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Precision Centrifuge Testing of an Accelerometer

Prepared by G. H. Neugebauer
Technology Division

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San Bernardino Operations
THE AEROSPACE CORPORATION

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FOREWORD

This report by The Aerospace Corporation, San Bernardino Operations has been prepared under Contract No. F04701-71-C-0172 as TR-0172(S2970-10)-1

This report, which documents research carried out by the author during the period from October 1970 through September 1971 was submitted on 18 October 1971 to Capt. Donald Depree (RNF) for review and approval.

Approved

R. O. Bock, Director
Guidance and Control Subdivision

Publication of this report does not constitute Air Force approval of the report's findings or conclusions. It is published only for the exchange and stimulation of ideas.

D. C. Depree, Capt. USAF (RNF)
Project Officer

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SECTION I

INTRODUCTION

Accurate knowledge of accelerometer nonlinearities is required in the application of guidance control to today's missiles and space vehicles. Though some attempts have been made to determine these coefficients on a precision linear shake table, the only successful method, thus far, has been by use of a precision centrifuge. However, the use of a precision centrifuge does not guarantee that the nonlinear coefficients will be accurately determined since there are several significant error sources that must be controlled.

A data reduction procedure to minimize the effects of misalignments and radius arm uncertainties has been described by Evans and Fuhrman (Ref. 1) and is widely used by the industry. In this procedure, quadratic equations are separately fitted to accelerometer output data taken in two positions (acceleration vector along plus and minus input axis) by the method of least squares. The constant and linear terms, so obtained, are subtracted from the data and a best fit cubic equation is fitted to the resulting combined residuals.

Though the above procedure is mathematically incorrect, it works fairly well for determining the nonlinear coefficients because these coefficients are relatively small and do not need to be known with great precision.

Evans and Fuhrman have also outlined an iteration procedure to improve the accuracy with which the nonlinear coefficients may be obtained.

In this paper, a simpler, more direct, more accurate, and mathematically correct data reduction procedure is presented. It is a procedure in which all the data (bipolar) are combined at once for the optimum separation of even and odd terms. It does involve a larger matrix of simultaneous equations than the method of Evans and Fuhrman but with modern computers, even desk computers, this is no great problem.

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The data reduction procedure has been applied not only for obtaining the input axis nonlinear coefficients but also for obtaining all the other nonlinear coefficients of the assumed model equation, Eq. (2-1). Methods for establishing the uncertainties of the coefficients and the adequacy of the model equation for particular applications are also presented.

It is shown that the radial acceleration gradient, inherent to a centrifuge, has no effect on a rigid body pendulum but that the rotation may affect the results due to product of inertia torques. In addition, there is a discussion on the severe limitations imposed on the determination of these coefficients by:

(1) misalignment angles

(2) uncertainties in the length of the radius arm

(3) relative magnitudes of the coefficients

(4) angular compliance of the centrifuge arm and of the mounting fixture about all three principal axes.

The above limitations are reasonably independent of the data reduction procedures. Limitations (1) and (2), above, were discussed by Evans and Fuhrman with regard to obtaining bias and scale factor but not with regard to their effects on the determination of the nonlinear coefficients, in general.

The misalignments include not only the misalignments of the true input axis with respect to the accelerometer mounting surface(s) but also the misalignments of fixture mounting surface(s) and the centrifuge mounting surface(s) with respect to the centrifuge and gravity acceleration vectors. In addition, misalignments can be introduced by tightening of mounting screws, by dirt on the mounting surfaces, or by other causes.

Depending on the magnitude of the nonlinear coefficients, it was found that the angular compliance of the centrifuge arm and mounting fixture should not exceed about 1 or 2 arc second per g about any axis.

SECTION II

MODEL EQUATION

For convenience in this presentation, the model equation for a linear, pendulous, nongyroscopic, torque balanced accelerometer is expressed in terms of the applied acceleration components along the true input, pendulous, and output axes rather than along the reference axes as given in Ref. 2. The misalignments δ_p and δ_o (Ref. 2) of the true input axis with respect to the input reference axis are included later in the misalignments of the accelerometer axes with respect to the centrifuge and gravity acceleration vectors.

$$A = \frac{E}{K_{1i}} = K_0 + a_i + K_{2i}a_i^2 + K_{3i}a_i^3 + K_{2p}a_p^2 + K_{3p}a_p^3 + K_{2o}a_o^2 + K_{3o}a_o^3 + K_{ip}a_ia_p + K_{po}a_pa_o + K_{oi}a_oa_i + K_t|a_i|a_i \tag{2-1}$$

where

A	=	acceleration indicated by the accelerometer – g[1]
E	=	accelerometer output-output units
a_i, a_p, a_o	=	applied acceleration[2] components along the true input, pendulous, and output (pivot) or flexure axes, respectively – g.
K_0	=	bias – g
K_{1i}	=	scale factor – output units/g
K_{2i}	=	second order input axis coefficient – g/g^2

---

[1] g is a unit of acceleration. For convenience, it is usually taken to be the local value of gravity though some standard value may be chosen, if desired.

[2] Applied acceleration refers to nongravitational acceleration since an accelerometer cannot sense the acceleration of free fall. The attractive force of gravity acting on an earthbound accelerometer is equivalent in its effect to an upward applied acceleration of one g.

K_{3i} = third order input axis coefficient – g/g^3

K_{2p}, K_{2o} = second order cross-axis coefficients – g/g^2

K_{3p}, K_{3o} = third order cross-axis coefficients – g/g^3

K_{ip}, K_{po}, K_{oi} = cross-coupling coefficients – g/g^2

K_t = torquing power (heating) coefficient – g/g^2

The output units may be in volts, mA, pulses/sec or other convenient units. Linear cross-axis sensitivity terms, $K_p a_p$ and $K_o a_o$ have not been included in the model equation since they cannot be distinguished from misalignments of the input axis about the pendulous and the output axes.

The input reference axis (IRA), the pendulous reference axis (PRA), and the output reference axis (ORA) form a cartesian, orthogonal coordinate system which is generally indicated by case markings and/or mounting surface(s). The true input axis (IA), pendulous axis (PA), and output axis (OA), to which Eq. (2-1) refers, are generally slightly misaligned with respect to the reference axes and can only be determined by test. The output axis (OA) is mechanically determined by the flex or pivot axis and the positive direction is chosen arbitrarily. The positive direction for PA is orthogonal to OA and in the direction from the flex or pivot axis through the centroid of the proof mass when the proof mass is at its null position. The three axes form a right-hand orthogonal, coordinate system with the origin at the centroid of the proof mass and such that

$$\overline{OA} \times \overline{IA} = \overline{PA} \tag{2-2}$$

where $\overline{OA}$, $\overline{IA}$, and $\overline{PA}$ are unit vectors along OA, IA, and PA, respectively.

If the torquing power is kept constant, as it is in some pulse torqued systems, then the coefficient K_t of Eq. (2-1) may be zero.[3] Terms may be added to or deleted from Eq. (2-1) as appropriate for the type of accelerometer and the requirements of the application. The data reduction principles outlined herein may be applied to any reasonable model equation.

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[3] There may be other sources than torquing power for this term.

SECTION III

ACCELERATION GRADIENT EFFECTS

In centrifuge testing of an accelerometer, it is evident that the acceleration field is nonuniform; it varies linearly with the radius from the axis of rotation. The question is frequently asked: What effects, if any, does this linear acceleration gradient have on the accelerometer?

Construct a right-hand, orthogonal coordinate system XYZ that rotates with the centrifuge as shown in Fig. 1. The Z axis coincides with the rotation axis and the X axis passes through the center of gravity, C, of the proof mass (the proof mass includes all pendulous parts and, in general, must be treated as a compound pendulum).

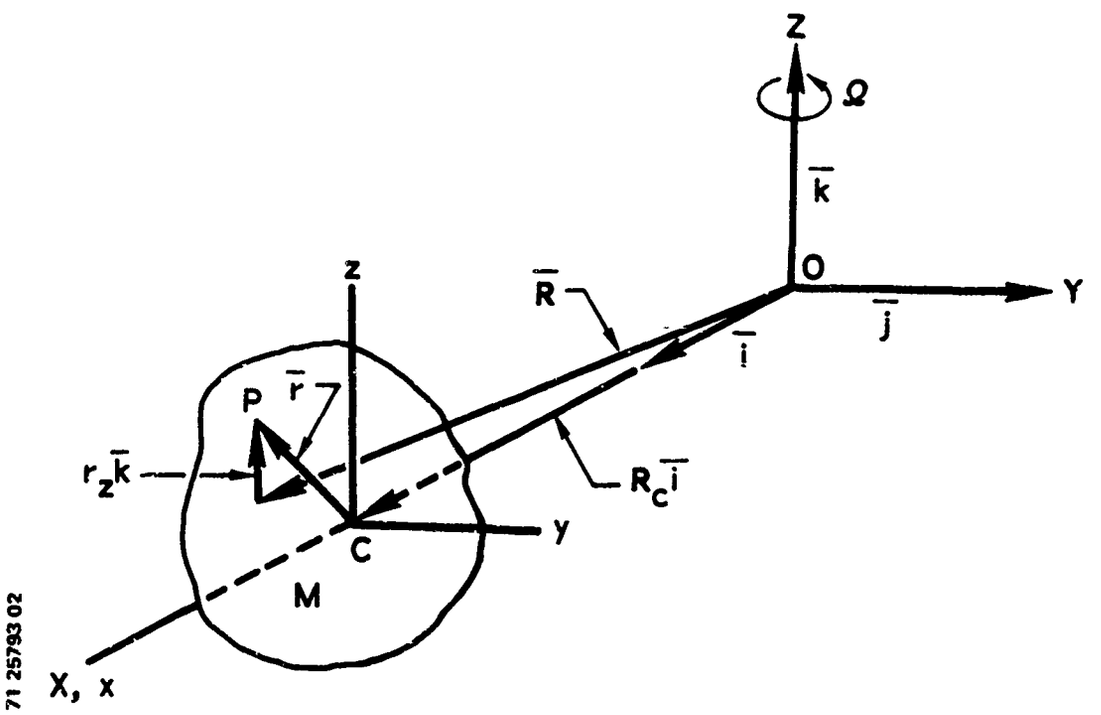

Figure 1. Proof Mass

Construct an xyz coordinate system with the center of gravity, C, of the proof mass as the origin and parallel to the XYZ system as shown in Fig. 1.

Let

$\bar{i}, \bar{j}, \bar{k}$	=	unit vectors along X, Y, Z axes, respectively. Also along x, y, z axes, respectively
$R_c \bar{i}$	=	radius vector from the Z axis to the center of gravity, C, of the proof mass.
$\bar{R}$	=	projection on XY plane of the radius vector from the Z axis to any point P of the proof mass
$\bar{r}$	=	radius vector from point C to point P
r_x, r_y, r_z	=	components of $\bar{r}$ parallel to the x, y, and z axes respectively
Ω	=	steady-state angular velocity of the centrifuge about the Z axis with respect to inertial space
M	=	mass of the proof mass.

Consider a differential mass, dm, at point P. The centripetal force acting on the differential mass is

$$\overline{dF} = -\Omega^2 \bar{R}\, dm \tag{3-1}$$

But, from Fig. 1, it is evident that

$$\bar{R} = R_c \bar{i} + \bar{r} - r_z \bar{k} \tag{3-2}$$

Therefore, if we substitute Eq. (3-2) in Eq. (3-1) and integrate over the entire body of the proof mass, we find

$$\bar{F} = -\Omega^2 \left[\bar{i} R_c \int dm + \int \bar{r}\, dm - \bar{k} \int r_z\, dm \right]$$

But $\int dm = M$, the total mass of the proof mass, and the other two integrals define the center of gravity of the proof mass and are equal to zero. Therefore

$$\bar{F} = -\Omega^2 R_c M \bar{i} \tag{3-3}$$

and the acceleration gradient has no effect on the central force vector acting at the centroid of the proof mass.

Let us now determine if the acceleration gradient causes any moments to act on the proof mass. The moment of the centripetal force acting on the differential mass about the centroid, C, is

$$\overline{dT} = \bar{r} \times \overline{dF} = -\Omega^2 \, dm \, \bar{r} \times \bar{R} \tag{3-4}$$

Substitute Eq. (3-2) in Eq. (3-4) and integrate over the entire body of the proof mass.

$$\bar{T} = -\Omega^2 \left[R_c \int \bar{r} \times \bar{i} \, dm + \int \bar{r} \times \bar{r} \, dm - \int \bar{r} \times r_z \, \bar{k} \, dm \right]$$

or

$$\bar{T} = -\Omega^2 \left[-R_c \, \bar{k} \int r_y \, dm + R_c \, \bar{j} \int r_z \, dm + \bar{j} \int r_z \, r_x \, dm - \bar{i} \int r_y \, r_z \, dm \right] \tag{3-5}$$

The first two integrals in Eq. (3-5) define the centroid and, therefore, are equal to zero. The next two integrals define products of inertia of the proof mass. Therefore

$$\bar{T} = (I_{yz} \, \bar{i} - I_{zx} \, \bar{j}) \, \Omega^2 \tag{3-6}$$

where I_{yz} and I_{zx} are products of inertia of the proof mass. This torque is a dynamic unbalance torque produced by rotation about the Z axis or an axis parallel to the Z axis. If the z axis is a principal axis of the proof mass, then $I_{yz} = I_{zx} = 0$.

The construction of most accelerometers is such that IA, PA, and OA are nominally principal axes of inertia of the proof mass. In all mounting positions considered in this paper, the centrifuge rotation axis is nominally parallel to IA, PA, or OA and no product of inertia effects are included. In general, only product of inertia torque components about OA need be considered and these vary linearly with acceleration.

SECTION IV

ACCELERATION COMPONENTS

In this section, the effects of earth's rate, nonverticality of the centrifuge axis, angular deflections of the centrifuge radius arm, and misalignments of the true accelerometer axes with respect to the accelerometer reference axes are considered. It is assumed that the accelerometer reference axes are defined with respect to the centrifuge and gravity acceleration vectors.

The following notation, in addition to that previously defined, will be used:

a_c	=	centripetal acceleration at centroid of proof mass – $\underline{g}$
a_x, a_y, a_z	=	components of applied acceleration along the x, y, and z axes, respectively – $\underline{g}$
a_u, a_v, a_w	=	components of applied acceleration along the u, v, and w axes, respectively – $\underline{g}$
a_i, a_p, a_o	=	components of applied acceleration along IA, PA, and OA, respectively – $\underline{g}$
G	=	magnitude of $\underline{g}$ – ft/sec^2
k_x, k_y, k_z	=	compliance coefficients of mounting fixture and centrifuge about the x, y, and z axes, respectively – rad/g
T	=	period of revolution with respect to earth-sec
γ_x, γ_y, γ_z	=	angular deflections of mounting fixture and centrifuge about the x, y, and z axes, respectively, as a function of a_c – rad
ϵ_{zv}	=	angle between Z axis and local vertical – rad
θ	=	angular displacement of the centrifuge arm with respect to an earth-fixed reference plane containing the Z axis – rad

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ϕ_i, ϕ_p, ϕ_o = small successive Euler angle displacements about IRA, PRA, and ORA to transform acceleration components along accelerometer reference axes to components along the true axes – rad

λ = astronomic latitude at centrifuge – deg

Ω_r = angular velocity of the centrifuge relative to earth-rad/sec

ω_e = angular velocity of earth relative to inertial space-rad/sec

The earth sidereal rate is quite small ($\omega_e = 72.9211 \times 10^{-6}$ rad/sec) and may be neglected in the calibration of some accelerometers, except on centrifuges with long radius arms and at low centrifuge rates of rotation. The angular velocity of the centrifuge with respect to inertial space is

$$\Omega = \Omega_r + \omega_e \sin \lambda$$

and the centripetal acceleration at the proof mass centroid is

$$a_c = \frac{R_c \Omega^2}{G} = \frac{R_c \left(\Omega_r + \omega_e \sin \lambda\right)^2}{G} \approx \frac{R_c \Omega_r^2}{G} \left(1 + \frac{2\omega_e \sin \lambda}{\Omega_r}\right) \tag{4-1}$$

Thus the vertical component of earth rate at the middle latitudes would result in an error of approximately $10^{-2}/\sqrt{a_c}\%$ for $R_c = 32$ ft and varies approximately as the square root of the radius.

The centripetal acceleration is a function of the angular velocity and the radius arm. The angular velocity, corrected for earth rate, if necessary, may be determined quite precisely by measuring its period of rotation, T, over several revolutions. The radius to the centroid of the proof mass is more difficult to obtain. Some centrifuges, such as the 100-inch centrifuges at Holloman AFB, have a cage rotating about a vertical axis which permits a fairly accurate determination of the radius arm. An error of 0.1 percent in the radius arm will result in an error of approximately 0.2 percent in a quadratic term and about 0.3 percent in a cubic term, which is usually quite acceptable. The centripetal acceleration at the proof mass centroid is

$$a_c \simeq \frac{4\pi^2 R_c}{GT^2}\left(1 + \frac{T\omega_e \sin\lambda}{\pi}\right) \tag{4-2}$$

Due to rotational stresses that vary as the square of the centrifuge rate, the radius arm length may change. It would probably be safe to assume that arm stretch, whether positive or negative, is proportional to Ω_r^2. It is comparatively easy to accurately measure arm stretch, so it will be assumed that this is done and used in the data reduction procedure, if required. In general, measurements of arm stretch do not include changes that are due to bending of the mounting fixture.

The centrifuge axis in a well-installed and adjusted centrifuge should be very close to vertical so that $\epsilon_{zv} \ll 1$. Assuming that this angle is independent of Ω_r, which is not necessarily true, ϵ_{zv} may be determined by observing a low threshold, high resolution, precision bubble level mounted parallel to the X axis. The bubble level reading should vary sinusoidally as the centrifuge is rotated from one position to another, i.e.

$$\phi_b = \phi_o + \epsilon_{zv} \cos(\theta + \beta)$$

where

ϕ_b = angular reading of bubble - rad

ϕ_o = offset angle (bias) of bubble - rad

β = phase angle with respect to reference plane - rad

The magnitude of ϵ_{zv} is equal to one-half the variation in ϕ_b over a complete revolution.

As mentioned previously, the centrifuge axis of rotation may vary with speed but probably not by significant amounts. The acceleration components in the xyz system are

$$\left.\begin{aligned} a_x &\simeq -a_c + \epsilon_{zv}\cos(\theta + \beta) \\ a_y &\simeq \epsilon_{zv}\sin(\theta + \beta) \\ a_z &= \cos\epsilon_{zv} \end{aligned}\right\} \tag{4-3}$$

From Eqs. (4-3), it is seen that the x and y components of acceleration arising from nonverticality of the centrifuge axis average out to zero over an integral number of revolutions. However, it is best to have the axis vertical within one or two arc minutes so that the output reading is not necessarily tied down to averaging over an integral number of revolutions and any rectification effects are negligible. It will now be assumed that the centrifuge axis is adjusted to within one or two arc minutes of the vertical. The average acceleration components along the x, y, and z axes, in g units, are:

$$\left.\begin{aligned} a_x &= -a_c \\ a_y &= 0 \\ a_z &\approx 1 \end{aligned}\right\} \qquad (4\text{-}4)$$

Due to bending of the centrifuge arm and of the mounting fixture induced by dynamic unbalance, aerodynamic effects, or heating effects, the accelerometer mount may deflect angularly as a function of acceleration level. As a first approximation, it will be assumed that angular deflections will be proportional to the acceleration level, i.e.

$$\left.\begin{aligned} \gamma_x &= k_x a_c \\ \gamma_y &= k_y a_c \\ \gamma_z &= k_z a_c \end{aligned}\right\} \qquad (4\text{-}5)$$

In a well constructed, dynamically balanced centrifuge with a rigid mounting fixture, the angular compliance should be less than two arc seconds per g. For example, the compliance of the MIT/CSDL 32-foot arm centrifuge was found to be less than 0.2 arc seconds per g. The compliance of the Holloman 100-inch arm centrifuge about the y (tangential) axis has been found to be less than 0.25 arc seconds per g on some centrifuge runs but as much as 4 arc seconds per g on other runs. The latter variations may be due to variations in the dynamic unbalance of the birdcage for the different setups. It should be noted that the contribution of the mounting fixture to the angular compliance may vary as a function of the accelerometer mounting position.

Construct a right-hand, orthogonal coordinate system uvw derived from the xyz system (see Fig. 2) by the Euler angle rotations γ_x, γ_y, γ_z. The uvw system coincides with the xyz system when $a_c = 0$.

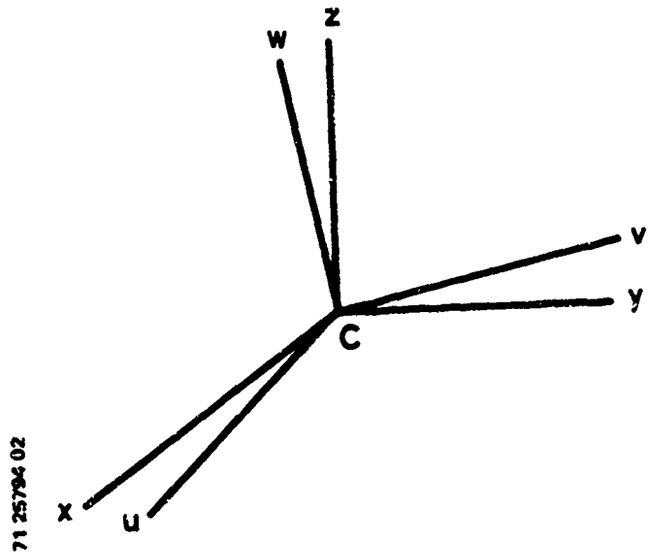

Figure 2. Coordinate Systems

Though angular displacements about orthogonal axes are not commutative, they may be treated as commutative when they are very small, say less than five arc minutes. Using small angle approximations, the acceleration components in the uvw system are:

$$\left.\begin{aligned} a_u &\simeq a_x + a_y\gamma_z - a_z\gamma_y \\ a_v &\simeq a_y + a_z\gamma_x - a_x\gamma_z \\ a_w &\simeq a_z + a_x\gamma_y - a_y\gamma_x \end{aligned}\right\} \tag{4-6}$$

From Eqs. (4-4) thru (4-6)

$$\left.\begin{aligned} a_u &\simeq -(1 + k_y)\, a_c \\ a_v &\simeq k_x a_c + k_z a_c^2 \\ a_w &\simeq 1 - k_y a_c^2 \end{aligned}\right\} \tag{4-7}$$

Next use small successive Euler angle displacements ϕ_i, ϕ_p, ϕ_o to transform acceleration components along the accelerometer reference axes to acceleration components along the true accelerometer axes. The resulting relationships are given in Eqs. (4-8).

$$\left.\begin{aligned}
a_i &= A_i \cos\phi_p \cos\phi_o + A_p (\sin\phi_i \sin\phi_p \cos\phi_o + \sin\phi_o \cos\phi_i) \\
&\quad + A_o (\sin\phi_o \sin\phi_i - \cos\phi_i \sin\phi_p \cos\phi_o) \\
a_p &= A_p (\cos\phi_o \cos\phi_i - \sin\phi_i \sin\phi_p \sin\phi_o) + A_o (\cos\phi_o \sin\phi_i \\
&\quad + \cos\phi_i \sin\phi_p \sin\phi_o) - A_i \cos\phi_p \sin\phi_o \\
a_o &= A_o \cos\phi_i \cos\phi_p + A_i \sin\phi_p - A_p \sin\phi_i \cos\phi_p
\end{aligned}\right\} \quad (4\text{-}8)$$

Let us specify that the magnitudes of the misalignment angles (ϕ_i, ϕ_p, ϕ_o) be limited to 0.01 rad (34.4'), i.e. $|\phi_j| \leq 0.01$ rad, where $j = i, p, o$. Recall that these misalignment angles include misalignments of the mounting fixture and of the centrifuge mounting surfaces with respect to the centripetal and gravity vectors but not misalignments due to angular compliance. We may now use the small angle approximations:

$\sin\phi_j \simeq \phi_j$ with an absolute error less than 3.3×10^{-7}

$\cos\phi_j \simeq 1 - \frac{1}{2}\phi_j^2$ with an absolute error less than 4×10^{-10}

Substitute the above approximations in Eqs. (4-8) and neglect third order errors, i.e. triple products of ϕ_j, $j = i, p, o$ and products of ϕ_j and the compliance coefficients k_x, k_y, k_z.

$$\left.\begin{aligned}
a_i &= A_i (1 - \tfrac{1}{2}\phi_p^2 - \tfrac{1}{2}\phi_o^2) + A_p (\phi_o + \phi_i \phi_p) - A_o (\phi_p - \phi_o \phi_i) \\
a_p &= A_p (1 - \tfrac{1}{2}\phi_o^2 - \tfrac{1}{2}\phi_i^2) + A_o (\phi_i + \phi_p \phi_o) - A_i \phi_o \\
a_o &= A_o (1 - \tfrac{1}{2}\phi_i^2 - \tfrac{1}{2}\phi_p^2) + A_i \phi_p - A_p \phi_i
\end{aligned}\right\} \quad (4\text{-}9)$$

SECTION V

MOUNTING POSITIONS AND LIMITS

In order to determine the coefficients of the model equation, it is necessary to test the accelerometer in a number of different mounting positions and at many acceleration levels. The chosen positions (others are possible) are shown in Figs. 3 and 4. Recall that the centripetal acceleration vector $\bar{a}_c$ is in the minus x direction and the effect of gravity is equivalent to an upward acceleration of one g.

Table I shows the model equation coefficients that will be determined in each position pair of Figs. 3 and 4.

Table I

Position Pair	Coefficient
1 and 2	K_{2i}, K_t, K_{3i}
3 and 4	K_{2p}, K_{3p}
5 and 6	K_{2o}, K_{3o}
7 and 8	K_{ip}
9 and 10	K_{po}
11 and 12	K_{oi}

Based on past experience with this type of torque balanced pendulous accelerometer, we may conservatively assume that the model equation coefficients satisfy the following limits:

$$\left.\begin{array}{l} |K_{2i}|, |K_{2p}|, |K_{2o}|, |K_{ip}|, |K_{oi}|, |K_t| \leq 10^{-4}\ g/g^2 \\ |K_{3i}|, |K_{3p}|, |K_{3o}| \leq 10^{-4}\ g/g^3 \\ |k_x|, |k_y|, |k_z| \leq 10^{-5}\ \text{rad/g (1 arc sec/g)} \end{array}\right\} \qquad (5\text{-}1)$$

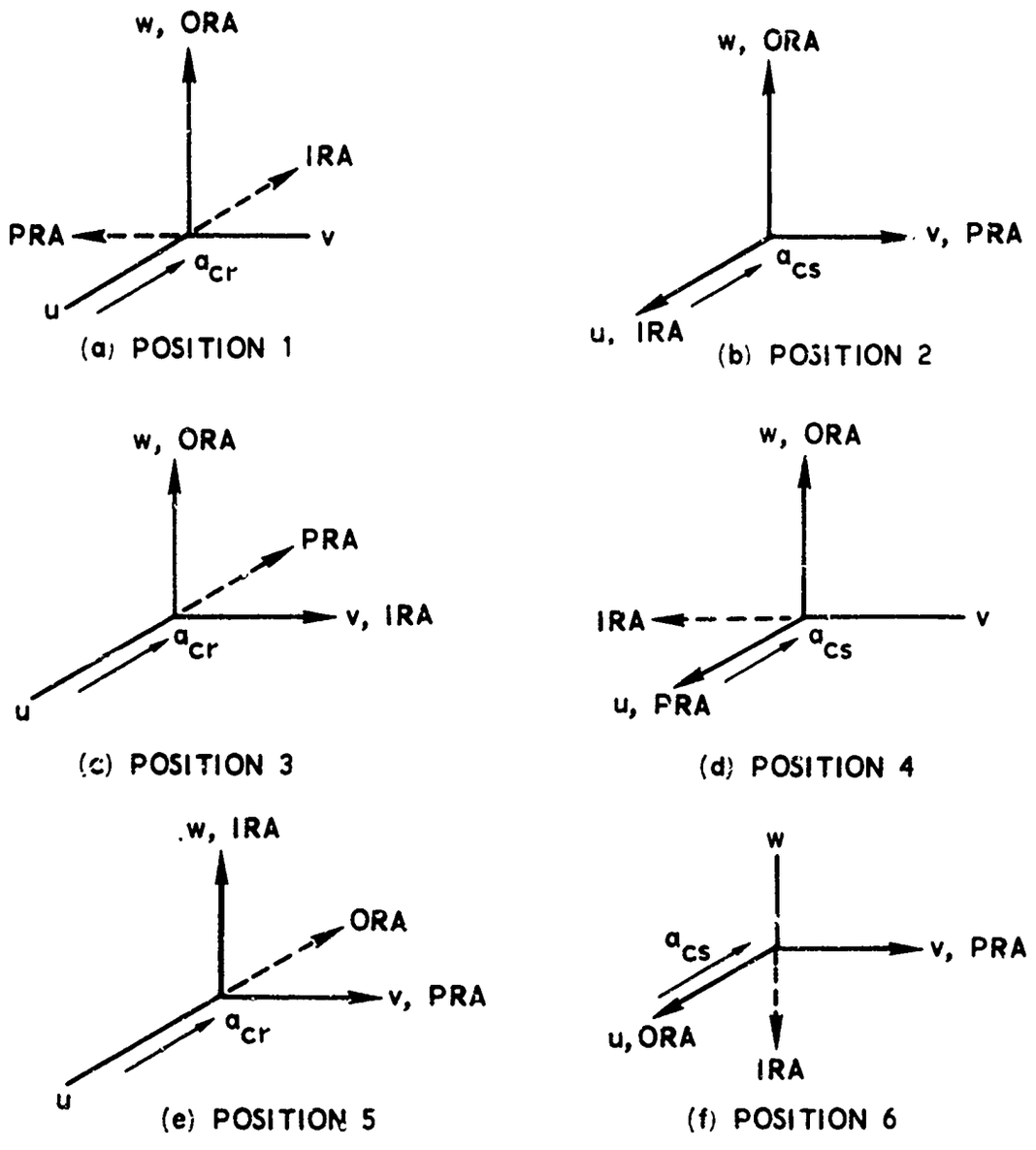

Figure 3. Accelerometer Mounting Positions

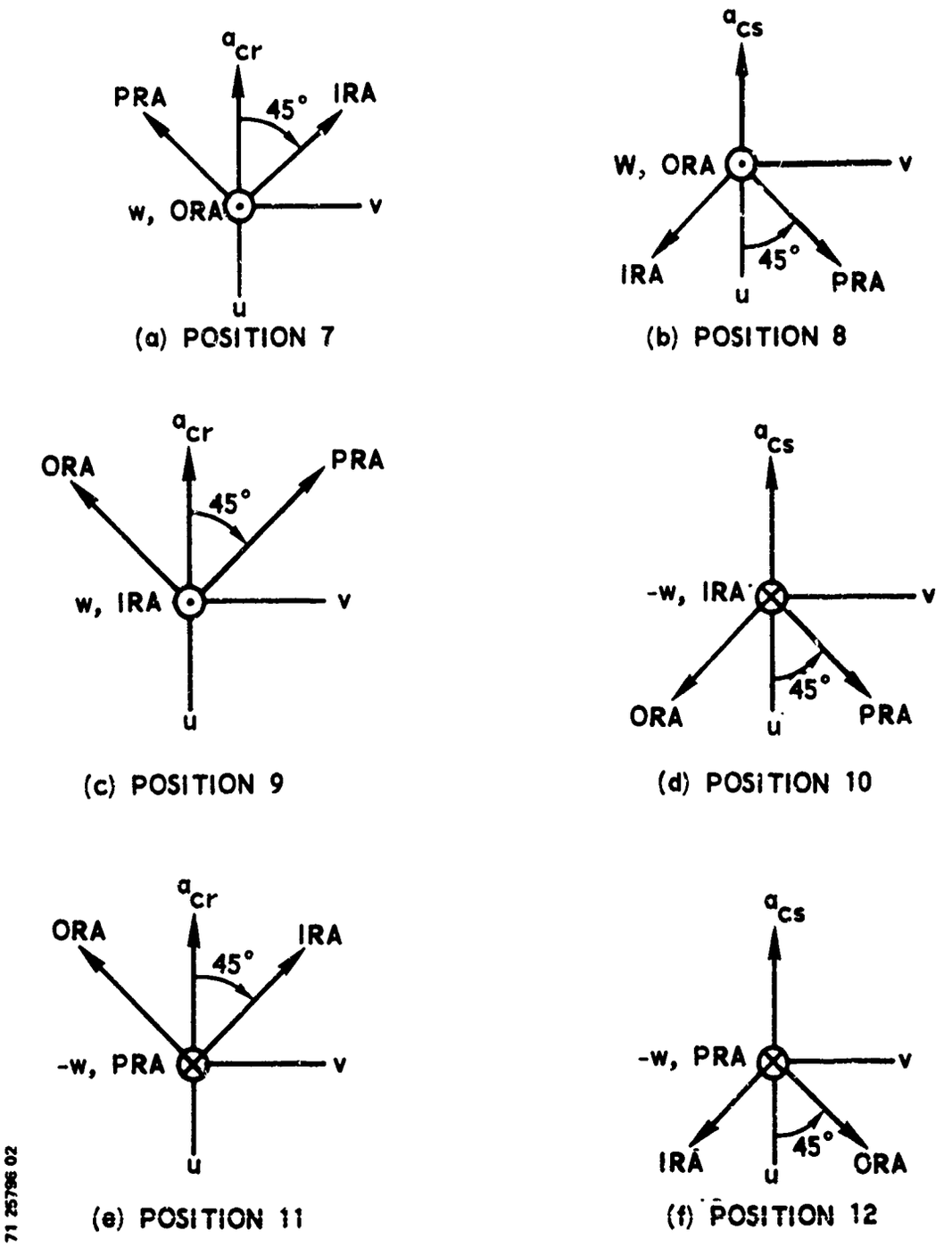

Figure 4. Accelerometer Mounting Positions (Top Views)

Recall that we have already limited the misalignment angles to

$$|\phi_j| \leq 0.01 \text{ rad } (34.4'), \; j = i, p, o.$$

Further, in our development of the centrifuge equations, let us neglect errors less than the following magnitudes:

$$\left.\begin{array}{ll} \text{Bias terms} & < 10^{-6}\,\underline{g} \\ \text{Linear terms} & < 10^{-6}\,\underline{g}/\underline{g} \\ \text{Quadratic terms} & < 10^{-7}\,\underline{g}/\underline{g}^2 \\ \text{Cubic terms} & < 10^{-8}\,\underline{g}/\underline{g}^3 \end{array}\right\} \qquad (5\text{-}2)$$

SECTION VI

INPUT AXIS NONLINEAR COEFFICIENTS K_{2i}, K_t, K_{3i}

6.1 MOUNTING AND DATA TAKING

For maximum accuracy in the separation of the quadratic and cubic terms, data should be taken with the input acceleration along both the positive and the negative input reference axis such as positions 1 and 2 of Fig. 3.

Mount the accelerometer on its fixture in position 1 and check for proper operation. Align the input axis (IA not IRA) to the radial direction on the centrifuge within 0.01 rad (34 $\widehat{\text{min.}}$) about the pendulous and output axes. Align the output axis (OA) to the vertical within 0.01 rad. Determine the radius arm to the centroid of the proof mass by whatever procedure is appropriate for the particular facility.

Measure and record the nominal bias and scale factor by the two-point static test method (IA up and IA down) before and after the group of runs in each position. Preferably, these should be made in the centrifuge environment. The measurements are to be used as a check on the drift or shift in the accelerometer bias and scale factor due to centrifuge action or other environmental changes. Any drift or shift of these parameters will affect the accuracy of the centrifuge tests.

The accelerometer should be tested at a number of acceleration levels which are approximately evenly spaced over the input range in each mounting position. The centrifuge speed should be smoothly and unidirectionally increased from one acceleration level to the next and allowed to stabilize at each level since there may be heating effects at the different torquing levels. When the accelerometer has been tested at its peak acceleration level, decrease the centrifuge speed smoothly and unidirectionally from one acceleration level to the next. Test at approximately the same levels as for increasing speed. In each mounting position the accelerometer should be tested at a minimum of 11 acceleration levels: at 5 acceleration levels each for increasing and decreasing speeds plus the peak acceleration level. More acceleration levels are desirable since redundancy reduces the uncertainties in the coefficients determined by the centrifuge tests.

The output of the accelerometer should be averaged over a number of revolutions sufficient to minimize quantization errors; quantization may be a function of the instrumentation as well as of the accelerometer. The period of the centrifuge relative to earth, the output of the accelerometer, the radius arm, the arm stretch, and the angular compliance (if available) must be determined as precisely as possible and recorded. It is common practice to make three sets of measurements at each acceleration level and to use either the median or the mean of the three sets of measurements in the data reduction.

6.2 OUTPUT EQUATIONS

In position 1 of Fig. 3, the acceleration components along the input, pendulous, and output reference axes, see Eqs. (4-7), are:

$$A_{i1} = -a_u = (1 + k_y)a_{cr}$$

$$A_{p1} = -a_v = -(k_x a_{cr} + k_z a_{cr}^2)$$

$$A_{o1} = a_w = 1 - k_y a_{cr}^2$$

Substitute the above values in Eqs. (4-9). Let a subscript 1 denote the misalignments and outputs in position 1 and a subscript r denote the variable acceleration level for this position. Let n equal the number of acceleration test levels in position 1 so that r = 1, 2, 3, ..., n. From the limits set in Eqs. (5-1) and (5-2), it is evident that we may neglect products of the misalignment angles by the compliance coefficients, e.g. $k_x \phi_{o1}$.

$$\left.\begin{aligned} a_{i1} &= a_{cr}\left(1 - \tfrac{1}{2}\phi_{p1}^2 - \tfrac{1}{2}\phi_{o1}^2 + k_y\right) - (\phi_{p1} - \phi_{o1}\phi_{i1}) \\ a_{p1} &= -a_{cr}^2 k_z - a_{cr}(\phi_{o1} + k_x) + (\phi_{i1} + \phi_{p1}\phi_{o1}) \\ a_{o1} &= -a_{cr}^2 k_y + a_{cr}\phi_{p1} + \left(1 - \tfrac{1}{2}\phi_{i1}^2 - \tfrac{1}{2}\phi_{p1}^2\right) \end{aligned}\right\} \qquad (6\text{-}1)$$

Substitute Eqs. (6-1) in the model equation, Eq. (2-1), and apply the limits of Eqs. (5-1) and (5-2). The acceleration indicated by the accelerometer in position 1 is given by Eq. (6-2).

$$
\begin{aligned}
A_{1r} = {} & \left[K_0 - \phi_{p1} + \phi_{o1}\,\phi_{i1} + K_{2o} + K_{3o} + K_{po}\,\phi_{i1} - K_{oi}\,\phi_{p1}\right] \\
& + a_{cr}\left[1 - \frac{1}{2}\phi_{p1}^2 - \frac{1}{2}\phi_{o1}^2 + k_y - (2K_{2i} + 2K_t - 2K_{2o} - 3K_{3o})\,\phi_{p1}\right. \\
& \left. + K_{ip}\,\phi_{i1} - K_{po}\,\phi_{o1} + K_{oi}\right] + a_{cr}^2\left[K_{2i} + K_t + (K_{oi} - 3K_{3i})\,\phi_{p1}\right. \\
& \left. - K_{ip}\,\phi_{o1}\right] + K_{3i}\,a_{cr}^3
\end{aligned}
\tag{6-2}
$$

In position 2 of Fig. 2 the acceleration components along the accelerometer reference axes are:

$$A_{i2} = a_u = -(1 + k_y)\,a_{cs}$$

$$A_{p2} = a_v = k_x\,a_{cs} + k_z\,a_{cs}^2$$

$$A_{o2} = a_w = 1 - k_y\,a_{cs}^2$$

Substitute the above values in Eqs. (4-9). Let a subscript 2 denote the misalignments and outputs in position 2 and a subscript s denote the variable acceleration for this position. Let m equal the number of acceleration test levels in position 2 so that $s = 1, 2, 3, \ldots, m$. Note that the misalignment angles with the accelerometer mounted in position 2 will not be the same as the misalignment angles with the accelerometer mounted in position 1, in general.

$$
\left.
\begin{aligned}
a_{i2} &= -a_{cs}\left(1 - \frac{1}{2}\phi_{p2}^2 - \frac{1}{2}\phi_{o2}^2 + k_y\right) - (\phi_{p2} - \phi_{o2}\,\phi_{i2}) \\
a_{p2} &= k_z\,a_{cs}^2 + a_{cs}\,(\phi_{o2} + k_x) + (\phi_{i2} + \phi_{p2}\,\phi_{o2}) \\
a_{o2} &= -a_{cs}^2\,k_y - a_{cs}\,\phi_{p2} + \left(1 - \frac{1}{2}\phi_{i2}^2 - \frac{1}{2}\phi_{p2}^2\right)
\end{aligned}
\right\}
\tag{6-3}
$$

Substitute Eqs. (6-3) in Eq. (2-1) and apply the limits of Eqs. (5-1) and (5-2). The indicated acceleration is given by Eq. (6-4).

$$
\begin{aligned}
A_{2s} = {} & \left[K_0 - \phi_{p2} + \phi_{o2}\,\phi_{i2} + K_{2o} + K_{3o} + K_{po}\,\phi_{i2} - K_{oi}\,\phi_{p2}\right] \\
& - a_{cs}\left[1 - \frac{1}{2}\phi_{p2}^2 - \frac{1}{2}\phi_{o2}^2 + K_y - (2K_{2i} + 2K_t - 2K_{2o} - 3K_{3o})\,\phi_{p2}\right. \\
& \left. + K_{ip}\,\phi_{i2} - K_{po}\,\phi_{o2} + K_{oi}\right] + a_{cs}^2\left[K_{2i} - K_t + (K_{oi} - 3K_{3i})\,\phi_{p2}\right. \\
& \left. - K_{ip}\,\phi_{02}\right] - K_{3i}\,a_{cs}^3
\end{aligned}
\tag{6-4}
$$

It is evident that the unknown misalignment angles and the uncertainty in the radius arm, which affects the value of a_{cr} and a_{cs}, makes it impossible to precisely determine either the bias or the scale factor from Eq. (6-2) and/or Eq. (6-4). If $(K_{2i} + K_t)$ and $(K_{2i} - K_t)$ are to be determined from Eqs. (6-2) and (6-4) with an error less than 10 percent (aside from noise in the data), then the magnitudes of the terms $3K_{3i}$, K_{ip}, and K_{oi} may not exceed the magnitude of K_{2i} or K_t, whichever is larger, by more than a factor of two or three. If the magnitude of these coefficients exceed those limits or if the allowable error is less than 10 percent, then the misalignment angles ϕ_j (j = i, o, p) must be reduced proportionately. If the above conditions are met, then Eqs. (6-2) and (6-4) may be simplified to:

$$A_{1r} = B_0 + B_1\,a_{cr} + B_2\,a_{cr}^2 + K_{3i}\,a_{cr}^3 \tag{6-5}$$

$$A_{2s} = C_0 - C_1\,a_{cs} + C_2\,a_{cs}^2 - K_{3i}\,a_{cs}^3 \tag{6-6}$$

where

B_0, C_0 = constant terms of Eqs. (6-2) and (6-4), respectively

B_1, C_1 = linear terms of Eqs. (6-2) and (6-4), respectively

$B_2 = K_{2i} + K_t$

$C_2 = K_{2i} - K_t$

Then

$$K_{2i} = (B_2 + C_2)/2 \tag{6-7}$$

$$K_t = (B_2 - C_2)/2 \tag{6-8}$$

The effect of arm stretch and earth rate must be considered in determining a_{cr} and a_{cs}, if necessary. Remember that both a_{cr} and a_{cs} are the centripetal accelerations and are always positive.

6.3 DETERMINATION OF THE COEFFICIENTS

In the proposed data reduction procedure, the method of least squares is applied to all the data at once rather than in the three stage method of Evans and Fuhrman. In order to accurately determine the coefficients for the quadratic and cubic terms, it is vital to have the cubic coefficient, at least, be common to both positions as in Eqs. (6-5) and (6-6). Let the best fit combined outputs of positions 1 and 2 as obtained from Eqs. (6-5) and (6-6) be

$$\begin{aligned}\bar{A}_{jk} = {} & (B_0 + B_1 a_{cr} + B_2 a_{cr}^2)\delta_r + (C_0 - C_1 a_{cs} + C_2 a_{cs}^2)\delta_s \\ & + K_{3i}(a_{cr}^3 \delta_r - a_{cs}^3 \delta_s)\end{aligned} \tag{6-9}$$

where

$\bar{A}_{jk}$ = best fit value of output for jk = 1r or 2s

A_{jk} = measured value of output for jk = 1r or 2s

$\delta_r = 1$ and $\delta_s = 0$ when k = r.

$\delta_s = 1$ and $\delta_r = 0$ when k = s.

The residual at centripetal acceleration level a_{ck} (k = r or s) is:

$$\begin{aligned}r_k = A_{jk} - \bar{A}_{jk} = {} & (A_{1r}\delta_r + A_{2s}\delta_s) - \Big[(B_0 + B_1 a_{cr} + B_2 a_{cr}^2)\delta_r \\ & + (C_0 - C_1 a_{cs} + C_2 a_{cs}^2)\delta_s + K_{3i}(a_{cr}^3 \delta_r - a_{cs}^3 \delta_s)\Big]\end{aligned} \tag{6-10}$$

and the sum of the squares of all the residuals is

$$\sum_{r,s} r_k^2 = \sum_{r=1}^{n} \sum_{s=1}^{m} \left\{ \left(A_{1r}\,\delta_r + A_{2s}\,\delta_s \right) - \left[\left(B_0 + B_1\,a_{cr} + B_2\,a_{cr}^2 \right)\delta_r + \left(C_0 - C_1\,a_{cs} + C_2\,a_{cs}^2 \right)\delta_s + K_{3i}\left(a_{cr}^3\,\delta_r - a_{cs}^3\,\delta_s \right) \right] \right\}^2 \tag{6-11}$$

In the method of least squares (Ref. 3), the sum of the squares of the residuals is minimized by setting the partial derivatives of Eq. (6-11) with respect to each unknown coefficient (B_0, B_1, B_2, C_0, C_1, C_2, and K_{3i}) equal to zero. The resulting normal equations are given in matrix form by Eq. (6-12). Recall that a_{cr} and a_{cs} are centripetal accelerations and are always considered to be positive.

Solve Eqs. (6-12) and (6-10) for the unknown coefficients and each residual, respectively. The coefficients B_0 and C_0 should closely approximate the bias determined from a two or four position test, within approximately 0.01g. The coefficients B_1 and C_1 should closely approximate unity with an error less than 0.15 percent if the radius arm has been determined within 0.1 percent. The coefficients K_{2i} and K_t of the model equation are determined from Eqs. (6-7) and (6-8). A plot of the residuals vs. g level will be used to estimate the adequacy of the model equation. The residuals will also be used to determine the uncertainty of the coefficients.

If $K_t = 0$, as may occur in pulse torqued systems of constant power input, then $B_2 = C_2 = K_{2i}$ and the best fit equation becomes:

$$\bar{A}_{jk} = (B_0 + B_1\,a_{cr})\delta_r + (C_0 - C_1\,a_{cs})\,\delta_s + K_{2i}(a_{cr}^2\,\delta_r + a_{cs}^2\,\delta_s) + K_{3i}(a_{cr}^3\,\delta_r - a_{cs}^3\,\delta_s) \tag{6-13}$$

$$
\begin{vmatrix}
n & o & \sum_{r=1}^{n} a_{cr} & o & \sum_{r=1}^{n} a_{cr}^{2} & o & \sum_{r=1}^{n} a_{cr}^{3} \\
o & m & o & -\sum_{s=1}^{m} a_{cs} & o & \sum_{s=1}^{m} a_{cs}^{2} & -\sum_{s=1}^{n} a_{cs}^{3} \\
\sum_{r=1}^{n} a_{cr} & o & \sum_{r=1}^{n} a_{cr}^{2} & o & \sum_{r=1}^{n} a_{cr}^{3} & o & \sum_{r=1}^{n} a_{cr}^{4} \\
o & \sum_{s=1}^{m} a_{cs} & o & -\sum_{s=1}^{m} a_{cs}^{2} & o & \sum_{s=1}^{m} a_{cs}^{3} & -\sum_{s=1}^{m} a_{cs}^{4} \\
\sum_{r=1}^{n} a_{cr}^{2} & o & \sum_{r=1}^{n} a_{cr}^{3} & o & \sum_{r=1}^{n} a_{cr}^{4} & o & \sum_{r=1}^{n} a_{cr}^{5} \\
o & \sum_{s=1}^{m} a_{cs}^{2} & o & -\sum_{s=1}^{m} a_{cs}^{3} & o & \sum_{s=1}^{m} a_{cs}^{4} & -\sum_{s=1}^{m} a_{cs}^{5} \\
\sum_{r=1}^{n} a_{cr}^{3} & -\sum_{s=1}^{m} a_{cs}^{3} & \sum_{r=1}^{n} a_{cr}^{4} & \sum_{s=1}^{m} a_{cs}^{4} & \sum_{r=1}^{n} a_{cr}^{5} & -\sum_{s=1}^{m} a_{cs}^{5} & \left(\sum_{r=1}^{n} a_{cr}^{6} - \sum_{s=1}^{m} a_{cs}^{6}\right)
\end{vmatrix}
\times
\begin{vmatrix}
B_0 \\ C_0 \\ B_1 \\ C_1 \\ B_2 \\ C_2 \\ K_{3i}
\end{vmatrix}
=
\begin{vmatrix}
\sum_{r=1}^{n} A_{1r} \\
\sum_{s=1}^{m} A_{2s} \\
\sum_{r=1}^{n} A_{1r}\, a_{cr} \\
\sum_{s=1}^{m} A_{2s}\, a_{cs} \\
\sum_{r=1}^{n} A_{1r}\, a_{cr}^{2} \\
\sum_{s=1}^{m} A_{2s}\, a_{cs}^{2} \\
\sum_{r=1}^{n} A_{1r}\, a_{cr}^{3} - \sum_{s=1}^{m} A_{2s}\, a_{cs}^{3}
\end{vmatrix}
\tag{6-12}
$$

$$
\begin{vmatrix}
n & o & \sum_{r=1}^{n} a_{cr} & o & \sum_{r=1}^{n} a_{cr}^{2} & \sum_{r=1}^{n} a_{cr}^{3} \\
o & m & o & -\sum_{s=1}^{m} a_{cs} & \sum_{s=1}^{m} a_{cs}^{2} & -\sum_{s=1}^{m} a_{cs}^{3} \\
\sum_{r=1}^{n} a_{cr} & o & \sum_{r=1}^{n} a_{cr}^{2} & o & \sum_{r=1}^{n} a_{cr}^{3} & \sum_{r=1}^{n} a_{cr}^{4} \\
o & \sum_{s=1}^{m} a_{cs} & o & -\sum_{s=1}^{m} a_{cs}^{2} & \sum_{s=1}^{m} a_{cs}^{3} & -\sum_{s=1}^{m} a_{cs}^{4} \\
\sum_{r=1}^{n} a_{cr}^{2} & \sum_{s=1}^{m} a_{cs}^{2} & \sum_{r=1}^{n} a_{cr}^{3} & -\sum_{s=1}^{m} a_{cs}^{3} & \left(\sum_{r=1}^{n} a_{cr}^{4} + \sum_{s=1}^{m} a_{cs}^{4}\right) & \left(\sum_{r=1}^{n} a_{cr}^{5} - \sum_{s=1}^{m} a_{cs}^{5}\right) \\
\sum_{r=1}^{n} a_{cr}^{3} & -\sum_{s=1}^{m} a_{cs}^{3} & \sum_{r=1}^{n} a_{cr}^{4} & \sum_{s=1}^{m} a_{cs}^{4} & \left(\sum_{r=1}^{n} a_{cr}^{5} - \sum_{s=1}^{m} a_{cs}^{5}\right) & \left(\sum_{r=1}^{n} a_{cr}^{6} + \sum_{s=1}^{m} a_{cs}^{6}\right)
\end{vmatrix}
\times
\begin{vmatrix} B_0 \\ C_0 \\ B_1 \\ C_1 \\ K_{2i} \\ K_{3i} \end{vmatrix}
=
\begin{vmatrix}
\sum_{r=1}^{n} A_{1r} \\
\sum_{s=1}^{m} A_{2s} \\
\sum_{r=1}^{n} A_{1r}\, a_{cr} \\
\sum_{s=1}^{m} A_{2s}\, a_{cs} \\
\sum_{r=1}^{n} A_{1r}\, a_{cr}^{2} + \sum_{s=1}^{m} A_{2s}\, a_{cs}^{2} \\
\sum_{r=1}^{n} A_{1r}\, a_{cr}^{3} - \sum A_{2s}\, a_{cs}^{3}
\end{vmatrix}
\qquad (6\text{-}15)
$$

The residuals are:

$$r_k = A_{jk} - \overline{A}_{jk} = (A_{1r}\,\delta_r + A_{2s}\,\delta_s) - \Big[(B_0 + B_1\,a_{cr})\,\delta_r + (C_0 - C_1\,a_{cs})\,\delta_s + K_{2i}(a_{cr}^2\,\delta_r + a_{cs}^2\,\delta_s) + K_{3i}(a_{cr}^3\,\delta_r - a_{cs}^3\,\delta_s)\Big] \tag{6-14}$$

The normal equations are obtained, as before, by taking partial derivatives of the sum of the squares of the residuals with respect to each unknown coefficient (B_0, B_1, C_0, C_1, K_{2i}, and K_{3i}) and setting each derivative equal to zero. The normal equations are given in matrix form by Eq. (6-15).

Solve Eqs. (6-15) and (6-14) for the unknown coefficients and each residual, respectively.

6.4 STANDARD DEVIATION OF THE RESIDUALS

The standard deviation of the residuals is a measure of the fit of the mode' equation to the measured accelerometer output. It is not necessarily a measure of the accuracy or completeness of the model equation in describing the phenomena.

The unbiased estimate of the standard deviation of the residuals is given by the formula of Eq. (6-16) (Ref. 3).

$$\hat{\sigma}(r_k) = \sqrt{\frac{\sum r_k^2}{(n+m) - q}} \tag{6-16}$$

where

$\hat{\sigma}(r_k)$ = unbiased estimate of the standard deviation of the residuals r_k ($k = r, s$) - g

$\sum r_k^2$ = sum of the squares of all the residuals - g^2

$n + m$ = total number of data points for k equal r and s

q = total number of coefficients B_0, C_0, etc. E.g., $q = 7$ in Eq. (6-12)

6.5 UNCERTAINTY OF THE COEFFICIENTS

The unbiased estimate of the uncertainty of the coefficients (Ref. 3) obtained from Eq. (6-12) or (6-15) are

$$\hat{\sigma}(Q) = \sigma(r_k)\sqrt{\frac{M(Q)}{D}} \tag{6-17}$$

where

$\hat{\sigma}(Q)$ = uncertainty of coefficient Q

Q = one of the coefficients in the second (column) matrix of Eq. (6-12) or (6-15), such as B_2

D = determinant of the left-hand matrix in Eq. (6-12) or (6-15)

M (Q) = minor of the determinant D for the coefficient Q. E.g., $M(B_2)$ is obtained by deleting column and row indicated by asterisks in Eq. (6-12)

It is quite possible that a coefficient of Eq. (6-12), or of a similar equation, should actually be zero but, because of the small number of observations, it is practically certain that we would obtain a nonzero value. We may determine if a coefficient Q is likely to be zero by assuming the null hypothesis, i.e., Q = 0, and applying a statistical significance test, such as the Student's t statistic to the <u>significance ratio</u> $|Q| / \hat{\sigma}(Q)$. The Student's t statistic varies with the number of degrees of freedom (f = n+m-q) and the chosen rejection level for the null hypothesis. The rejection level depends on the application and is a matter of judgment; however, a 5 percent rejection level is commonly used. By a 5 percent rejection level, we mean that the null hypothesis will be rejected if there is only a 5 percent chance that Q = 0 if $|Q| / \hat{\sigma}(Q)$ of our sample exceeds a certain value. This is equivalent to saying that we can be 95 percent confident that the coefficient is significant if the significance ratio exceeds the Student's t statistic.

At the 5 percent rejection level (2 sided), the t statistic varies from 3.182 for f = 3 to 1.96 for f = ∞. Since we should have a minimum of 22 data points ($n+m \geq 22$) and since the maximum number of coefficients in Eq. (6-12) is 7 (q = 7),

the minimum number of degrees of freedom is $f = n+m-q = 15$. At the 5 percent rejection level and $f = 15$, the Student's t statistic is 2.13. With our assumptions, the t statistic can vary only from 2.13 for $f = 15$ to 1.96 for $f = \infty$. Therefore, we may apply the simple rule that the null hypothesis will be rejected if $|Q| / \hat{\sigma}(Q) \geq 2$. For a more extensive discussion on the null hypothesis and the various significance tests, see Ref. 4, or equivalent.

If a coefficient is set equal to zero as a result of the t test, Eqs. (6-2) and (6-4) thru (6-15) should be modified appropriately.

6.6 "GOODNESS" OF THE MODEL EQUATION

6.6.1 Introduction

There are no general rules for defining the "goodness" of a model equation though we may apply one or more criteria such as:

(1) Is it derived from basic physical principles?

(2) Is it simple?

(3) How well does it fit the observational data?

(a) What is the value of the standard deviation of the residuals?

(b) What is the magnitude of the peak residual?

(4) Would a different model equation fit the observational data better?

It is apparent that the "goodness" of a model equation is a subjective question which can be settled only by each person for each application. There are some statistical criteria that we may apply, but they must be applied with judgment.

Thus, in Section 6.5, we showed how to apply the null hypothesis and the Student's t significance test to the coefficients of our best-fit equation, but at an arbitrary 5 percent rejection level, which was a matter of judgment. In examples 1 and 2, below, we will illustrate some of the decisions or judgments that may be required in adopting a model equation.

It is obvious that observational data must be valid, i.e. without systematic error and with a minimum of noise, if we are to obtain a "good" model equation. Care must be taken in the choice of instrumentation, the test setup (including temperature control), the mounting fixture(s), the data-taking procedures, and data processing (such as smoothing) in order to avoid systematic errors and excessive noise.

Use of one model equation for all accelerometers in all applications is to be deplored. A model equation which is satisfactory for a particular model of accelerometer, in a particular application, may be quite unsatisfactory for a different accelerometer and/or a different application. Equally deplorable is a data reduction program which grinds out coefficients, even programs with statistical criteria, without the intervention of human interpretation and judgment.

6.6.2 Example 1

The effects of an incorrect model equation will now be illustrated with an artificial example. For acceleration inputs only along plus and minus IA, let the true and exact output equation be:

$$\bar{A} = 2\times10^{-3} + a_i + 10^{-5} a_i^2 + 10^{-6} a_i^3 + 10^{-7} a_i^4 \qquad (6\text{-}18)$$

The calculated outputs $\bar{A}$ over the input range from +30 g to -30 g in 2 g steps is given in column 2 of Table II with the results rounded off at six decimal places. The tabulated inputs and outputs of columns 1 and 2 will now be treated as observational data to which we will fit a cubic equation by the method of least squares. The resulting cubic equation is:

$$\bar{A}' = -5.83326 \times 10^{-3} + a_i + 9.2\times10^{-5} a_i^2 + 10^{-6} a_i^3 \qquad (6\text{-}19)$$

Note that the linear and cubic coefficients are unchanged, as would be expected, since it is only the symmetric fourth degree term which has been omitted. Also note that the K_2 coefficient is an order of magnitude larger than that of Eq. (6-18).

Table II

1	2	3	4	5
a_i	$\bar{A}$	$\bar{A}'$	$r' = \bar{A}' - \bar{A}$	$r'' = \bar{A}'' - \bar{A}$
30	30.119000	30.10396674	-0.01503326	-0.007200
28	28.093258	28.08824674	-0.00501126	0.002822
26	26.072034	26.07393474	0.00190074	0.009734
24	24.054762	24.06098274	0.00622074	0.014054
22	22.040914	22.04934274	0.00842874	0.016262
20	20.030000	20.03896674	0.00896674	0.016800
18	18.021570	18.02980674	0.00823674	0.016070
16	16.015210	16.02181474	0.00660474	0.014438
14	14.010546	14.01494274	0.00439674	0.012230
12	12.007242	12.00914274	0.00190074	0.009734
10	10.005000	10.00436674	-0.00063326	0.007200
8	8.003562	8.00056674	-0.00299526	0.004838
6	6.002706	5.99769474	-0.00501126	0.002822
4	4.002250	3.99570274	-0.00654726	0.001286
2	2.002050	1.99454274	-0.00750726	0.000326
0	0.002000	-0.00583326	-0.00783326	0.000000
-2	-1.997966	-2.00547326	-0.00750726	0.000326
-4	-3.997878	-4.00442526	-0.00654726	0.001286
-6	-5.997726	-6.00273726	-0.00501126	0.002822
-8	-7.997462	-8.00045726	-0.00299526	0.004838
-10	-9.997000	-9.99763326	-0.00063326	0.007200
-12	-11.996214	-11.99431326	0.00190074	0.009734
-14	-13.994942	-13.99054526	0.00439674	0.012230
-16	-15.992982	-15.98637726	0.00660474	0.014438
-18	-17.990094	-17.98185726	0.00823674	0.016070
-20	-19.986000	-19.97703326	0.00896674	0.016800
-22	-21.980382	-21.97195326	0.00842874	0.016262
-24	-23.972886	-23.96666526	0.00622074	0.014054
-26	-25.963118	-25.96121726	0.00190074	0.009734
-28	-27.950646	-27.95565726	-0.00501126	0.002822
-30	-29.935000	-29.95003326	-0.01503326	-0.007200
		Rms Value =	6.9×10^{-3} $\Sigma r' = 1 \times 10^{-6}$	10.5×10^{-3}

The outputs, $\overline{A}'$, and the corresponding residuals, $\overline{A}' - \overline{A}$, were calculated to 8 decimal places and are tabulated in columns 3 and 4 of Table II. The sum of the residuals is not quite zero due to round-off errors.

Using Eq. (6-16), the unbiased estimate of the standard deviation of the residuals is $\hat{\sigma}(r') = 7.422 \times 10^{-3}$ g. The coefficients and the uncertainties of the coefficients of Eq. (6-19) are listed along with the significance ratio $|Q| / \hat{\sigma}(Q)$ in Table III.

Table III

Q	$\hat{\sigma}(Q)$	$\lvert Q\rvert / \hat{\sigma}(Q)$
$K_0 = -5.83326 \times 10^{-3}$	1.333×10^{-3}	4.37
$K'_1 = 1$	0.187×10^{-3}	5.35
$K_2 = 9.2 \times 10^{-5}$	0.467×10^{-5}	19.7
$K_3 = 1 \times 10^{-6}$	0.298×10^{-6}	3.36

Note that K'_1 is not the scale factor as defined in Eq. (2-1) but is in g/g.

The application of the Student's t statistic for f = 31-4=27 degrees of freedom shows that all four coefficients are significantly different than zero at a 99 percent or higher confidence level. (1 percent rejection level) However, two of the coefficients are far different than the corresponding coefficients of the exact model equation, Eq. (6-18).

Let us not stop here but let us plot the residuals given in the fourth column of Table II, see Fig. 5. Even a casual glance shows that the residuals are not randomly distributed, therefore, the residuals are not primarily due to noise in the observational data but are largely systematic. From the shape of the curve (3 maxima), we should suspect that a fourth degree term should be added to the model equation. If this is done, we know that we would get Eq. (6-18) except for slight computational errors. If there had been four maxima in Fig. 5 instead of three, we would have suspected that the model equation required a fifth degree term.

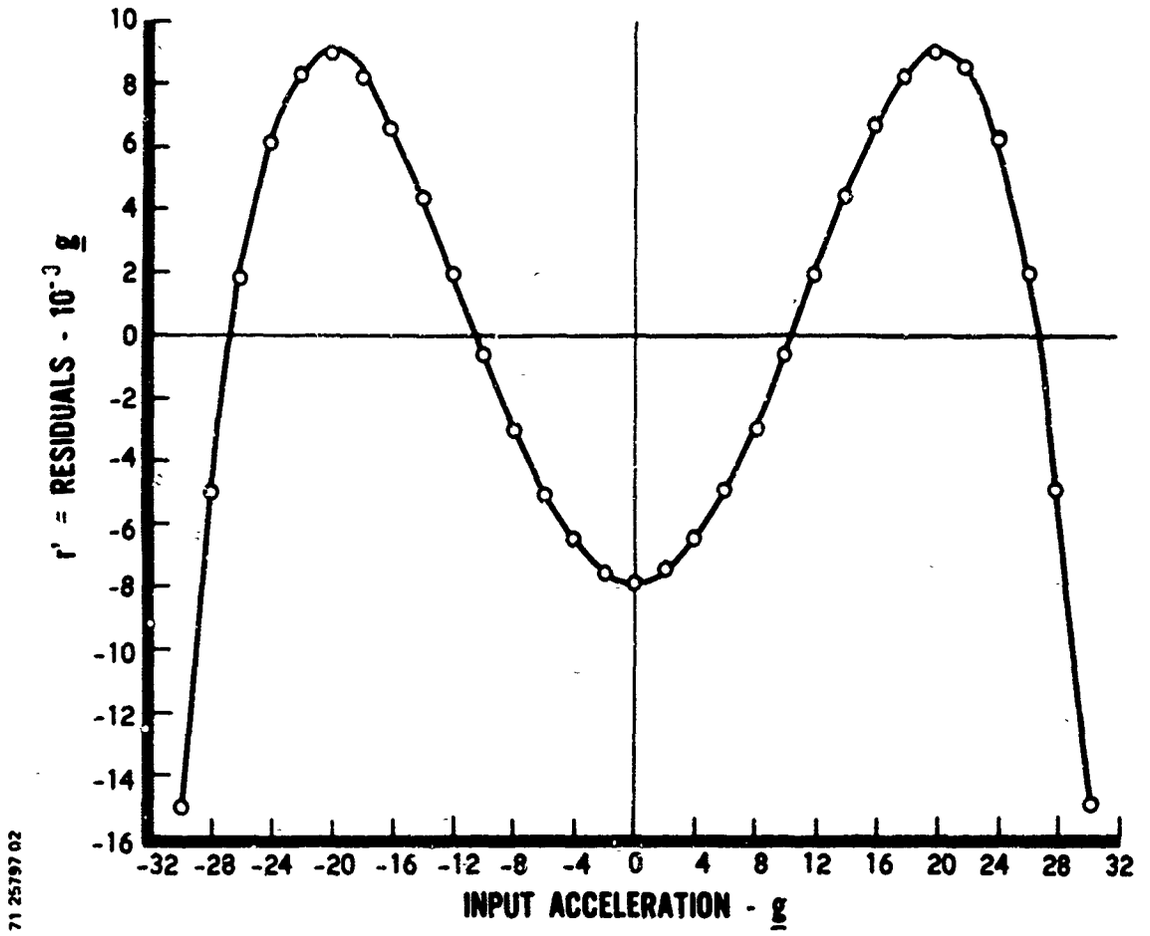

Figure 5. Residuals, Ex. 1

It should be remembered that this illustration is an ideal case with no noise in the data except that due to rounding-off errors. In an actual case, the residual plot would be distorted and roughened by noise.

Let us now consider one more fact that is best illustrated by this artificial example. As indicated previously, it is presumed that the bias and scale factor obtained by reduction of centrifuge data is probably seriously in error due to misalignments and uncertainty in the radius arm to the centroid of the proof mass. The bias and scale factor are generally obtained from a two, four, or six point static calibration test in a one g field. In this example we would find it to be those of Eq. (6-18), i.e. $K_0 = 0.002\,g$ and $K'_1 = 1\ g/g$. The nonlinear coefficients would be those of Eq. (6-19). Thus, it would be presumed that the output equation is:

$$\overline{A}'' = 2 \times 10^{-3} + a_i + 9.2\times10^{-5}\, a_i^2 + 10^{-6}\, a_i^3 \qquad (6\text{-}20)$$

In many applications, the bias is subtracted from the output but the other errors are tolerated; the errors, for this example, would be:

$$r = \overline{A} - (a_i + 0.002) \qquad (6\text{-}21)$$

In other applications, not only the bias but also the errors due to the nonlinear terms are removed by a computer. The errors would be the difference between the predicted output of Eq. (6-20) and the true output of Eq. (6-18), i.e. the errors would be:

$$r'' = \overline{A}'' - \overline{A} \qquad (6\text{-}22)$$

The residuals r'' are given in column 5 of Table II. As would be expected, the rms value of r'' is larger than the rms value of r': in this example, over 50 percent larger.

Example 1 illustrates that it is not sufficient to look at the standard deviation of the residuals nor to compute the statistical significance of the coefficients to determine if we have a "good" model equation; we must also look at a plot of the residuals. If the residuals look as if they are randomly distributed, we probably cannot obtain a better fit model equation, but we might try a polynomial of lower degree to determine if it is satisfactory. If the residuals are not randomly distributed,

we might try a more complex model equation, or we might decide that the model equation is satisfactory for the particular application, i.e. the errors are acceptable. However, we should also recognize that the actual residuals are larger than the calculated residuals as illustrated by the difference between columns 5 and 4 of Table II.

6.6.3 Example 2

In this example, we will use actual experimental data and obtain certain coefficients of the model equation, Eq. (2-1): first, with the assumption $K_t \neq 0$ and second with the assumption that $K_t = 0$. The input and output data of the binary pulse torque accelerometer (constant torquing power) are listed in Tables IV and V. The input data have been partly processed to save time and space, i.e. the input acceleration has been derived from the period of the centrifuge and the radius arm with corrections for arm stretch and earth rate. At the end of the increasing run, the centrifuge was run beyond 45 g before starting the decreasing run.

The data in Tables IV and V are substituted in Eqs. (6-12) and (6-15) to determine the coefficients B_0, C_0, etc. The resulting coefficients, the unbiased estimate of the standard deviation of the residuals from Eq. (6-16), the uncertainties of the coefficients from Eq. (6-17), the number of degrees of freedom, and the significance ratios are given in Table VI. The residuals were determined using Eqs. (6-10) and (6-14) and are plotted in Figs. 6 and 7.

Normally, we would expect the unbiased estimate of the standard deviation of the residuals for Eq. (6-12) to be less than for Eq. (6-15) since Eq. (6-12) has more coefficients and, therefore, it should give a better fit. Actually, the rms of the residuals is less for Eq. (6-12) by approximately 1 percent but this is more than offset by the effect of the reduction in the degrees of freedom from 30 to 29.

The significance ratio for B_2, C_2, and K_{3i} of Eq. (6-12) indicate that these coefficients may not be significantly different from zero. On the other hand, all the coefficients of Eq. (6-15) are significantly different from zero at a 99 percent or higher confidence level. This is not too surprising when we consider that B_2 and C_2 are of opposite sign and of different magnitudes; therefore, the combination has some of the characteristics of both quadratic and cubic terms. Thus, we have three terms

Table IV

POSITION 1 OF FIGURE 3 ($a_i > 0$)

Increasing Data		Decreasing Data	
Input a_{cr}	Output A_{1r}	Input a_{cr}	Output A_{1r}
5.000281	5.034095	45.048050	45.119699
10.020010	10.059407	40.040700	40.103608
15.018229	15.063017	35.022121	35.086046
20.030342	20.080320	30.020392	30.080080
25.015052	25.069831	25.015060	25.070161
30.020392	30.079824	20.030342	20.080584
35.022121	35.085875	15.018229	15.063135
40.040700	40.103535	10.020008	10.059338
45.048050	45.119553	5.000280	5.033769

Table V

POSITION 2 OF FIGURE 3 ($a_i < 0$)

Increasing Data		Decreasing Data	
Input a_{cs}	Output A_{2s}	Input a_{cs}	Output A_{2s}
5.000281	-4.976889	45.048050	-45.055412
10.020010	-10.000542	40.040717	-40.044992
15.018225	-15.002753	35.022108	-35.022991
20.030336	-20.018835	30.020392	-30.018014
25.015060	-25.007460	25.015052	-25.008900
30.020392	-30.016623	20.030336	-20.020096
35.022108	-35.022125	15.018229	-15.003737
40.040717	-40.044235	10.020010	-10.001131
45.048050	-45.054973	5.000281	-4.976946

Table VI

COEFFICIENTS AND THEIR UNCERTAINTIES

Eq. (6-12), $K_t \neq 0$, f = 36-7=29

$\hat{\sigma}$ (r) = 0.373259×10^{-3}

Eq. (6-15), K_t 0, f 36-6 30

$\hat{\sigma}$ (r) = 0.371817×10^{-3}

Coefficient		Uncert.	Signif. Ratio	Coefficient		Uncert.	Signif. Ratio
Q	Value	$\hat{\sigma}$ (Q)	Q / $\hat{\sigma}$ (Q)	Q	Value	$\hat{\sigma}$ (Q)	Q / $\hat{\sigma}$ (Q)
B_0	0.0277396	0.4015×10^{-3}	69.1	B_0	0.0280632	0.3091×10^{-3}	90.8
C_0	0.0277998	0.4015×10^{-3}	69.2	C_0	0.0274761	0.3091×10^{-3}	88.9
B_1	1.00120610	0.6956×10^{-4}	14,390	B_1	1.00114626	0.2531×10^{-4}	39,560
C_1	1.00088233	0.6956×10^{-4}	14,390	C_1	1.00082249	0.2531×10^{-4}	39,540
B_2	-0.4174×10^{-5}	0.3018×10^{-5}	1.38	K_{2i}	-0.1412×10^{-5}	0.4226×10^{-6}	3.34
C_2	0.1349×10^{-5}	0.3018×10^{-5}	0.447	K_{3i}	-0.5864×10^{-7}	0.557×10^{-8}	10.52
K_{3i}	-0.2259×10^{-7}	0.394×10^{-7}	0.573				

performing the function of two terms, at least partially, and all three are weakened by this process. In other words, it is difficult to separate an odd square term (a quadratic term with change of sign at 0) such as the $K_t|a_i|a_i$ term from a cubic term.

It is apparent that Eq. (6-15) is a more appropriate equation to use than Eq. (6-12) for this example since it is simpler, the unbiased estimate of the standard deviation of the residuals is smaller (though not significantly so), and the coefficients are statistically more significant as shown by the significance ratios.

Though we have shown that Eq. (6-15) is better than Eq. (6-12) for this case, we have not demonstrated that it is the "best" equation. The plots of the residuals in Figs. 6 and 7 confirm our previous conclusion that Eqs. (6-12) and (6-15) fit the data about equally well and they also show that the residuals are definitely not random.

However, unlike our Example 1, we cannot reduce the residuals significantly by going to a higher-order polynomial since the residual errors are primarily due to hysteresis for negative input accelerations.

At this time, the mechanism causing the hysteresis is unknown. The hysteresis could be due to a number of causes of which some may be inherent to the accelerometer design or it could be due to the fixturing (including the centrifuge) or to the instrumentation.

If we had blindly used Eq. (6-12) and accepted the derived coefficients, we would have determined values of K_{2i} and K_t by means of Eqs. (6-7) and (6-8). These would be $-1.412\ \mu\ g/g$ and $-2.761\ \mu\ g/g^2$, respectively. However, when we applied the Student's t statistic to the significance ratio, we showed that, statistically, those coefficients were not significantly different than zero. Physically, we should not be surprised if $K_t = 0$ since this is a binary, pulse-torqued accelerometer with constant power input.

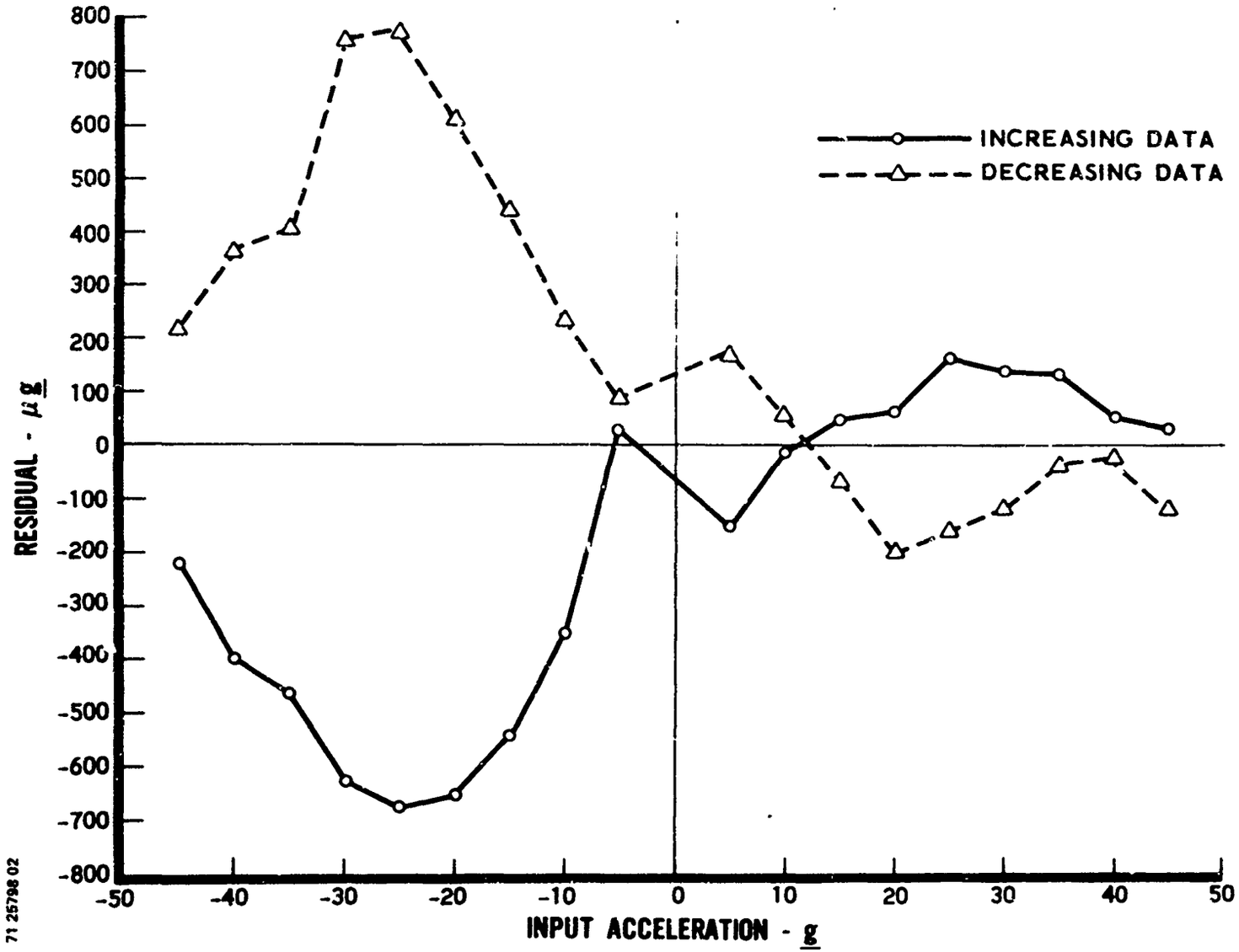

Figure 6. Residuals of Example 2 with $K_t \neq 0$, Eq. (6-12)

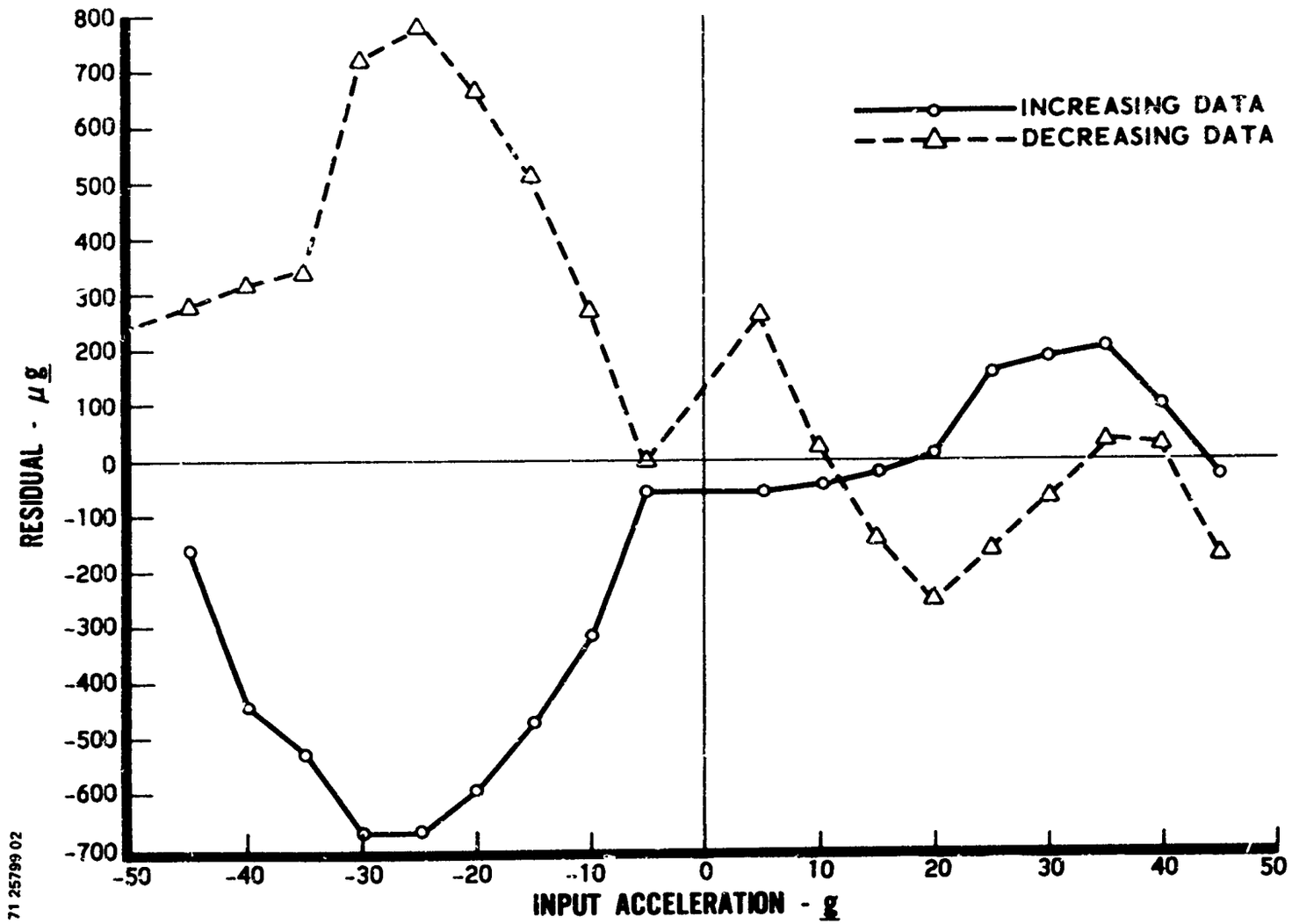

Figure 7. Residuals of Example 2 with $K_t = 0$, Eq. (6-15)

SECTION VII

PENDULOUS AXIS, NONLINEAR COEFFICIENT K_{2p} AND K_{3p}

Positions 3 and 4 of Fig. 3 will be used to obtain the value of K_{2p} and K_{3p}. The mounting and data taking procedures for the accelerometer in these positions would be similar to those outlined in Section 6.1.

In position 3 of Fig. 3, the acceleration components along the accelerometer reference axes are:

$$A_{i3} = a_v = k_x a_{cr} + k_z a_{cr}^2$$

$$A_{p3} = -a_u = (1 + k_y) a_{cr}$$

$$A_{o3} = a_w = 1 - k_y a_{cr}^2$$

Substitute the above values in Eqs. (4-9). Let a subscript 3 denote the outputs in position 3 and a subscript r denote the variable acceleration level for this position. Let n equal the number of acceleration test levels in position 3 so that r = 1, 2, 3, ... n. Neglect second order errors as was done in Section 6.2. From here on, it is to be understood that the misalignments ϕ_j (j = i, p, o) will be different, in general, for each mounting position. For convenience, the subscript numbers on the misalignments will be dropped.

$$\left.\begin{aligned} a_{i3} &= a_{cr}^2 k_z + a_{cr} (\phi_o + \phi_i \phi_p + k_x) - (\phi_p - \phi_o \phi_i) \\ a_{p3} &= a_{cr} (1 - \tfrac{1}{2} \phi_o^2 - \tfrac{1}{2} \phi_i^2 + k_y) + (\phi_i + \phi_p \phi_o) \\ a_{o3} &= -a_{cr}^2 k_y - a_{cr} \phi_i + (1 - \tfrac{1}{2} \phi_i^2 - \tfrac{1}{2} \phi_p^2) \end{aligned}\right\} \qquad (7\text{-}1)$$

Substitute Eqs. (7-1) in the model equation and apply the limits of Eqs. (5-1) and (5-2). Since the bias and linear terms are so uncertain because of misalignments,

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etc., we will not list the individual factors in these terms. The acceleration indicated by the accelerometer may be expressed in the following form:

$$A_{3r} = D_0 + D_1 a_{cr} + \left[K_{2p} + K_{ip}\,\phi_o + (3K_{3p} - K_{po})\phi_i + k_z \right] a_{cr}^2 + K_{3p} a_{cr}^3 \quad (7\text{-}2)$$

In position 4 of Fig. 3, the acceleration components along the accelerometer reference axes are:

$$A_{i4} = -a_v = -(k_x a_{cs} + k_z\, a_{cs}^2)$$

$$A_{p4} = a_u = -(1 + k_y)\, a_{cs}$$

$$A_{o4} = a_w = 1 - k_y\, a_{cs}^2$$

Substitute the above values in Eqs. (4-9). Let the subscript 4 denote the outputs in position 4 and a subscript s denote the variable acceleration level for this position. Let m equal the number of acceleration test levels in position 4 so that s = 1, 2, 3, ..., m. Neglect second order errors. The acceleration components along the true axes are:

$$\left.\begin{aligned} a_{i4} &= -a_{cs}^2\, k_z - a_{cs}(\phi_o + \phi_i\,\phi_p + k_x) - (\phi_p - \phi_o\,\phi_i) \\ a_{p4} &= -a_{cs}(1 - \tfrac{1}{2}\,\phi_o^2 - \tfrac{1}{2}\,\phi_i^2 + k_y) + (\phi_i + \phi_p\,\phi_o) \\ a_{o4} &= -a_{cs}^2\, k_y + a_{cs}\,\phi_i + (1 - \tfrac{1}{2}\,\phi_i^2 - \tfrac{1}{2}\,\phi_p^2) \end{aligned}\right\} \quad (7\text{-}3)$$

Substitute Eqs. (7-3) in Eq. (2-1) and apply the limits of Eqs. (5-1) and (5-2). The indicated acceleration in position 4 is

$$A_{4s} = E_0 - E_1\, a_{cs} + \left[K_{2p} + K_{ip}\,\phi_o + (3K_{3p} - K_{po})\,\phi_i - k_z \right] a_{cs}^2 - K_{3p}\, a_{cs}^3 \quad (7\text{-}4)$$

If K_{2p} is to be obtained from Eqs. (7-2) and (7-4) with an error of no more than 10 percent (aside from noise in the data), then the magnitudes of the products of the accelerometer coefficients by the misalignment angles which appear in those equations

must be smaller than K_{2p}. If those products are sufficiently small, then Eqs. (7-2) and (7-4) may be simplified.

$$A_{3r} = D_0 + D_1 a_{cr} + D_2 a_{cr}^2 + K_{3p} a_{cr}^3 \tag{7-5}$$

$$A_{4s} = E_0 - D_1 a_{cs} + E_2 a_{cs}^2 - K_{3p} a_{cs}^3 \tag{7-6}$$

where

$$D_2 = K_{2p} + k_z$$

$$E_2 = K_{2p} - k_z$$

The coefficients D_0, E_0, D_i, etc., and their uncertainties are determined by the methods outlined in Section 6. If it is assumed that the compliance coefficient k_z is the same for both positions 3 and 4 (depends on construction of mounting fixture) then

$$K_{2p} = \frac{1}{2}(D_2 + E_2) \tag{7-7}$$

The residuals should be determined and plotted to see if there are systematic errors that indicate the model equation is incomplete. The standard deviation of the residuals and the uncertainty of the coefficients are obtained in a manner similar to that outlined in Sections 6.4 and 6.5.

Suppose that in place of positions 3 and 4 of Fig. 3, we were to use the alternate positions of 3' and 4' of Fig. 8 for the determination of K_{2p} and K_{3p}. The output for the latter positions are:

$$A'_{3r} = D'_0 + D_1 a_{cr} + \left[K_{2p} + (K_{ip} + 3K_{3p})\,\phi_o - K_{po}\,\phi_o + k_y \right] a_{cr}^2 + K_{3p} a_{cr}^3 \tag{7-8}$$

$$A'_{4s} = E'_0 - E_1 a_{cs} + \left[K_{2p} + (K_{ip} - 3K_{3p})\,\phi_o - K_{po}\,\phi_i - k_y \right] a_{cs}^2 - K_{3p} a_{cs}^3 \tag{7-9}$$

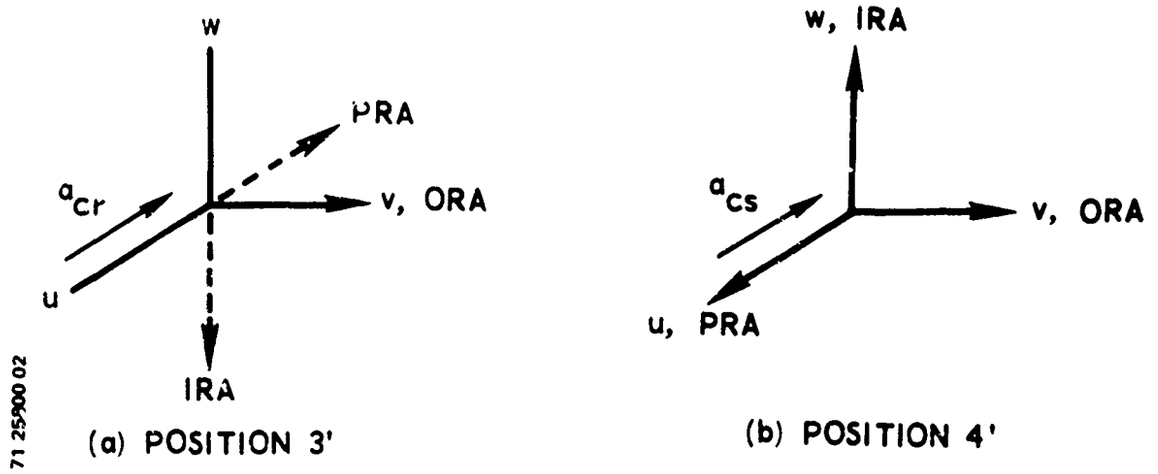

Figure 8. Alternate Mounting Positions

The only significant difference between the above equations and Eqs. (7-2) and (7-4) is that the compliance coefficient k_y has replaced the coefficient k_z in the quadratic term. It would be desirable to use Eqs. (7-8) and (7-9) instead of Eqs. (7-2) and (7-4) if:

(a) k_y is measured or known but k_z is not

(b) if $k_y < k_z$

(c) if k_y is the same for positions 3' and 4' but k_z is not the same for positions 3 and 4.

SECTION VIII

OUTPUT AXIS, NONLINEAR COEFFICIENT K_{2o} AND K_{3o}

Positions 5 and 6 of Fig. 3 will be used to obtain the value of K_{2o} and K_{3o}. The mounting and data taking procedures for the accelerometer in these positions would be similar to thouse outlined in Section 6.1.

In position 5 of Fig. 3, the acceleration components along the accelerometer reference axes are:

$$A_{i5} = a_w = 1 - k_y a_{cr}^2$$

$$a_{p5} = a_v = k_x a_{cr} + k_z a_{cr}^2$$

$$A_{o5} = -a_u = (1 + k_y) a_{cr}$$

Substitute the above values in Eqs. (4-9). Let a subscript 5 denote the outputs in position 5 and a subscript r denote the variable acceleration level for this position. Again neglect second order errors.

$$\left.\begin{aligned} A_{i5} &= -a_{cr}^2 k_y - a_{cr}(\phi_p - \phi_o \phi_i) + (1 - \tfrac{1}{2}\phi_p^2 - \tfrac{1}{2}\phi_o^2) \\ a_{p5} &= a_{cr}^2 k_z + a_{cr}(\phi_i + \phi_p \phi_o + k_x) - \phi_o \\ a_{o5} &= a_{cr}(1 - \tfrac{1}{2}\phi_i^2 - \tfrac{1}{2}\phi_p^2 + k_y) + \phi_p \end{aligned}\right\} \qquad (8\text{-}1)$$

Substitute Eqs. (8-1) in the model equation and apply the limits of Eqs. (5-1) and (5-2).

$$A_{5r} = F_0 + F_1 a_{cr} + \left[K_{2o} + K_{po}\phi_i + (3K_{3o} - K_{oi})\phi_p - k_y\right] a_{cr}^2 + K_{3o} a_{cr}^3 \qquad (8\text{-}2)$$

In order to minimize the uncertainty in K_{2o} introduced by the compliance k_y, position 6 has been chosen which is obtained by rotating about the v axis rather than

the w axis as was done to obtain positions 2 and 4. The acceleration components along the accelerometer reference axes are:

$$A_{i6} \quad -a_w = -(1 - k_y a_{cs}^2)$$

$$A_{p6} = a_v = k_x a_{cs} + k_z a_{cs}^2$$

$$A_{o6} = a_u = -(1 + k_y) a_{cs}$$

In the usual manner, we find the acceleration components along the true axes are:

$$\left.\begin{aligned} a_{i6} &= a_{cs}^2 k_y + a_{cs} (\phi_p - \phi_o \phi_i) - (1 - \tfrac{1}{2} \phi_p^2 - \tfrac{1}{2} \phi_o^2) \\ a_{p6} &= a_{cs}^2 k_z - a_{cs} (\phi_i + \phi_p \phi_o - k_x) + \phi_o \\ a_{o6} &= -a_{cs} (1 - \tfrac{1}{2} \phi_i^2 - \tfrac{1}{2} \phi_p^2 + k_y) - \phi_p \end{aligned}\right\} \tag{8-3}$$

Substitute Eqs. (8-3) in Eq. (2-1) and apply the limits of Eqs. (5-1) and (5-2).

$$A_{6s} = G_0 - G_1 a_{cs} + \left[K_{2o} + K_{po} \phi_i - (3K_{3o} + K_{oi}) \phi_p + k_y \right] a_{cs}^2 - K_{3o} a_{cs}^3 \tag{8-4}$$

If K_{2o} is to be obtained from Eqs. (8-2) and (8-4) with an error of no more than 10 percent (aside from noise in the data), then the magnitudes of the products of the accelerometer coefficients by the misalignment angles which appear in those equations must be much smaller than K_{2o}. If those products are sufficiently small, then Eqs. (8-2) and (8-4) may be simplified.

$$A_{5r} = F_0 + F_1 a_{cr} + F_2 a_{cr}^2 + K_{3o} a_{cr}^3 \tag{8-5}$$

$$A_{6s} = G_0 - G_1 a_{cs} + G_2 a_{cs}^2 - K_{3o} a_{cs}^3 \tag{8-6}$$

where

$$F_2 = K_{2o} - k_y$$

$$G_2 = K_{2o} + k_y$$

Assuming that k_y is the same for positions 3 and 4, then

$$K_{2o} = \frac{1}{2} (F_2 + G_2) \tag{8-7}$$

The values of K_{2o} and K_{3o} are obtained in a manner similar to that used for the determination of K_{2p} and K_{3p} in Section 7.

If there is any advantage to having the compliance coefficient k_z instead of k_y in the quadratic term use alternate positions in which IRA is along the plus v axis when ORA is along the minus u axis and in the second position IRA is along the minus v axis when ORA is along the positive u axis.

SECTION IX

CROSS COUPLING COEFFICIENT K_{ip}

For the assumed model equation, the best mounting positions for determining K_{ip} are those in which the acceleration components along IRA and PRA are both reversed in sign. This results in a common cubic term which is vital for accuracy in separating the quadratic and cubic terms of the output equations. In order to separate the compliance term $\sqrt{2}k_z$ from K_{ip}, it is necessary to choose positions in which one and only one of these terms reverse sign. Positions 7 and 8 of Fig. 4 is one set of positions which satisfy the above criteria providing k_z is the same for the two positions.

The acceleration components along the accelerometer reference axes in these positions are:

$$\left.\begin{aligned}
A_{i7} &= -\frac{1}{\sqrt{2}}(a_u - a_v) = \frac{1}{\sqrt{2}}\left[a_{cr}^2 k_z + a_{cr}(1 + k_x + k_y)\right] \\
A_{p7} &= -\frac{1}{\sqrt{2}}(a_u + a_v) = -\frac{1}{\sqrt{2}}\left[a_{cr}^2 k_z - a_{cr}(1 - k_x + k_y)\right] \\
A_{o7} &= a_w = 1 - k_y a_{cr}^2
\end{aligned}\right\} \qquad (9\text{-}1)$$

$$\left.\begin{aligned}
A_{i8} &= \frac{1}{\sqrt{2}}(a_u - a_v) = -\frac{1}{\sqrt{2}}\left[a_{cs}^2 k_z + a_{cs}(1 + k_x + k_y)\right] \\
A_{p8} &= \frac{1}{\sqrt{2}}(a_u + a_v) = \frac{1}{\sqrt{2}}\left[a_{cs}^2 k_z - a_{cs}(1 - k_x + k_y)\right] \\
a_{o4} &= a_w = 1 - k_y a_{cs}^2
\end{aligned}\right\} \qquad (9\text{-}2)$$

Substitute Eqs. (9-1) and (9-2) in Eqs. (4-9) to obtain the acceleration components along the true axes of the accelerometer, neglect second order errors. These components

are then substituted in the model equation, Eq. (2-1). Using the criteria of Eqs. (5-1) and (5-2), the output equations are:

$$A_{7r} = H_0 + H_1 a_{cr} + \frac{a_{cr}^2}{2} \Big[(K_{2i} + K_t)(1 + 2\phi_o) - 3K_{3i}\phi_p + K_{2p}(1 - 2\phi_o) + 3K_{3p}\phi_i$$

$$- (K_{po} + K_{oi})(\phi_i - \phi_p) + \sqrt{2}\, k_z + K_{ip} \Big] + (K_{3i} + K_{3p}) \frac{a_{cr}^3}{2\sqrt{2}} \tag{9-3}$$

$$A_{8s} = I_0 - I_1 a_{cs} + \frac{a_{cs}^2}{2} \Big[(K_{2i} - K_t)(1 + 2\phi_o) - 3K_{3i}\phi_p + K_{2p}(1 - 2\phi_o)$$

$$+ 3K_{3p}\phi_i - (K_{po} + K_{oi})(\phi_i - \phi_p) - \sqrt{2}\, k_z + K_{ip} \Big] - (K_{3i} + K_{3p}) \frac{a_{cs}^3}{2\sqrt{2}} \tag{9-4}$$

Equations (9-3) and (9-4) show that, due to the magnitude of the allowable misalignment angles, the value of K_{ip} can be determined from the quadratic terms only if the magnitudes of the products of the accelerometer coefficients by the misalignment angles are small compared to the magnitude of K_{ip}. It may be found necessary to tighten the tolerance on the misalignment angles. It should be recalled at this point that the misalignment angles ϕ_j ($j = i, p, o$) are unknown and, in general, are different in each mounting position.

If the above criteria are satisfied, then Eqs. (9-3) and (9-4) may be simplified to:

$$A_{7r} = H_0 + H_1 a_{cr} + H_2 a_{cr}^2 + K_{3ip} a_{cr}^3 \tag{9-5}$$

$$A_{8s} = I_0 - I_1 a_{cs} + I_2 a_{cs}^2 - K_{3ip} a_{cs}^3 \tag{9-6}$$

where

$$H_2 = \frac{1}{2} \Big[K_{2i} + K_t + K_{2p} + \sqrt{2}\, k_z + K_{ip} \Big] \tag{9-7}$$

$$I_2 = \frac{1}{2} \Big[K_{2i} - K_t + K_{2p} - \sqrt{2}\, k_z + K_{ip} \Big] \tag{9-8}$$

$$K_{3ip} = (K_{3i} + K_{3p}) / 2\sqrt{2}$$

The coefficients H_0, I_0, H_1, etc., and their uncertainties are determined by the methods outlined in Section 6. The value of K_{ip} is obtained from Eqs. (9-7) and (9-8) and the previously determined values of K_{2i} and K_{2p}, i.e.

$$K_{ip} = H_2 + I_2 - (K_{2i} + K_{2p}) \tag{9-9}$$

Even though the parameters of the output equations and their uncertainties are such that K_{ip} cannot be determined from these tests, nevertheless, the centrifuge test may be useful for determining if there are buckling or other unexpected effects. As usual, the residuals should be plotted to see if they are randomly or systematically distributed.

SECTION X

CROSS PRODUCT COEFFICIENT K_{po}

The best mounting positions for obtaining the cross product coefficient K_{po} are those in which the acceleration components along PRA and ORA are reversed in sign. This is necessary in order to have a common cubic term for optimum separation of the quadratic and cubic terms. The compliance term in the quadratic coefficient of the output equations are reversed in the two positions relative to the cross coupling term.

Positions 9 and 10 of Fig. 4 satisfy the above criteria. The acceleration components along the reference axes are:

$$\left.\begin{aligned} A_{i9} &= a_w = 1 - k_y a_{cr}^2 \\ A_{p9} &= -\frac{1}{\sqrt{2}}(a_u - a_v) = \frac{1}{\sqrt{2}}\left[a_{cr}^2 k_z + a_{cr}(1 + k_x + k_y)\right] \\ A_{o9} &= -\frac{1}{\sqrt{2}}(a_u + a_v) = -\frac{1}{\sqrt{2}}\left[a_{cr}^2 k_z - a_{cr}(1 - k_x + k_y)\right] \end{aligned}\right\} \qquad (10\text{-}1)$$

and

$$\left.\begin{aligned} A_{i10} &= -a_w = -(1 - k_y a_{cs}^2) \\ A_{p10} &= \frac{1}{\sqrt{2}}(a_u + a_v) = \frac{1}{\sqrt{2}}\left[a_{cs}^2 k_z - a_{cs}(1 - k_x + k_y)\right] \\ A_{o10} &= \frac{1}{\sqrt{2}}(a_u - a_v) = -\frac{1}{\sqrt{2}}\left[a_{cs}^2 k_z + a_{cs}(1 + k_x + k_y)\right] \end{aligned}\right\} \qquad (10\text{-}2)$$

Substitute Eqs. (10-1) and (10-2) in Eqs. (4-9) to obtain the acceleration components along the true axes for each position. These components are then substituted in Eq. (2-1) in accordance with the criteria of Eqs. (5-1) and (5-2). The output equations are:

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$$A_{9r} = J_0 + J_1 a_{cr} + \frac{a_{cr}^2}{2} \Big[K_{2p} (1 + 2\phi_i) - 3K_{3p} \phi_o + K_{2o} (1 - 2\phi_i)$$

$$+ 3K_{3o} \phi_p - (K_{ip} + K_{oi}) (\phi_p - \phi_o) - 2k_y + K_{po} \Big] + (K_{3p} + K_{3o}) \frac{a_{cr}^3}{2\sqrt{2}} \qquad (10\text{-}3)$$

$$A_{10s} = L_0 - L_1 a_{cs} + \frac{a_{cs}^2}{2} \Big[K_{2p} (1 + 2\phi_i) + 3K_{3p} \phi_o + K_{2o} (1 - 2\phi_i) - 3K_{3o} \phi_p$$

$$- (K_{ip} + K_{oi}) (\phi_p - \phi_o) + 2k_y + K_{po} \Big] - (K_{3p} + K_{3o}) \frac{a_{cs}^3}{2\sqrt{2}} \qquad (10\text{-}4)$$

Because of the magnitude of the allowable misalignment angles, the value of K_{po} in the quadratic terms of Eqs. (10-3) and (10-4) can be determined only if the magnitudes of the products of the accelerometer coefficients by the allowable misalignment angles are small compared to the magnitude of K_{po}. As previously mentioned, it may be necessary to tighten the tolerance on the misalignment angles.

If the above criteria are satisfied, then Eqs. (10-3) and (10-4) may be simplified to:

$$A_{9r} = J_0 + J_1 a_{cr} + J_2 a_{cr}^2 + K_{3po} a_{cr}^3 \qquad (10\text{-}5)$$

$$A_{10s} = L_0 - L_1 a_{cs} + L_2 a_{cs}^2 - K_{3po} a_{cs}^3 \qquad (10\text{-}6)$$

where

$$J_2 = \frac{1}{2} \Big[K_{2p} + K_{2o} - 2k_y + K_{po} \Big] \qquad (10\text{-}7)$$

$$L_2 = \frac{1}{2} \Big[K_{2p} + K_{2o} + 2k_y + K_{po} \Big] \qquad (10\text{-}8)$$

$$K_{3po} = (K_{3p} / K_{3o}) / 2\sqrt{2}$$

The coefficients J_0, L_0, J_1, etc., and their uncertainties are determined by the methods outlined in Section 6. The value of K_{po} is obtained from Eqs. (10-7) and (10-8) and the previously determined values of K_{2p} and K_{2o}, i.e.

$$K_{po} = J_2 + L_2 - (K_{2p} + K_{2o}) \qquad (10\text{-}9)$$

SECTION XI

CROSS PRODUCT COEFFICIENT K_{oi}

Using the same reasoning as in Sections 9 and 10, the mounting positions 11 and 12 of Fig. 4 were chosen for determining K_{oi}. Other combinations of positions may also satisfy our criteria. The acceleration components along the reference axes for these positions are:

$$\left.\begin{aligned}
A_{i11} &= -\frac{1}{\sqrt{2}}(a_u - a_v) = \frac{1}{\sqrt{2}}\left[a_{cr}^2 k_z + a_{cr}(1 + k_x + k_y)\right] \\
A_{p11} &= -a_w = -(1 - k_y a_{cr}^2) \\
A_{o11} &= -\frac{1}{\sqrt{2}}(a_u + a_v) = -\frac{1}{\sqrt{2}}\left[a_{cr}^2 k_z - a_{cr}(1 - k_x + k_y)\right]
\end{aligned}\right\} \quad (11\text{-}1)$$

$$\left.\begin{aligned}
A_{i12} &= \frac{1}{\sqrt{2}}(a_u - a_v) = -\frac{1}{\sqrt{2}}\left[a_{cs}^2 k_z + a_{cs}(1 + k_x + k_y)\right] \\
A_{p12} &= -a_w = -(1 - k_y a_{cs}^2) \\
A_{o12} &= \frac{1}{\sqrt{2}}(a_u + a_v) = \frac{1}{\sqrt{2}}\left[a_{cs}^2 k_z - a_{cs}(1 - k_x + k_y)\right]
\end{aligned}\right\} \quad (11\text{-}2)$$

Substitute Eqs. (11-1) and (11-2) in Eqs. (4-9) to obtain the acceleration components along the true axes for each position. These components are then substituted in Eq. (2-1) in accordance with the criteria of Eqs. (5-1) and (5-2). The output equations are:

$$A_{11r} = M_0 + M_1 a_{cr} + \frac{1}{2}\Big[(K_{2i} + K_t)(1 - 2\phi_p) - 3K_{3i}\phi_o + K_{2o}(1 + 2\phi_p)$$
$$+ 3K_{3o}\phi_i - (K_{ip} + K_{po})(\phi_o - \phi_i) + \sqrt{2}\,k_z + K_{oi}\Big] a_{cr}^2 + (K_{3o} + K_{3i})\frac{a_{cr}^3}{2\sqrt{2}} \quad (11\text{-}3)$$

$$A_{12s} = N_0 - N_1 a_{cs} + \frac{1}{2}\Big[(K_{2i} - K_t)(1 - 2\phi_p) - 3K3_i \phi_o + K_{2o}(1 + 2\phi_p) + 3K_{3o}\phi_i$$

$$- (K_{ip} + K_{po})(\phi_o + \phi_i) - \sqrt{2}\, k_z + K_{oi}\Big] a_{cs}^2 - (K_{3o} + K_{3i}) \frac{a_{cs}^3}{2\sqrt{2}} \qquad (11\text{-}4)$$

As in the determination of K_{ip} and K_{po}, the products of the accelerometer coefficients by the allowable misalignment angles in the quadratic terms of Eqs. (11-3) and (11-4) must be small compared to the magnitude K_{oi}. If these products are small, then Eqs. (11-3) and (11-4) may be simplified to:

$$A_{11r} = M_0 + M_1 a_{cr} + M_2 a_{cr}^2 + K_{3oi} a_{cr}^3 \qquad (11\text{-}5)$$

$$A_{12s} = N_0 - N_1 a_{cs} + N_2 a_{cs}^2 - K_{3oi} a_{cs}^3 \qquad (11\text{-}6)$$

where

$$M_2 = \frac{1}{2}\Big[K_{2i} + K_t + K_{2o} + \sqrt{2}\, k_z + K_{oi}\Big] \qquad (11\text{-}7)$$

$$N_2 = \frac{1}{2}\Big[K_{2i} - K_t + K_{2o} - \sqrt{2}\, k_z + K_{oi}\Big] \qquad (11\text{-}8)$$

$$K_{3oi} = (K_{3o} + K_{3i}) / 2\sqrt{2}$$

The coefficients M_0, N_0, M_1, etc., and their uncertainties are determined by the method of least squares as outlined in Section 6. The value of K_{oi} is obtained from Eqs. (11-7), (11-8) and the previously determined values of K_{2i} and K_{2o}, i.e.

$$K_{oi} = M_2 + N_2 - (K_{2i} + K_{2o})$$

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