, DECIMAL

3.7417360057E-02	1.1267293990E-01	1.8921020839E-01	2.6794919243E-01	3.4991513394E-01
4.3629553996E-01	5.2851543101E-01	6.2834164536E-01	7.3803358710E-01	8.6056958507E-01

C1 ARRAY, OCTAL

.0231206	.0715404	.1407003	.2111412	.2631201	.3373042	.4164631	.5015540	.5716771	.6704711

C2 ARRAY, DECIMAL

7.4730093586E-02	7.4312172588E-02	7.3478667777E-02	7.2234240450E-02	7.0585849952E-02
6.8542714749E-02	6.6116260883E-02	6.3320058063E-02	6.0169743792E-02	5.6682935910E-02

C2 ARRAY, OCTAL

.0462060	.0460304	.0454760	.0447676	.0441075	.0430600	.0416640	.0403270	.0366351	.0350131

88 SIDED POLYGON, ERROR LIMITS +EN,-0

	EN	RN	SP	CO
DECIMAL	6.3758058241E-04	1.0004250447E 00	1.9004677734E-04	3.5715090578E-02
OCTAL	.0002471	1.0001573	.0000617	.0222224

C1 ARRAY, DECIMAL

7.1521411558E-02	1.4377829399E-01	2.1753668456E-01	2.9362649294E-01	3.7298071632E-01
4.5668469790E-01	5.4604131170E-01	6.4266097716E-01	7.4859062328E-01	8.6650493252E-01

C1 ARRAY, OCTAL

.0444747 .1114725 .1573017 .2262543 .2767565 .3516451 .4274454 .5110256 .5772164 .6735151 .

C2 ARRAY, DECIMAL

7.1248185241E-02	7.0703124903E-02	6.9797777651E-02	6.8536756923E-02	6.6926488590E-02
6.4975178217E-02	6.2692769222E-02	6.0090892246E-02	5.7182805862E-02	5.3983329005E-02

C2 ARRAY, OCTAL

.0443652 .0441463 .0435711 .0430564 .0422103 .0412107 .0400624 .0366104 .0352161 .0335073 .

```
92 SIDED POLYGON, ERROR LIMITS +0,-FP

                    FP                 HP                 SP                CO
DECIMAL    5.8297760552E-04   9.9980566655E-01   1.7389330017E-04               0
OCTAL              .0002307           .7777150           .0000555               0

C1 ARRAY, DECIMAL
   3.4161025301E-02   1.0280311701E-01   1.7241741743E-01   2.4369333116E-01   3.1738573342E-01
   3.9434883333E-01   4.7557407186E-01   5.6227193272E-01   6.5590001906E-01   7.5832393750E-01
   8.7195478480E-01

C1 ARRAY, OCTAL
 .0213731  .0645052  .1302162  .1746126  .2424006  .3117201  .3633762  .4377042  .5176442  .6042061  .6763416

C2 ARRAY, DECIMAL
   6.8242413363E-02   6.7924235732E-02   6.7289363953E-02   6.6340758101E-02   6.5082841014E-02
   6.3521477676E-02   6.1663947884E-02   5.9518912305E-02   5.7096372078E-02   5.4407622208E-02
   5.1465198889E-02

C2 ARRAY, OCTAL
 .0427413  .0426157  .0423474  .0417567  .0412451  .0404136  .0374447  .0363624  .0351674  .0336665  .0322632
```

```
96 SIDED POLYGON, ERROR LIMITS +E14-0

                        EN                    PN                    SP                    C0
DECIMAL      5.3569930606F-04     1.0003571265E 00     1.5985356781E-04     3.2736610412E-02
OCTAL                .0002143             1.0001355             .0000517              .0206056

C1 ARRAY, DECIMAL
   6.5543462813E-02     1.3165249758E-01     1.9891236738E-01     2.6794919243E-01     3.3945425886E-01
   4.1421356236E-01     4.9314542601E-01     5.7735026918E-01     6.6817863788E-01     7.6732698796E-01
   8.7697646297E-01

C1 ARRAY, OCTAL
 .0414357  .1033177  .1456575  .2111412  .2556317  .3240475  .3743731  .4474647  .5260670  .6106761  .7010061

C2 ARRAY, DECIMAL
   6.5333037636E-02     6.4913088342E-02     6.4215171244E-02     6.3242274924E-02     6.1998565474E-02
   6.0489368647E-02     5.8721147059E-02     5.6701472503E-02     5.4439993535E-02     5.1943398438E-02
   4.9225373738E-02

C2 ARRAY, OCTAL
 .0413465  .0411705  .0407015  .0403025  .0375744  .0367607  .0360413  .0350200  .0336767  .0324605  .0311501
END JOB
ELAPSED JOB TIME 00 HR 02 MIN 07 SEC
```

APPLIED
RESEARCH
LABORATORIES